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Column and row operator spaces over QSL_p-spaces and their use in abstract harmonic analysis

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ABSTRACT

The notions of column and row operator space were extended by A. Lambert from Hilbert spaces to general Banach spaces. In this paper, we use column and row spaces over quotients of subspaces of general L_p-spaces to equip several Banach algebras occurring naturally in abstract harmonic analysis with canonical, yet not obvious operator space structures that turn them into completely bounded Banach algebras. We use these operator space structures to gain new insights on those algebras.

0. Introduction

The Fourier algebra $A(G)$ of a general locally compact group G was introduced by P. Eymard in [7]. If G is abelian with dual group $\hat{G}$, then $A(G)$ is just $L_1(\hat{G})$ via the Fourier transform. As the predual of the group von Neumann algebra, $A(G)$ has a canonical structure as an abstract operator space (see [6,27], or [29] for the theory of operator spaces), turning it into a completely contractive Banach algebra. In the past decade and a half, operator space theoretic methods have given new momentum to the study of $A(G)$ (see [18,26], or [30], for example), yielding new insights, even if the problem in question seemed to have nothing to do with operator spaces ([11] or [12]).

The definition of $A(G)$ can be extended to an L_p-context: instead of restricting oneself to the left regular representation of G on $L_2(G)$, one considers the left regular representation of G on $L_p(G)$ for general $p \in (1,\infty)$. This approach leads to the *Figà-Talamanca–Herz algebras* $A_p(G)$: they were introduced as mere linear spaces by A. Figà-Talamanca [10], were recognized by C. Herz in [15] as Banach algebras, and further studied in [8,16], for instance. Ever since, the Figà-Talamanca–Herz algebras have been objects of independent interest in abstract harmonic analysis. At the first glance, it may seem that the passage from $L_2(G)$ to $L_p(G)$ for $p \neq 2$ is of little significance, and, indeed, many (mostly elementary) properties of $A(G)$ can be established for $A_p(G)$ with $p \neq 2$ along the same lines. However, the lack of von Neumann algebraic methods for operator algebras on L_p-spaces for $p \neq 2$ has left other problems, which have long been solved for $A(G)$, wide open for $A_p(G)$. For instance, any closed subgroup of G is a set of synthesis for $A(G)$ [34] whereas the corresponding statement for $A_p(G)$ with $p \neq 2$ is still wide open.

As the Figà-Talamanca–Herz algebras have no obvious connections with operator algebras on Hilbert space, it appears at first glance that operator space theoretic methods are of very limited use when dealing with $A_p(G)$ for $p \neq 2$. There is a

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notion of p-completely boundedness for general $p \in (1,\infty)$ with 2-complete boundedness just being usual complete boundedness, and an abstract theory based on p-complete boundedness can be developed—called p-operator space theory in [4]—that parallels operator space theory [24]. There are indeed applications of p-complete boundedness to Figà-Talamanca–Herz algebras (see [9] and [4]). Alas, as pointed out in [4], there is no suitable Hahn–Banach theorem for p-completely bounded maps, so that the duality theory of p-operator spaces has to be fairly limited.

In [23], A. Lambert and the authors pursued a different approach to putting operator spaces to work on Figà-Talamanca–Herz algebras. In his doctoral thesis [22], Lambert extended the notions of column and row operator space, which are canonical over Hilbert space, to general Banach spaces. This allows, for $p \in (1,\infty)$, to equip $\mathcal{B}(L_p(G))$ for any $p \in (1,\infty)$ with an operator space structure, which, for $p = 2$, is the canonical one. This, in turn, can be used to equip $A_p(G)$—for any $p \in (1,\infty)$—with an operator space structure in the usual sense, making it a completely bounded Banach algebra. With respect to this operator space structure, [30, Theorem 3.6] extends to Figà-Talamanca–Herz algebras: G is amenable if and only if $A_p(G)$ is operator amenable for one—and, equivalently, all—$p \in (1,\infty)$.

In the present paper, we continue the work begun in [23] and link it with the paper [32] by the second author. Most of it is devoted to extending operator space theoretic results known to hold for the Fourier algebra and (reduced) Fourier–Stieltjes algebra of a locally compact group to the suitable generalizations in a general L_p-context. In particular, we show that, for any $p \in (1,\infty)$, the Banach algebra $B_p(G)$ introduced in [32] can be turned into a completely bounded Banach algebra in a canonical manner, and we obtain an L_p-generalization of [33, Theorem 4.4].

1. Preliminaries

In this section, we recall some background from [23] and [32]. We shall throughout rely heavily on those papers, and the reader is advised to have them at hand.

1.1. Column and row operators spaces over Banach spaces

The notions of column and row operator space over a Hilbert space are standard in operator space theory [6, 3.4]. In [22], Lambert extended these notions to general Banach spaces. As his construction is fairly involved, we will only sketch it very briefly here and refer to [23, Sections 2 and 3] instead (and to [22] for more details). Throughout the paper, we adopt the notation from [23].

Lambert introduces a category—called *operator sequence spaces*—that can be viewed as an intermediary between Banach spaces and operator spaces, and defines functors

$$\min, \max : \{\text{Banach spaces}\} \to \{\text{operator sequence spaces}\}$$

and

$$\mathrm{Min}, \mathrm{Max} : \{\text{operator sequence spaces}\} \to \{\text{operator spaces}\}$$

such that $\mathrm{Min} \circ \min = \mathrm{MIN}$ and $\mathrm{Max} \circ \max = \mathrm{MAX}$. He then defines

$$\mathrm{COL}, \mathrm{ROW} : \{\text{Banach spaces}\} \to \{\text{operator spaces}\}$$

as

$$\mathrm{COL} := \mathrm{Min} \circ \max \quad \text{and} \quad \mathrm{ROW} := \mathrm{Max} \circ \min.$$

For any Banach space E, the operator spaces $\mathrm{COL}(E)$ and $\mathrm{ROW}(E)$ are homogeneous and satisfy

$$\mathrm{COL}(E)^* = \mathrm{ROW}(E^*) \quad \text{and} \quad \mathrm{ROW}(E)^* = \mathrm{COL}(E^*).$$

By [25], these definitions coincide with the usual ones in the case of a Hilbert space.

1.2. Representations on QSL_p-spaces

By a *representation* of a locally compact group G on a Bananach we mean a pair (π, E), where E is a Banach space and π is homomorphism from G into the group of invertible isometries on E which is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{B}(E)$. This is somewhat more restrictive than the usual use of the term, but is in line with the usage of [32]. We shall follow [32] mostly concerning our terminology for representations, and whenever we deviate from [32], we shall indicate it.

In this paper, we are interested in representations on QSL_p-spaces with $p \in (1,\infty)$, i.e., on Banach spaces that are isometrically isomorphic to quotients of subspaces—or, equivalently, subspace of quotients—of the usual L_p-spaces. By [21, Section 4, Theorem 2], these are precisely the p-spaces of [16]. For a locally compact group G and $p \in (1,\infty)$, we denote by $\mathrm{Rep}_p(G)$ the collection of all (equivalence classes of) representations of G on a QSL_p-space.

For the following definition, recall that, for $p \in (1,\infty)$, any QSL_p-space E is reflexive, so that $\mathcal{B}(E)$ is a dual Banach space in a canonical way, so that we can speak of a weak* topology.

Definition 1.1. Let G be a locally compact group, let $p \in (1,\infty)$, and let $(\pi, E) \in \operatorname{Rep}_p(G)$. Then:

(a) the algebra $\mathrm{PF}_{p,\pi}(G)$ of *p-pseudofunctions associated with* (π, E) is the norm closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$;
(b) the algebra $\mathrm{PM}_{p,\pi}(G)$ of *p-pseudomeasures associated with* (π, E) is the weak* closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$.

If $(\pi, E) = (\lambda_p, L_p(G))$, i.e., the left regular representation of G on $L_p(G)$, we simply speak of p-pseudofunctions and p-pseudomeasures, as is standard usage, and write $\mathrm{PF}_p(G)$ and $\mathrm{PM}_p(G)$, respectively.

Let G be a locally compact group. Then $\mathrm{PF}_2(G)$ and $\mathrm{PM}_2(G)$ are the reduced group C^*-algebra $C_r^*(G)$ and the group von Neumann algebra $\mathrm{VN}(G)$, respectively. As C^*-subalgebras of $\mathcal{B}(L^2(G))$, they are operator spaces in a canonical manner. For any Hilbert space $\mathfrak{H}$, the operator spaces $\mathcal{B}(\mathfrak{H})$ and $\mathcal{CB}(\mathrm{COL}(\mathfrak{H}))$ are completely isometrically isomorphic.

We thus define:

Definition 1.2. Let G be a locally compact group, let $p \in (1,\infty)$, and let $(\pi, E) \in \operatorname{Rep}_p(G)$. Then the canonical operator space structure of $\mathrm{PF}_{p,\pi}(G)$ and $\mathrm{PM}_{p,\pi}(G)$, respectively, is the one inherited as a subspace of $\mathcal{CB}(\mathrm{COL}(E))$.

2. $\mathrm{PF}_{p'}(G)$ and $B_p(G)$ as completely bounded Banach algebras

Let G be a locally compact group. Then $A(G) = \mathrm{VN}(G)_*$, $B_r(G) = C_r^*(G)^*$—the *reduced Fourier–Stieltjes algebra* of G, and the *Fourier–Stieltjes algebra* $B(G) = C^*(G)^*$, where $C^*(G)$ denotes the full group C^*-algebra, all have canonical operator space structures turning them into completely contractive Banach algebras.

For $p \in (1,\infty)$, the embedding $A_p(G) \subset \mathrm{PM}_{p'}(G)^*$ turns $A_p(G)$ into a *completely bounded Banach algebra*, i.e., turns it into an operator space such that multiplication is completely bounded, albeit not necessarily completely contractive (see [23] for details).

For any $p \in (1,\infty)$, the space $\mathrm{PF}_{p'}(G)^*$ consists of continuous functions on G and is a Banach algebra under pointwise multiplication (see [17] and [2]). Moreover, in [32], the second author defined a unital, commutative Banach algebra $B_p(G)$ containing $\mathrm{PF}_{p'}(G)^*$ [32, Theorem 6.6(i)], which, for $p = 2$, is just $B(G)$. In this section, we will adapt the construction from [23] to equip both $\mathrm{PF}_{p'}(G)$ and $B_p(G)$ with canonical operator space structures—generalizing those for $B_r(G)$ and $B(G)$ in the $p = 2$ case—such that they become completely bounded Banach algebras.

We begin the following proposition:

Proposition 2.1. *Let $p \in (1,\infty)$, and let E and F be QSL_p-spaces. Then there is a norm $\|\cdot\|_p$ on the algebraic tensor product $E \otimes F$ with the following properties*:

(i) *$\|\cdot\|_p$ is a cross norm dominating the injective tensor norm*;
(ii) *the completion $E \tilde{\otimes}_p F$ of $(E \otimes F, \|\cdot\|_p)$ is a QSL_p-space*;
(iii) *if G is a locally compact group with $(\pi, E), (\rho, F) \in \operatorname{Rep}_p(G)$, then $(\pi \otimes \rho, E \tilde{\otimes}_p F) \in \operatorname{Rep}_p(G)$*;
(iv) *the bilinear maps*

$$\mathrm{COL}(E) \times \mathrm{COL}(F) \to \mathrm{COL}(E \tilde{\otimes}_p F), \quad (\xi, \eta) \mapsto \xi \otimes \eta$$

and

$$\mathrm{ROW}(E) \times \mathrm{ROW}(F) \to \mathrm{ROW}(E \tilde{\otimes}_p F), \quad (\xi, \eta) \mapsto \xi \otimes \eta$$

are completely bounded with cb*-norm at most $K_{\mathbb{G}}$, the* complex Grothendieck constant.

Moreover, if $E = L^p(X)$ for some measure space X, we can choose $\|\cdot\|_p$ as the norm $L^p(X) \otimes F$ inherits as a subspace of $L^p(X, F)$.

Proof. (i), (ii), and (iii) just summarize [32, Theorem 3.1] and the "moreover" part is clear from an inspection of the proof of that theorem.

(iv) follows from [23, Theorems 5.5 and 5.8] and the construction of $\|\cdot\|_p$ in [32]. □

Given a locally compact group G, $p \in (1,\infty)$, and $(\pi, E) \in \operatorname{Rep}_{p'}(G)$, let $\mathrm{PM}_{p,,\pi}(G)_*$ denote the canonical predual of $\mathrm{PM}_{p',\pi}(G)$; we shall consider it with the operator space structure inherited from $\mathrm{PM}_{p,\pi}(G)^*$. It is immediate that $\mathrm{PM}_{p,\pi}(G)_*$ consists of continuous functions on G. We have:

Lemma 2.2. *Let G be a locally compact group, let $p \in (1,\infty)$, let $(\pi, E), (\rho, F) \in \operatorname{Rep}_{p'}(G)$, and let $(\pi \otimes \rho, E \tilde{\otimes}_p F)$ be as in Proposition* 2.1. *Then pointwise multiplication induces a completely bounded map from $\mathrm{PM}_{p',\pi}(G)_* \hat{\otimes} \mathrm{PM}_{p',\rho}(G)_*$ into $\mathrm{PM}_{p',\pi\otimes\rho}(G)_*$ with* cb*-norm at most $K_{\mathbb{G}}^2$.*

Proof. It follows from [32, Corollary 3.2] that pointwise multiplication of two functions in $\mathrm{PM}_{p',\pi}(G)_*$ and $\mathrm{PM}_{p',\rho}(G)_*$, respectively, does indeed yield a function in $\mathrm{PM}_{p',\pi\otimes\rho}(G)_*$.

A diagram chase just as in the proof of [23, Lemma 6.2]—invoking Proposition 2.1(iv)—then shows that the induced bilinear map is indeed completely bounded with norm at most $K_{\mathbb{G}}^2$. □

We can now prove:

Proposition 2.3. *Let G be a locally compact group, let $p \in (1, \infty)$, let $(\pi, E), (\rho, F) \in \mathrm{Rep}_{p'}(G)$, and let $(\pi \otimes \rho, E \tilde{\otimes}_p F)$ be as specified in Proposition* 2.1*. Then pointwise multiplication induces a completely bounded, bilinear map from $\mathrm{PF}_{p',\pi}(G)^* \times \mathrm{PF}_{p',\rho}(G)^*$ into $\mathrm{PF}_{p',\pi\otimes\rho}(G)^*$ with* cb*-norm at most $K_{\mathbb{G}}^2$. Moreover, this multiplication is separately continuous with respect to the weak* topologies involved.*

Proof. Let

$$m : \mathrm{PM}_{p',\pi}(G)_* \times \mathrm{PM}_{p',\rho}(G)_* \to \mathrm{PM}_{p',\pi\otimes\rho}(G)_*$$

denote pointwise multiplication and recall from Lemma 2.2 that $\|m\|_{\mathrm{cb}} \leqslant K_{\mathbb{G}}^2$. As a bilinear map between Banach spaces, m has two Arens extensions

$$m_1^{**} : \mathrm{PM}_{p',\pi}(G)^* \times \mathrm{PM}_{p',\rho}(G)^* \to \mathrm{PM}_{p',\pi\otimes\rho}(G)^* \quad \text{and}$$

$$m_2^{**} : \mathrm{PM}_{p',\pi}(G)^* \times \mathrm{PM}_{p',\rho}(G)^* \to \mathrm{PM}_{p',\pi\otimes\rho}(G)^*.$$

(This construction is mostly done only for the product of a Banach algebra—see [3]—, but works as well for general bilinear maps: see [14].) It is routinely checked that m_1^{**} and m_2^{**} are both completely bounded with $\|m_j^{**}\|_{\mathrm{cb}} \leqslant K_{\mathbb{G}}^2$ for $j = 1, 2$.

For $\sigma \in \{\pi, \rho, \pi \otimes \rho\}$, let $Q_\sigma : \mathrm{PM}_{p',\sigma}(G)^* \to \mathrm{PF}_{p',\sigma}(G)^*$ denote the restriction map, and note that it is a complete quotient map. We claim that

$$Q_{\pi\otimes\rho} \circ m_1^{**} : \mathrm{PM}_{p',\pi}(G)^* \times \mathrm{PM}_{p',\rho}(G)^* \to \mathrm{PF}_{p',\pi\otimes\rho}(G)^*$$

drops to a map

$$\tilde{m} : \mathrm{PF}_{p',\pi}(G)^* \times \mathrm{PF}_{p',\rho}(G)^* \to \mathrm{PF}_{p',\pi\otimes\rho}(G)^*,$$

which is easily seen to be pointwise multiplication and clearly satisfies $\|\tilde{m}\| \leqslant K_{\mathbb{G}}^*$. (We could equally well work with m_2.)

For $\sigma \in \{\pi, \rho, \pi \otimes \rho\}$, let $\iota_\sigma : \mathrm{PM}_{p',\sigma}(G)_* \to L_\infty(G)$ and $\tilde{Q}_\sigma : \mathrm{PM}_{p',\sigma}(G)^* \to L_\infty(G) = L_1(G)^*$ denote the canonical inclusion and restriction maps, respectively. Also, let $Q : L_\infty(G)^{**} \to L_\infty(G)$ be the canonical restriction map, and note that it is an algebra homomorphism.

As the diagram

$$\begin{tikzcd}
\mathrm{PM}_{p',\pi}(G)_* \arrow[d, "\iota_\pi"'] & \times & \mathrm{PM}_{p',\rho}(G)_* \arrow[r, "m"] \arrow[d, "\iota_\rho"] & \mathrm{PM}_{p',\pi\otimes\rho}(G)_* \arrow[d, "\iota_{\pi\otimes\rho}"] \\
L_\infty(G) & \times & L_\infty(G) \arrow[r] & L_\infty(G),
\end{tikzcd}$$

where the bottom row is pointwise multiplication in $L_\infty(G)$ commutes, so does

$$\begin{tikzcd}
\mathrm{PM}_{p',\pi}(G)^* \arrow[d, "Q \circ \iota_\pi^{**}"'] & \times & \mathrm{PM}_{p',\rho}(G)^* \arrow[r, "m_1^{**}"] \arrow[d, "Q \circ \iota_\rho^{**}"] & \mathrm{PM}_{p',\pi\otimes\rho}(G)^* \arrow[d, "Q \circ \iota_{\pi\otimes\rho}^{**}"] \\
L_\infty(G) & \times & L_\infty(G) \arrow[r] & L_\infty(G).
\end{tikzcd} \tag{1}$$

As

$$Q \circ \iota_\sigma^{**} = \tilde{Q}_\sigma \quad \bigl(\sigma \in \{\pi, \rho, \pi \otimes \rho\}\bigr),$$

this entails the commutativity of

$$\begin{tikzcd}
\mathrm{PM}_{p',\pi}(G)^* \arrow[d, "\tilde{Q}_\pi"'] & \times & \mathrm{PM}_{p',\rho}(G)^* \arrow[r, "m_1^{**}"] \arrow[d, "\tilde{Q}_\rho"] & \mathrm{PM}_{p',\pi\otimes\rho}(G)^* \arrow[d, "\tilde{Q}_{\pi\otimes\rho}"] \\
L_\infty(G) & \times & L_\infty(G) \arrow[r] & L_\infty(G),
\end{tikzcd}$$

and thus of

$$\begin{tikzcd}
\mathrm{PM}_{p',\pi}(G)^* \arrow[d, "Q_\pi"'] & \times & \mathrm{PM}_{p',\rho}(G)^* \arrow[r, "m_1^{**}"] \arrow[d, "Q_\rho"] & \mathrm{PM}_{p',\pi\otimes\rho}(G)^* \arrow[d, "Q_{\pi\otimes\rho}"] \\
\mathrm{PF}_{p',\pi}(G)^* & \times & \mathrm{PF}_{p',\rho}(G)^* \arrow[r] & \mathrm{PF}_{p',\pi\otimes\rho}(G)^*
\end{tikzcd}$$

with the bottom row being the desired map $\tilde{m}$.

Finally, since the weak* topology of $\mathrm{PF}_{p',\sigma}(G)^*$ for $\sigma \in \{\pi, \rho, \pi \otimes \rho\}$ coincides with the weak* topology of $L_\infty(G)$ on norm bounded subsets and since multiplication in $L_\infty(G)$ is separately weak* continuous, the commutativity of (1) and the Kreĭn–Šmulian theorem [5, Theorem V.5.7] establish the separate weak* continuity of pointwise multiplication from $\mathrm{PF}_{p',\pi}(G)^* \times \mathrm{PF}_{p',\rho}(G)^*$ to $\mathrm{PF}_{p',\pi\otimes\rho}(G)^*$. □

Following [33], we call a completely bounded Banach algebra a *dual, completely bounded Banach algebra* if it is a dual operator space such that multiplication is separately weak* continuous.

We can finally state and prove the first theorem of this section:

Theorem 2.4. *Let G be a locally compact group G, and let $p \in (1, \infty)$. Then $\mathrm{PF}_{p'}(G)^*$ is a dual, completely bounded Banach algebra with multiplication having the* cb*-norm at most $K_{\mathbb{G}}^2$.*

Proof. By Proposition 2.3, pointwise multiplication

$$\mathrm{PF}_{p'}(G)^* \times \mathrm{PF}_{p'}(G)^* \to \mathrm{PF}_{p',\lambda_{p'}\otimes\lambda_{p'}}(G)^*$$

is completely bounded with cb-norm at most $K_{\mathbb{G}}^2$ and separately weak* continuous.

From [23, Theorem 4.6] and [32, Proposition 5.1], we conclude that $\mathrm{PF}_{p'}(G)$ and $\mathrm{PF}_{p',\lambda_{p'}\otimes\lambda_{p'}}(G)$ are canonically completely isometrically isomorphic. This completes the proof. □

We shall now turn to the task of turning $B_p(G)$—the p-analog of the Fourier–Stieltjes algebra introduced in [32]—into a completely bounded Banach algebra. As in [32], a difficulty arises due to the fact that $\mathrm{Rep}_{p'}(G)$ is not a set, but only a class; we circumvent the problem, by imposing size restrictions on the spaces involved:

Definition 2.5. Let G be a locally compact group, and let $p \in (1, \infty)$. We call $(\pi, E) \in \mathrm{Rep}_{p'}(G)$ *small* if $\mathrm{card}(E) \leqslant \mathrm{card}(L_1(G))^{\aleph_0}$.

Remarks.

1. The left regular representation $(\lambda_p, L_p(G))$ is small, as are the cyclic representations used in [32].
2. Unlike $\mathrm{Rep}_p(G)$, the class of all small representation in $\mathrm{Rep}_p(G)$ is indeed a set.

Let G be a locally compact group, let $p \in (1, \infty)$, and let $(\pi, E), (\rho, F) \in \mathrm{Rep}_p(G)$ be such that $(\rho, F) \subset (\pi, E)$. Then we have a canonical complete contraction from $\mathrm{PF}_{p,\pi}(G)$ to $\mathrm{PF}_{p,\rho}(G)$. Consequently, if $((\rho_\alpha, F_\alpha))_\alpha$ is a family of representations contained in (π, E), we have a canonical complete contraction from $\mathrm{PF}_{p,\pi}(G)$ to $\ell_\infty\text{-}\bigoplus_\alpha \mathrm{PF}_{p,\rho_\alpha}(G)$.

We note:

Proposition 2.6. *Let G be a locally compact group, let $p \in (1, \infty)$, let $(\pi, E) \in \mathrm{Rep}_p(G)$, and let $((\rho_\alpha, F_\alpha))_\alpha$ be the family of all small representations contained in (π, E). Then the canonical map from $\mathrm{PF}_{p,\pi}(G)$ to $\ell_\infty\text{-}\bigoplus_\alpha \mathrm{PF}_{p,\rho_\alpha}(G)$ is a complete isometry.*

Proof. We need to show the following: for each $n, m \in \mathbb{N}$, each $n \times n$ matrix $[f_{j,k}] \in M_n(L_1(G))$, and each $\epsilon > 0$, there is a closed subspace F of G invariant under $\pi(G)$ with $\mathrm{card}(F) \leqslant \mathrm{card}(L_1(G))^{\aleph_0}$ such that

$$\big\|\big[\pi(f_{j,k})^{(m)}|_{M_m(F)}\big]\big\|_{\mathcal{B}(M_m(F),M_{nm}(F))} \geqslant \big\|\big[\pi(f_{j,k})^{(m)}\big]\big\|_{\mathcal{B}(M_m(E),M_{nm}(E))} - \epsilon.$$

Let $n, m \in \mathbb{N}$, $[f_{j,k}] \in M_n(L_1(G))$, and $\epsilon > 0$. Trivially, there is $[\xi_{\nu,\mu}] \in M_m(E)$ with $\|[\xi_{\nu,\mu}]\|_{M_m(E)} \leqslant 1$ such that

$$\big\|\big[\pi(f_{j,k})\xi_{\nu,\mu}\big]\big\|_{M_{nm}(E)} \geqslant \big\|\big[\pi(f_{j,k})^{(m)}\big]\big\|_{\mathcal{B}(M_m(E),M_{nm}(E))} - \epsilon.$$

Let F be the closed linear span of $\{\pi(f)\xi_{\nu,\mu}\colon\ f \in L_1(G),\ \nu, \mu = 1, \ldots, m\}$; it clearly has the desired properties. □

Definition 2.7. Let G be a locally compact group, and let $p \in (1, \infty)$. We say that $(\pi_u, E_u) \in \mathrm{Rep}_p(G)$ *p-universal* if it contains every small representation in $\mathrm{Rep}_p(G)$. We write $\mathrm{UPF}_p(G)$ instead of $\mathrm{PF}_{p,\pi_u}(G)$ and call the elements of $\mathrm{UPF}_p(G)$ *universal p-pseudofunctions*.

Remarks.

1. Since cyclic representations in the sense of [32] are small, a p-universal representation according to Definition 2.7 is also p-universal in the sense of [32, Definition 4.5]. We do not know if the converse is also true.
2. There are indeed p-universal representations: this can be seen as in the example immediately after [32, Definition 4.5].

3. If $(\pi_u, E_u) \in \operatorname{Rep}_p(G)$ is p-universal and $(\rho, F) \in \operatorname{Rep}_p(G)$ is arbitrary, then Proposition 2.6 shows that we have a canonical complete contraction from $\mathrm{UPF}_p(G)$ to $\mathrm{PF}_{p,\rho}(G)$. In particular, the operator space structure of $\mathrm{UPF}_p(G)$ does not depend on a particular p-universal representation.

Let G be a locally compact group, and let $p \in (1, \infty)$. As every p'-universal representation of G is also p'-universal in the sense of [32], [32, Theorem 6.6(ii)] remains valid, and we can identity $B_p(G)$ with the Banach space dual of $\mathrm{UPF}_{p'}(G)$. As $\mathrm{UPF}_{p'}(G)$ is an operator space by virtue of Definition 1.2, we define the canonical operator space structure of $B_p(G)$ as the one it has as the dual space of $\mathrm{UPF}_{p'}(G)$.

Theorem 2.8. *Let G be a locally compact group, and let $p \in (1, \infty)$. Then*:

(i) *$B_p(G)$ is a dual, completely bounded Banach algebra*;
(ii) *the canonical image of $\mathrm{PF}_{p'}(G)^*$ in $B_p(G)$ is an ideal of $B_p(G)$.*

Proof. Let $(\pi_u, E_u) \in \operatorname{Rep}_{p'}(G)$ be p'-universal. By Proposition 2.3, pointwise multiplication

$$B_p(G) \times B_p(G) \to \mathrm{PF}_{p',\pi_u \otimes \pi_u}(G)^* \tag{2}$$

is completely bounded. Since (π_u, E_u) is p'-universal, we have a canonical complete contraction from $\mathrm{UPF}_{p'}(G)$ to $\mathrm{PF}_{p',\pi_u \otimes \pi_u}(G)$. Composing the adjoint of this map with (2), we obtain pointwise multiplication on $B_p(G)$, which is thus completely bounded. That multiplication in $B_p(G)$ is separately weak* continuous is seen as in the proof of Theorem 2.4. This proves (i).

(ii) follows from [32, Proposition 5.1]. □

Remarks.

1. Unless $p = 2$, we it is well possible that the canonical map from $\mathrm{PF}_{p'}(G)^*$ to $B_p(G)$ fails to be an isometry: see the remark immediately after [32, Corollary 5.3].
2. We even have that $\mathrm{PF}_{p'}(G)^*$ is a $B_p(G)$ module with completely bounded module actions. Since the canonical map from $\mathrm{PF}_{p'}(G)^*$ to $B_p(G)$ need not be a (complete) isometry, this is somewhat stronger than Theorem 2.8(ii).

3. Herz–Schur and completely bounded multipliers of $A_p(G)$

Let G be a locally compact group, let $p, q \in (1, \infty)$, and—as in [6] and [23]—let $\otimes^\gamma$ stand for the projective tensor product of Banach spaces. Even though $L^p(G) \otimes^\gamma L^q(G)$ does not consist of functions on $G \times G$, but rather of equivalence classes of functions, it still makes sense to speak of multipliers of $L^p(G) \otimes^\gamma L^q(G)$: by a multiplier of $L^p(G) \otimes^\gamma L^q(G)$, we mean a continuous function f on $G \times G$, so such that the corresponding multiplication operator M_f induces a bounded linear operator on $L^p(G) \otimes^\gamma L^q(G)$.

For $p \in (1, \infty)$, we write $\mathcal{V}_p(G)$ to denote the pointwise multipliers of $L^p(G) \otimes^\gamma L^{p'}(G)$. For any function $f : G \to \mathbb{C}$, we write $K(f)$ for the function

$$G \times G \to \mathbb{C}, \quad (x, y) \mapsto f\big(xy^{-1}\big).$$

We define the *Herz–Schur multipliers* of $A_p(G)$ as

$$\mathcal{M}_{\mathrm{HS}}\big(A_p(G)\big) := \big\{f : G \to \mathbb{C}\colon\ K(f) \in \mathcal{V}_p(G)\big\}.$$

As $\mathcal{V}_p(G)$ is a closed subspace of $\mathcal{B}(L^p(G) \otimes^\gamma L^{p'}(G))$, and since the map $\mathcal{M}_{\mathrm{HS}}(A_p(G)) \ni f \mapsto K(f)$ is injective, we can equip $\mathcal{M}_{\mathrm{HS}}(A_p(G))$ with a natural norm turning it into a Banach space.

In [1], M. Bożejko and G. Fendler showed that $\mathcal{M}_{\mathrm{HS}}(A(G))$ and $\mathcal{M}_{\mathrm{cb}}(A(G))$ are isometrically isomorphic (see also [20]), and in [9], Fendler showed that, for general $p \in (1, \infty)$, the Herz–Schur multipliers of $A_p(G)$ are precisely the p-completely bounded ones.

In this section, we investigate how $\mathcal{M}_{\mathrm{HS}}(A_p(G))$ and $\mathcal{M}_{\mathrm{cb}}(A_p(G))$ relate to each other for general $p \in (1, \infty)$, but with $A_p(G)$ having the operator space structure introduced in [23]. We start with a lemma:

Lemma 3.1. *Let $p \in (1, \infty)$, let X and Y be measure spaces, and let E be a QSL_p-space. Then the map*

$$\begin{aligned} &\big(L^p(X) \otimes E\big) \otimes \big(L^{p'}(Y) \otimes E^*\big) \to L^p(X) \otimes L^{p'}(Y), \\ &(f \otimes \xi) \otimes (g \otimes \phi) \mapsto \langle \xi, \phi \rangle (f \otimes g) \end{aligned}$$

extends to a complete quotient map

$$\operatorname{tr}_E : \mathrm{ROW}\big(L^p(X, E)\big) \hat{\otimes} \mathrm{COL}\big(L^{p'}(Y, E^*)\big) \to \mathrm{ROW}\big(L^p(X)\big) \hat{\otimes} \mathrm{COL}\big(L^{p'}(Y)\big).$$

Proof. Since E^* is a $QSL_{p'}$-space, this follows from [23, Theorem 4.6] through taking adjoints. □

Proposition 3.2. *Let $p \in (1,\infty)$, and let G be a locally compact group. Then, for any $f \in \mathcal{M}_{\mathrm{HS}}(A_p(G))$, the multiplication operator $M_{K(f)} : L^p(G) \otimes^\gamma L^{p'}(G) \to L^p(G) \otimes^\gamma L^{p'}(G)$ is completely bounded on $\mathrm{ROW}(L^p(G)) \hat{\otimes} \mathrm{COL}(L^{p'}(G))$ such that*

$$\|M_{K(f)}\|_{\mathrm{cb}} = \|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))}.$$

Proof. Let $f \in \mathcal{M}_{\mathrm{HS}}(A_p(G))$, and let $\epsilon > 0$. Then, by [13] (see also [9, Theorem 4.4]), there is a QSL_p-space E along with bounded continuous functions $L : G \to E$ and $R : G \to E^*$ such that

$$K(f)(x,y) = \langle L(x), R(y) \rangle \quad (x, y \in G)$$

and

$$\|L\|_\infty \|R\|_\infty < \|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))} + \epsilon, \tag{3}$$

where

$$\|L\|_\infty := \sup_{x \in G} \|L(x)\| \quad \text{and} \quad \|R\|_\infty := \sup_{x \in G} \|R(x)\|.$$

Define $\tilde{L} : L^p(G) \to L^p(G, E)$ by letting

$$(\tilde{L}\xi)(x) := \xi(x)L(x) \quad \bigl(\xi \in L^p(G),\ x \in G\bigr).$$

Then $\tilde{L}$ is linear and bounded with $\|\tilde{L}\| = \|L\|_\infty$. Similarly, one defines a bounded linear map $\tilde{R} : L^{p'}(G) \to L^{p'}(G, E^*)$ with $\|\tilde{R}\| = \|R\|_\infty$ by letting

$$(\tilde{R}\eta)(x) := \eta(x)R(x) \quad \bigl(\eta \in L^{p'}(G),\ x \in G\bigr).$$

Since the row and the column spaces over any Banach space are homogeneous, it is clear that $\tilde{L} : \mathrm{ROW}(L^p(G)) \to \mathrm{ROW}(L^p(G,E))$ and $\tilde{R} : \mathrm{COL}(L^{p'}(G)) \to \mathrm{COL}(L^{p'}(G,E^*))$ are completely bounded with $\|\tilde{L}\|_{\mathrm{cb}} = \|L\|_\infty$ and $\|\tilde{R}\|_{\mathrm{cb}} = \|R\|_\infty$. From [6, Corollary 7.1.3], it thus follows that

$$\tilde{L} \otimes \tilde{R} : \mathrm{ROW}\bigl(L^p(G)\bigr) \hat{\otimes} \mathrm{COL}\bigl(L^{p'}(G)\bigr) \to \mathrm{ROW}\bigl(L^p(G,E)\bigr) \hat{\otimes} \mathrm{COL}\bigl(L^{p'}(G,E^*)\bigr)$$

is completely bounded as well with $\|\tilde{L} \otimes \tilde{R}\|_{\mathrm{cb}} \leqslant \|L\|_\infty \|R\|_\infty$. Since

$$\mathrm{tr}_E : \mathrm{ROW}\bigl(L^p(G,E)\bigr) \hat{\otimes} \mathrm{COL}\bigl(L^{p'}(G,E^*)\bigr) \to \mathrm{ROW}\bigl(L^p(G)\bigr) \hat{\otimes} \mathrm{COL}\bigl(L^{p'}(G)\bigr)$$

as in Lemma 3.1 is a complete contraction, we conclude that $\mathrm{tr}_E \circ (\tilde{L} \otimes \tilde{R})$ is completely bounded with cb-norm at most $\|L\|_\infty \|R\|_\infty$.

From the definitions of tr_E, $\tilde{L}$, and $\tilde{R}$, it straightforward to verify that $\mathrm{tr}_E \circ (\tilde{L} \otimes \tilde{R}) = M_{K(f)}$. In view of (3), we thus obtain that

$$\|M_{K(f)}\|_{\mathrm{cb}} \leqslant \|L\|_\infty \|R\|_\infty < \|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))} + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this yields $\|M_{K(f)}\|_{\mathrm{cb}} \leqslant \|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))}$. By definition,

$$\|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))} = \|M_{K(f)}\| \leqslant \|M_{K(f)}\|_{\mathrm{cb}}$$

holds, so that have equality as claimed. □

Passing to quotients we thus obtain:

Theorem 3.3. *Let $p \in (1,\infty)$, and let G be a locally compact group. Then $\mathcal{M}_{\mathrm{HS}}(A_p(G))$ is contained in $\mathcal{M}_{\mathrm{cb}}(A_p(G))$ such that the inclusion is a contraction.*

Remarks.

1. By Proposition 3.2, the linear map $\mathcal{M}_{\mathrm{HS}}(A_p(G)) \ni f \mapsto M_{K(f)}$ is an isometric embedding into the operator space $\mathcal{CB}(\mathrm{ROW}(L^p(G)) \hat{\otimes} \mathrm{COL}(L^{p'}(G)))$ and can be used to equip $\mathcal{M}_{\mathrm{HS}}(A_p(G))$ with a canonical operator space structure. We do not know whether Theorem 3.3 can be improved to yield a completely contractive—or at least completely bounded—inclusion map.
2. For amenable G, the algebras $\mathrm{PF}_{p'}(G)^*$, $B_p(G)$, $\mathcal{M}_{\mathrm{cb}}(A_p(G))$, and $\mathcal{M}_{\mathrm{HS}}(A_p(G))$ are easily seen to be isometrically isomorphic (see [2,17,32]). We do not know whether theses isometric isomorphisms are, in fact, completely isometric; for some of them, this seems to be open even in the case where $p = 2$.

For $p \in (1, \infty)$ and a locally compact group G, any $f \in A_p(G)$ is a cb-multiplier of $A_p(G)$, simply because $A_p(G)$ is a completely bounded Banach algebra. However, as $A_p(G)$ is not known to be completely contractive, this does not allow us to conclude that $\|f\|_{\mathcal{M}_{\mathrm{cb}}(A_p(G))} \leqslant \|f\|_{A_p(G)}$, but only that $\|f\|_{\mathcal{M}_{\mathrm{cb}}(A_p(G))} \leqslant K_{\mathbb{G}}^2 \|f\|_{A_p(G)}$.

Theorem 3.3, nevertheless, allows us to obtain a better norm estimate:

Corollary 3.4. *Let $p \in (1, \infty)$, and let G be a locally compact group. Then we have*

$$\|f\|_{\mathcal{M}_{\mathrm{cb}}(A_p(G))} \leqslant \|f\|_{A_p(G)} \quad \big(f \in A_p(G)\big).$$

Proof. Let $f \in A_p(G)$. By [28, Proposition 10.2], we have $\|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))} \leqslant \|f\|_{A_p(G)}$ and thus

$$\|f\|_{\mathcal{M}_{\mathrm{cb}}(A_p(G))} \leqslant \|f\|_{\mathcal{M}_{\mathrm{HS}}(A_p(G))} \leqslant \|f\|_{A_p(G)}$$

by Theorem 3.3. □

4. $B_p(G)$, $\mathrm{PF}_{p'}(G)^*$, and the amenability of G

A classical amenability criterion due to R. Godement asserts that a locally compact group G is amenable if and only if its trivial representation is weakly contained in $(\lambda_2, L_2(G))$ (see [28, Theorem 8.9], for instance). In terms of Fourier–Stieltjes algebras this means that G is amenable if and only if $B_r(G) = B(G)$ (the equality is automatically a complete isometry).

The following theorem generalizes this to a general L_p-context:

Theorem 4.1. *The following are equivalent for a locally compact group G:*

(i) *the canonical map from $\mathrm{PF}_{p'}(G)^*$ into $B_p(G)$ is surjective for each $p \in (1, \infty)$;*
(ii) *there is $p \in (1, \infty)$ such that $1 \in \mathrm{PF}_{p'}(G)^*$;*
(iii) *G is amenable.*

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii). An inspection of the proof of [2, Theorem 5] shows that $1 \in \mathrm{PF}_{p'}(G)^*$ for just one $p \in (1, \infty)$ is possible only if G is amenable.

(iii) ⇒ (i). This follows from [32, Theorem 6.7]. □

Remark. Unless $p = 2$, we cannot say for amenable G whether or not $\mathrm{PM}_{p'}(G)^* = B_p(G)$ holds completely isometrically. By [32, Theorem 6.7], we do have an isometric isomorphism, but this is all we can say. Due to the lack of an inverse mapping theorem for completely bounded maps, we even do not know for general $p \in (1, \infty)$ whether the completely bounded bijective map from $\mathrm{PF}_{p'}(G)^*$ onto $B_p(G)$ has a completely bounded inverse.

In [30], Z.-J. Ruan adapted the notion of an amenable Banach algebra due to B.E. Johnson ([19]) to an operator space context.

Given a completely bounded Banach algebra $\mathfrak{A}$ and a completely bounded Banach $\mathfrak{A}$-bimodule E, i.e., a Banach $\mathfrak{A}$-bimodule which is also an operator space such that the module actions are completely bounded, the dual operator space E^* because a completely bounded Banach $\mathfrak{A}$-bimodule in its own right via

$$\langle \xi, a \cdot \phi \rangle := \langle \xi \cdot a, \phi \rangle \quad \text{and} \quad \langle \xi, \phi \cdot a \rangle := \langle a \cdot \xi, \phi \rangle \quad (\xi \in E,\ \phi \in E^*,\ a \in \mathfrak{A}),$$

and $\mathfrak{A}$ is said to be *operator amenable* if and only if, for each completely bounded Banach $\mathfrak{A}$-bimodule E, every completely bounded derivation $D : \mathfrak{A} \to E^*$ is inner. Ruan showed that a locally compact group G is amenable if and only if $A(G)$ is operator amenable, and in [23], Lambert and the authors extended this result to $A_p(G)$ for arbitrary $p \in (1, \infty)$.

Suppose that $\mathfrak{A}$ is a *dual*, completely bounded Banach algebra. If E is a completely bounded Banach $\mathfrak{A}$-bimodule, we call E^* *normal* if the bilinear maps

$$\mathfrak{A} \times E^* \to E^*, \qquad (a, \phi) \mapsto \begin{cases} a \cdot \phi, \\ \phi \cdot a \end{cases}$$

are separately weak* continuous. Following [33], we say that $\mathfrak{A}$ is *operator Connes-amenable* if, for every completely bounded Banach $\mathfrak{A}$-bimodule E such that E^* is normal, every weak*-weak*-continuous, completely bounded derivation $D : \mathfrak{A} \to E^*$ is inner.

Extending [33, Theorem 4.4] in analogy with [23, Theorem 7.3], we obtain:

Theorem 4.2. *The following are equivalent for a locally compact group G:*

(i) *G is amenable*;
(ii) *$P_{p'}(G)^*$ is operator Connes-amenable for every $p \in (1, \infty)$*;
(iii) *$B_r(G)$ is operator Connes-amenable*;
(iv) *there is $p \in (1, \infty)$ such that $\mathrm{PF}_{p'}(G)^*$ is operator Connes-amenable.*

Proof. (i) ⇒ (ii). Let $p \in (1, \infty)$ be arbitrary. Then [23, Theorem 7.3] yields the operator amenability of $A_p(G)$. Since the inclusion of $A_p(G)$ into $\mathrm{PF}_{p'}(G)^*$ is a completely contractive algebra homomorphism with weak* dense range, the operator space analog of [31, Proposition 4.2(i)] yields the operator Connes-amenability of $\mathrm{PF}_{p'}(G)^*$.

(ii) ⇒ (iii) ⇒ (iv) are trivial.

(iv) ⇒ (i). Let $p \in (1, \infty)$ be such that $\mathrm{PF}_{p'}(G)^*$ is operator Connes-amenable. The operator space analog of [31, Proposition 4.1] then yields that $\mathrm{PF}_{p'}(G)^*$ has an identity, so that Theorem 4.1(ii) is satisfied. By Theorem 4.1, this means that G is amenable. □

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