Searching for semifield flocks

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Received 5 April 1999; received in revised form 26 June 2001; accepted 24 September 2001

Abstract

Searches are performed in PG(3,q), q odd, for finding semifield and likeable flocks. For
small values of q exhaustive searches are performed; limited searches are done for larger q. The
main result of this paper is: In PG(3,27) any semifield flock is isomorphic either to the linear,
Kantor–Knuth or Ganley flock. No new semifield or likeable flocks have been found.

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1. Introduction

In PG(3,q) let K be a quadratic cone \( x_0x_1 = x_2^2 \) with vertex \( v=(0,0,0,1) \). A flock \( \mathcal{F} \) of K is a set of q disjoint conics partitioning \( K\setminus\{v\} \). Since each conic determines
a unique plane containing it we usually describe \( \mathcal{F} \) by a set of q planes, \( \pi_t \), of the
form \( atx_0 + btx_1 + ctx_2 + x_3 = 0 \), \( t \in GF(q) \), with \( t \mapsto at \) and \( t \mapsto bt \) being bijections. The
q planes on an external line to K partitioning \( K\setminus\{v\} \) give rise to a flock of K called
a linear flock of K.

Flocks are of interest to finite geometers since they tie together a large number of
areas: generalized quadrangles, spreads, translation planes, herds of ovals, ovoids, inver-
sive planes, designs, BLT-sets (for a nonexhaustive collection see [3,6,9,12,15,17,20]).

Following [20], we let \( \mathcal{F}(f, g) \) be a set of q planes

\[
\begin{align*}
    tx_0 + f(t)x_1 + g(t)x_2 + x_3 &= 0,
\end{align*}
\]

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PII: S0012-365X(02)00379-5

* The author acknowledges the funding of a CNR Borsa di Studio and funding by the research bodies
‘Incidence Geometry’ (RUG) and ‘Fundamental Methods and Techniques in Mathematics’ (FWO—Flanders).
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where \( f, g \in \text{GF}(q)[x] \) with \( f(0) = g(0) = 0 \). (Note in the papers of Johnson et al. [9,11], Eq. (1) would need \( f(t) \) to be replaced with \(- f(t)\) for \( \mathcal{F}(f, g) \) to be a flock.) When \( q \) is odd, \( \mathcal{F} \) is a flock of \( K \) if and only if for all \( s \neq t \)

\[
(g(t) - g(s))^2 - 4(t - s)(f(t) - f(s))
\]

is a nonsquare [20]. We remark that when \( f \) and \( g \) are additive then condition (2) becomes: for all \( t \neq 0 \)

\[
g^2(t) - 4f(t)t
\]

is a nonsquare for \( \mathcal{F}(f, g) \) to be a flock of \( \text{PG}(3, q) \).

We say two flocks \( \mathcal{F}(f, g) \) and \( \mathcal{F}(f', g') \) of \( K \) are equivalent, denoted \( \mathcal{F}(f, g) \simeq \mathcal{F}(f', g') \), if there is a collineation of \( \text{PG}(3, q) \) that leaves \( K \) invariant and maps the planes of \( \mathcal{F}(f, g) \) onto the planes of \( \mathcal{F}(f', g') \) setwise.

2. The search space

There are a few ways of constructing flocks of a quadratic cone in \( \text{PG}(3, q) \). Geometrically, we could try and find the \( q + 1 \) planes partitioning the quadratic cone \( K \). But this would only work for very small values of \( q \) since we need to go through \((q^3 + q^2 + q + 1)\) combinations of planes. As one would expect, this (very) naive approach would only work for very small values of \( q \). A geometric approach is possible though by looking at the relationship between flocks and BLT sets [3] as done by Penttila and Royle [17].

In this paper we search for particular functions \( f \) and \( g \) such that \( \mathcal{F}(f, g) \) is a flock of \( K \), that is, functions \( f \) and \( g \) satisfying condition (2). Given a finite field \( \text{GF}(q) \) a function \( f : \text{GF}(q) \to \text{GF}(q) \) can be uniquely represented by a polynomial \( f' \in \text{GF}(q)[x] \) of degree \(< q - 1 \) as

\[
f'(x) = \sum_{i=0}^{q-2} a_i x^i,
\]

where the \( a_i \in \text{GF}(q) \) for \( i = 0, \ldots, q - 2 \). Hence we can suppose, from now on, that the two functions \( f \) and \( g \) satisfying condition (2) are uniquely represented by \( f' \) and \( g' \in \text{GF}(q)[x] \) of degree \(< q - 1 \): thus \( \mathcal{F}(f, g) = \mathcal{F}(f', g') \).

Even though the search space for all functions over \( \text{GF}(q) \) is huge, we will attempt a search in this space by restricting the set of functions over \( \text{GF}(q) \) to the set of additive functions over \( \text{GF}(q) \). By [14] all additive functions over a finite field \( \text{GF}(q) \), with \( q = p^h \), \( p \) prime, are of the form

\[
f(x) = \sum_{i=0}^{h-1} a_i x^i \]
where \( a_i \in \text{GF}(q) \) for \( i = 0, \ldots, h - 1 \). Hence the space of additive functions is of size \( q^h \) (compared to \( q^h \) for the space of all functions).

There are special classes of fields associated with additive functions. If \( f \) and \( g \) are additive and \( \mathcal{F}(f, g) \) is a flock, we say \( \mathcal{F} \) is a semifield flock (see Section 3). Given an additive function \( J \) such that \( \mathcal{F}(-\frac{1}{4}t^3 + J(t), -t^2) \) is a flock, we say \( \mathcal{F} \) is a likeable flock (see Section 5). In this paper we will search for such additive functions.

Semi-field flocks are associated with a remarkable set of geometric objects: Translation generalised quadrangles [11], translation duals of generalised quadrangles [20, 21], translation ovoids [19], translation planes defined over a semi-field [6, 9, 23] and 2–(\( q^3, q, 1 \)) translation designs [15]. Likeable flocks are associated with likeable translation planes [12].

3. Semi-field flocks

It is well known [20, 23] that for every flock \( \mathcal{F} \) of a quadratic cone in \( \text{PG}(3, q) \) there corresponds a spread \( S(\mathcal{F}) \) of \( \text{PG}(3, q) \). When the translation plane constructed using \( S(\mathcal{F}) \) is a semifield plane, then we say \( \mathcal{F} \) is a semifield flock. When \( q \) is even, Johnson [11] has shown that all semifield flocks are linear. Hence our searches for semifield flocks will be for \( q \) odd. Johnson and Gevaert [9] have shown that a flock \( \mathcal{F}(f, g) \) is a semifield flock if and only if \( f \) and \( g \) are additive.

**Theorem 1** (Thas [22, Corollary 17; 21, Theorem 6.9].) In \( \text{PG}(3, p^2) \) for any prime \( p \) the semifield flocks are linear or Kantor–Knuth semifield.

Hence to construct new semifield flocks or to do a nontrivial classification we require \( h > 2 \) for \( q = p^h \).

**Lemma 2** (Bader and Lunardon [1]). For \( q \) odd, let \( \mathcal{F}(f, g) \) be a semifield flock of \( K \). If \( \alpha \) and \( \beta \) are elements of \( \text{GF}(q) \), with \( \alpha \neq 0 \), and if \( g'(t) = g(\alpha t) + 2\beta t \) and \( f'(t) = \alpha f(\alpha t) - \beta g(\alpha t) - \beta^2 t \) then \( \mathcal{F}(f', g') = \mathcal{F}(f, g) \).

Condition (3) gives the following:

**Lemma 3.** For \( q \) odd, if \( \mathcal{F}(f, g) \) is a flock then so are \( \mathcal{F}(f^{-1}, g \circ f^{-1}) \) and \( \mathcal{F}(f, -g) \). In fact, they are equivalent.

**Proof.** The cone preserving maps (see [10]) \( [t, f(t), g(t), 1] \mapsto [f(t), t, g(t), 1] \) and \( [t, f(t), g(t), 1] \mapsto [t, f(t), -g(t), 1] \) map the flock \( \mathcal{F}(f, g) \) to \( \mathcal{F}(f^{-1}, g \circ f^{-1}) \) and \( \mathcal{F}(f, -g) \), respectively. Since both maps preserve the cone and map the set of \( q \) planes partitioning the cone minus its vertex onto another set of planes partitioning the cone minus its vertex, we have \( \mathcal{F}(f, g) \cong \mathcal{F}(f^{-1}, g \circ f^{-1}) \) and \( \mathcal{F}(f, g) \cong \mathcal{F}(f, -g) \). \( \square \)
There are three classes of nonlinear semifield flocks known.

1. The Kantor–Knuth semifield flocks [9,13]: Let \( q \) be odd. The planes \( \pi_t, t \in \text{GF}(q) \),

\[ \pi_t: tx_0 - mt\sigma x_1 + x_3 = 0, \]

where \( m \) a nonsquare and \( \sigma \) an automorphism of the field, define a semifield flock of \( K \). The flock is linear if and only if \( \sigma = 1 \).

The next two theorems give a characterisation of the linear and Kantor–Knuth semifield flocks.

**Theorem 4** (Thas [20]). Every flock of \( K \) for which the planes of the \( q \) conics all contain precisely one common exterior point of \( K \) is a Kantor–Knuth semifield flock.

**Theorem 5** (Thas [20]). For a semifield flock \( F(f,g) \) of \( \text{PG}(3,q) \), the points \((t,f(t),g(t))\) for nonzero \( t \in \text{GF}(q) \) are collinear in \( \text{PG}(2,q) \) if and only if \( F \) is either linear or a Kantor–Knuth semifield flock.

2. The Ganley semifield flocks [7,9]: Let \( q = 3^r, r > 2 \). The planes \( \pi_t, t \in \text{GF}(q) \),

\[ \pi_t: tx_0 - (mt + m^{-1}\sigma^3 t)x_1 - t^3x_2 + x_3 = 0, \]

where \( m \) is a given nonsquare, define a semifield flock of \( K \).

3. The Penttila–Williams semifield flock [2,18]. Let \( q = 243 = 3^5 \). The planes \( \pi_t, t \in \text{GF}(q) \),

\[ \pi_t: tx_0 + 2t^3x_1 + t^7x_2 + x_3 = 0 \]

define a semifield flock of \( K \).

3.1. Searches for semifield flocks

We say that the flock \( F(f,g) \) in \( \text{PG}(3,q) \) is over \( \text{GF}(q_0) \) when the coefficients of the polynomials \( f \) and \( g \) are in the subfield \( \text{GF}(q_0) \) of \( \text{GF}(q) \), that is, \( f, g \in \text{GF}(q_0)[x] \).

The searches for semifield flocks were done by finding two functions \( f \) and \( g \in \text{GF}(q) \), satisfying condition (3). Since from [9] these functions must be additive we need to find 2h elements \( a_0, a_1, \ldots, a_{h-1}, b_0, b_1, \ldots, b_{h-1} \) of \( \text{GF}(p^h), q = p^h, p \) prime, such that \( f(t) = \sum_{i=0}^{h-1} a_i t^p \) and \( g(t) = \sum_{i=0}^{h-1} b_i t^p \) satisfy condition (3). The below theorems are proved by finding all such functions.

Due to Theorem 1 we only search in \( \text{PG}(3, p^h) \), for \( h > 2 \). Since we deal with \( \text{PG}(3, 3^3) \) in Section 4 the next smallest case is for \( q = 5^3 \).

**Computer Result 1.** A semifield flock \( F(f,g) \) in \( \text{PG}(3, 125) \) over \( \text{GF}(5) \) is equivalent to the linear or Kantor–Knuth semifield flock.
Proof. There are 70 \((f, g)\) pairs with coefficients in \(\text{GF}(5)\) such that \(f\) and \(g\) satisfy condition (3). Hence there are 70 semifield flocks over \(\text{GF}(5)\) in \(\text{PG}(3, 125)\). To determine which family each semifield flock belongs to (if any) we use the maps shown in Lemmas 2 and 3 to construct equivalent semifield flocks. For instance, the linear flock \((t, t)\) under these maps, is equivalent to the linear flocks \((t, 4t), (3t, 3t), (2t, t), (3t, 2t), (2t, 4t), (2t, 0), (3t, 0), (4t, 2t)\) and \((4t, 3t)\) (where 0 denotes the zero function). Similarly, the Kantor–Knuth semifield \((2t^2 + 5t; 0)\) is equivalent to \((3t^2 + 5t^2; 0), (2t + 2t^2 + 3t^2 + t^3; 2t), (4t + 2t^2 + 4t^2; 4t), (3t + 2t^2 + t^3 + 2t^4 + 4t^5), (4t + 2t^2 + 3t^2 + t^3 + t^5; 3t^2 + 2t^4 + t^5), (3t^2, 0), (2t^2, 0)\) and so on. With repeated use of these maps we determine equivalence classes of semifield flocks corresponding to the known families of semifield flocks. We find of the 70 \((f, g)\) pairs, 60 are equivalent to the Kantor–Knuth semifield flock and 10 are equivalent to the linear flock. \(\square\)

**Computer Result 2.** A semifield flock \(\mathcal{F}(f, g)\) in \(\text{PG}(3, 243)\) over \(\text{GF}(3)\) is equivalent to either the linear, Kantor–Knuth, Ganley or Penttila–Williams semifield flocks.

**Proof.** There are 75 semifield flocks over \(\text{GF}(3)\) in \(\text{PG}(3, 243)\), that is, pairs \((f, g)\) satisfying condition (3). As for Computer Results 1 we can use the maps of Lemmas 2 and 3 to determine equivalence classes of semifield flocks corresponding to the known families. Of the 75 semifield flocks, 24 are Kantor–Knuth, 3 are linear, 24 are Ganley and 24 are equivalent to the Penttila–Williams flock. \(\square\)

**Computer Result 3.** There are no semifield flocks \(\mathcal{F}(f, g)\) in \(\text{PG}(3, 3^4)\) and \(\text{PG}(3, 3^6)\) over \(\text{GF}(3)\) and in \(\text{PG}(3, 5^4)\) over \(\text{GF}(5)\).

**Proof.** Since all nonzero elements of \(\text{GF}(3)\) as a subfield of \(\text{GF}(3^4)\) or \(\text{GF}(3^6)\) are squares, any semifield flock in \(\text{PG}(3, 3^4)\) or \(\text{PG}(3, 3^6)\) over \(\text{GF}(3)\) must be new. Similarly for semifield flocks over \(\text{GF}(5)\) in \(\text{PG}(3, 5^4)\). None were found. \(\square\)

**Computer Result 4.** A semifield flock \(\mathcal{F}(f, g)\) in \(\text{PG}(3, 3^7)\) over \(\text{GF}(3)\) is equivalent to either the linear, Kantor–Knuth or Ganley semifield flock.

**Proof.** There are 63 semifield flocks over \(\text{GF}(3)\) in \(\text{PG}(3, 2187)\). There are 34 equivalent to the Kantor–Knuth semifield flock, 26 equivalent to the Ganley semifield flocks and 3 equivalent to the linear flock. \(\square\)

**Computer Result 5.** A semifield flock \(\mathcal{F}(f, g)\) in \(\text{PG}(3, 5^5)\) over \(\text{GF}(5)\) is equivalent to either the linear or Kantor–Knuth flock.

**Proof.** There are 130 semifield flocks over \(\text{GF}(5)\) in \(\text{PG}(3, 3125)\). Of these, 10 are equivalent to the linear flock and 120 equivalent to the Kantor–Knuth semifield flock. \(\square\)
4. Semifield flocks in PG(3, 27)

In this section we restrict ourselves to the classification of semifield flocks in PG(3, 27).

Since PG(3, 27) is too large for current isomorphism packages such as nauty [16], the classification of semifield flocks in PG(3, 27) is done through three methods. Firstly, we use Theorem 5 to determine which of the semifield flocks are nonlinear and non-Kantor–Knuth. Secondly, we construct an F-profile of each flock (also using Theorem 4 to determine non-Kantor–Knuth semifield flocks). Thirdly, we calculate all the flocks \( \mathcal{F}(f, g) \) that are equivalent to the Ganley flock. If each of the equivalent Ganley flocks matches all of the nonlinear and non-Kantor–Knuth flocks found then all the constructed flocks are known.

Let \( F = F(f, g) \) be a flock of PG(3, q). An F-profile [17] of \( F \) is a vector of \( q + 1 \) integers \((x_0, x_1, \ldots, x_q)\), where \( x_i \geq 0 \) is the number of points of PG(3, q) that lie on precisely \( i \) planes of the flock (usually only nonzero values are stated). The F-profile of a flock is a coarse isomorphism invariant, that is, different F-profiles imply nonisomorphic flocks, though the converse is not necessarily true (hence its coarseness). This invariance is a computationally quick way of determining nonisomorphic flocks. In PG(3, 27) the F-profile of the linear flock is: \( x_0 = 729, x_1 = 19,683, x_27 = 28 \); the F-profile of the Kantor–Knuth flock is: \( x_0 = 6345, x_1 = 10,935, x_3 = 3159 \) and \( x_27 = 1 \); and the F-profile of the Ganley flock is: \( x_0 = 5929, x_1 = 11,664, x_3 = 2808 \) and \( x_9 = 39 \).

From Theorem 4 we can see that the F-profile characterises the linear and Kantor–Knuth semifield flocks in PG(3, q) since they are the only flocks with \( x_q \) nonzero, or more precisely, \( x_q = 1 \) if and only if the F-profile is of a Kantor–Knuth semifield flock [20].

Lemma 6. All semifield flocks in PG(3, 27) have F-profiles equivalent to the linear, Kantor–Knuth and Ganley semifield flocks.

Proof. A flock \( F(f, g) \) is a semifield flock if and only if \( f \) and \( g \) are additive. We determine all pairs of additive functions \((f, g)\) that satisfy condition (3). There are 74,061 \((f, g)\) pairs of additive functions satisfying condition (3). We conclude there are 74061 semifield flocks in PG(3, 27) and no more.

For each \((f, g)\) satisfying condition (3) we determine its F-profile. The F-profile of each of these semifield flocks is the same as either a linear, Kantor–Knuth or Ganley F-profile. Hence there are at least 3 nonequivalent semifield flocks in PG(3, 27).

Theorem 7. All semifield flocks in PG(3, 27) are equivalent to either the linear, Kantor–Knuth or Ganley flock.

Proof. The search produces 74,061 semifield flocks of PG(3, 27). From the characterisation of the Kantor–Knuth flocks by Theorem 4 we have 9828 Kantor–Knuth flocks from their F-profiles. From Theorem 5 we can show that there are 10,179 semifield flocks that are either linear or Kantor; hence we have 351 linear flocks. This leaves us with 63,882 semifield flocks not equivalent to either the linear nor the Kantor–Knuth
semi-field flock. These semi-field flocks all have the same F-profile: the F-profile of the Ganley flock. Since the F-profile is a coarse isomorphism invariant we need to determine that a semi-field flock $F$ with an F-profile of a Ganley semi-field flock in $\text{PG}(3,27)$ implies that $F$ is a Ganley semi-field flock.

We do this by counting the number of Ganley flocks in $\text{PG}(3,27)$. To calculate all flocks equivalent to a semi-field flock $F(f,g)$ we need only calculate the images of $[t;f(t);g(t);1]$, for all $t \in \text{GF}(q)$, under the stabiliser of the cone of a particular plane, not on the vertex of the cone. Let $G'$ be the homography group of the cone $K': x_0x_2 = x_1^2$ in $\text{PG}(3,q)$. Then the stabiliser of the hyperplane $[0;0;0;1]$ in $G'$, as a point map, is

$$
\begin{pmatrix}
a^2 & 2ac & c^2 & 0 \\
ab & ad - bc & cd & 0 \\
b^2 & 2bd & d^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

with matrices acting on the left and $ad - bc \neq 0$. If $F(f,g)$ is the Ganley flock of the cone $K: x_0x_1 = x_2^2$ with planes $\pi_i: [t,f(t),g(t),1]$ then $F(f,g)$ is equivalent to the flock $F'(f',g')$ with planes, $\pi'_i = [t,f'(t),g'(t),1]$, defined as

$$
\begin{pmatrix}
t \\
f'(t) \\
g'(t) \\
1
\end{pmatrix} =
\begin{pmatrix}
a^2 & c^2 & 2ac & 0 \\
ab & cd & ad - bc & 0 \\
b^2 & d^2 & 2bd & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}^T
\begin{pmatrix}
t^\sigma \\
f(t)^\sigma \\
g(t)^\sigma \\
1
\end{pmatrix}
$$

with $\sigma \in \text{Aut GF}(q)$. By mapping the Ganley flock under all possibilities for map (5) we construct 511 056 equivalent images of the Ganley flock.

Let $F$ be one of the 63 882 flocks constructed from the exhaustive search which are not linear nor Kantor–Knuth. Each $F$ appears as an image of the Ganley flock constructed from (5). Hence $F$ is a Ganley semi-field flock. (Furthermore, every image appears in the exhaustive search. As expected.) Hence there are at most 3 nonequivalent semi-field flocks in $\text{PG}(3,27)$. From Lemma 6 we have the conclusion. \(\Box\)

5. Likeable functions

In $\text{GF}(q)$ with $q > 3$ and $q = p^e$ with $p$ a prime and $p \neq 3$, we say a function $J: \text{GF}(q) \to \text{GF}(q)$ is likeable if it satisfies the following conditions:

1. $J$ is additive over $\text{GF}(q)$, and
2. if $u^2 = t^2 - \frac{1}{2} t^4 + t J(t)$ then $t = u = 0$. 
An additive function $J$ is likeable if and only if the equation
\[ x^2 - x + \frac{1}{3} \frac{J(a)}{a^3} = 0 \]
has no solution for all $x \in \text{GF}(q)$ and nonzero $a \in \text{GF}(q)$. In particular, if $q$ is odd then $J$ is likeable if and only if
\[ \frac{J(a)}{a^3} - \frac{1}{12} \]
is a nonsquare for all nonzero $a \in \text{GF}(q)$ [12].

Given a likeable function $J$, $\mathcal{F}(\frac{-1}{2}t^3 + J(t), -t^2)$ is a flock [9], called a likeable flock.

There are two classes of known likeable functions:

The likeable function $J \equiv 0$ where $q \equiv 2 \pmod{3}$ associated with the Fisher–Thas–Walker–Kantor–Betten (FTWKB) objects [4,6,13,23]. (Fisher and Thas constructed the flock, Walker and Betten constructed the translation plane and Kantor constructed the generalised quadrangle associated with the likeable function.) For $q$ even the only likeable flock occurs with $J \equiv 0$ [8,9].

The Kantor likeable functions $J(t) = n^{-1}t + nt^5$ for a nonsquare $n$ with $q = 5^e > 5$ [12].

Again we use (4) to reduce the search space of functions we need to consider. Since we only need to find one additive function, namely $a_0, \ldots, a_{h-1} \in \text{GF}(p^h)$ with $J(t) = \sum_{i=0}^{h-1} a_it^{p^i}$, we can search in spaces with much larger $q$ than for the semifield case.

The main result on likeable flocks is Computer Result 11.

5.1. Searches for likeable functions

**Computer Result 6.** All the likeable functions in $\text{GF}(5^2)$ are equivalent to the Kantor likeable functions.

**Proof.** There are 12 likeable functions all of the form $n^{-1}t + nt^5$ with $n$ a nonsquare. Hence all Kantor. □

**Computer Result 7.** There are no likeable functions in $\text{GF}(7^2)$.

**Computer Result 8.** There are no likeable functions in $\text{GF}(11^2)$.

**Computer Result 9.** All the likeable functions in $\text{GF}(5^3)$ are those equivalent to either the FTWKB or Kantor likeable function.

**Proof.** There are 62 likeable functions of the form $n^{-1}t + nt^5$, with $n$ a nonsquare, and the likeable function $J \equiv 0$. Hence 62 Kantor likeable functions and one FTWKB likeable function. □
Computer Result 10. There are no likeable functions in GF(7^3).

Computer Result 11. All likeable functions in GF(5^4) are equivalent to either the FTWKB or Kantor likeable functions.

Proof. There are 312 likeable functions all of the form \( n^{-1} t + nt^5 \) with \( n \) a nonsquare. Hence all Kantor. \( \square \)

Computer Result 12. All the likeable functions in GF(11^3) are equivalent to the FTKWB likeable function.

Proof. One likeable function is found: \( J \equiv 0 \), the FTWKB likeable function. \( \square \)

Computer Result 13. There are no likeable functions in GF(7^4) over GF(7) and over GF(49).

Computer Result 14. All likeable functions in GF(5^5) over GF(5) are equivalent to either the FTWKB or Kantor likeable functions.

Proof. There are 2 (Kantor) likeable functions, \( J(t) = 2t + 3t^5 \) and \( J(t) = 3t + 2t^5 \), and the (FTWKB) likeable function \( J \equiv 0 \). \( \square \)

A function, \( f \), is binomial if it has the form \( f(t) = at^i + bt^j \) for some \( a, b \in GF(q) \) and for appropriate \( i \) and \( j \) where \( 0 < i, j < q \).

Computer Result 15. All binomial likeable functions over GF(5^4), GF(5^5) and GF(5^6) are known.

Proof. In GF(15625) there are 7812 binomial likeable functions all Kantor. In GF(3125) there are 1562 binomial likeable functions that are Kantor and 10 binomial functions \( (a = b = 0 \) in the definition of binomial function) that are FTWKB. In GF(625) there are 312 binomial likeable functions all Kantor. Hence all binomial likeable functions for \( q = 625, 3125 \) and 15 625 are known. \( \square \)

6. Remarks

The searches for new semifield and likeable flocks were inspired by two events: firstly, realising that condition (3) gives an extremely quick way of determining ‘flock-ness’, and, secondly, that the recently constructed Penttila–Williams semifield flock could have been found by these types of simple minded searches (though showing that it was new would have been extremely difficult). From the results in this paper one could say that any new semifield or likeable flock would live in a very large field or not be of a type where the coefficients of the functions are in the subfield. In fact, one could go so far as to say that there are no other examples of likeable functions, but possibly more examples of semifield flocks.
Any further new results in searches for semifield and likeable flocks will be posted at Bill Cherowitzo’s *Flocks of Cones* web page [5].

**Acknowledgements**

I would like to thank, in no particular order, Laura Bader, Frank De Clerck, Marialuisa de Resmini, Guglielmo Lunardon and Tim Penttila for encouragement and inspiration and The University of Ghent for the use of their computer facilities.

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