Module containment property for linear equations
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Abstract
A general module containment property is proved for almost periodic linear systems of differential equations, in both finite and infinite dimensions.
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1. The problem
Some time ago a friend of mine, Rafael Ortega, asked me what can be said a priori about the periodicity type of the solutions to a linear differential equations in $\mathbb{R}^N$

$$\dot{x} = A(t)x + f(t),$$

where the matrix $A(t)$ and the vector $f(t)$ are almost periodic in time. Indeed, though much is known in many particular cases, no general answers seem to be available in the literature: to provide one of them is the aim of this paper.

The correct way to state the problem is to use the so-called module of an almost periodic function $u(t)$, call it $\text{mod}(u)$. This is the smallest additive subgroup of $\mathbb{R}$ containing all the
characteristic frequencies appearing in the Fourier expansion of $u(t)$, namely the real $\lambda$’s for which

$$\hat{u}(\lambda) = \lim_{T \to +\infty} \int_0^T u(t)e^{-i\lambda t} dt$$

is different from zero. Parseval type arguments show this may happen for an at most a countable set of frequencies. The module of an almost periodic function encodes its periodicity type. For instance, saying that a (continuous) function $u(t)$ is $T$-periodic rewrites as $\text{mod}(u) \subset \omega \mathbb{Z}$, where $\omega = 2\pi/T$. More generally, $u(t)$ is quasi-periodic with basic frequencies $\omega_1, \ldots, \omega_k$ if and only if $\text{mod}(u) \subset \omega_1 \mathbb{Z} + \cdots + \omega_k \mathbb{Z}$. All these facts may be found in every book dealing with almost periodic functions. As notation, when $\mathcal{U}$ is a set of almost periodic functions, $\text{mod}(\mathcal{U})$ is defined as before, but considering the characteristic frequencies of every element of $\mathcal{U}$.

The problem then becomes to provide the best a priori estimate for $\text{mod}(x)$, where $x$ is any given almost periodic solution to (1), if any of course exists.

Quite clearly, the characteristic frequencies of the coefficients of the equation must enter this estimate. New frequencies possibly arise from the almost periodic solutions of the associated homogeneous equation

$$\dot{y} = A(t)y.$$  

In general, these frequencies are unrelated to those of $A(t)$, and the well-known failure of the Floquet theory in the almost periodic context (see for instance [4]) makes even more difficult any prediction about them. Summing up, denoting by $S$ the class (in fact, a finite dimensional linear space) of the almost periodic solutions to (2), the best one can expect to prove is that

$$\text{mod}(x) \subset \text{mod}(S, A, f)$$  

holds for every almost periodic solution to (1).

The validity of this inclusion may be checked rather directly in many cases, like for instance when Eq. (2) exhibits an exponential dichotomy, or when both $A(t)$ and $f(t)$ are $T$-periodic. In the former case, indeed, $S = 0$ and Eq. (1) has a unique bounded solution: its integral representation shows it is in fact almost periodic and fulfills (3). In the latter, the conclusion follows from a remark in a classical paper by Massera [7], which ensures the existence of a $T$-periodic solution if a bounded solution exists. It should be mentioned that in this case, however, a result more stringent than (3) is available, which also works for nonlinear equations: Cartwright proved in [2] that the most general almost periodic solution is indeed a quasi-periodic one, with at most $N$ basic frequencies, one of which is of course $\omega = 2\pi/T$.

In fact, the general validity of the inclusion (3) is a straightforward consequence of the following result, which seems to have gone unnoticed in the literature.

**Theorem 1.** If (1) admits almost periodic solutions, then at least one of them satisfies

$$\text{mod}(x) \subset \text{mod}(A, f).$$  

Roughly speaking, these solutions are the most natural answer to the forced oscillations of the system: they are classically said to enjoy the *module containment property*, with respect to
the coefficients of the equation. Imagine one has to prove or to disprove the existence of almost periodic solutions when, for instance, $A(t)$ and $f(t)$ are quasi-periodic of frequencies $\omega_1, \ldots, \omega_m$ and $\omega_{m+1}, \ldots, \omega_N$, respectively: the previous result says it suffices to look for quasi-periodic solutions of frequencies $\omega_1, \ldots, \omega_N$.

The celebrated Favard theory, which has its origin in [3], also applies to get the same type of almost periodic solutions, reducing their existence to an a priori estimate: they do exist as soon as Eq. (1) admits bounded solutions. Though the practical relevance of Theorem 1 is not comparable with that of Favard theory, the latter does not cover the former because of the Favard condition. Indeed, for Favard theory to work, some structural condition has to be satisfied by the class of homogeneous equations

$$\dot{y} = B(t)y, \quad B \in H(A),$$

where $H(A)$, the hull of $A(t)$, is the uniform (in time) closure of the set of all the matrices obtained from $A$ by translation. Precisely, for every equation in (5), every nontrivial bounded solution $y(t)$ must be uniformly bounded away from zero, in the sense that $\inf_t |y(t)| > 0$. A systematic approach to Favard theory and to its many applications may be found, for instance, in [4].

Coming back to the module containment property of Theorem 1, it should be noticed that the scalar case is quite exceptional: when $N = 1$, indeed, it is automatically satisfied for every almost periodic solution, even in the more general framework of nonlinear equations (see [4]).

The opposite case $N = \infty$, namely when (1) is a differential equation in an infinite dimensional Banach space $E$, rather than $\mathbb{R}^N$, seems also to be exceptional from the point of view of Favard theory. It is well know, indeed, that almost periodic functions may now exist, whose primitive is bounded but not almost periodic in the classical (also called strong) sense. Starting with the pioneering work of Amerio, it became clear that the relation between bounded and almost periodic solutions requires, to survive, some restrictions of geometrical nature on $E$. These restrictions may be to a large extent removed if one accepts weak almost periodic solutions (see [8]), instead of classical ones. A detailed exposition of these and related topics may be found in the beautiful books [1] and [6], written by some of the main contributors to the researches in the field.

On the contrary, Theorem 1 extends unconditionally to the infinite dimensional context, but with a more delicate proof. To be precise, assume that $f(t)$ and $A(t)$ are almost periodic functions with values, respectively, in the Banach space $E$, and in the space of bounded linear operators on $E$. The norm here is the usual operatorial one, associated to the given norm on $E$.

**Theorem 2.** For any Banach space $E$, if (1) admits almost periodic solutions, then at least one of them has the module containment property.

The novelty of Theorems 1 and 2 lies in their full generality. The solution which the former alludes to, is ‘the (unique) almost periodic solution of minimal $L^\infty$ norm,’ while the latter needs a more involved selection. The original contribution here reduces to the intuition that the notion of minimal almost periodic solution makes sense in any situation, while the rest, like for most of the existing literature on the subject, is a nowadays standard application of the pioneering approach introduced by Favard.

Before concluding, it is worth mentioning a version of the classical Favard theory, which holds again in every Banach space $E$. The price to pay for such a generality, is the need to know
the existence of a relatively compact solution to (1), that is a solution with a relatively compact image, which unfortunately is a hard task in general.

**Theorem 3.** Assume that every nontrivial relatively compact solution $y(t)$ to (5) satisfies $\inf_t |y(t)| > 0$. If Eq. (1) admits a relatively compact solution, then it also admits an almost periodic solution with the module containment property.

The proof mimics the classical finite dimensional one, but for the variants already introduced for Theorem 2. Though the result is the natural extension to general linear systems of the well-known Bochner infinite dimensional version of the Bohl–Bohr theorem on the primitives of almost periodic functions (see [1]), I was not able to trace it in the existing literature.

**Notations and prerequisites.** The symbol $|\ |$ denotes the norm of $E$. When $E = \mathbb{R}^N$, the Euclidean norm is considered, to make use of the parallelogram law.

The almost periodicity is always intended in the classical sense of Bohr. Definition, by means of approximate periods, and main properties may be found, for instance, in [4] or [6]: both the monographs are especially concerned with differential equations, the second one also in the infinite dimensional setting. Some of these properties, related to Fourier analysis, have been already discussed in the introduction. Hereafter, some additional ones will be summarized, which are of interest for the present paper.

Every almost periodic function $f : \mathbb{R} \to E$ is uniformly continuous on all the real line, and its image is a relatively compact subset of $E$. Denote by $f_\tau$ the translated function $f_\tau(t) = f(t + \tau)$. The almost periodicity may be equivalently restated as the compactness of $H(f) = \{f_\tau : \tau \in \mathbb{R}\}$, the so-called hull of $f$, where the closure is intended in the topology of the uniform convergence over the real line. This statement is usually referred as the Bochner criterium for almost periodicity. Since almost periodicity is preserved under uniform limits, $H(f)$ is made of almost periodic functions: quite clearly, all of them share with $f$ their hull and their module.

When more almost periodic functions $f_1, \ldots, f_m$ are considered, $\text{mod}(f_1, \ldots, f_m)$ and $H(f_1, \ldots, f_m)$ are defined as the module and the hull of $f = (f_1, \ldots, f_m)$, respectively.

A key point in the applications is to decide whether an arbitrary function $g$ on $\mathbb{R}$ is almost periodic and satisfies $\text{mod}(g) \subset \text{mod}(f)$. This is equivalent to say that $g$ may be represented over $H(f)$, in the sense that $g(t) = U(f_t)$ holds for every $t$, where $U$ is some suitable continuous function on $H(f)$. A sometimes convenient way to test this fact is to show that, for any sequence of translation factors $\tau_k$, if $\|f_{\tau_k} - f\|_\infty \to 0$ then $\|g_{\tau_k} - g\|_\infty \to 0$ is also true.

**2. Proof of Theorem 1**

Define by $\varphi$ the existing almost periodic solution to (1), and set

$$\lambda = \inf \{\|x\|_\infty : x \text{ almost periodic solution to (1)}\} \leq \|\varphi\|_\infty.$$ 

Here $\|x\|_\infty = \sup_t |x(t)|$ where $|\ |$ denotes the Euclidean norm on $\mathbb{R}^N$.

To prove that $\lambda$ is attained, take a minimizing sequence $x_k$ for $\lambda$, and decompose $x_k = \varphi + z_k$, where $z_k \in S$, the linear space of the almost periodic solutions to the homogeneous equation (2). By construction, $\|z_k\|_\infty$ is bounded, and then (possibly passing to a subsequence) one has

$$\|z_k - z^*\|_\infty \to 0.$$
for some suitable \( z^\ast \). This is because \( S \) is finite dimensional. Of course, \( z^\ast \) is a solution to (1), which is almost periodic inasmuch as it is a uniform limit of almost periodic functions. Hence \( z^\ast \in S \) and \( \lambda \) is attained at \( x^\ast = \varphi + z^\ast \).

Next argument shows that \( \lambda \) is uniquely attained: it goes back to Favard [3]. Suppose by contradiction that \( \lambda \) is also attained at a different almost periodic solution \( y^\ast \), and set \( w^\ast = (x^\ast - y^\ast)/2 \). Clearly \( w^\ast \) is an almost periodic solution to (2). Since it is nontrivial by construction, one knows it is in fact bounded away from zero, namely that

\[
|w^\ast(t)| \geq \delta \quad \forall t
\]

for some suitable \( \delta > 0 \). This property is well known in literature, with the name of separation property for (2). For the proof, see for instance [4] or, in the next section, the corresponding particular case of Lemma 4.

Look now at the convex combination \( v^\ast = (x^\ast + y^\ast)/2 \), which is again an almost periodic solution to (1). The parallelogram identity for the Euclidean norm in \( \mathbb{R}^N \) implies

\[
|v^\ast(t)|^2 + \delta^2 \leq |v^\ast(t)|^2 + |w^\ast(t)|^2 \leq \frac{|x^\ast(t)|^2 + |y^\ast(t)|^2}{2} \leq \lambda^2 \quad \forall t,
\]

which yields \( \|v^\ast\|_\infty^2 \leq \lambda^2 - \delta^2 < \lambda^2 \), contradicting the definition of \( \lambda \).

Summing up, the minimum value \( \lambda \) must be uniquely attained, say at \( x^\ast \). It remains to prove that \( x^\ast \) enjoys the module containment property (4). To do this, one has to take any translation factor \( \tau_k \) for which

\[
\|A_{\tau_k} - A\|_\infty + \|f_{\tau_k} - f\|_\infty \to 0,
\]

and show that, possibly passing to a subsequence,

\[
\|x^\ast_{\tau_k} - x^\ast\|_\infty \to 0.
\]

In this case indeed, since \( x^\ast \) is the only limit point of \( x^\ast_{\tau_k} \), the entire sequence tends to it. Then the desired module containment follows from a standard characterization, given at the end of the previous section.

Since \( x^\ast \) is almost periodic, the Bochner criterium says that, possibly passing to a subsequence, there exists \( y^\ast \) such that \( \|x^\ast_{\tau_k} - y^\ast\|_\infty \to 0 \). Hence \( y^\ast \) is almost periodic too, and of course solves (1). On the other hand,

\[
\|x^\ast\|_\infty = \|x^\ast_{\tau_k}\|_\infty \to \|y^\ast\|_\infty,
\]

showing that \( \lambda \) is attained at \( y^\ast \). Hence \( y^\ast = x^\ast \) by uniqueness, which concludes the proof.

3. Proof of Theorem 2

When trying to reproduce the finite dimensional scheme, one is faced with two evident problems. The first one is that \( S \), the space of the almost periodic solutions to (2), is now possibly infinite dimensional. The second one is that the norm \( \| \| \) on the Banach space \( E \) needs not satisfy the parallelogram identity. Both problems ask for a more refined selection of the ‘minimal
solution \( x^* \): while the former may be dealt with quite standard instruments, the latter needs a trick due to Zhikov [8].

From now on, the almost periodicity of maps with values in \( E \) will be always intended in the strong sense, with respect to the norm \( | | \). Denote again by \( \varphi \) the existing almost periodic solution to (1), and consider the set

\[
K_\varphi = \overline{\text{co}(\text{Im}\varphi)}
\]

(6)

where ‘co’ stays for the convex hull. Since \( \varphi \) is almost periodic, \( \text{Im}\varphi \) is totally bounded and hence, due to the completeness of \( E \), \( K_\varphi \) is a compact set.

Now let \( S_\varphi \) be the set of all the almost periodic solutions \( x \) which satisfy

\[
\text{Im} x \subset K_\varphi,
\]

\[
\|x_\tau - x\|_\infty \leq \|\varphi_\tau - \varphi\|_\infty \quad \forall \tau.
\]

(7)

It is not difficult to check that the set \( S_\varphi \) is a bounded, convex and closed subset of \( L^\infty(\mathbb{R}; E) \), endowed with the topology of the uniform convergence over the real line.

In fact, \( S_\varphi \) is also a compact subset of \( L^\infty(\mathbb{R}; E) \). Notice indeed that \( \varphi \) is uniformly continuous on \( \mathbb{R} \), so that the second condition in (7) is in fact, for small \( \tau \), an equicontinuity assumption on the family \( S_\varphi \). Moreover, the same condition guarantees that the approximate periods of \( \varphi \) also work for every element of \( S_\varphi \). An appropriate version of the Ascoli–Arzelà theorem (see for instance Lyusternik’s theorem in [6]) then applies to conclude.

The naïve idea of taking \( x^* \) as ‘the unique element of minimal \( L^\infty(\mathbb{R}; E) \) norm in \( S_\varphi \)’ fails because uniqueness, which is certainly granted if \( E \) is an Hilbert space and \( | | \) is its natural norm, may be false in general. Here is where the Zhikov idea comes into the play.

Denote by \( E_\varphi \) the smallest linear closed subspace of \( E \) containing \( \text{Im}\varphi \), and then also \( K_\varphi \). Since \( \text{Im}\varphi \) is totally bounded, \( E_\varphi \) must be separable. Choose an Hilbert space \( H \) and an injective, bounded linear operator \( \iota : E_\varphi \rightarrow H \). For instance, one may take \( H = L^2([0, 1]) \) with the usual integral norm: indeed a classical result (see [5]) says that every separable Banach space is norm-isomorphic to a closed subspace of \( C([0, 1]) \), endowed with the \( L^\infty \) norm.

If \( x : \mathbb{R} \rightarrow E_\varphi \) is almost periodic, then the same is true for \( \iota \circ x \): this holds, in particular, for all the elements of \( S_\varphi \). In the same spirit of [8], the idea is now to consider the following minimization problem

\[
\lambda = \inf_{x \in S_\varphi} \|\iota \circ x\|_\infty.
\]

Since \( S_\varphi \) was proved to be a compact in the \( L^\infty(\mathbb{R}; E) \) topology, the same happens to \( \{\iota \circ x : x \in S_\varphi\} \) in the \( L^\infty(\mathbb{R}; H) \) topology, so that the infimum is attained. On the other hand, one may now take advantage of the parallelogram identity in \( H \) in order to prove that \( \lambda \) is attained at a unique point \( x^* \in S_\varphi \), exactly as in the proof of Theorem 1. The only delicate point concerns the separation property, which takes now the following form.

**Lemma 4.** Assume \( z \) is an almost periodic solution to (2), with values in \( E_\varphi \). If \( z \neq 0 \) then

\[
\inf_{t} |\iota(z(t))| > 0.
\]
**Proof.** Assume by contradiction that \( \iota(z(\tau_k)) \to 0 \) in \( H \) for some \( \tau_k \). By almost periodicity, it is not restrictive to assume that \( \|z_{\tau_k} - w\|_\infty + \|A_{\tau_k} - B\|_\infty \to 0 \) holds for some suitable \( w \) and \( B \).

Of course, \( B \) is an almost periodic operator and \( w \) is an almost periodic solution to
\[ \dot{y} = B(t)y. \]

Since \( \|z\|_\infty = \|z_{\tau_k}\|_\infty \to \|w\|_\infty \), \( w \) cannot be the trivial solution. On the other hand, since \( \iota \) is continuous, one has \( \iota(z(\tau_k)) \to \iota(w(0)) \) and hence \( \iota(w(0)) = 0 \). Thus \( w(0) = 0 \) due to injectivity of \( \iota \); this forces \( w \) to be the trivial solution, which contradicts the previous conclusion.

The rest of the proof, namely the verification that \( x^* \) enjoys (4), is unchanged with respect to Theorem 1.

4. **Proof of Theorem 3**

Denote by \( \varphi \) the existing relatively compact solution to (1), namely a solution whose range \( \text{Im} \varphi \) is relatively compact in \( E \), and take \( K_\varphi \) as in (6). Of course, \( K_\varphi \) is again a compact subset of \( E \). Define now \( S^{B,g}_\varphi \) to be, for every \( (B, g) \in H(A, f) \), the convex set of all the solutions to
\[ \dot{x} = B(t)x + g(t) \] (8)
which satisfy both the conditions in (7), and collect all of them into
\[ S_\varphi = \bigcup_{(B, g) \in H(A, f)} S^{B,g}_\varphi. \]

Though the set \( S_\varphi \) is a bounded and closed subset of \( L^\infty(\mathbb{R}; E) \), compactness is out of question in this space. The right space to work with is now \( L^\infty_{\text{loc}}(\mathbb{R}; E) \), endowed with the topology of the uniform convergence over the compact subsets of the real line. This topology is weak enough to admit many compacts but, at the same time, strong enough to preserve the key property that limits of solutions are again solutions.

Both the conditions in (7) are closed with respect to the convergence in \( L^\infty_{\text{loc}}(\mathbb{R}; E) \). This is trivial for the first condition, since \( K_\varphi \) itself is closed in \( E \). Concerning the second condition, one has to use the fact that one may loose but not gain mass, when taking limits in \( L^\infty_{\text{loc}}(\mathbb{R}; E) \): more precisely, \( \|\cdot\|_\infty \) is lower semicontinuous with respect to the uniform convergence on compact sets. This property, which may be easily tested, will be implicitly used in the rest of the proof.

As a straightforward consequence, all the sets \( S^{B,g}_\varphi \) are also closed in the \( L^\infty_{\text{loc}}(\mathbb{R}; E) \) topology. The same may be proved for \( S_\varphi \), by using the Bochner criterium: the compactness of \( H(A, f) \), in the topology of the uniform convergence over the real line, allows to construct the limit equation which is needed during the proof.

The next step is to prove the compactness of \( S_\varphi \), and a posteriori of every \( S^{B,g}_\varphi \), with respect to the \( L^\infty_{\text{loc}}(\mathbb{R}; E) \) topology. Since \( \varphi \) is uniformly continuous on \( \mathbb{R} \), the second condition in (7) guarantees that \( S_\varphi \) is an equicontinuous family. Hence, the compactness of \( K_\varphi \) allows one to apply the Ascoli–Arzelà theorem on every compact subset of the real line. To conclude, one has just to use a diagonal extraction argument, on a nested sequence of compact sets covering all the real line.
Finally, notice that every $S_{\varphi}^{B,g}$ is nonempty. To see this, start from $\varphi$ and act on it with a sequence of translation factors $\tau_k$ satisfying

$$\|A_{\tau_k} - B\|_{\infty} + \|f_{\tau_k} - g\|_{\infty} \to 0. \quad (9)$$

The compactness of $S_{\varphi}$ provides an element of $S_{\varphi}^{B,g}$, as the limit of a suitable subsequence of $\varphi_{\tau_k}$.

Let now $E_{\varphi}$ be the separable space already introduced in the previous section, and insert it in a suitable Hilbert space by means of $\iota$: $E_{\varphi} \to H$. Then consider, for every $(B,g) \in H(A,f)$, the following minimization problem

$$\lambda_{\varphi}^{B,g} = \inf_{x \in S_{\varphi}^{B,g}} \|\iota \circ x\|_{\infty}. \quad (10)$$

The very same compactness arguments used above, which are clearly preserved under the action of $\iota$, show that all these infima are attained and indeed coincide, i.e.

$$\lambda_{\varphi}^{B,g} = \lambda_{\varphi}^{A,f} =: \lambda_{\varphi} \quad \forall (B,g) \in H(A,f). \quad (11)$$

To verify the last sentence, one can start from any given $\psi \in S_{\varphi}^{A,f}$, and act on it with a sequence $\tau_k$ as in (9). By compactness, a subsequence of $\psi_{\tau_k}$ tends to an element of $S_{\varphi}^{B,g}$, uniformly on compact sets. Since $\psi$ is arbitrary, this yields the inequality $\lambda_{\varphi}^{B,g} \leq \lambda_{\varphi}^{A,f}$. The reverse inequality is obtained in the same way, but using the translation factors $-\tau_k$.

The key point is that each minimum in (10) is uniquely attained. The reason is again the parallelogram identity in $H$, as in the previous section. Here is the point where the Favard condition comes into the play, directly providing the necessary separation condition.

Let $x_{B,g}$ denote this unique minimum, and define a map $U: H(A,f) \to E$ by

$$U(B,g) = x_{B,g}(0).$$

This map is continuous, when $H(A,f)$ is endowed with the topology of the uniform convergence over the real line. Assume indeed that $\|B_n - B\|_{\infty} + \|g_n - g\|_{\infty} \to 0$ and notice that, by compactness, a subsequence of $x_{B_n,g_n}$ converges to some $y \in S_{\varphi}^{B,g}$ uniformly on compact sets. This implies

$$\|\iota \circ y\|_{\infty} \leq \lambda_{\varphi}$$

and hence $y = x_{B,g}$, by uniqueness. Since moreover the limit is always the same, independently of the considered convergent subsequence of $x_{B_n,g_n}$, the entire sequence tends to $x_{B,g}$ in $L_{\text{loc}}^\infty(\mathbb{R}; E)$. The desired continuity of $U$ follows by taking $t = 0$.

To conclude, define

$$x^*(t) = U(A_t, f_t) \quad \forall t.$$ 

Due to a standard characterization, given at the end of Section 1, the map $x^*$ is almost periodic and enjoys the right module containment property. It only remains to show that it solves (1). Indeed notice that

$$x_{B_{t+},g_{t+}}^*(t) = x_{B,g}^*(t + \tau) \quad \forall (B,g), \forall t, \tau.$$
This follows from the uniqueness of the minimum, by using (11). Hence

\[ x^*(t) = x^{A, f_1}(0) = x^{A, f}(t) \]

namely \( x^* \) is a solution to the right equation.

References