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PROBLEMS OF SUBSTITUTION AND ADMISSIBILITY IN THE MODAL SYSTEM Grz AND IN INTUITIONISTIC PROPOSITIONAL CALCULUS

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Questions connected with the admissibility of rules of inference and the solvability of the substitution problem for modal and intuitionistic logic are considered in an algebraic framework. The main result is the decidability of the universal theory of the free modal algebra $\mathcal{F}_\omega(\mathrm{Grz})$ extended in signature by adding constants for free generators. As corollaries we obtain: (a) there exists an algorithm for the recognition of admissibility of rules with parameters (hence also without them) in the modal system Grz, (b) the substitution problem for Grz and for the intuitionistic calculus H is decidable, (c) intuitionistic propositional calculus H is decidable with respect to admissibility (a positive solution of Friedman's problem). A semantical criterion for the admissibility of rules of inference in Grz is given.

The need for simplification of derivations in formal systems has led to the consideration of the class of all inference rules that do not increase the set of provable formulas. The rules of this class are called admissible rules of inference.

Investigations into admissible rules of inference have mostly dealt with intuitionistic propositional calculus H. A number of conditions for admissibility and derivability of rules in H have been obtained in [15, 16, 36, 37]; a description of quasi-characteristic admissible rules in H has been given in [34]; the structurally pre-complete extensions of H have been given in [35]. The connection between admissible rules in extensions of H and admissible rules of modal logics was observed in [17]. This connection draws attention to admissible rules of modal logics. A number of results regarding admissible rules in modal logics have been presented in [17, 18, 19].

The general problem of finding an algorithm which recognizes admissibility of rules in H was posed by Friedman [5, problem 40]. Kuznetsov formulated a related problem: Does there exist a finite basis of admissible rules of H? The positive solution of Friedman's problem is given in [20, 23, 27]. Kuznetsov's problem and it's analogues for the modal systems $S4$, Grz have negative solutions [21, 24]. The approach to the solution of these problems is based on properties of universal theories of free topo-boolean and pseudo-boolean algebras. The problem of substitution (or the problem of logical equations) may also be formulated in terms of properties of universal theories.

The *substitution problem* (or *problem of logical equations*) for a propositional logic λ consists in recognizing for an arbitrary formula $A(x_i, p_j)$ (the x_i are variables, the p_j are propositional letters) whether there exist formulas B_i such that $A(B_i, p_j)$ is a theorem of the logic λ. The problem of logical equations has so far not been investigated satisfactorily. The decidability of the substitution problem for the modal system $S4$ has been obtained recently [25, 26].

The aim of this article is a proof of the algorithmical decidability of admissibility in modal systems Grz of rules of inference (with parameters) of generalized form, and a proof of decidability of the substitution problem for Grz and for intuitionistic logic H.

The main result of the paper is a proof of the decidability of the universal theory of the free modal algebra $\mathcal{F}_\omega(\mathrm{Grz})$ (the variety of modal algebras corresponding to the logic Grz), in signature extended by adding constants for free generators. As simple corollaries of this result we obtain the following facts: (a) there exists an algorithm for recognition of admissibility in Grz of rules of inference with parameters (hence also without parameters), (b) the substitution problem for Grz and H is decidable, (c) the algorithmical decidability of admissible rules in H (so we have another positive solution to Friedman's problem). A semantical criterion for the admissibility of rules of inference in Grz is found and examples of its use are given.

We suppose the reader to be familiar with the main principles and conventions of first-order theories. Familiarity with Kripke semantic for modal logic is required. In Section 1 we will review some definitions and notations. All undefined terms can be found in [29, 2].

1. Introduction

As usual we understand by a *modal logic* (m.l.) a set of modal propositional formulas containing all axioms of the minimal normal system K and which is closed under substitution, modus ponens and rule of necessitation: $A/\Box A$. Similarly a *superintuitionistic logic* (s.l.) is a set of propositional formulas containing all axioms of Heyting's intuitionistic calculus H and which is closed under modus ponens and substitution.

We shall use a combination of the algebraic semantics and the relational semantics of Kripke. A *modal algebra* (m.a.) is a boolean algebra with an additional unary operation $\Box$ satisfying the equations:

$$\Box 1 = 1, \qquad \neg\Box(\neg x \vee y) \vee (\neg\Box x \vee \Box y) = 1.$$

Let $\varphi(p_1, \ldots, p_n)$ be a modal propositional formula with propositional letters $p_1, \ldots, p_n$. The formula φ is said to be valid in the m.a. $\mathfrak{B}$ (notation: $\mathfrak{B} \models \varphi$) if for all tuples $(a_1, \ldots, a_n)$ it is true that

$$\mathfrak{B} \models (\varphi(a_1, \ldots, a_n) = 1) \qquad (a_i \in \mathfrak{B}).$$

A pseudo-boolean algebra (p.b.a.) $\mathfrak{A}$ is a distributive lattice with smallest and greatest elements 0 and 1, and such that for arbitrary elements $a, b \in \mathfrak{A}$ there is a relative pseudo-complement $a \supset b$ (that is a greatest element x such that $a \cap x \leqslant b$, this element x is denoted by $a \supset b$). The element $a \supset 0$ is called the *pseudo-complement* of the element a and is denoted by $\neg a$. The definition of validity in $\mathfrak{A}$ for propositional formulas φ is similar to the modal case.

If $\mathfrak{B}$ is a m.a. (p.b.a.) then $\lambda(\mathfrak{B}) := \{\varphi \mid \mathfrak{B} \vDash \varphi\}$ is its m.l. (s.l.). According to the completeness theorem [8, 12] (which is based on the Lindenbaum algebra) for each m.l. (s.l.) λ there exists an m.a. (p.b.a.) $\mathfrak{B}$ such that $\lambda = \lambda(\mathfrak{B})$.

Let λ be a m.l. (s.l.), then $\mathrm{Var}(\lambda)$ denotes the algebraic variety of m.a.'s (p.b.a.'s) $\{\mathfrak{B} \mid \forall\varphi \in \lambda\,(\mathfrak{B} \vDash \varphi)\}$. By the completeness theorem we have

$$\varphi \in \lambda \quad \Leftrightarrow \quad \forall \mathfrak{B} \in \mathrm{Var}(\lambda)(\mathfrak{B} \vDash \varphi).$$

Now we review the basics of Kripke's semantics [7, 10]. A *frame* $\mathcal{T} = \langle T, R \rangle$ is a pair where T is a nonempty set and R is a binary relation on T. Let P be some set of propositional letters. A *model* $\mathfrak{M} = \langle T, R, V \rangle$ is an 3-tuple where $\langle T, R \rangle$ is a frame and V (*valuation*) is a function, mapping P into the set of all subsets of the set T.

The validity (or truth) of modal propositional formulas on elements $x \in T$ is defined by induction on the formula:

$$x \Vdash_V p_i \quad \Leftrightarrow \quad x \in V(p_i),$$

$$x \Vdash_V (A \wedge B) \quad \Leftrightarrow \quad (x \Vdash_V A) \;\&\; (x \Vdash_V B),$$

$$x \Vdash_V (A \vee B) \quad \Leftrightarrow \quad (x \Vdash_V A) \vee (x \Vdash_V B),$$

$$x \Vdash_V \neg A \quad \Leftrightarrow \quad \neg(x \Vdash_V A),$$

$$x \Vdash_V \Box A \quad \Leftrightarrow \quad (\forall y \in T)((x R y) \Rightarrow y \Vdash_V A),$$

$$x \Vdash_V \Diamond A \quad \Leftrightarrow \quad (\exists y \in T)((x R y) \;\&\; (y \Vdash_V A)),$$

$$x \Vdash_V (A \to B) \quad \Leftrightarrow \quad (x \Vdash_V B) \vee \neg(x \Vdash_V A).$$

A formula φ with propositional letters from P is said to be valid in the model $\mathfrak{M}$ ($\mathfrak{M} \Vdash \varphi$) iff $\forall x \in T\ (x \Vdash_V \varphi)$. A formula φ is called valid in the frame $\mathcal{T}$ ($\mathcal{T} \Vdash \varphi$) iff φ is true for all valuations V of its propositional letters.

The set $\lambda(\mathcal{T}) := \{\varphi \mid \mathcal{T} \Vdash \varphi\}$ for a frame $\mathcal{T} = \langle T, R \rangle$ is a modal logic. A m.l. λ is said to be Kripke-complete [4, 33] if there exists a frame $\mathcal{T}$ with the property $\lambda(\mathcal{T}) = \lambda$. Fine [4] and Thomasson [33] showed that there exist modal logics which are not Kripke-complete. However, Kripke semantics and its modifications (for example refined first-order semantics [32]) turned out to be very convenient.

A few words about first-order semantics in the style of Kripke [10, 23, 32]. Let $\langle W, R \rangle$ be a frame. We assign to this frame the associated modal algebra $\langle W, R \rangle^+$, where $\langle W, R \rangle^+$ is the boolean algebra of all subsets of the set W and $\Box$ is the operation defined by the following equation:

$$\Box X := \{a \mid a \in W, \forall b \in W((a R b) \Rightarrow b \in X)\}.$$

Let $X_1, \ldots, X_n \in \langle W, R\rangle^+$. By $\langle W, R\rangle^+ (X_1, \ldots, X_n)$ we denote the subalgebra of the algebra $\langle W, R\rangle^+$ generated by the elements $X_1, \ldots, X_n$. Arbitrary elements of this subalgebra have the form $\varphi(X_1, \ldots, X_n)$, where φ is a term. Let us define the valuation of $\{p_1, \ldots, p_n\}$ on $\langle W, R\rangle$ by $V(p_i) := X_i$. If $\varphi(X_1, \ldots, X_n)$ is an element of $\langle W, R\rangle^+$ (where φ is some term), then by $\varphi(p_1, \ldots, p_n)$ we mean the formula obtained from φ by substituting letters p_i for X_i and logical connectives for the corresponding operations.

The following lemma is well-known (it is proved by induction on φ).

Lemma 1. *For arbitrary* $x \in W$,

$$x \in \varphi(X_1, \ldots, X_n) \quad \Leftrightarrow \quad x \Vdash_V \varphi(p_1, \ldots, p_n).$$

The associated frame $\mathfrak{B}^+$ of a m.a. $\mathfrak{B}$ is given by $\mathfrak{B}^+ := \langle T_{\mathfrak{B}}, R\rangle$, where $T_{\mathfrak{B}}$ is the set of all ultrafilters on $\mathfrak{B}$ and

$$\forall x \in \mathfrak{B}\,(\Box x \in \nabla_1 \Rightarrow \Box x \in \nabla_2) \quad \Leftrightarrow \quad \nabla_1 R \nabla_2, \quad \nabla_1, \nabla_2 \in T_{\mathfrak{B}}.$$

According to the inclusion theorem of Jonsson–Tarski–Stone [8] the mapping $i: \mathfrak{B} \rightarrow \mathfrak{B}^+$ where $i(a) := \{\nabla \mid \nabla \in T_{\mathfrak{B}}, a \in \nabla\}$ is a monomorfism "in". If $\mathfrak{B}$ is finite, then i is a mapping "onto".

The m.l. λ is said to have the *finite model property* if $\lambda = \bigcap_{i \in I} \lambda(\mathfrak{B}_i)$ where the $\mathfrak{B}_i$ are finite m.a.'s. Lemmon [10] showed that this definition is equivalent to $\lambda = \bigcap_{i \in I} \lambda(\mathcal{F}_i)$ where the $\mathcal{F}_i$ are finite frames.

Now we proceed to the rules of inference. Let λ be a m.l. (s.l.) and A_j, B formulas in the language of this logic; $p_1, \ldots, p_n$ are propositional letters from these formulas and $x_1, \ldots, x_n$ are distinct variables.

An expression of the form

$$A_1(x_1, \ldots, x_n), \ldots, A_m(x_1, \ldots, x_n)/B(x_1, \ldots, x_n) \tag{1}$$

is called a rule of inference. We note that in Polish mathematical literature [1, 11, 12, 31, 38] a more general notion of rule of inference is used. A rule is mapping the set of all α-tuples of formulas into the set of formulas. If $\alpha < \omega$, then the rule is called finite. If the rule is closed with respect to substitution, then the rule is called structural. Each finite structural rule is a set-theoretic union of rules which are defined by rules of the form (1), that is mappings which assign formulas $B(C_1, \ldots, C_n)$ to tuples $A_1(C_1, \ldots, C_n), \ldots, A_m(C_1, \ldots, C_n)$. A rule of the form (1) is said to be (finite) *sequential* or *standard.* According to the Los'–Suzko representation theorem [12], an arbitrary standard logical consequence operation is generated by a countable set of standard rules. Therefore we mean in this paper by rules of inference rules of the form (1) only.

The rule (1) is said to be *admissible* in the logic λ iff for all formulas $B_1, \ldots, B_n$

$$\forall_j\, 1 \leqslant j \leqslant m \quad A_j(B_1, \ldots, B_n) \in \lambda \quad \text{implies} \quad B(B_1, \ldots, B_n) \in \lambda.$$

The rule (1) is called *derivable* in m.l. λ (s.l. λ) iff from $A_1, \ldots, A_m$ and the set of theorems of the logic λ the formula B is derivable with the help of the necessitation rule and modus ponens (modus ponens only). It is clear that derivability implies admissibility. Harrop's rule [7]

$$(\neg p \supset (q \vee r))/(\neg p \supset q) \vee (\neg p \supset r)$$

is an example of an admissible but inderivable rule in Heyting's intuitionistic calculus H.

There exists an algebraic approach to admissibility. Let λ be a m.l. (s.l.). Let $g_j(x_i)$, $f_j(x_i)$, $g(x_i)$, $f(x_i)$ be terms of the signature of the variety $\mathrm{Var}(\lambda)$ (the constants 0 and 1 are also terms). The first-order formula of the form

$$\forall \bar{x} \bigwedge_{j=1}^{n} (g_j(x_i) = f_j(x_i)) \Rightarrow (f(x_i) = g(x_i))$$

is called a quasi-identity. This quasi-identity is said to be valid in the algebra $\mathfrak{B}$ iff for all $a_i \in \mathfrak{B}$, $\forall j\, (1 \leq j \leq n)\, g_j(a_i) = f_j(a_i)$ in $\mathfrak{B}$ implies $g(a_i) = f(a_i)$. Let $r: A_1(x_i), \ldots, A_m(x_i)/B(x_i)$ be a rule of inference. We assign to r the quasi-identity r^* of the form

$$\left(\bigwedge_{j=1}^{m} A_j(x_i)\right) = 1 \quad \Rightarrow \quad B(x_i) = 1.$$

The following well-known proposition belongs to folklore and goes back to the Polish method of contracting logical calculi and logical consequence operations [11, 12, 31]. Let $\mathscr{F}_\alpha(\lambda)$ denote the free algebra of rank α over $\mathrm{Var}(\lambda)$.

Lemma 2. *The rule r is admissible in logic λ iff the quasi-identity r^* is valid in the free algebra $\mathscr{F}_\omega(\lambda)$.*

Proof. Let us suppose that r is not admissible in λ. Then for some formulae C_i we have $A_1(C_i) \in \lambda, \ldots, A_m(C_i) \in \lambda$ and $B(C_i) \notin \lambda$. Therefore the identities $A_1(C_i) = 1, \ldots, A_m(C_i) = 1$ are valid on $\mathrm{Var}(\lambda)$. If we view formulas as elements of the algebra $\mathscr{F}_\omega(\lambda)$ and regard propositional letters as free generators of $\mathscr{F}_\omega(\lambda)$, then we obtain $B(C_i) \neq 1$. Hence the quasi-identity r^* is not valid on $\mathscr{F}_\omega(\lambda)$.

On the other hand let us assume that r^* is not valid on $\mathscr{F}_\omega(\lambda)$. Then $A_1(C_i) = 1, \ldots, A_m(C_i) = 1$ and $B(C_i) \neq 1$ in $\mathscr{F}_\omega(\lambda)$, where the C_i are elements from $\mathscr{F}_\omega(\lambda)$. If we regard, as above, $B(C_i)$, $A_j(C_i)$ as formulas, then $A_j(C_i) \in \lambda$, $1 \leq j \leq m$, $B(C_i) \notin \lambda$. Thus r is not admissible in λ. $\square$

Note also that to every quasi-identity $q: \bigwedge_{j=1}^{n} (g_j(x_i) = f_j(x_i)) \Rightarrow (f(x_i) = g(x_i))$ these corresponds a rule q^* of the form $\bigwedge_{j=1}^{n} (g_j \Leftrightarrow f_j)/(f \Leftrightarrow g)$. It is also easy to see that q is valid in $\mathscr{F}_\omega(\lambda)$ iff the rule q^* is admissible in λ.

We now turn to the substitution problem and its algebraic treatment. Let us remind that the substitution problem (or problem of logical equations) for a logic

λ (m.l. or s.l.) consists in the recognition for arbitrary formulas $A(x_i, p_j)$ (where the p_j are propositional letters in A, the x_i are variables substituted for other letters) whether there exist formulas B_i such that $A(B_i, p_j) \in \lambda$. As above we view formulas as elements of the free algebra $\mathscr{F}_\omega(\lambda)$ and letters as free generators of $\mathscr{F}_\omega(\lambda)$.

Lemma 3. *There exist formulas B_i such that $A(B_i, p_j) = 1$ iff the equation $A(x_i, p_j) = 1$ is solvable in the free algebra $\mathscr{F}_\omega(\lambda)$.*

Proof. Suppose that $A(B_i, p_j) \in \lambda$. Then we regard the formulas B_i as elements of $\mathscr{F}_\omega(\lambda)$ and because $A(B_i, p_j) \in \lambda$, the identity $A(x_i, p_j) = 1$ holds in $\mathrm{Var}(\lambda)$. Moreover $\mathscr{F}_\omega(\lambda) \in \mathrm{Var}(\lambda)$. Therefore the B_i are solutions of $A(x_i, p_j) = 1$ in $\mathscr{F}_\omega(\lambda)$. Now let $A(x_i, p_j) = 1$ have a solution in $\mathscr{F}_\omega(\lambda)$. Then there exist $C_i \in \mathscr{F}_\omega(\lambda)$ with the property $A(C_i, p_j) = 1$. Because $\mathscr{F}_\omega(\lambda)$ is a free algebra on $\mathrm{Var}(\lambda)$, we have $A(C_i, p_j) = 1$ in $\mathrm{Var}(\lambda)$, and by the completeness theorem $A(C_i, p_j) \in \lambda$. $\Box$

Thus questions about the decidability of the substitution problem in logics λ are reduced to problems of the solvability of equations in free algebras $\mathscr{F}_\omega(\lambda)$. Hence the problems of logical equations and of admissibility of rules (Lemma 2) are reduced to questions concerning the universal (or dually, existential) theory of the algebra $\mathscr{F}_\omega(\lambda)$ extended in signature by the addition of constants for free generators.

Gödel's translation T provides a connection between admissible rules of s.l. and of m.l. We remind that the Gödel translation T of propositional formulas into modal propositional formulas is defined by induction on the length:

$$T(p_i) = \Box p_i,$$

$$T(A \wedge B) = T(A) \wedge T(B), \qquad T(A \vee B) = T(A) \vee T(B),$$

$$T(A \supset B) = \Box(T(A) \rightarrow T(B)),$$

$$T(\neg A) = \Box\neg\Box T(A).$$

Let λ be a s.l. The modal associate for λ is a m.l. λ_1 (arbitrary m.l.) such that $\forall A\ (A \in \lambda \Leftrightarrow T(A) \in \lambda_1)$.

By Dummet–Lemmon's strengthening [3] of Gödel's translation theorem, for an arbitrary s.l. $H + X$ (where H is Heyting's intuitionistic calculus and X a set of formulas, and $H + X$ the smallest s.l. containing $H \cup X$)

$$A \in H + X \quad \Leftrightarrow \quad T(A) \in S4 + T(X).$$

Thus $S4 + T(X)$ is the smallest modal associate for the s.l. $H + X$ (among the extensions of $S4$). There exists a greatest modal associate (among extensions of the m.l. $S4$) for each s.l. Let us turn to its construction.

Let $\mathfrak{A}$ be a p.b.a. The wrapping modal algebra $S(\mathfrak{A})$ is constructed as follows [cf. 13]: it is the boolean algebra defined by the set $\mathfrak{A}$ as generating elements and with the lattice identifies true in $\mathfrak{A}$ as the generators' relations. On this boolean algebra the operation $\Box$ is given by

$$\Box((\neg a_1 \vee b_1) \wedge \cdots \wedge (\neg a_n \vee b_n)) := ((a_1 \supset b_1) \wedge \cdots \wedge (a_n \supset b_n)).$$

The correctness of this is easily seen from the fact that $a_i \supset b_i$ is the greatest element in $\mathfrak{A}$ which is less than $\neg a_i \vee b_i$. Also it is obvious that $S(\mathfrak{A})$ as an algebra is generated by elements of the form $\Box a$ where $a \in \mathfrak{A}$. Maksimova proved the following lemma.

Lemma 4 [13]. *For arbitrary s.l. λ the modal logic*

$$\sigma(\lambda) := \{C \mid \forall \mathfrak{A}\, (\mathfrak{A} \in \mathrm{Var}(\lambda) \Rightarrow S(\mathfrak{A}) \Vdash C)\}$$

is the greatest modal associate for λ (among extensions of $S4$).

We need next

Lemma 5. *Let $A(p_i)$ be a modal propositional formula. There exists a propositional formula C such that the formula $\Box A(\Box p_i)$ is equivalent to $T(C)(p_i)$ in $S4$.*

Proof. Induction on the length of A. If $A = p_i$ then $T(p_i) = \Box p_i$. Thus we can take C to be p_i. Suppose that for formulas with length less than length of A the lemma is proved. For $A = \Box B$ the induction step is obvious. Another case A is constructed by applying connectives $\neg$, $\rightarrow$, $\wedge$, $\vee$ to subformulas of A of the form $\Box B_j$ and to propositional letters p_i. Thus

$$A(p_i) = D(\Box B_1, \ldots, \Box B_k, p_l, \ldots, p_t)$$

where D is a formula without occurrences of $\Box$. Thus the formula $\Box A(\Box p_i)$ has the form

$$\Box D(\Box B_1(\Box p_i), \ldots, \Box B_k(\Box p_i), \Box p_l, \ldots, \Box p_t).$$

We transform D to a conjunctive normal form and obtain a formula F, where

$$F = \Box \bigwedge_m \left(\left(\bigvee_{r=1}^{n_m} \neg \Box E_r^m(\Box p_i) \right) \vee \left(\bigvee_{r=1}^{k_m} \Box G_r^m(\Box p_i) \right) \right),$$

here E_r^m, G_r^m are distinct formulas from $\{B_1, \ldots, B_k, p_l, \ldots, p_t\}$. It is clear that F is equivalent to $\Box A(\Box p_i)$ in $S4$. But F is also equivalent to the formula

$$L = \bigwedge_m \Box \left(\bigwedge_{r=1}^{m_n} \Box E_r^m(\Box p_i) \rightarrow \bigvee_{r=1}^{k_m} \Box G_r^m(\Box p_i) \right).$$

By induction hypothesis, the formulas $\Box E_r^m(\Box p_i)$ and $\Box G_r^m(\Box p_i)$ are equivalent in $S4$ to the formulas $T(C_r^m)(p_i)$ and $T(D_r^m)(p_i)$, for some C_r^m and D_r^m. Therefore

L is equivalent to

$$\bigwedge_m \Box\Big(\bigwedge_{r=1}^{m_n} T(C_r^m)(p_i) \to \bigvee_{r=1}^{k_m} T(D_r^m)(p_i)\Big),$$

and this formula is equivalent to

$$T\Big(\bigwedge_m \Big(\bigwedge_{r=1}^{m_n} C_r^m(p_i) \to \bigvee_{r=1}^{k_m} D_r^m(p_i)\Big)\Big).$$

Now the lemma is proved. $\Box$

Now we can make the connection between admissibility in s.l. and m.l. Since for arbitrary m.l. or s.l. λ it is true that $A \wedge B \in \lambda \Leftrightarrow (A \in \lambda) \wedge (B \in \lambda)$, we will consider henceforth the rules with one premisse only.

Theorem 6. *The rule A/B is admissible in the s.l. λ iff the rule $T(A)/T(B)$ is admissible in the greatest modal associate $\sigma(\lambda)$ of λ.*

Proof. Let us assume $T(A)/T(B)$ is admissible in $\sigma(\lambda)$. Suppose that $A(B_i) \in \lambda$. Then $T(A(B_i)) \in \sigma(\lambda)$ as $\sigma(\lambda)$ is a modal associate of λ. But $T(A(B_i))$ is equivalent in $S4$ (and $S4 \subseteq \sigma(\lambda)$) to $T(A)(T(B_i))$. Therefore the last formula is a theorem of $\sigma(\lambda)$, that is $T(A)(T(B_i)) \in \sigma(\lambda)$. By our assumption about admissibility we have $T(B)(T(B_i)) \in \sigma(\lambda)$ and $T(B(B_i)) \in \sigma(\lambda)$. Hence $B(B_i) \in \sigma(\lambda)$.

Now let A/B be admissible in λ and let $T(A)(C_i) \in \sigma(\lambda)$ but $T(B)(C_i) \notin \sigma(\lambda)$. By the definition of $\sigma(\lambda)$ there exists $\mathfrak{A} \in \mathrm{Var}(\lambda)$ such that $S(\mathfrak{A}) \Vdash \neg(T(B)(C_i) = 1)$. As we noted, the algebra $S(\mathfrak{A})$ is generated by elements of the form $\Box a$ where $\Box a \in \mathfrak{A}$. Therefore there exist $\Box x_j \in \mathfrak{A}$ and terms D_k such that $T(B)(C_i(D_k(\Box x_j))) \neq 1$.

In order to avoid complications in notation we shall denote the formulas in the language of λ and the corresponding terms of the variety $\mathrm{Var}(\lambda)$ by the same symbols. According to Lemma 5 the formulas $\Box C_i(D_k(\Box p_j))$ are equivalent in $S4$ to the formulas $T(A_i)(p_j)$. From this we get by regarding the $T(A_i)$ as terms,

$$S(\mathfrak{A}) \Vdash (\Box C_i(D_k(\Box x_j)) = T(A_i)(x_j)).$$

Hence

$$T(B)(C_i(D_k(\Box x_j)) = T(B)(T(A_i)(x_j)).$$

Therefore $T(B)(A_i)(x_j) \neq 1$. Since we know that all formulas from $\sigma(\lambda)$ are valid in $S(\mathfrak{A})$, we get $T(B(A_i)) \notin \sigma(\lambda)$, that is $B(A_i) \notin \lambda$. This conclusion gives us $A(A_i) \notin \lambda$ in view of our assumption about admissibility of A/B. Since $\sigma(\lambda)$ is a modal associate of λ we obtain $T(A(A_i)) \notin \sigma(\lambda)$.

But we have assumed that $T(A)(C_i) \in \sigma(\lambda)$ and this together with closure of $\sigma(\lambda)$ under substitution gives us

$$T(A)(\Box C_i(D_k(\Box p_j))) \in \sigma(\lambda).$$

We obtain $T(A)(T(A_i)) \in \sigma(\lambda)$, by using the equivalence of $\Box C_i(D_k(\Box p_j))$ and $T(A_i)(p_j)$, which contradicts $T(A(A_i)) \notin \sigma(\lambda)$. $\Box$

The problem of finding an algorithm for the recognition of admissible rules in intuitionistic propositional calculus H was posed in Friedman's paper [5, problem 40]. From this problem and Theorem 6 it follows that the greatest modal associate for H, i.e. $\sigma(H)$, is interesting. It is well known that $\sigma(H) = \mathrm{Grz}$ where

$$\mathrm{Grz} := S4 + \Box(\Box(p \to \Box p) \to \Box p) \to \Box p.$$

is the modal system of Grzegorczyk (this follows for example from the finite model property of Grz [29] and from σ preserving arbitrary intersections of logics [13]). Therefore from Theorem 6 we obtain:

Corollary 7. *The rule A/B is admissible in H iff the rule $T(A)/T(B)$ is admissible in Grzegorczyk's modal system* Grz.

The problem of logical equations (substitution problem) for the calculus H may also be reduced to the corresponding problem for Grz. Let, as above, $\mathscr{F}_\alpha(\lambda)$ be the free algebra of rank α over the variety $\mathrm{Var}(\lambda)$ (here λ is m.l. or s.l.). By Σ_f we denote the signature of $\mathrm{Var}(\lambda)$ extended by a countable set of constants for free generators of $\mathscr{F}_\omega(\lambda)$.

Lemma 8. *The equation $A(x_i, p_j) = 1$ is solvable in the free algebra $\mathscr{F}_\omega(H)$ iff in the algebra $\mathscr{F}_\omega(\mathrm{Grz})$ the equation $T(A)(\Box x_i, \Box p_j) = 1$ is solvable.*

Proof. Let $A(B_i, p_j) = 1$ in $\mathscr{F}_\omega(H)$ for some $B_i \in \mathscr{F}_\omega(H)$. Then we have $A(B_i, p_j) \in H$. The m.l. Grz is a modal associate for H, therefore we have $T(A)(T(B_i), \Box p_j) \in \mathrm{Grz}$. From this property we obtain $T(A)(T(B_i), \Box p_j) = 1$ in the free modal algebra $\mathscr{F}_\omega(H)$. But it is clear that $T(B_i) = \Box T(B_i)$. Therefore the equation $T(A)(\Box x_i, \Box p_j) = 1$ is solvable in $\mathscr{F}_\omega(\mathrm{Grz})$.

Conversely, let $T(A)(\Box C_i, \Box p_j) = 1$ in $\mathscr{F}_\omega(\mathrm{Grz})$. Then $T(A)(\Box C_i, \Box p_j) \in \mathrm{Grz}$. The m.l. Grz is closed under substitution and we obtain $T(A)(\Box C_i(\Box p_\xi), \Box p_j) \in \mathrm{Grz}$. According to Lemma 5 there exist formulas D_i which are constructed effectively from $\Box C_i(\Box p_\xi)$ and $\Box C_i(\Box p_\xi) \leftrightarrow T(D_i) \in S4$. Therefore (using that $S4 \subseteq \mathrm{Grz}$) we obtain $T(A)(T(D_i), \Box p_j) \in \mathrm{Grz}$. By the property of the modal associate we have $A(D_i, p_j) \in H$, that is $A(D_i, p_j) = 1$ in $\mathscr{F}_\omega(H)$. $\Box$

Lemmas 3 and 8 give us a reduction of the substitution problem for H to the substitution problem for Grz.

Thus the problems of admissibility and substitution for H, on basis of Lemmas 2, 3, 8 and Corollary 7, are reduced to properties of the free modal algebra $\mathscr{F}_\omega(\mathrm{Grz})$. This leads us to investigate the structure of this algebra.

2. Description of the structure of $\mathscr{F}_n$(Grz)

The well-known method [6, 17, 18, 30] for the description of the free algebra from Var(λ), where λ is a m.l., by means of models is as follows. Let $\mathscr{T} = \langle W, R, V\rangle$ be a model where $V: P_n \to 2^W$. The model $\mathscr{T}$ is said to be *n-characteristic* for the m.l. λ iff for an arbitrary formula A with propositional letters from P_n ($P_n := \{p_1, \ldots, p_n\}$), $A \in \lambda \Leftrightarrow \langle W, R, V\rangle \vDash A$.

From Lemma 1 it immediately follows that:

Lemma 9. *Let $\langle W, R, V\rangle$ be an n-characteristic model for the m.l. λ. The free algebra $\mathscr{F}_n(\lambda)$ is isomorphic to a subalgebra of $\langle W, R\rangle^+(V(P_n))$ of the modal algebra $\langle W, R\rangle^+$ and this subalgebra is freely generated by elements $V(p_i)$, $1 \leq i \leq n$.*

Thus the description of $\mathscr{F}_n(\lambda)$ depends on the choice of the n-characteristic model.

Let us fix some more notation and definitions. Let $\mathscr{T} = \langle W, R, V\rangle$ be a model and $X \subseteq W$. We denote by $\langle X\rangle$ the set $\{b \mid \exists a \in X\ (aRb)\}$. If $X = \{a\}$ we write $\langle a\rangle$ for $\langle\{a\}\rangle$. Let X be a subset of W and $\langle X\rangle = X$. The set X with the related R inherited from $\mathscr{T}$ (i.e. the pair $\langle X, R\rangle$) is called an *open subframe* of the frame $\langle W, R\rangle$. The tuple $\langle X, R, V'\rangle$ where $V'(p_i) := V(p_i) \cap X$ is called an *open submodel* of the model $\mathscr{T}$.

The main property of open submodels is the following: for each formula A with letters from P, $V: P \to 2^W$ and $a \in X$ it holds that $a \Vdash_{V'} A \Leftrightarrow a \Vdash_V A$ (in model $\mathscr{T}$). The proof of this property is easily obtained by induction on A.

A subset X of a reflexive transitive frame $\langle W, R\rangle$ (henceforth we shall often identify frames and their sets of elements) is called a circle (or cluster) if $\exists x\, \forall y\ ((xRy)\ \&\ (yRx) \Leftrightarrow y \in X)$. The depth of $a \in W$ is the maximal length of chains of circles starting with the circle including a. By $\mathscr{D}_n(\langle W, R\rangle)$ we denote the set of all elements of W with depth $\leq n$. $\mathscr{L}_n(\langle W, R\rangle)$ denotes the set of all elements of W which have depth n (this set is called the *n-layer* of W).

Now we turn to the construction of a n-characteristic model for Grz. We shall construct a sequence of models $U_k = \langle U_k, \leq_k, V_k\rangle$, where $\leq_k$ is a partial order and $(\leq_k) = (\leq_{k+1}) \cap U_k^2$, $U_k = \mathscr{D}_k(U_{k+1})$ and V_{k+1} restricted to U_k coincides with V_k (that is $\langle U_k, \leq_k, V_k\rangle$ is an open submodel of the model U_{k+1}).

A subset of an arbitrary partially ordered set is said to be an *anti-chain* if every two elements of this subset are incomparable. π_i denotes the projection of the Cartesian product on the ith factor of the product.

Let $P_n := \{p_1, \ldots, p_n\}$ be a set of propositional letters. We introduce the set $U_1 := \{0\} \times 2^{P_n} \times \{1\}$ and assume that U_1 is an anti-chain with respect to $\leq_1$. The valuation V_1 of the set P_n in U_1 is given by

$$\forall a \in U_1\ (a \in V_1(p_i) \Leftrightarrow p_i \in \pi_2(a)).$$

Suppose that the models $U_1, \ldots, U_k$ with the desired properties have already been constructed. Let us denote by T the set of all anti-chains from U_k containing at least one element from $\mathscr{L}_k(U_k)$. Consider the set $T \times 2^{P_n} \times \{k+1\}$. The tuples of this set we shall add to U_k; the first element of these tuples shows to which anti-chain this element corresponds, the second element of the tuple shows its valuation. We choose a subset $\mathscr{D}$ of $T \times 2^{P_n} \times \{k+1\}$, where

$$\mathscr{D} := \{\langle x, Y, \{k+1\}\rangle \mid x = \{b\}, b \in \mathscr{L}_k(U_k), \pi_2(b) = Y\}$$

(i.e. in tuples from $\mathscr{D}$ the first element is a one-element anti-chain and this element has depth k in U_k and the valuation V_k on b is the same as it would be on these tuples. Define the set

$$\theta_{k+1} = (T \times 2^{P_n} \times \{k+1\}) \setminus \mathscr{D}.$$

We denote by U_{k+1} the set $U_k \cup \theta_{k+1}$. The relation $\leqq_{k+1}$ on U_{k+1} is given by

$$\forall x \in \theta_{k+1}\, \forall y \in U_{k+1}$$
$$x \leqq_{k+1} y \quad \Leftrightarrow \quad (x = y) \vee (y \in U_k \wedge \exists z \in \pi_1(x)(z \leqslant_k y)).$$

Let

$$(\leqslant_{k+1}) = (\leqslant_k) \cup (\leqq_{k+1}),$$
$$\forall x \in \theta_{k+1}\, (x \in V_{k+1}(p_i) \Leftrightarrow p_i \in \pi_2(x)),$$
$$V_{k+1}(p_i) \cap U_k = V_k(p_i),$$
$$U_{k+1} = \langle U_{k+1}, \leqslant_{k+1}, V_{k+1}\rangle.$$

Lemma 10. *The relation $\leqslant_{k+1}$ is a partial order.*

Proof. It is clear that $\leqslant_{k+1}$ is reflexive and anti-symmetric. If $x \leqslant_{k+1} y \leqslant_{k+1} t$ and $x \notin \theta_{k+1}$, then $x \leqslant_k y \leqslant_k t$ and $x \leqslant_k t$ as $\leqslant_k$ is transitive. Let us suppose that $x \leqq_{k+1} y \leqslant_{k+1} t$, then $y \in U_k$ and $y \leqslant_k t$. By definition of $\leqq_{k+1}$ there exists $z \in \pi_1(x)$ such that $z \leqslant_k y$. But $\leqslant_k$ is transitive, therefore $z \leqslant_k t$, and by definition of $\leqq_{k+1}$ we obtain $x \leqq_{k+1} t$. This proves the lemma. □

Moreover, it is easy to see that elements of θ_{k+1} form an anti-chain with respect to $\leqslant_{k+1}$ and that the depth of the elements of θ_{k+1} is $k+1$. That is, $U_k = \mathscr{D}_k(U_{k+1})$ and U_{k+1} has the required properties.

We introduce the model $U(n) = \langle U(n), \leqslant, V\rangle$ by constructing a sequence of models U_k, $k < \omega$, where

$$U(n) := \bigcup_{k=1}^{\infty} U_k, \qquad (\leqslant) := \bigcup_{k=1}^{\infty} (\leqslant_k), \qquad V := \bigcup_{k=1}^{\infty} V_k.$$

Let $\langle W, \leqslant, V\rangle$ be a model, where $\leqslant$ is a partial order. The element $b \in W$ is called a duplicate of the element $a \in W$ if a is the smallest element in $\langle b\rangle \backslash \{b\}$ and $V(a) = V(b)$ (that is $\forall p_i\, a \Vdash_V p_i \Leftrightarrow b \Vdash_V p_i$). It is clear that in the model $\langle W, \leqslant, V\rangle$ the formula A is valid in a iff A is valid in b (induction on A). If we change the model by removal of the duplicate b from the model $\langle W, \leqslant, V\rangle$, then the validity of formulas on elements of the resulting model will coincide with validity in the initial model. (This property is easy to prove by induction on the length of formulas, using the preceding proposition). We shall use these facts in proving the next theorem.

Theorem 11. *The model $U(n)$ is n-characteristic for the modal system* Grz.

Proof. First we note that every formula of the set of theorems of Grz is valid in U_k as U_k is a finite poset. This implies that such formulas are valid in $U(n)$.

Let us assume that $A(p_1, \ldots, p_n)$ is not a theorem of Grz, that is, $A \notin$ Grz. By the finite model property of Grz [29] there exists a finite model (poset) $\mathscr{X} = \langle \mathscr{X}, \leqslant, V\rangle$ on which A is not valid. Let $\mathscr{X}_0 = \mathscr{X}$ and suppose that the model $\mathscr{X}_i = \langle \mathscr{X}_i, \leqslant, V\rangle$ such that $\mathscr{D}_i(\mathscr{X}_i)$ is an open submodel of $U(n)$ and such that the depth of $\mathscr{X}_i$ is not more than the depth of $\mathscr{X}$ and such that $A(p_1, \ldots, p_n)$ is not valid in $\mathscr{X}_i$ has already been constructed. We construct the model $\mathscr{X}_{i+1}$ as follows:

First we remove from $\mathscr{X}_i$ all duplicates to start with the minimum all the way to the top and we obtain a model $\langle \mathscr{X}_i'), \leqslant, V\rangle$, $\mathscr{X}_i' \subseteq \mathscr{X}_i$ in which A is also not valid. But $\mathscr{X}_i'$ has no duplicates. Since $U(n)$ has no duplicates, our removal does not concern the elements of $\mathscr{D}_i(\mathscr{X}_i)$ and $\mathscr{D}_i(\mathscr{X}_i) = \mathscr{D}_i(\mathscr{X}_i')$.

On the model $\mathscr{X}_i'$ we introduce an equivalence relation $\sim_i$, where two elements from $\mathscr{X}_i'$ are equivalent under $\sim_i$ if they both have depth $i+1$, and both have the same sets of strictly larger elements and the valuation V on them is the same.

Construct the factor-set $\mathscr{X}_i'/{\sim_i}$ under this equivalence relation. On $\mathscr{X}_i'/{\sim_i}$ the relation $\leqslant$ is inherited from $\mathscr{X}_i'$:

$$[a]_{\sim_i} \leqslant [b]_{\sim_i} \quad \Leftrightarrow \quad \exists r \in [a]_{\sim_i} \exists f \in [b]_{\sim_i} (r \leqslant f)$$

and the valuation V is also copied from $\mathscr{X}_i'$. We obtain a model $\mathscr{X}_{i+1} = \langle \mathscr{X}_{i+1}, \leqslant, V\rangle$. It is easy to see that $\leqslant$ is a partial order and that $\mathscr{D}_i(\mathscr{X}_{i+1}) = \mathscr{D}_i(\mathscr{X}_i')$, thus $\mathscr{D}_i(\mathscr{X}_{i+1})$ is an open submodel of the model $U(n)$, and for every two elements from $\mathscr{L}_{i+1}(\mathscr{X}_{i+1})$ which have the same sets of strictly larger elements, they differ one from another under the valuation V (choice of $\sim_i$). Moreover, the duplicates in $\mathscr{X}_{i+1}$, if any have depth more than $i+1$. Therefore $\mathscr{D}_{i+1}(\mathscr{X}_{i+1})$ is an open submodel of the model $U(n)$.

The depth of $\mathscr{X}_{i+1}$ coincides with the depth of $\mathscr{X}_i'$, hence the depth of $\mathscr{X}_{i+1}$ is not more than the depth of $\mathscr{X}$. By induction on the length of the formula B it is not difficult to check that

$$\forall x \in \mathscr{X}_i' \; ([x]_{\sim_i} \Vdash_V B \Leftrightarrow x \Vdash_V B).$$

Therefore A is not valid in $\mathscr{X}_{i+1}$. Continuing the construction of the models $\mathscr{X}_j$, we obtain the model $\mathscr{X}_m$ where m is the depth of $\mathscr{X}_0$. Then $\mathscr{D}_m(\mathscr{X}_m) = \mathscr{X}_m$ and $\mathscr{X}_m$ is an open submodel of the model $U(n)$ and A is not valid in $\mathscr{X}_m$. Thus A is not valid in $U(n)$. □

The element x of an arbitrary model $\langle W, R, V\rangle$ is said to be *expressible* if there exists a formula A such that $\forall y \in W\ (y \Vdash_V A \Leftrightarrow (x = y))$. Similarly $X \subseteq W$ is called *expressible* if $\forall y \in W\ (y \Vdash_V A \Leftrightarrow y \in X)$ for a certain formula A.

We need next

Lemma 12. *All elements of the model $U(n)$ are expressible.*

Proof. For $a \in U(n)$ we denote by $p(a)$ the set $\{i \mid a \Vdash_V p_i\}$. Let A_a be the formula

$$\bigwedge_{i \in p(a)} \Box p_i \wedge \bigwedge_{i \notin p(a)} \Box\neg p_i.$$

It is easy to see that $\forall a \in \mathscr{D}_1(U(n))$, $a \Vdash_V A_a$ holds and if $b \in \mathscr{D}_1(U(n))$, $b \neq a$, then $\neg(b \Vdash_V A_a)$. If $b \in U(n)$ and b has depth more than 1, then either there exist elements b_1, b_2 with depth 1, $b_1 \neq b_2$, b_1, $b_2 \in \langle b\rangle$ and obviously $b \Vdash_V \neg A_a$ or there is a maximal element in $\langle b\rangle$ (only one: b_1). In the last case there exists an element c with depth 2 such that $b \leqslant c < b_1$. By the construction of $U(n)$, c has no duplicates, hence $p(c) \neq p(b_1)$. Therefore $b \Vdash_V \neg A_a$. Thus all depth-1 elements are expressible. Suppose that all depth $\leqslant k$ elements are expressible, and let us denote by $f(x)$ the formula which defines x. Let e be some element of $\mathscr{L}_{k+1}(U(n))$. We introduce some formulas:

$$B_e = \bigwedge_{i \in P(e)} p_i \wedge \bigwedge_{i \notin P(e)} \neg p_i,$$

$$F_e = B_e \wedge \bigwedge_{x > e} (\Diamond f(x) \wedge \neg f(x)),$$

$$E_e = F_e \wedge \Box\Big(\bigvee_{x>e} f(x) \vee F_e\Big) \wedge \neg\Big(\bigvee_{x \not> e} \Diamond f(x)\Big).$$

It is clear that $e \Vdash_V E_e$, and if $a \in U(n)$ then

$$a \Vdash_V E_e \quad \Leftrightarrow \quad \langle a\rangle \cap \mathscr{D}_k(U(n)) = \langle e\rangle \cap \mathscr{D}_k(U(n)).$$

Therefore if $a \Vdash_V E_e$, then the sets of minimal elements in $\langle e\rangle \backslash \{e\}$ and in $(\langle a\rangle \backslash \{a\}) \cap \mathscr{D}_k(U(n))$ coincide. If $a \in \mathscr{L}_{k+1}(U(n))$, then the valuation V is the same on a and on e. By the construction of $U(n)$ we have $a = e$.

Assume that $a \in \mathscr{L}_{k+m}(U(n))$ and $m \geqslant 2$. Then there exists an element b with depth $k+2$ such that $a \leqslant b$. We consider the set $\langle b\rangle \backslash \{b\}$. If among the minimal elements of this set there are two elements with depth $k+1$, then one of them, say d, is distinct from e. In this case $a \Vdash_V E_e$ implies $\langle d\rangle \cap U_k = \langle e\rangle \cap U_k$ and

$d \Vdash_V B_e$. Then $p(d) \neq p(e)$, by the construction of $U(n)$, therefore $d \Vdash_V \neg B_e$. Contradiction.

Now suppose that there is only one element d which is minimal in the set $\langle b \rangle \backslash \{b\}$. In view of $a \Vdash_V E_e$ we have

$$\langle d \rangle \cap U_k = \langle b \rangle \cap U_k, \qquad d \Vdash_V B_e, \qquad b \Vdash_V B_e.$$

Then $p(d) = p(b)$, which contradicts the construction of elements of depth $k+1$ in the model $U(n)$ (it has no duplicates). So every element of the model $U(n)$ is expressible. $\square$

Note that Lemma 9 and Theorem 11 immediately imply:

Theorem 13. *The free modal algebra of rank n in the variety* Var(Grz) *is isomorphic to the subalgebra* $\langle U(n), \leqslant \rangle^+(V(P_n))$ *of the algebra* $\langle U(n), \leqslant \rangle^+$ *which is freely generated by the set* $V(P_n)$.

3. The universal theory of the free modal algebra $\mathscr{F}_\omega$(Grz)

As above, if $\mathscr{F}_\omega(\lambda)$ is the free algebra from Var(λ) then Σ_f denotes the signature of $\mathscr{F}_\omega(\lambda)$ extended by constants for free generators. In this section quasi-identity is always in the signature Σ_f of the algebra $\mathscr{F}_\omega$(Grz).

Two quasi-identities are called equivalent if they are equivalent as universal formulas in the class Var(Grz). First we show that it is sufficient to consider only quasi-identities in a special, rather simple form. In this part it is convenient to use the modal operator $\diamond$ as basic (keeping the identities $\square x = \neg\diamond\neg x$, $\diamond x = \neg\square\neg x$ in mind). It is easy to see that an arbitrary quasi-identity of the form $\bigwedge_i (f_i = g_i) \rightarrow (f = g)$ is equivalent to a quasi-identity of the form $(\bigwedge_i (f_i \leftrightarrow g_i)) = 1 \Rightarrow (f \leftrightarrow g) = 1$. Therefore it is sufficient to consider only quasi-identities of the form $A = 1 \Rightarrow B = 1$. Let us denote $x^0 := x$, $x^1 := \neg x$.

Theorem 14. *There exists an algorithm which constructs for an arbitrary quasi-identity q an equivalent quasi-identity r(q) of the form*

$$r(q) = \left[\left(\bigvee_j \varphi_j \right) = 1 \quad \Rightarrow \quad \neg\diamond x_0 = 1 \right],$$

where

$$\varphi_j = \bigwedge_{i=1}^{m} x_i^{k(j,i,1)} \wedge \bigwedge_{i=0}^{m} (\diamond x_i)^{k(j,i,2)},$$

$k(j, i, 1)$, $k(j, i, 2) \in \{0, 1\}$, x_i are either variables or constants from Σ_f. Moreover, r(q) and q have the same constants and all variables from q are variables from r(q). If r(q) is not valid in $\mathfrak{B} \in$ Var(Grz) where the variables from r(q), x_i, take values $a_i \in \mathfrak{B}$, then q is also not valid in $\mathfrak{B}$ when its variables x_i take the same value $a_i \in \mathfrak{B}$.

Proof. Let $q = (A = 1) \Rightarrow (B = 1)$ be a quasi-identity. Then q is equivalent to the quasi-identity $A = 1 \Rightarrow \Box B = 1$. By introducing the additional variable x_0 we obtain the quasi-identity

$$[(A \wedge (\Diamond\neg B \leftrightarrow \Diamond x_0)) = 1 \quad \Rightarrow \quad \neg\Diamond x_0 = 1$$

which is equivalent to the preceding one. Thus we have that q is equivalent to a quasi-identity of the form

$$f = 1 \quad \Rightarrow \quad \neg\Diamond x_0 = 1 \tag{2}$$

in which the variable x_0 occurs only in the subterm $\Diamond x_0$. We transform the premise of (2). We introduce a variable x_t for each subterm of the premise of (2). If t is a variable or a constant we take $x_t = t$. Let us consider the quasi-identity

$$x_f \wedge \bigwedge_{t=t_1*t_2} (x_t \leftrightarrow x_{t_1} * x_{t_2}) \wedge \bigwedge_{t=*t_1} (x_t \leftrightarrow * x_{t_1}) = 1 \quad \Rightarrow \quad \neg\Diamond x_0 = 1. \tag{3}$$

From the construction of (3) we easily obtain that (2) and (3) are equivalent. The use of this method of transformation of premise goes back to Waisberg, as it seems. Note that x_0 in (3), as in (2), occurs only in subterms $\Diamond x_0$. Thus q is equivalent to a quasi-identity of the form

$$\mathscr{V}(\Diamond y_j, y_k, \Diamond x_0) = 1 \quad \Rightarrow \quad \neg\Diamond x_0 = 1, \tag{4}$$

where $\mathscr{V}$ is a boolean term, y_j, y_k, $k \neq 0$ are either variables or constants. We transform the premise of (4) into disjunctive normal form. Thus we obtain the equivalent quasi-identity

$$\bigvee_i \theta_i = 1 \quad \Rightarrow \quad \neg\Diamond x_0 = 1. \tag{5}$$

We can assume that every variable and every constant (of (5)) x_ξ and $\Diamond x_\xi$, $\Diamond x_0$ occur in each of the disjunct θ_i. Otherwise, we replace a disjunct θ_i by disjuncts that are obtained from the original by prefixing conjunctions of the missing elements with all possible distributions of $\neg$. It is clear that the resulting quasi-identity

$$r(q) = \left[\left(\bigvee_j \varphi_j = 1\right) \Rightarrow \neg\Diamond x_0 = 1\right] \tag{6}$$

is equivalent to (5) and has the required form.

Let $r(q)$ not be valid in $\mathfrak{B} \in \mathrm{Var(Grz)}$ when $x_i = a_i$ and when the constants also take fixed values. It is easy to see that under the chosen values of the variables the quasi-identities (5), (4), (3) also are not valid in $\mathfrak{B}$. The fact that (3) is not valid when $x_t = a_t$ gives us $a_t = * a_{t_1}$ and $a_t = a_{t_1} * a_{t_2}$. Then $a_f = 1$ and (2) is not valid in $\mathfrak{B}$ when $x_i = a_i$. So $A(a_i) = 1$, $B(a_i) \neq 1$ in $\mathfrak{B}$. $\Box$

We call the quasi-identity $r(q)$ the *reduced form* (notation: R.F. or r.f.) of q. If a quasi-identity has the form $r(q)$, we say that it has reduced form or that it is in reduced form.

If q is a quasi-identity, then $P(q)$ is the set of constants from Σ_f which occur in q.

Let $q = [\bigvee \varphi_j = 1 \Rightarrow \neg\Diamond x_0 = 1]$ be a quasi-identity in r.f. We introduce the additional notations:

$$\theta_1(\varphi_j) := \{x_i \mid (k(j, i, 1) = 0 \;\&\; i > 0) \vee (i = 0 \;\&\; k(j, 0, 2) = 0)\},$$

$$\theta_2(\varphi_j) := \{x_i \mid k(j, i, 2) = 0\}.$$

$\mathscr{D}(q)$ denotes the set of disjunction members of the premise of q which have the property $\theta_2(\varphi_j) \supseteq \theta_1(\varphi_j)$.

$P_1(\varphi_j)$ and $P_2(\varphi_j)$ denote the sets $\theta_1(\varphi_j) \cap P(q)$ and $\theta_2(\varphi_j) \cap P(q)$ respectively.

If $\langle W, R, V\rangle$ is an arbitrary model, where the domain of the valuation V contains the set $P(q)$ (which we now consider as consisting of propositional letters), then $P(a)$ denotes the set

$$\{p_i \mid p_i \in P(q) \;\&\; a \Vdash_V p_i\} \quad \text{for } a \in W.$$

We turn to basics for further results in constructing models on the members of $\mathscr{D}(q)$.

Models on subsets of $\mathscr{D}(q)$

Let $\mathscr{X} \subseteq \mathscr{D}(q)$. We introduce the model $\langle \mathscr{X}, \lhd, V\rangle$, where $\forall \varphi_i, \varphi_j \in \mathscr{X}$

$$\varphi_i \lhd \varphi_j \quad \Leftrightarrow \quad (\varphi_i = \varphi_j) \vee (\theta_2(\varphi_i) \supsetneqq \theta_2(\varphi_j))$$

and the valuation V on the set $P(q)$ and the set of all variables from q (both these sets we consider as consisting of propositional letters) is determined by equality:

$$V(x_i) := \{\varphi_j \mid x \in \theta_1(\varphi_j)\}.$$

The reader will note that the relation $\lhd$ is a partial order, that is $\langle \mathscr{X}, \lhd\rangle$ is poset.

Let for each $\varphi_j \in \mathscr{X}$ the subset $T(\varphi_j)$ of the set $\mathscr{D}(q)$ be fixed (e.g. $T(\varphi_j) = \emptyset$ is allowed) such that $\varphi_j \notin T(\varphi_j)$ and

$$\forall \varphi_k \in T(\varphi_j)\,(\theta_2(\varphi_k) = \theta_2(\varphi_j)).$$

We consider elements from distinct $T(\varphi_j)$ and $\mathscr{X}$ as distinct elements: $T(\varphi_j) \cap T(\varphi_i) = \emptyset$, $T(\varphi_j) \cap \mathscr{X} = \emptyset$ (even if these sets had non-empty intersections, for the sake of notation).

On the set $\mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)]$ the relation $\leqslant$ is determined such that: $\leqslant$ is the reflexive, transitive closure of the relation $(\lhd) \cup (\leqslant_1)$ where

$$\forall \varphi_k \in T(\varphi_j)\,(\varphi_k \leqslant_1 \varphi_j).$$

It is easy to see that $\leqslant$ is a partial order. On the frame $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant\rangle$ the valuation V is defined as above: $V(x_i) = \{\varphi_j \mid x \in \theta_1(\varphi_j)\}$.

Let us recall that the constants from $P(q)$ are interpreted on $\mathcal{F}_k(\mathrm{Grz})(\mathcal{F}_\omega(\mathrm{Grz}))$ as distinct free generators.

Theorem 15. *If the quasi-identity q has reduced form and is not valid in $\mathcal{F}_\omega(\mathrm{Grz})$, then there exists a set $\mathcal{X}$, where $\mathcal{X} \subseteq \mathcal{D}(q)$, and for each $\varphi_j \in \mathcal{X}$ there exists the set $T(\varphi_j) \subseteq \mathcal{D}(q)$ such that $\forall \varphi_k \in T(\varphi_j)(\theta_2(\varphi_k) = \theta_2(\varphi_j))$ and the model $\langle \mathcal{X} \cup [\bigcup_{\varphi_j \in \mathcal{X}} T(\varphi_j)], \leqslant, V \rangle$ satisfies:*

(1) *There exists $\varphi_j \in \mathcal{X}$ such that $k(j, 0, 2) = 0$.*

(2) *For each element φ_j from this model, $\varphi_j \Vdash_V \varphi_j$.*

(3) *If $\mathcal{H}$ is a subset of this model and A is a subset of the $P(q)$, then there exists an element $\varphi(\mathcal{H}, A)$ from this model such that $P_1(\varphi(\mathcal{H}, A)) = A$ and*

$$\theta_2(\varphi(\mathcal{H}, A)) = \theta_1(\varphi(\mathcal{H}, A)) \cup \Big(\bigcup_{\varphi_\beta \in \mathcal{H}} \theta_2(\varphi_\beta) \Big).$$

Proof. If q is not valid in $\mathcal{F}_\omega(\mathrm{Grz})$, then q is also not valid in $\mathcal{F}_k(\mathrm{Grz})$ for some k, where k is greater than the number of elements in $P(q)$. By Theorem 13 we then see that q is not valid in $\langle U(k), \leqslant \rangle^+$. So there exist subsets B_i of the set $U(k)$ such that

$$\bigvee_j \varphi_j(B_i) = 1, \qquad \Diamond B_0 \neq 0. \tag{7}$$

Let Y be the set $\{\varphi_j \mid \varphi_j(B_i) \neq \emptyset\}$. Then $Y \subseteq \mathcal{D}(q)$. For each $\varphi_j \in Y$ we introduce the set $[\varphi_j] \subseteq U(k)$ where

$$[\varphi_j] := \bigcup \{\varphi_m(B_i) \mid \varphi_m \in Y \,\&\, \theta_2(\varphi_m) = \theta_2(\varphi_j)\}.$$

We denote by $\max([\varphi_j])$ the set of maximal elements of the set $[\varphi_j]$ (in the frame $U(k)$).

Each element a of $U(k)$ is included in a unique set $\varphi_l(B_i)$ (recall that the sets $\varphi_m(B_i)$ are disjoint). We fix for each element $a \in \max([\varphi_j])$ the set $\varphi_l(B_i)$ such that $a \in \varphi_l(B_i)$, $\varphi_l(B_i) \subseteq [\varphi_j]$, $\varphi_l \in Y$, and denote this φ_l by φ_a (that is $a \in \varphi_a(B_i)$).

For an arbitrary member φ_j of the set Y we introduce the set $\bar{C}(\varphi_j)$,

$$\bar{C}(\varphi_j) := \{\varphi_a \mid a \in \max([\varphi_j])\}.$$

Now we may define the set $\mathcal{X}$:

$$\mathcal{X} := \bigcup_{\varphi_j \in Y} \bar{C}(\varphi_j).$$

For each $\varphi_j \in \mathcal{X}$ let $T(\varphi_j)$ denote the set

$$\{\varphi_k \mid \varphi_k \in Y, \theta_2(\varphi_k) = \theta_2(\varphi_j)\} \setminus \{\varphi_j\}.$$

Now our aim is to prove that the model $\langle \mathcal{X} \cup [\bigcup_{\varphi_j \in \mathcal{X}} T(\varphi_j)], \leqslant, V \rangle$ has the desired properties.

By (7) we have $B_0 \neq \emptyset$. Let $x \in B_0$, then by (7), $x \in \varphi_j(B_i)$ for some j. So $k(j, 0, 2) = 0$ and $\varphi_j \in Y$. If we take an arbitrary $\varphi_\xi \in \bar{C}(\varphi_j)$, then $k(\xi, 0, 2) = 0$ and $\varphi_\xi \in \mathscr{X}$. Thus property (1) is true.

Before we turn to property (2), we shall prove the following result.

Lemma 16. *In the frame $\langle \mathscr{X}, \lhd \rangle$, $\theta_1(\varphi_j) = \theta_2(\varphi_j)$ holds for each $\varphi_j \in \mathscr{D}_1(\mathscr{X})$.*

Proof. Let φ_j be a member of $\mathscr{D}_1(\mathscr{X})$. By the construction of $\mathscr{X}$, φ_j is a member of $\bar{C}(\varphi_j)$ and $\varphi_j = \varphi_a$, $a \in \max([\varphi_j])$, $a \in \varphi_a(B_i) = \varphi_j(B_j)$.

Consider the maximal element v of the frame $\langle U(k), \leqslant \rangle$ such that $a \leqslant v$. By (7) there exists $\varphi_\xi \in Y$ such that $v \in \varphi_\xi(B_j)$. If $x_r \in \theta_2(\varphi_\xi)$, then $v \in \Diamond B_r$. This implies that $v \in B_r$ as v is a maximal element of $U(k)$. But $v \in \varphi_\xi(B_i)$, therefore $x_r \in \theta_1(\varphi_\xi)$. Consequently $\theta_1(\varphi_\xi) = \theta_2(\varphi_\xi)$.

We claim that $\varphi_\xi = \varphi_j$. Let x_r be a member of $\theta_2(\varphi_\xi)$. Then we have $v \in \Diamond B_r$, as $v \in \varphi_\xi(B_i)$. Therefore $a \leqslant v$ implies $a \in \Diamond B_r$. We recall that $a \in \varphi_a(B_i)$, therefore $\Diamond x_r \in \theta_2(\varphi_a)$. Since $\varphi_a = \varphi_j$, we obtain $x_r \in \theta_2(\varphi_j)$. Hence $\theta_2(\varphi_j) \supseteq \theta_2(\varphi_\xi)$. Let us assume $\theta_2(\varphi_j) \nsubseteq \theta_2(\varphi_\xi)$. Then v is clearly maximal in $[\varphi_\xi]$. Hence $\varphi_\xi \in \bar{C}(\varphi_\xi)$ and $\varphi_\xi \in \mathscr{X}$. Then from the assumption $\theta_2(\varphi_j) \nsubseteq \theta_2(\varphi_\xi)$ it follows that $\varphi_j \lhd \varphi_\xi$, $\varphi_j \neq \varphi_\xi$ which contradicts $\varphi_j \in \mathscr{D}_1(\mathscr{X})$.

So $\theta_2(\varphi_j) = \theta_2(\varphi_\xi)$ and $[\varphi_\xi] = [\varphi_j]$. Therefore from $a \in \max[\varphi_\xi]$, $a \leqslant v$, $v \in [\varphi_\xi]$ we conclude that $a = v$. Then from $a \in \varphi_j(B_i)$ and $v \in \varphi_\xi(B_i)$ we conclude that $\varphi_\xi = \varphi_j$ and $\theta_1(\varphi_j) = \theta_2(\varphi_j)$. $\square$

Now we need the next lemma.

Lemma 17. *In the model $\langle \mathscr{X}, \lhd, V \rangle$ $\varphi_\alpha \Vdash_V \varphi_\alpha$ holds for each $\varphi_\alpha \in \mathscr{X}$.*

Proof. We prove this lemma by induction on the depth of the element φ_α in the model $\langle \mathscr{X}, \lhd, V \rangle$. If φ_α has depth 1 then by Lemma 16, $\varphi_\alpha \Vdash_V \varphi_\alpha$. Suppose that the claim of lemma holds for all $\varphi_\xi \in \mathscr{D}_k(\mathscr{X})$. Let φ_α be a member of $\mathscr{L}_{k+1}(\mathscr{X})$. By definition of the model $\langle \mathscr{X}, \lhd, V \rangle$,

$$\forall \varphi_j \in \mathscr{X} \quad \varphi_j \Vdash_V x_r \quad \Leftrightarrow \quad x_r \in \theta_1(\varphi_j).$$

Therefore in φ_α the nonmodal part of the conjunction of φ_α is true. Let $\varphi_\alpha \Vdash_V \Diamond x_r$. Then there exists $\varphi_\beta \in \mathscr{X}$ such that $\varphi_\alpha \lhd \varphi_\beta$ and $\varphi_\beta \Vdash_V x_r$, that is $x_r \in \theta_1(\varphi_\beta)$. If $\varphi_\beta = \varphi_\alpha$, then $x_r \in \theta_1(\varphi_\alpha)$ and $x_r \in \theta_2(\varphi_\alpha)$. Now we assume that $\varphi_\alpha \neq \varphi_\beta$. Then $\varphi_\beta \in \mathscr{D}_k(\mathscr{X})$ and by the induction hypothesis $\varphi_\beta \Vdash_V \varphi_\beta$.

Since the relation $\lhd$ is reflexive we obtain $\varphi_\beta \Vdash_V \Diamond x_r$ from $\varphi_\beta \Vdash_V x_r$. Then $x_r \in \theta_2(\varphi_\beta)$ follows from $\varphi_\beta \Vdash_V \varphi_\beta$. At the same time $\theta_2(\varphi_\beta) \subseteq \theta_2(\varphi_\alpha)$ and we have $x_r \in \theta_2(\varphi_\beta)$, and $x_r \in \theta_2(\varphi_\beta)$ gives us $x_r \in \theta_2(\varphi_\alpha)$.

Assume now that $x_r \in \theta_2(\varphi_\alpha)$. If $x_r \in \theta_1(\varphi_\alpha)$, then by definition of V on $\mathscr{X}$ we have $\varphi_\alpha \Vdash_V x_r$ and $\varphi_\alpha \Vdash_V \Diamond x_r$. Let now $x_r \in \theta_2(\varphi_\alpha) \backslash \theta_1(\varphi_\alpha)$; by definition of $\mathscr{X}$ it is true that

$$\varphi_\alpha = \varphi_a, \qquad a \in \max([\varphi_\alpha]), \qquad a \in \varphi_a(B_i).$$

Corrollary: $a \in \Diamond B_r$ and $a \notin B_r$. Then there exists $b \in U(k)$ such that $a < b$ and $b \in B_r$. By (7) there exists $\varphi_j \in Y$ such that $b \in \varphi_j(B_i)$. The relation $a < b$ gives us $\theta_2(\varphi_a) \supseteq \theta_2(\varphi_j)$. Moreover the relation $a < b$ implies $\theta_2(\varphi_j) \supsetneqq \theta_2(\varphi_\alpha)$ (as $a \in \max([\varphi_\alpha])$), that is, $\theta_2(\varphi_\alpha) \supsetneqq \theta_2(\varphi_j)$.

Let φ_β be an arbitrary member of $\bar{C}(\varphi_j)$. Then $\theta_2(\varphi_\beta) = \theta_2(\varphi_j)$ and $\varphi_\alpha \lhd \varphi_\beta$, $\varphi_\alpha \neq \varphi_\beta$. From $b \in \varphi_j(B_i)$ and $b \in B_r$ we conclude $x_r \in \theta_1(\varphi_j)$, $x_r \in \theta_2(\varphi_j)$. But $\theta_2(\varphi_\beta) = \theta_2(\varphi_j)$, consequently $x_r \in \theta_2(\varphi_\beta)$. By means of $\varphi_\alpha \lhd \varphi_\beta$, $\varphi_\alpha \neq \varphi_\beta$, $\varphi_\alpha \in \mathscr{L}_{k+1}(\mathscr{X})$, we conclude that $\varphi_\beta \in \mathscr{D}_k(\mathscr{X})$. Then by induction hypothesis we may conclude that $\varphi_\beta \Vdash_V \varphi_\beta$.

Hence, by $x_r \in \theta_2(\varphi_\beta)$, it is true that $\varphi_\beta \Vdash_V \Diamond x_r$. Therefore $\varphi_\alpha \lhd \varphi_\beta$ implies $\varphi_\alpha \Vdash_V \Diamond x_r$. Thus we proved

$$x_r \in \theta_2(\varphi_\alpha) \quad \Leftrightarrow \quad \varphi_\alpha \Vdash_V \Diamond x_r.$$

This conclusion and the observation that on φ_α the nonmodal part of the conjunction φ_α is valid gives us that $\varphi_\alpha \Vdash_V \varphi_\alpha$. □

On the basis of this lemma we may prove next:

Lemma 18. *For each element φ_α of the model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ we have $\varphi_\alpha \Vdash_V \varphi_\alpha$.*

Proof. Let φ_j be an element of the set $\mathscr{X}$. Since $\langle \mathscr{X}, \lhd, V \rangle$ is an open submodel of the given model, we get $\varphi_j \Vdash_V \varphi_j$ by Lemma 17 and we may conclude that $\varphi_j \Vdash_V \varphi_j$ in our model.

Assume that φ_k is an element of the set $T(\varphi_j)$. By definition of V on φ_k the nonmodal part of the conjunction φ_k is valid.

If $x_r \in \theta_2(\varphi_k)$, then $x_r \in \theta_2(\varphi_j)$. We showed above $\varphi_j \Vdash_V \varphi_j$. Hence $\varphi_j \Vdash_V \Diamond x_r$. But $\varphi_k \leqslant \varphi_j$, therefore $\varphi_k \Vdash_V \Diamond x_r$.

Assume that $\varphi_k \Vdash_V \Diamond x_r$. First suppose that $\varphi_k \Vdash_V x_r$. Then $x_r \in \theta_1(\varphi_k)$, and from $\theta_1(\varphi_k) \subseteq \theta_2(\varphi_k)$ we obtain $x_r \in \theta_2(\varphi_k)$. Next suppose that $\varphi_k \Vdash_V \neg x_r$. In this case there exists φ_l: $\varphi_k < \varphi_l$, where $\varphi_l \Vdash_V x_r$ and $x_r \in \theta_1(\varphi_l)$. There exists only one immediate successor for φ_k: φ_j. Therefore $\varphi_j \Vdash_V \Diamond x_r$. As we noted above $\varphi_j \Vdash_V \varphi_j$, it follows that $x_r \in \theta_2(\varphi_j)$. Since φ_k is an element of the set $T(\varphi_j)$ we obtain $x_r \in \theta_2(\varphi_k)$. Thus $x_r \in \theta_2(\varphi_k) \Leftrightarrow \varphi_k \Vdash_V \Diamond x_r$. Therefore $\varphi_k \Vdash_V \varphi_k$. □

According to Lemma 18, the model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ has property (2) from Theorem 15.

Lemma 19. *The model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ has property* (3) *from Theorem* 15.

Proof. Let $\mathscr{H}$ be a subset of our model and let A be a subset of the $P(q)$. We may without loss of generality assume that $\mathscr{H} \subseteq \mathscr{X}$ such that $\forall \varphi_k \in T(\varphi_j)$ we have $\theta_2(\varphi_j) = \theta_2(\varphi_k)$. By definition of $\mathscr{X}$ we have

$$\mathscr{H} = \{\varphi_{a_j} \mid j \leqslant k\}, \qquad a_j \in \max([\varphi_{a_j}]), \qquad a_j \in \varphi_{a_j}(B_i).$$

Consider the set $\{a_j \mid j \leqslant k\}$. Let $\{a_j \mid j \leqslant m\}$ be the set of minimal (in the frame $\langle U(k), \leqslant \rangle$) elements of the set $\{a_j \mid j \leqslant k\}$. According to the construction of the model $U(k)$ there exists $x \in U(k)$ such that

$$\langle x \rangle = \{x\} \cup \langle \{a_j \mid j \leqslant m\} \rangle, \qquad P_1(x) = A. \tag{8}$$

By (7), $x \in \varphi_\alpha(B_i)$ for certain $\varphi_\alpha \in Y$. Since $P_1(x) = A$, $x \in \varphi_\alpha(B_i)$ implies $P_1(\varphi_\alpha) = A$. We claim that

$$\theta_2(\varphi_\alpha) \supseteq \theta_1(\varphi_\alpha) \cup \Big(\bigcup_{\varphi_{\alpha_j} \in \mathscr{H}} \theta_2(\varphi_{\alpha_j}) \Big).$$

Indeed, let x_r be a member of $\theta_2(\varphi_{\alpha_j})$, $\varphi_{\alpha_j} \in \mathscr{H}$. By the choice of x we have $x \leqslant a_j$. Thus from $a_j \in \varphi_{a_j}(B_i)$ and from $x \in \varphi_\alpha(B_i)$ it follows that $a_j \in \Diamond B_r$ and $x \in \Diamond B_r$. Thus we have $x_r \in \theta_2(\varphi_\alpha)$, which was required.

Now we assume that $x_r \in \theta_2(\varphi_\alpha) \backslash \theta_1(\varphi_\alpha)$. From $x \in \varphi_\alpha(B_i)$ and from $x \notin B_r$, $x \in \Diamond B_r$ follows. But these assumptions by (8) imply that there exists an a_j, where $j \leqslant k$, such that $a_j \in \Diamond B_r$. Therefore we have $x_r \in \theta_2(\varphi_{a_j})$, since $a_j \in \varphi_{a_j}(B_i)$. Thus we conclude

$$\theta_2(\varphi_\alpha) = \theta_1(\varphi_\alpha) \cup \Big(\bigcup_{\varphi_{\alpha_j} \in \mathscr{H}} \theta_2(\varphi_{a_j}) \Big).$$

It remains to note that $\varphi_\alpha \in Y$ and either $\varphi_\alpha \in \mathscr{X}$ or $\varphi_\alpha \in (\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j))$ and that φ_α is some element of our model. Corollary: we may choose the element φ_α for $\varphi(\mathscr{H}, A)$. $\square$

With Lemma 19 the proof of Theorem 15 is completed. We proved that the model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ has properties (1)–(3) from the formulation of Theorem 15.

The next theorem is basic for the remaining results of this paper.

Theorem 20. *Let q be a quasi-identity in reduced form (in the signature Σ_f). If there exists a set $\mathscr{X}$, where $\mathscr{X} \subseteq \mathscr{D}(q)$, and for each $\varphi_j \in \mathscr{X}$ there exists a set $T(\varphi_j) \subseteq \mathscr{D}(q)$, where $\forall \varphi_k \in T(\varphi_j)(\theta_2(\varphi_k) = \theta_2(\varphi_j))$, $\varphi_j \notin T(\varphi_j)$, such that the model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ has properties (1)–(3) from Theorem 15, then q is not valid in $\mathscr{F}_\omega(\mathrm{Grz})$.*

Proof. Suppose that $\mathscr{X}$, $T(\varphi_j)$, $\varphi_j \in \mathscr{X}$ with properties from the condition of the theorem have been chosen. We take an n-characteristic model $\langle U(k), \leqslant, V \rangle$ for Grz (see Theorem 11) where k is the sum of the number of elements in

$\mathscr{X} \cup [\bigcup T(\varphi_j)]$, $\varphi_j \in \mathscr{X}$ and of the elements in $P(q)$. We stipulate that $P(q)$ is included in the domain of V, that is constants from q under valuation V are mapped into subsets of the set $U(k)$. First we prove the following lemma.

Lemma 21. *The frame* $\mathscr{X}_1 = \langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant \rangle$ *may be included in the frame* $\langle U(k), \leqslant, V \rangle$ *as an open subframe. Moreover,*

$$\forall \varphi_j \in \Big(\mathscr{X} \cup \Big[\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)\Big]\Big), \quad \forall p_i \in P(q)$$
$$p_i \in P_1(\varphi_j) \quad \Leftrightarrow \quad \varphi_j \in V(P_i),$$

where V is the valuation on $U(k)$.

Proof. We shall prove by induction on n that the claim of the lemma is true for $\mathscr{D}_n(\mathscr{X}_1)$. First we assign to members φ_j of the frame $\mathscr{X}_1$ distinct propositional letters $p(\varphi_j)$ from $(P_k \backslash P(q))$ (where $V: P_k \to 2^{U(k)}$). Consider the set $\mathscr{D}_1(\mathscr{X}_1)$ of elements of depth 1. Recall that if $c \in U(k)$, then by $P(c)$ we denote the set $\{p \mid p \in P(q),\ c \Vdash_V p\}$. By the construction of $U(k)$ and U_{k1} there exist elements $a_j \in \mathscr{D}_1(U(k))$ such that $P(a_j) = P_1(\varphi_j)$ and $\forall p_\xi \in (P_k \backslash P(q))$, $a_j \Vdash_V p_\xi \Leftrightarrow p_\xi = P(\varphi_j)$.

Therefore $\mathscr{D}_1(\mathscr{X}_1)$ may be considered an open subframe of the frame $\mathscr{D}_1(U(k))$. Suppose that $\mathscr{D}_n(\mathscr{X}_1)$ satisfies the claim of the lemma. Let $\varphi_j \in \mathscr{L}_{n+1}(\mathscr{X}_1)$. By our assumption the set $\langle \varphi_j \rangle \backslash \{\varphi_j\}$ is a subset of the set $\mathscr{D}(U(k))$. By the construction of $U(k)$ we may conclude that there exists $a_j \in \mathscr{L}_{n+1}(U(k))$ such that

$$\langle a_j \rangle \backslash \{a_j\} = \langle \varphi_j \rangle \backslash \{\varphi_j\},$$
$$P(a_j) = \{p_\xi \mid a_j \Vdash_V p_\xi,\ p_\xi \in P(q)\} = P_1(\varphi_j),$$
$$\forall p_\xi \in (P_k \backslash P(q)) \quad a_j \Vdash_V p_\xi \Leftrightarrow p_\xi = P(\varphi_j)$$

(a_j is not a duplicate so that $\forall b \in \langle q_j \rangle\ b \Vdash_V P(\varphi_j) \Leftrightarrow b = a_j$).

We assign to φ_j an element $a_j \in \mathscr{L}_{n+1}(U(k))$ for each $\varphi_j \in \mathscr{L}_{n+1}(\mathscr{X}_1)$. We obtain that $\mathscr{D}_{n+1}(\mathscr{X}_1)$ is an open subframe of the frame $\mathscr{D}_{n+1}(U(k))$ and the desired properties are true. $\square$

Now we fix the inclusion of $\mathscr{X}_1$ into $U(k)$ (as an open subframe) which exists by Lemma 21. Thus we regard the $\varphi_\alpha \in \mathscr{X}_1$ as elements of the model $U(k)$.

We now turn to the construction of a special sequence of subsets $\varphi^t_\alpha \subseteq U(k)$, $\varphi_\alpha \in \mathscr{X}_1$, $t \in \mathbb{Z}$, $-1 \leqslant t \leqslant m_1$ where m_1 is the number of elements in the set $\mathscr{X}_1$. This sequence will have the properties:

(a) If $\alpha \neq \beta$, then $\varphi^t_\alpha \cap \varphi^t_\beta = \emptyset$. (9)

(b) For each φ_α, $\varphi^t_\alpha \subseteq \varphi^{t+1}_\alpha$. (10)

(c) $\forall a \in \varphi^t_\alpha \quad P(a) = P_1(\varphi_\alpha)$. (11)

(d) The sets φ^t_α are expressible in $U(k)$. (12)

(e) If $t \geqslant 0$, then $\forall x \in U(k)\ (x \notin \bigcup \varphi^t_\alpha)$, $\varphi_\alpha \in \mathscr{X}_1$, implies that there exist distinct $\varphi^t_{j_1}, \ldots, \varphi^t_{j_{t+1}}$ such that $x \in \Diamond \varphi^t_{j_r}$, $1 \leqslant r \leqslant t+1$. (13)

Let $(\forall \varphi_\alpha \in \mathscr{X}_1)$ φ_α^{-1} be $\{\varphi_\alpha\} \subseteq U(k)$. By Lemma 21 we have $P_1(\varphi_\alpha) = P(\varphi_\alpha)$, where we recall that $P(\varphi_\alpha) := \{p \mid p \in P(q),\ \varphi_\alpha \Vdash_V p\}$. Therefore (11) is true. By Lemma 12 all elements of $U(k)$ are expressible, so is therefore φ_α^{-1}, and (12) is also true.

Consider the first layer of the model $U(k)$ without elements of the chosen sets φ_α^{-1}. Let $x \in \mathscr{L}_1(U(k)) \backslash \bigcup \{\varphi_\alpha^{-1} \mid \varphi_\alpha \in \mathscr{X}_1\}$. By property (3) from the conditions of Theorem 15, there exists for $\mathscr{H} = \emptyset$ and $A = P_1(x)$ a $\varphi(\mathscr{H}, A)$ such that

$$\theta_2(\varphi(\emptyset, A)) = \theta_1(\varphi(\emptyset, A)), \quad P_1(\varphi(\mathscr{H}, A)) = A.$$

We fix such a $\varphi(\emptyset, A)$ for each $A = P(x)$ and we put

$$\varphi(\emptyset, A)^0 := \{\varphi(\emptyset, A)\} \cup \{y \mid y \in (\mathscr{L}_1(U(k)) \backslash \bigcup \{\varphi_\alpha^{-1} \mid \varphi_\alpha \in \mathscr{X}_1\}),\ P(y) = A\}.$$

If $\varphi_i \neq \varphi(\emptyset, A)$ for all the above fixed $\varphi(\emptyset, A)$, then we make $\varphi_i^0 = \varphi_i^{-1}$. It is obvious that φ_i^{-1}, φ_i^0, $\varphi_i \in \mathscr{X}_1$ have the properties (9) and (10). Moreover, if $x \in \varphi(\emptyset, A)^0$, then $P(x) = A = P_1(\varphi(\emptyset, A))$, so that (11) is true for $\varphi(\emptyset, A)^0$. Therefore φ_i^0, $\varphi_i \in \mathscr{X}_1$ also have the property (11). Property (12) follows for φ_i^0, $\varphi_i \in \mathscr{X}_1$ from the finiteness of $\mathscr{L}_1(U(k))$ and Lemma 12.

If $x \in U(k)$, then there exists $y \in \mathscr{L}_1(U(k))$ such that $x \leqslant y$. By the construction of φ_α^0, $\varphi_\alpha \in \mathscr{X}_1$, we have $y \in \varphi_j^0$ for some $\varphi_j \in \mathscr{X}_1$. Therefore $x \in \Diamond \varphi_j^0$. Thus property (13) holds for φ_j^0, $\varphi_j \in \mathscr{X}_1$.

Let us assume that subsets φ_α^t, $\varphi_\alpha \in \mathscr{X}_1$, with the required properties (9)–(13) have been constructed $(t < m_1)$.

Let $\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}$ be some members of the set $\mathscr{X}_1$. We introduce the set

$$[\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}] := \Big(\neg \bigvee_{\varphi_\alpha \in \mathscr{X}_1} f(\varphi_\alpha^t)\Big) \wedge \Big(\bigwedge_{1 \leqslant i \leqslant t+1} \Diamond(f(\varphi_{\alpha_i}^t)) \wedge \Big(\bigwedge_{\beta \neq \alpha 1, \ldots, \alpha(t+1)} \neg\Diamond(f(\varphi_\beta^t)\Big),$$

where from now on $f(X)$ denotes the formula defining the expressible set X (φ_α^t is expressible by our assumption). In order to simplify notation we shall often denote the formula and what it defines by the same symbols. The formula at the right-hand side of the equation is denoted by $\psi(\mathscr{H})$ (where $\mathscr{H} := \{\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}\}$), the set $[\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}]$ is denoted by $\bar{\mathscr{H}}$.

Our goal now is to construct the sets φ_j^{t+1}, $\varphi_j \in \mathscr{X}$. An important point is to obtain the fulfilment of (13). So, we must "force out" the elements of $U(k)$ not belonging to $\bigcup \varphi_j^{t+1}$ to the set of elements each of which has at least $t+2$ attainable by $\leqslant$ sets φ_α^{t+1}. For this reason the construction below is called the force-out method.

Force-out method

We consider all sets $\mathscr{H}$ of the form $\{\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}\}$ and all subsets A of the set $P(q)$. By property (3) of Theorem 15 for all $\mathscr{H}$ and all A there exists

$\varphi(\mathcal{H}, A) \in \mathcal{X}_1$ such that property (3) is fulfilled We fix $\varphi(\mathcal{H}, A)$ for all $\mathcal{H}$ and A. In the next stage of the proof we consider only $\mathcal{H}$ for which the following holds.

(A) $\exists A \subseteq P(q)$ such that $\varphi(\mathcal{H}, A) \in \mathcal{H}$.

For $A \subseteq P(q)$ we denote by $\varphi(A)$ the formula

$$\bigwedge_{i \in A} p_i \wedge \bigwedge_{p_i \in (P(q) \setminus A)} \neg p_i$$

If $A \subseteq P(q)$ and $\varphi(\mathcal{H}, A) \in \mathcal{H}$ then put

$$T(\mathcal{H}, A) := \varphi(A) \wedge \psi(\mathcal{H}) \wedge \Box(\psi(\mathcal{H}) \to \bigvee_{\varphi(\mathcal{H}, B) \in \mathcal{H}} \varphi(B). \tag{14}$$

If $A \subseteq P(q)$ and $\varphi(\mathcal{H}, A) \notin \mathcal{H}$ then put

$$W(\mathcal{H}, A) = \varphi(A) \wedge \psi(\mathcal{H}) \wedge \Box(\psi(\mathcal{H}) \to \varphi(A) \vee \theta), \tag{15}$$

where θ is the disjunction of all formulas of the form (14) related to the same $\mathcal{H}$.

We introduce the sets $(\varphi_j^t)'$, $\varphi_j \in \mathcal{X}_1$, by setting

$$(\varphi_j^t)' := \varphi_j^t \cup \Bigg[\bigcup_{\substack{\varphi(\mathcal{H}, A) = \varphi_j \\ \varphi(\mathcal{H}, A) \in \mathcal{H}}} T(\mathcal{H}, A) \Bigg] \cup \Bigg[\bigcup_{\substack{\varphi(\mathcal{H}, A) = \varphi_j \\ \varphi(\mathcal{H}, A) \notin \mathcal{H}}} W(\mathcal{H}, A) \Bigg].$$

The obtained sets are expressible since the new sets (14), (15) are expressible and there is only a finite number of these sets. We need:

Lemma 22. *In the case $\mathcal{H}$ satisfies* (A), *the arbitrary maximal element from $\bar{\mathcal{H}}$ is included in the set $\bigcup (\varphi_j^t)'$, $\varphi_j \in \mathcal{X}_1$.*

Proof. Let x be a maximal element of $\bar{\mathcal{H}}$ in the frame $U(k)$. Then $x \Vdash_V \psi(\mathcal{H})$. Consider $P(x)$ where

$$P(x) = \{p_i \mid p_i \in P(q), x \vDash_V p_i\} \subseteq P(q).$$

If $\varphi(\mathcal{H}, P(x)) \in \mathcal{H}$ then x is included in the set defined by (14), therefore $x \in (\varphi(\mathcal{H}, P(x))^t)'$.

If $\varphi(\mathcal{H}, P(x)) \notin \mathcal{H}$, then x is included in the set defined by (15), therefore $x \in W(\mathcal{H}, P(x)) \subseteq (\varphi(\mathcal{H}, P(x))^t)'$. $\square$

Lemma 23. *If $\varphi_\alpha \neq \varphi_\beta$, then $(\varphi_\alpha^t)' \cap (\varphi_\beta^t)' = \emptyset$.*

Proof. If $\mathcal{H}_1$, $\mathcal{H}_2$ are $t+1$-element subsets of the set $\mathcal{X}_1$ and $\mathcal{H}_1 \neq \mathcal{H}_2$ and both satisfy (A), then $\mathcal{H}_1$ and $\mathcal{H}_2$ are disjoint. For this reason the sets added to φ_j^t which are defined by formulas $T(\mathcal{H}_1, A)$, $W(\mathcal{H}_1, A)$ and $T(\mathcal{H}_2, A)$, $W(\mathcal{H}_2, A)$ have no common elements as subsets of the sets $\bar{\mathcal{H}}_1$ and $\bar{\mathcal{H}}_2$ respectively.

If $A_1, A_2 \subseteq P(q)$ and $A_1 \neq A_2$, then in view of the formulas $\varphi(A_1)$ and $\varphi(A_2)$, the sets $T(\mathcal{H}, A_1)(W(\mathcal{H}, A_1))$ and $T(\mathcal{H}, A_2)(W(\mathcal{H}, A_2))$ also have no common elements.

Thus all added sets are disjoint. All added sets are subsets of the sets of the form $\psi(\mathscr{H})$. But $\psi(\mathscr{H})$ and $\bigcup \varphi'_\alpha$, $\varphi_\alpha \in \mathscr{X}_1$, have no common elements. Therefore, if $\varphi_\alpha \neq \varphi_\beta$, then $(\varphi'_\alpha)' \cap (\varphi'_\beta)' = \emptyset$. □

Lemma 25. *If $\mathscr{H}$ is $t+1$-element subset of the set $\mathscr{X}_1$ and $\mathscr{X}$ has the property* (A) *and $x \in \bar{\mathscr{X}}$ and $x \notin \bigcup (\varphi'_\alpha)'$, $\varphi_\alpha \in \mathscr{X}_1$, then there exist $(t+2)$ members φ_{α_j}, $1 \leq j \leq t+2)$, of the set $\mathscr{X}_1$ such that*

$$\forall_j \, (1 \leq j \leq t+2) \quad x \in \Diamond((\varphi'_{\alpha_j})').$$

Proof. It is easy to see that it is sufficient to prove the lemma assuming that x is a maximal (in $U(k)$ under $\leq$) element satisfying the condition of the lemma. Thus let x be maximal, then by Lemma 22, $(\langle x\rangle \setminus \{x\}) \cap \bar{\mathscr{H}}$ is not empty.

Consider the set $(\langle x\rangle \setminus \{x\}) \cap \bar{\mathscr{H}}$. If y is an element of this set, then y is included in one of the sets in (14) or (15) (by the assumption of maximality for x).

First we assume that there exists a $y \in (\langle x\rangle \setminus \{x\}) \cap \bar{\mathscr{H}}$ such that $\varphi(\mathscr{H}, P(y)) \notin \mathscr{H}$. Then y is not included in a set of the form (14), therefore y is an element of a set of the form (15). Therefore $y \in (\varphi(\mathscr{H}, P(y))')'$, but $\varphi(\mathscr{H}, P(y)) \notin \mathscr{H}$ and $x \leq y$ imply the conclusion of our lemma.

Now let

$$\forall y \in (\langle x\rangle \setminus \{x\}) \cap \bar{\mathscr{H}} \quad \varphi(\mathscr{H}, P(y)) \in \mathscr{H}. \tag{16}$$

Then each such y is included in a set defined by a formula of the form (14). Hence $y \Vdash_V \theta$, where θ is a formula from (15).

If $P(x)$ is such that $\varphi(\mathscr{H}, P(x)) \notin \mathscr{H}$, then by $y \Vdash_V \theta$ and by (15) it follows that x is an element of the set defined by formula $W(\mathscr{H}, P(x))$. Thus we have $x \notin (\varphi(\mathscr{H}, P(x))')'$ which contradicts $x \notin \bigcup (\varphi'_\alpha)'$, $\varphi_\alpha \in \mathscr{X}_1$.

If $P(x)$ is such that $\varphi(\mathscr{H}, P(x)) \in \mathscr{H}$, then we again obtain a contradiction. Indeed in this case $x \Vdash_V \varphi(P(x)) \wedge \psi(\mathscr{H})$. If $x < y$ and $y \Vdash_V \psi(\mathscr{H})$, then as we noted above, by assumption (16) we get $y \Vdash_V \theta$. Then $y \Vdash_V \varphi(A)$ for some $A \subseteq P(q)$, where $\varphi(\mathscr{H}, A) \in \mathscr{H}$. Thus

$$y \Vdash_V \bigvee_{\varphi(\mathscr{H}, B) \in \mathscr{H}} \varphi(B).$$

This fact and the above observation give us

$$x \Vdash_V T(\mathscr{H}, P(x)).$$

Then x was included in $(\varphi(\mathscr{H}, P(x))')'$ and this is in contradiction to $x \notin \bigcup (\varphi'_\alpha)'$, $\varphi_\alpha \in \mathscr{X}_1$. Thus (16) is impossible. □

We now turn to the proof of Theorem 20. Now we make the second step of the construction of the set φ'^{t+1}_α. We consider only those $t+1$-element subsets of the set $\mathscr{X}_1$ which do not satisfy (A). That is, we consider all $\mathscr{H} \subseteq \mathscr{X}_1$, $\bar{\bar{\mathscr{H}}} = t+1$ satisfying the condition

(B) $\forall A \subseteq P(q) \quad \varphi(\mathscr{H}, A) \notin \mathscr{H}$.

We fix for each such $\mathscr{H}$ and for $A \subseteq P(q)$ the formula $E(\mathscr{H}, A)$ of the form

$$E(\mathscr{H}, A) := \varphi(A) \wedge \psi(\mathscr{H}) \wedge \Box(\psi(\mathscr{H}) \rightarrow \varphi(A)) \tag{17}$$

where the formula $\varphi(A)$ is defined just before (14).

We introduce the sets $(\varphi_j^t)''$, $\varphi_j \in \mathscr{X}_1$, by putting

$$(\varphi_j^t)'' := \varphi_j^t \cup \Bigg[\bigcup_{\substack{\varphi(\mathscr{H}, A) = \varphi_j \\ \varphi(\mathscr{H}, A) \notin \mathscr{H}}} E(\mathscr{H}, A) \Bigg].$$

Note that formula (17) differs from (15) only by the absence of θ. Therefore Lemmas 26–29 are proved similarly as Lemmas 22–25. We had the possibility not to split the cases (A) and (B) by the corresponding union of (15) and (17). But our construction is rather complicated and difficult to trace. For this reason we have chosen to break the construction into a number of simple steps, although we see a certain repetition.

By the construction of $(\varphi_j^t)''$ all these sets are expressible.

Lemma 26. *If $\mathscr{H}$ satisfies* (B), *then every maximal element x of the set $\bar{\mathscr{H}}$ is included in the set $\bigcup \{(\varphi_j^t)'' \mid \varphi_j \in \mathscr{X}_1\}$.*

Proof. Let x be a maximal element in $\bar{\mathscr{H}}$ (in the frame $U(k)$ under $\leqslant$). Then $x \Vdash_V \varphi(P(x))$ and $x \Vdash_V \psi(\mathscr{H})$. Assume that $y \in U(k)$ and $x \leqslant y$, $x \neq y$. Then by the maximality of x in $\bar{\mathscr{H}}$, we have $y \notin \bar{\mathscr{H}}$ and $y \Vdash_V \neg\psi(\mathscr{H})$. This observation gives us $x \Vdash_V E(\mathscr{H}, P(x))$. As a consequence $x \in (\varphi(\mathscr{H}, P(x))^t)''$. □

Lemma 27. *If $\varphi_\alpha \neq \varphi_\beta$, then $(\varphi_\alpha^t)'' \cap (\varphi_\beta^t)'' = \emptyset$.*

The proof is a reformulation of the proof of Lemma 23.

Lemma 28. *If $a \in (\varphi_\alpha^t)'' \setminus \varphi_\alpha^t$, then $P(a) = P_1(\varphi_\alpha)$.*

Proof. For a satisfying the premise of the lemma we have $\varphi_\alpha = \varphi(\mathscr{H}, A)$, and $a \Vdash_V \varphi(A)$ by the construction of $(\varphi_\alpha^t)''$. By the definition of $\varphi(\mathscr{H}, A)$ in the condition of Theorem 15, we have $P_1(\varphi(\mathscr{H}, A)) = A$. From $a \Vdash_V \varphi(A)$ we obtain $P(a) = A$, $P(a) = P_1(\varphi(\mathscr{H}, A)) = P_1(\varphi_\alpha)$. □

Lemma 29. *If $\mathscr{H}$ is a $t+1$-element subset of the set $\mathscr{X}_1$ with property* (B), $x \in \bar{\mathscr{H}}$ *and $x \notin \bigcup \{(\varphi_\alpha^t)'' \mid \varphi_\alpha \in \mathscr{X}_1\}$, then there exists φ_β in the set $\mathscr{X}_1 \setminus \mathscr{H}$ such that $x \in \Diamond((\varphi_\beta^t)'')$.*

Proof. If x satisfies the condition of the lemma, then by Lemma 26 the set $(\langle x \rangle \setminus \{x\}) \cap \bar{\mathscr{H}}$ is non-empty. We consider a maximal element z of $\bar{\mathscr{H}}$ which is a member of the set $(\langle x\rangle \backslash \{x\})$. By Lemma 26, $z \in \bigcup \{(\varphi^t_\alpha)'' \mid \varphi_\alpha \in \mathscr{X}_1\}$, therefore $z \Vdash_V \psi(\mathscr{H})$ and $z \notin \bigcup \{(\varphi^t_\alpha) \mid \varphi_\alpha \in \mathscr{X}_1\}$.

Thus z was included in the set of the form (17). So $z \Vdash_V E(\mathscr{H}, A)$ and $z \in (\varphi(\mathscr{H}, A)^t)''$. But $\varphi(\mathscr{H}, A) \notin \mathscr{H}$, by (B). Moreover, $x \leq z$ and so we have $x \in \Diamond((\varphi(\mathscr{H}, A)^t)'')$. □

Thus we have constructed the sets $(\varphi^t_\alpha)'$, $(\varphi^t_\alpha)''$, $\varphi_\alpha \in \mathscr{X}_1$, and we have proved some of their properties. We define the sets φ^{t+1}_α, $\varphi_\alpha \in \mathscr{X}_1$, as follows

$$\varphi^{t+1}_\alpha := (\varphi^t_\alpha)' \cup (\varphi^t_\alpha)''.$$

Lemma 30. *The sequence of sets φ^r_α, $\varphi_\alpha \in \mathscr{X}_1$, $-1 \leq r \leq t+1$, has the properties* (9)–(13).

Proof. Since $(\varphi^t_\alpha)'$ and $(\varphi^t_\alpha)''$ are expressible, the sets φ^{t+1}_α are also expressible and (12) is true. The sets defined by (14), (15) and the sets defined by (17) have no common elements in view of the conjunct $\psi(\mathscr{H})$. For (14), (15) it is true that $\exists A \subseteq P(q)\ \varphi(\mathscr{H}, A) \in \mathscr{H}$, and for (17) it is true that $\forall A \subseteq P(q)\ \varphi(\mathscr{H}, A) \notin \mathscr{H}$ (that is: the $\mathscr{H}$ in (14), (15) and in (17) are distinct. Using this fact and Lemmas 23, 27 we obtain that (9) is true.

Property (10) is obvious and (11) follows from Lemmas 24, 28 and from the induction hypothesis.

Let us assume that $x \notin \bigcup \{\varphi^{t+1}_\alpha \mid \varphi_\alpha \in \mathscr{X}_1\}$. Then $x \notin \bigcup \{\varphi^t_\alpha \mid \varphi_\alpha \in \mathscr{X}_1\}$, and since (13) is true for φ^j_α, $\varphi_\alpha \in \mathscr{X}_1$, $-1 \leq j \leq t$, we get $x \in \Diamond\varphi^t_{ji}$, $1 \leq i \leq t+1$. Therefore either $x \in \Diamond\varphi^t_\alpha$, where $\alpha \notin \{j1, \ldots, j(t+1)\}$ and (13) is true for x and φ^{t+1}_α, or $x \Vdash_V \psi(\mathscr{H})$, where $\mathscr{H} := \{\varphi_{j1}, \ldots, \varphi_{j(t+1)}\}$. Let $x \Vdash_V \psi(\mathscr{H})$. If the set $\mathscr{H}$ satisfies (A), then by Lemma 25 there exists a $\varphi_\beta \in \mathscr{X}_1$, where $\varphi_\beta \notin \mathscr{H}$ and $x \in \Diamond(\varphi^t_\beta)'$. Then $x \in \Diamond\varphi^{t+1}_\beta$ and we obtain that (13) holds for x, and φ^{t+1}_α is true. Suppose that (A) is not satisfied for $\mathscr{H}$. Then (B) is satisfied. Then by Lemma 29 there exists a $\varphi_\beta \in \mathscr{X}_1 \backslash \mathscr{H}$ such that $x \in \Diamond(\varphi^t_\beta)$. Then $x \in \Diamond(\varphi^{t+1}_\beta)$ and we again obtain that (13) is true for x and φ^{t+1}_α. Thus (13) is true. □

Continuing the described construction we construct the sequence φ^t_α, $\varphi_\alpha \in \mathscr{X}_1$, $-1 \leq t \leq m_1$, with properties (9)–(13).

We claim that

$$\bigcup_{\varphi_\alpha \in \mathscr{X}_1} \varphi^{m_1}_\alpha = U(k). \tag{18}$$

Indeed let $x \in U(k)$ and $x \notin \bigcup \{\varphi^{m_1-1}_\alpha \mid \varphi_\alpha \in \mathscr{X}_1\}$. Then by (13), $x \in \Diamond\varphi^{m_1-1}_\beta$ for each $\varphi_\beta \in \mathscr{X}_1$. Therefore for $\mathscr{H} := \mathscr{X}_1$ we have $x \Vdash_V \psi(\mathscr{H})$. Note that for all $A \subseteq P(q)$ it is true that $\varphi(\mathscr{H}, A) \in \mathscr{H}$ (that is, at the construction of the sets $\varphi^{m_1}_\alpha$ (A) always takes place). We take $A := P(x)$ (where $P(x) := \{p_i \mid p_i \in P(q)\ \&\ x \Vdash_V p_i\}$). Then $x \Vdash_V \varphi(P(x)) \wedge \psi(\mathscr{H})$ and in this case $\bigvee_{\varphi(\mathscr{H}, B) \in \mathscr{H}} \varphi(B) =$

$\bigvee_{A\subseteq P(q)} \varphi(A)$. Therefore for $z \in U(k)$

$$z \Vdash_V \left(\psi(\mathscr{H}) \to \bigvee_{\varphi(\mathscr{H},B)\in\mathscr{H}} \varphi(B)\right)$$

holds, and therefore $x \Vdash_V T(\mathscr{H}, A)$.

So we have $x \in \varphi(\mathscr{H}, P(x))^{m_1}$ and (18) is proved.

So the "force-out" method is now complete.

We now introduce the special valuation S of the constants from $P(q)$ (which we consider here as propositional letters) and of the variables from q in the frame $U(k)$. As before we use the same notations—x_i—for variables from q and for constants. Let

$$\forall x \in \varphi_\alpha^{m_1} \quad x \Vdash_S x_i \quad \Leftrightarrow \quad x_i \in \theta_1(\varphi_\alpha).$$

In view of (11), the valuations S and V on the set $P(q)$ coincide. The correctness of the definition of S follows from (18) and (9).

Lemma 31. *In the model $\langle U(k), \leqslant, S\rangle$ we have $\forall x \in \varphi_\alpha^{m_1} (x \Vdash_S \varphi_\alpha)$.*

Proof. We shall conduct the proof by induction on the minimal t such that $x \in \varphi_\alpha^t$. Suppose that $x \in \varphi_\alpha^{-1}$ ($\varphi_\alpha \in \mathscr{X}_1$), that is $t = -1$. We recall that by Lemma 21, $\mathscr{X}_1$ is an open subframe of the frame $U(k)$ and $\forall p_i \in P(q)\, p_i \in P_1(\varphi_j) \Leftrightarrow \varphi_j \in V(p_i)$ where V is the valuation of the model $\langle U(k), \leqslant, V\rangle$.

If we consider $\varphi_j \in \mathscr{X}_1$ as elements of $U(k)$ and consider the valuation S on φ_j, then by definition of S, $\varphi_j \Vdash_S x_i$ iff $\varphi_j \Vdash_V x_i$ in the model $\langle \mathscr{X}_1, \leqslant, V\rangle$. In the last model, by property (2) from the condition of Theorem 15, we have $\varphi_j \Vdash_V \varphi_j$. As we noted above, the valuations V and S on elements of $\mathscr{X}_1$ ($\mathscr{X}_1 \subseteq U(k)$) coincide. From $\varphi_j \Vdash_V \varphi_j$ (in $\mathscr{X}_1$) and the fact that $\mathscr{X}_1$ is an open subframe of the frame $U(k)$ and the coinciding of V and S, we obtain $\varphi_j \Vdash_S \varphi_j$ in the model $\langle U(k), \leqslant, S\rangle$. But $x \in \varphi_\alpha^{-1}$ implies $x = \varphi_\alpha$ and $x \Vdash_S \varphi_\alpha$.

Let us assume that $t = 0$, $x \in (\varphi_\alpha^0 \backslash \varphi_\alpha^{-1})$. Then $x \in \mathscr{L}_1(U(k))$ and by the construction of φ_α^0 we have $\varphi_\alpha = \varphi(\emptyset, P(x))$. Moreover, $\theta_2(\varphi_\alpha) = \theta_1(\varphi_\alpha)$. Therefore $x \Vdash_S x_i$ iff $x_i \in \theta_2(\varphi_\alpha)$. Since x is maximal in $U(k)$ we obtain $x \Vdash_S \varphi_\alpha$.

Now let for all $y \in \bigcup\{\varphi_\alpha^t \mid \varphi_\alpha \in \mathscr{X}_1\}$, the claim of our lemma be true and let $x \in (\varphi_\xi^{t+1} \backslash \varphi_\xi)$. Then either $x \in (\varphi_\xi^t)'$, or $x \in (\varphi_\xi^t)''$. First we consider the case $x \in (\varphi_\xi^t)''$.

$$\text{I.} \qquad x \in ((\varphi_\xi^t)'' \backslash \varphi_\xi^t). \tag{19}$$

Then $x \Vdash_V E(\mathscr{H}, A)$ and $\varphi_\xi = \varphi(\mathscr{H}, A)$. Therefore we have $A = P(x)$ and $x \Vdash_V \psi(\mathscr{H})$. By the definition of S, $x \Vdash_S x_i \Leftrightarrow x_i \in \theta_1(\varphi_\xi)$. Therefore the nonmodal part of the conjunction φ_ξ is valid on x under S. Now consider the modal part.

Assume that $x \leqslant y$ and $y \Vdash_S x_i$. If $y \in \bigcup\{\varphi_\alpha^t \mid \varphi_\alpha \in \mathscr{X}_1\}$, then $y \in \varphi_\beta^t$, and by induction hypothesis $y \Vdash_S \varphi_\beta$. Therefore $y \Vdash_S x_i$ implies $x_i \in \theta_1(\varphi_\beta)$ and $x_i \in \theta_2(\varphi_\beta)$. Then $x \in \Diamond(\varphi_\beta^t)$, and from $x \Vdash_V \psi(\mathscr{H})$ we obtain $\varphi_\beta \in \mathscr{H}$. By Property (3) from the condition of Theorem 15 by choice of $\varphi(\mathscr{H}, A)$ we get

$$\theta_2(\varphi(\mathscr{H}, A)) = \theta_1(\varphi(\mathscr{H}, A)) \cup \Big(\bigcup_{\varphi_\beta \in \mathscr{H}} \theta_2(\varphi_\beta)\Big). \tag{20}$$

From (20) and $x_i \in \theta_2(\varphi_\beta)$ it follows that

$$x_i \in \theta_2(\varphi(\mathscr{H}, A)) = \theta_2(\varphi_\xi).$$

Now we assume that $y \notin \bigcup \varphi'_\alpha$, $\varphi_\alpha \in \mathscr{X}_1$.

Then by (13) we have $y \in \Diamond(\varphi'_{\alpha_j})$, $1 \leq j \leq t+1$. This property and $x \leq y$, $x \Vdash_V \psi(\mathscr{H})$ imply $\mathscr{H} = \{\varphi_{\alpha 1}, \ldots, \varphi_{\alpha(t+1)}\}$ and so imply that $y \Vdash_V \psi(\mathscr{H})$. From $x \Vdash_V E(\mathscr{H}, A)$ and $x \leq y$ and $y \Vdash_V \psi(\mathscr{H})$ we obtain $y \Vdash_V \varphi(A)$ and $y \Vdash_V E(\mathscr{H}, A)$. Thus $y \in (\varphi'_\xi)''$. Then from $y \Vdash_S x_i$ we obtain $x_i \in \theta_1(\varphi_\xi)$ and $x_i \in \theta_2(\varphi_\xi)$.

Thus we have shown that

$$x \Vdash_S \Diamond x_i \quad \Rightarrow \quad x_i \in \theta_2(\varphi_\xi).$$

Turn now to the coinverse implication. Let x_i be a member of the set $\theta_2(\varphi_\xi)$. Then by (20), $x_i \in \theta_1(\varphi_\xi)$ or $x_i \in \theta_2(\varphi_\beta)$, where $\varphi_\beta \in \mathscr{H}$. In the first case $x \Vdash_S x_i$ and $x \Vdash_S \Diamond x_i$. Assume that the second case takes place.

In view of $x \Vdash_V \psi(\mathscr{H})$, there exists a y such that $x \leq y$, $y \in \varphi'_\beta$. By the induction hypothesis, $y \Vdash_S \varphi_\beta$, therefore $x_i \in \theta_2(\varphi_\beta)$ implies $y \Vdash_S \Diamond x_i$ and then, of course, $x \Vdash_S \Diamond x_i$.

Thus we have proved $x_i \in \theta_2(\varphi_\xi) \Rightarrow x \Vdash_S \Diamond x_i$ and by using the converse implication proved above, we have $x_i \in \theta_2(\varphi_\xi) \Leftrightarrow x \Vdash_S \Diamond x_i$. This property and the remarks made by us above about the validity of the nonmodal part of θ_ξ allow us to conclude that $x \Vdash_S \varphi_\xi$. Consequently, in case (19) the claim of the lemma is true.

Let the second assumption be

$$\text{II.} \quad x \in ((\varphi'_\xi)' \setminus \varphi'_\xi). \tag{21}$$

As in case (19) by the definition of S, the validity in x of the nonmodal part of φ_ξ is evident. Consider the modal part. In case (21) x will be included in the set defined by (14) or (15). First we consider the case that x is an element of (14), that is

$$x \Vdash_V T(\mathscr{H}, A), \qquad \varphi_\xi = \varphi(\mathscr{H}, A). \tag{22}$$

Assume that $x \Vdash_S \Diamond x_i$, that is $x \leq y$, $y \Vdash_S x_i$ and $y \Vdash_S x_i$ for some y. Assume that $y \Vdash_V \psi(\mathscr{H})$. Then $y \Vdash_V \varphi(B)$ for a certain B where $\varphi(\mathscr{H}, B) \in \mathscr{H}$. Then it is easy to see that $y \Vdash_V T(\mathscr{H}, A)$. In this case, by the construction of $(\varphi(\mathscr{H}, B)')'$, y was included in the set $(\varphi(\mathscr{H}, B)')'$. Then, by the definition of S we have $x_i \in \theta_1(\varphi(\mathscr{H}, B))$. In view of the definition of $\varphi(\mathscr{H}, A)$, (20) is the case. Since $\varphi(\mathscr{H}, B) \in \mathscr{H}$, we obtain from (20) (as $x_i \in \theta_1(\varphi(\mathscr{H}, B))$), $x_i \in \theta_2(\varphi(\mathscr{H}, A)) = \theta_2(\varphi_\xi)$ (see (22)).

Assume now that $\neg(y \Vdash_V \psi(\mathscr{H}))$. If $y \notin \bigcup \{\varphi'_\alpha \mid \varphi_\alpha \in \mathscr{X}_1\}$, then by (13) we have

$$y \in \Diamond(\varphi'_{\alpha\rho}), \quad 1 \leq \rho \leq t+1.$$

Thus, $x \leq y$ and $x \Vdash_V \psi(\mathcal{H})$ imply $y \Vdash_V \psi(\mathcal{H})$, which contradicts our assumption. Therefore $y \in \bigcup \varphi'_\alpha$, $\varphi_\alpha \in \mathcal{X}_1$. So $y \in \varphi'_\alpha$, $\varphi_\alpha \in \mathcal{X}_1$. By induction hypothesis $y \Vdash_S \varphi_\alpha$, and from $y \Vdash_S x_i$ we obtain $x_i \in \theta_1(\varphi_\alpha)$ and $x_i \in \theta_2(\varphi_\alpha)$. From $x \leq y$ it follows that $x \in \Diamond(\varphi'_\alpha)$. By $x \Vdash_V \psi(\mathcal{H})$ this implies $\varphi_\alpha \in \mathcal{H}$. Then from (20) (by the definition of $\varphi(\mathcal{H}, A)$ (20) is true) and $x_i \in \theta_2(\varphi_\alpha)$, we may conclude that $x_i \in \theta_2(\varphi(\mathcal{H}, A)) = \theta_2(\varphi_\xi)$ (see (22)). Thus we obtain

$$x \Vdash_S \Diamond x_i \quad \Rightarrow \quad x_i \in \theta_2(\varphi_\xi).$$

Conversely, let x_i be an element of $\theta_2(\varphi_\xi)$. As we noted above, $x \Vdash_V \psi(\mathcal{H})$ (see (22)). Therefore there exists a y such that $x \leq y$ and $y \in \varphi(\mathcal{H}, A)^t$. By induction hypothesis, $y \Vdash_S \varphi(\mathcal{H}, A)$ holds. But we had $x_i \in \theta_2(\varphi_\xi) = \theta_2(\varphi(\mathcal{H}, A))$ (see (22)). Therefore $y \Vdash_S \Diamond x_i$ and then, of course, $x \Vdash_S \Diamond x_i$. Thus we proved that $x \Vdash_S \Diamond x_i \Leftrightarrow x_i \in \theta_2(\varphi_\xi)$. Therefore it is true that $x \Vdash_S \varphi_\xi$. The case (22) is completed.

Now we assume that x was included in the set defined by (15), that is

$$x \Vdash_V W(\mathcal{H}, A), \qquad \varphi(\mathcal{H}, A) = \varphi_\xi. \tag{23}$$

Assume that $x \Vdash_S \Diamond x_i$. Then there exists a y such that $x \leq y$ and $y \Vdash_S x_i$.

Consider the case $y \notin \bigcup \{\varphi'_\alpha \mid \varphi_\alpha \in \mathcal{X}_1\}$. By (13) we have $y \in \Diamond(\varphi'_{\alpha_i})$, $1 \leq i \leq t+1$, and in view of $x \Vdash_V \psi(\mathcal{H})$ (by (23)) we obtain $y \Vdash_V \psi(\mathcal{H})$. By the form of (15) we conclude that $y \Vdash_V W(\mathcal{H}, A)$ or $y \Vdash_V T(\mathcal{H}, B)$, where $\varphi(\mathcal{H}, B) \in \mathcal{H}$.

First we assume that the second case is true. Then $y \in ((\varphi(\mathcal{H}, B)^t)' \backslash \varphi(\mathcal{H}, B)^t)$ and we have case (22). As we showed above, in this case $y \Vdash_S \varphi(\mathcal{H}, B)$. But by assumption $y \Vdash_S x_i$, consequently $x_i \in \theta_1(\varphi(\mathcal{H}, B))$. Using (20) ((20) is true by the definition of $\varphi(\mathcal{H}, A)$) we obtain $x_i \in \theta_2(\varphi(\mathcal{H}, A)) = \theta_2(\varphi_\xi)$ (see (23)).

Now consider the case $y \Vdash_V W(\mathcal{H}, A)$. Then $y \in ((\varphi(\mathcal{H}, A)^t)' \backslash \varphi(\mathcal{H}, A)^t)$. From $y \Vdash_S x_i$ we obtain $x_i \in \theta_1(\varphi(\mathcal{H}, A))$, and by using (23) we conclude that $x_i \in \theta_2(\varphi_\xi)$.

Now we turn to the consideration of the case $y \in \bigcup \{\varphi'_\alpha \mid \varphi_\alpha \in \mathcal{X}_1\}$. Then $y \in \varphi'_\alpha$ for a certain $\varphi_\alpha \in \mathcal{X}_1$ and $x \in \Diamond(\varphi'_\alpha)$. From (23) we have $x \Vdash_V \psi(\mathcal{H})$, therefore $\varphi_\alpha \in \mathcal{H}$. By induction hypothesis we obtain $y \Vdash_S \varphi_\alpha$. Then $x_i \in \theta_1(\varphi_\alpha)$ and $x_i \in \theta_2(\varphi_\alpha)$. Using $\varphi_\alpha \in \mathcal{H}$ and (20), we obtain $x_i \in \theta_2(\varphi(\mathcal{H}, A)) = \theta_2(\varphi_\xi)$. Thus we completed the proof of the implication $x \Vdash_S \Diamond x_i \Rightarrow x_i \in \theta_2(\varphi_\xi)$.

Suppose that $x_i \in \theta_2(\varphi_\xi) = \theta_2(\varphi(\mathcal{H}, A))$; by (20) we have $x_i \in \theta_1(\varphi_\xi)$ or $x_i \in \theta_2(\varphi_\beta)$, $\varphi_\beta \in \mathcal{H}$. If the first case holds, then $x \Vdash_S x_i$ and $x \Vdash_S \Diamond x_i$. Let the second case be true. Then $x_i \in \theta_2(\varphi_\beta)$, where $\varphi_\beta \in \mathcal{H}$. Then from $x \Vdash_V \psi(\mathcal{H})$ (see (23)) it follows that $x \leq y$, $y \in \varphi'_\beta$, and $x \in \Diamond(\varphi'_\beta)$. By induction hypothesis, $y \Vdash_S \varphi_\beta$ and $x_i \in \theta_2(\varphi_\beta)$ implies $y \Vdash_S \Diamond x_i$ and $x \Vdash_S \Diamond x_i$. Thus we have proved that $x \Vdash_S \Diamond x_i$ iff $x_i \in \theta_2(\varphi_\xi)$. Consequently $x \Vdash_S \varphi_\xi$ in the case (23).

Thus we have proved that $\forall x \in ((\varphi^t_\xi)' \backslash (\varphi^t_\xi))\, x \Vdash_S \varphi_\xi$. By combining this result with the considered case (19), we obtain that $\forall x \in (\varphi^{t+1}_\xi \backslash \varphi^t_\xi)\, x \Vdash_S \varphi_\xi$. $\square$

Now turn to the completion of the proof of Theorem 20. By Lemma 1, for arbitrary $x \in U(k)$,

$$x \in \varphi_\alpha(S(x_i)) \iff x \Vdash_S \varphi_\alpha. \tag{24}$$

In view of Lemma 31 and (24) it is true that

$$\varphi_\alpha^{m_1} \subseteq \varphi_\alpha(S(x_i)).$$

Therefore (18) implies

$$\bigvee_{\varphi_\alpha \in \mathscr{X}_1} \varphi_\alpha(S(x_i)) = U(k).$$

According to property (1) from the conditions of Theorem 15 there exists a $\varphi_j \in \mathscr{X}_1$ such that $x_0 \in \theta_1(\varphi_j)$, and by the definition of the set $\varphi_j^{m_1}$ this set is non-empty, that is, there exists an $x \in \varphi_j^{m_1}$. Then by Lemma 31, $x \Vdash_S \Diamond x_0$ and by (24), $x \in \Diamond S(x_0)$, that is $\Diamond S(x_0) \neq \emptyset$. Consequently, the quasi-identity q is not valid on the algebra $\langle U(k), \leqslant\rangle^+$ when its variables x_i take values $S(x_i)$ and the constants p_i from $P(q)$ are interpreted as $S(p_i)$.

By the definition of S

$$S(x_i) = \bigvee \{\varphi_\alpha^{m_1} \mid \varphi_\alpha \in \{\varphi_\beta \mid x_i \in \theta_1(\varphi_\beta)\}\}.$$

According to (12), all sets $\varphi_\alpha^{m_1}$ are expressible. Consequently, by Lemma 1, all $S(x_i)$ are elements of the algebra $\langle U(k), \leqslant\rangle^+ (V(p_k))$, which is a subalgebra of the algebra $\langle U(k), \leqslant\rangle^+$. By Theorem 13 this subalgebra is isomorphic to the free algebra $\mathscr{F}_k(\mathrm{Grz})$ and the $V(P_k)$ are its free generators.

Let $p_i \in P(q)$ and $x \in S(p_i)$. By (18), $x \in \varphi_\alpha^{m_1}$ for some $\varphi_\alpha^{m_1}$, and $p_i \in \theta_1(\varphi_\alpha)$. By (11), $P(x) = P_1(\varphi_\alpha)$, that is, we obtain $x \Vdash_V p_i$ and $x \in V(p_i)$. Assume that conversely $p_i \in P(q)$ and $x \in V(p_i)$, that is, $x \Vdash_V p_i$. By (18), $x \in \varphi_\alpha^{m_1}$ for some $\varphi_\alpha \in \mathscr{X}_1$. Again, by (11), $P(x) = P_1(\varphi_\alpha)$ and from this equality we obtain $p_i \in \theta_1(\varphi_\alpha)$ and $x \in S(p_i)$. Thus for an arbitrary constant p_i from $P(q)$, $V(p_i) = S(p_i)$.

Therefore the $S(p_i)\,(S(p_i) = V(p_i))$ are free generators of the algebra $\langle U(k), \leqslant\rangle^+(V(P_k))$.

Consequently q is not valid on $\mathscr{F}_k(\mathrm{Grz})$ (when the $P(q)$ are interpreted as free generators). Now Theorem 20 is proved. □

Let λ be a modal logic (or superintuitionistic logic) and let Σ_f be the signature of the algebra $\mathscr{F}_\omega(\lambda)$, extended by constants for the free generators. An *obstacle* for the universal formula $A(\bar{x})$ in the signature Σ_f is a certain tuple $\bar{a}$ from $\mathscr{F}_\omega(\lambda)$ such that $\neg(\mathscr{F}_\omega(\lambda) \vDash A(\bar{a}))$.

Theorem 32. *The universal theory of the free algebra $\mathscr{F}_\omega(\mathrm{Grz})$ in the signature Σ_f is solvable. There exists an algorithm for the construction of obstacles for the universal formulas in the signature Σ_f which fail in $\mathscr{F}_\omega(\mathrm{Grz})$.*

Proof. Let $\forall \bar{x}\, \varphi$ be a universal formula in the signature Σ_f in prenex normal form. It is easy to see that this formula is equivalent to a formula A of the form

$$\forall \bar{x} \left(\bigwedge_i \left(B_i = 1 \Rightarrow \bigvee_j (B_j^i = 1) \right) \right).$$

Then $\forall \bar{x}\, \varphi$ is equivalent to the formula

$$\bigwedge_i \left(\forall \bar{x} \left(B_i = 1 \Rightarrow \bigvee_j (B_j^i = 1) \right) \right).$$

We recall that Grz has the so-called disjunction property, that is from $\Box C \vee \Box D \in \mathrm{Grz}$ it follows that $\Box C \in \mathrm{Grz}$ or $\Box D \in \mathrm{Grz}$ [29]. Then in $\mathscr{F}_\omega(\mathrm{Grz})$, $\Box a \vee \Box b = 1 \Leftrightarrow (\Box a = 1) \vee (\Box b = 1)$ and moreover $\Box a = 1 \Leftrightarrow a = 1$. Therefore the formulas $\forall \bar{x}\, (B_i = 1 \Rightarrow \bigvee_j (B_j^i = 1))$ are equivalent in $\mathscr{F}_\omega(\mathrm{Grz})$ to the quasi-identities

$$C_i := \left(\forall \bar{x} \left(B_i = 1 \Rightarrow \left(\bigvee_j \Box B_j^i \right) = 1 \right) \right).$$

Then in $\mathscr{F}_\omega(\mathrm{Grz})$ the formula $\forall \bar{x}\, \varphi$ is equivalent to the conjunction $\bigwedge_i C_i$. Moreover, these formulas have the same sets of variables and the same sets of constants and the sets of obstacles for these formulas coincide. Thus

$$\mathscr{F}_\omega(\mathrm{Grz}) \vDash \forall \bar{x}\, \varphi \quad \Leftrightarrow \quad \mathscr{F}_\omega(\mathrm{Grz}) \vDash \bigwedge_i C_i;$$

by Theorem 14, $\mathscr{F}_\omega(\mathrm{Grz}) \vDash C_i \Leftrightarrow \mathscr{F}_\omega(\mathrm{Grz}) \vDash r(C_i)$. According to Theorems 15 and 20, $\neg(\mathscr{F}_\omega(\mathrm{Grz}) \vDash r(C_i))$ iff there exist a set $\mathscr{X}$ where $\mathscr{X} \subseteq \mathscr{D}(r(C_i))$ and $\forall \varphi_j \in \mathscr{X}$ there exist sets $T(\varphi_j) \subseteq \mathscr{D}(r(C_i))$ where $\forall \varphi_k \in T(\varphi_j)\ \theta_2(\varphi_k) = \theta_2(\varphi_j)$, $\varphi_j \notin T(\varphi_j)$, such that in the model $\langle \mathscr{X} \cup [\bigcup_{\varphi_j \in \mathscr{X}} T(\varphi_j)], \leqslant, V \rangle$ the properties (1)–(3) from the condition of Theorem 15 are true. These properties yield effective tests and give us an algorithm for checking $\mathscr{F}_\omega(\mathrm{Grz}) \vDash r(C_i)$. Consequently, the universal theory of $\mathscr{F}_\omega(\mathrm{Grz})$ in the signature Σ_f is solvable.

Assume that $\forall \bar{x}\, \varphi$ is not valid in $\mathscr{F}_\omega(\mathrm{Grz})$. Then for certain i, C_i is not valid in $\mathscr{F}_\omega(\mathrm{Grz})$ and $\mathscr{F}_\omega(\mathrm{Grz}) \nvDash r(C_i)$. Now by Theorem 15 we may find a model with the required properties (1)–(3). The proof of Theorem 20 gives us an algorithm to construct an obstacle for $r(C_i)$:

$$S(x_j) := \bigvee \{ \varphi_\alpha^{m_1} \mid \varphi_\alpha \in \{ \varphi_\beta \mid \varphi_\beta \in \mathscr{X}_1,\ x_j \in \theta_1(\varphi_\beta) \} \},$$
$$S(p_j) := V(p_j), \quad \forall p_j \in P(r(C_i)).$$

By Theorem 14 the set of elements of this obstacle which correspond to variables and constants from C_i will be an obstacle for C_i and $\bigwedge_i C_i$. As we noted above, the sets of obstacles for $\forall \bar{x}\, \varphi$ and for $\bigwedge_i C_i$ coincide. Consequently, we obtain the obstacle for $\forall \bar{x}\, \varphi$. $\Box$

4. Admissibility and substitution problems

The above mentioned problems will be solved first for the modal system Grz; after all they will carry over to intuitionistic propositional calculus H.

Let $A(x_i)/B(x_i)$ be a rule of inference. Let us suppose that some variables x_j of this rule are replaced by propositional letters p_j, and then we have an expression of the form $A(x_i, p_j)/B(x_i, p_j)$. $A(x_i, p_j)/B(x_i, p_j)$ is called a rule of inference with parameters. We call this rule admissible in the logic λ iff $B(B_i, p_j) \in \lambda$ follows from $A(B_i, p_j) \in \lambda$ for arbitrary formulas B_i.

It is clear that the admissibility of rules with parameters generalizes the usual admissibility. If the rule $A(x_i)/B(x_i)$ is admissible in logic λ, then the rule with parameters $A(x_i, p_j)/B(x_i, p_j)$ is also admissible in λ. The converse is not true. For example, the rule with parameters $p/p \wedge \neg p$ is admissible in each nontrivial logic λ, but the rule $x/x \wedge \neg x$ is not admissible in λ.

The first main corollary of Theorem 32 is

Theorem 33. *The problem of admissibility of rules of inference with parameters (hence also without them) in the m.l.* Grz *is algorithmically solvable.*

The proof follows immediately from the analogue of Lemma 2 for rules with parameters: rule $A(x_i, p_j)/B(x_i, p_j)$ is admissible in the logic λ iff the quasi-identity $A(x_i, p_j) = 1 \Rightarrow B(x_i, p_j) = 1$ is valid in $\mathcal{F}_\omega(\lambda)$ where the p_j's are interpreted as free generators (the proof of Lemma 2 immediately carries over to this case), and from Theorem 32.

Theorem 33 generalizes the result about algorithmical recognition of admissibility rules (without parameters) in the m.l. Grz [27], which result was obtained on the basis of the decidability of the universal theory of the algebra $\mathcal{F}_\omega$(Grz) in a signature without constants. In [28] it was proved that the problem of admissibility of rules in the modal provability system G is also algorithmically decidable.

The second main corollary of Theorem 32 is the next theorem.

Theorem 34. *The substitution problem for the modal system* Grz *is algorithmically decidable. There exist an algorithm for the recognition of solvability in the free algebra* $\mathcal{F}_\omega$(Grz) *of equations (in the signature* Σ_f*) and for the construction of some solutions for solvable equations.*

Proof. The second part of the theorem immediately follows from Theorem 32 and the fact that $A(x_i, p_j) = 1$ is solvable in $\mathcal{F}_\omega$(Grz) iff $\mathcal{F}_\omega(\mathrm{Grz}) \nvDash \forall \bar{x}\, \neg(A(x_i, p_j) = 1)$. The first part of the theorem follows from the second part and Lemma 3. □

Now we turn to intuitionistic propositional calculus H.

Theorem 35. *There exists an algorithm for the recognition of solvability of equations in the free pseudo-boolean algebra $\mathscr{F}_\omega(H)$ (in the signature Σ_f) and for constructing some solution for solvable equations.*

Proof. According to Lemma 8 the equation $A(x_i, p_j) = 1$ is solvable in $\mathscr{F}_\omega(H)$ iff $T(A)(\Box x_i, \Box p_j) = 1$ has a solution in $\mathscr{F}_\omega(\mathrm{Grz})$. Therefore, by Theorem 34 there exists an algorithm for the recognition of solvability of equations in $\mathscr{F}_\omega(H)$.

If $A(x_i, p_j) = 1$ is solvable, then $T(A)(\Box x_i, \Box p_j) = 1$ is also solvable and by Theorem 34 we may effectively construct some solution for $T(A)(\Box x_i, \Box p_j) = 1$. As we proved in the second part of the proof of Lemma 8 this solution may be effectively transformed into a solution of the same equation, and the members of this solution have the form $T(D_i)$. Then the D_i form a solution for $A(x_i, p_j) = 1$. □

Theorem 35 and Lemma 3 imply

Theorem 36. *The substitition problem for intuitionistic logic H is decidable.*

The results of Theorems 35 and 36 were obtained in [25, 26] in the same way on the basis of analogues of the above obtained results for the modal system $S4$ (that is, solvability of the universal theory the free algebra $\mathscr{F}_\omega(S4)$ in signature Σ_f and solvability of the substitution problem for $S4$ and so on).

Let Σ be the signature of the algebra $\mathscr{F}_\omega(H)$ (without constants).

Theorem 37. *The universal theory of the free pseudo-boolean algebra $\mathscr{F}_\omega(H)$ in the signature Σ is decidable.*

Proof. Let $\forall \bar{x} A$ be a universal formula. The reader will easily note that this formula is equivalent to the conjunction of formulas of the form

$$\forall \bar{x}\, (f = 1 \Rightarrow (g_1 = 1) \vee \cdots \vee (g_n = 1)). \tag{25}$$

It is well known that intuitionistic logic H has the so-called disjunction property: $A \vee B \in H$ implies $A \in H$ or $B \in H$. Therefore formula (25) is equivalent in $\mathscr{F}_\omega(H)$ to the quasi-identity

$$f = 1 \;\Rightarrow\; \Big(\bigvee_{j=1}^{n} g_i\Big) = 1.$$

By Theorem 6 and Lemma 2 this quasi-identity is valid in $\mathscr{F}_\omega(H)$ iff the quasi-identity

$$T(f) = 1 \;\Rightarrow\; \Big(T\Big(\bigvee_{j=1}^{n} g_j\Big)\Big) = 1$$

is valid in $\mathscr{F}_\omega(\mathrm{Grz})$. According to Lemma 2 and Theorem 33 we obtain an algorithm for the recognition of validity of (25) on $\mathscr{F}_\omega(H)$. □

By Theorem 37 and Lemma 2 we obtain a positive solution of Friedman's problem:

Corollary 38. *There exists an algorithm for the recognition of the admissibility of rules of inference in intutionistic logic, H.*

This result was obtained earlier in the author's papers [20, 23, 27] on the basis of the decidability of the universal theory of the algebra $\mathcal{F}_\omega(H)$.

Remark 1. Despite the decidability of universal theories of the free algebras $\mathcal{F}_\omega(\lambda)$, $\lambda = \mathrm{Grz}, H$ (Theorems 32, 37) the first-order theories of these algebras are heriditarily undecidable [22].

Remark 2. We note that the universal theory of free pseudo-boolean algebra $\mathcal{F}_\omega(H)$ in the signature Σ_f (which is extended by constants for free generators) is decidable too. For this reason there exists an algorithm for the recognition of the admissibility of rules of inference with parameters in H. This result was obtained in [26].

Finding algorithms for the recognition of admissibility in Grz and H is rather complex. In suitable cases it is more convenient to use the following semantical criterion.

Theorem 39. *The rule A/B is admissible in* Grz *iff* $\langle U(n), \leqslant\rangle^+ \vDash (A = 1 \Rightarrow B = 1)$ *for each* $n \in \mathbb{N}$.

Proof. By Theorem 13 and Lemma 2, "sufficiency" is obvious.

Let $\langle U(n), \leqslant\rangle^+ \nvDash (A = 1 \Rightarrow B = 1)$. Then $r(q)$, where $q = (A = 1 \Rightarrow B = 1)$, according to Theorem 14, is not valid in $\langle U(n), \leqslant\rangle^+$. Note that the proof of Theorem 15 rests on just this assumption: $r(q)$ is not valid in $\langle U(n), \leqslant\rangle^+$ (see (7)). Therefore, by Theorem 15, there exist a model

$$\left\langle \mathcal{X} \cup \left[\bigcup_{\varphi_i \in \mathcal{X}} T(\varphi_i) \right], \leqslant, V \right\rangle$$

with properties (1)–(3) from the condition of Theorem 15.

Then, according to Theorem 20, $r(q)$ is not valid in $\mathcal{F}_\omega(\mathrm{Grz})$. Therefore by Theorem 14 we have $\neg(\mathcal{F}_\omega(\mathrm{Grz}) \vDash q)$.

Then by Lemma 2, A/B is not admissible in Grz. $\square$

For example, using this criterion it is not hard to deduce that the rule

$$(q \supset r) \supset (\neg r \vee q)/\neg\neg q \vee \neg r$$

is admissible in H. Of course, we first check that the translation T of this rule is admissible in Grz. This rule is not found in the literature, as it seems. From the admissibility of this rule in H the admissibility of the well-known rule of Scott–Jankov–Kuznetsov follows:

$$((\neg\neg x \supset x) \supset (x \vee \neg x))/\neg\neg x \vee \neg x.$$

The admissibility of the last rule and of Harrop's rule

$$(\neg x \supset z \vee y)/(\neg x \supset z) \vee (\neg x \supset y)$$

in H is also not hard to check by our semantical criterion of admissibility in Grz.

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