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The Order of Magnitude of the Moments of the Modulus of Continuity of Multiparameter Poisson and Empirical Processes

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In this note we derive the exact order of magnitude of the moments of the modulus of continuity for multiparameter Poisson and, almost as a corollary, for multivariate empirical processes.

1. INTRODUCTION AND MAIN RESULT

In Silverman [8] the convergence to zero of the expectation of the modulus of continuity of the weighted univariate empirical process appears to be an important tool for establishing weak convergence of empirical processes based on U-statistics. Apart from this practical motivation, the study of the moments of the modulus of continuity seems interesting for purely theoretical reasons as well. It is a complement to investigations concerning the a.s. properties of the modulus of continuity. For multiparameter Brownian motion the almost sure behavior of the modulus of continuity has been specified in Orey and Pruitt [5] and for the univariate empirical process in Mason, Shorack, and Wellner [4]; a related problem has been considered in Komlós, Major, and Tusnády [2].

In this note we derive the *exact* order of magnitude of the moments of the modulus of continuity for multiparameter Poisson processes and, almost as a corollary, for multivariate empirical processes. The remainder of this section is devoted to the formulation of the main result, the proof of which is deferred to Section 2. In Section 3 we briefly comment on the result.

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For $d \in \mathbb{N}$ let $X_1, X_2, \ldots,$ be i.i.d. random vectors in $[0, 1]^d$ with common d.f. F having Uniform $(0, 1)$ marginals. Let $\mathscr{F}$ be the class of these d.f.'s; note that any $F \in \mathscr{F}$ is continuous. The empirical process based on the first n vectors is written as

$$U_n = \{U_n(t) = n^{1/2}(\hat{F}_n(t) - F(t)), \qquad t \in [0, 1]^d\}, \tag{1.1}$$

where the empirical d.f. $\hat{F}_n$ is defined in the usual way by $n\hat{F}_n(t) = \#\{1 \leqslant i \leqslant n: X_i \in [0, t_1] \times \cdots \times [0, t_d]\}$. Let $N_n = \{N_n(t), t \in [0, \infty)^d\}$ be a Poisson process with

$$\mathscr{E} N_n(t) = nF(t); \qquad t \in [0, \infty)^d, n \in \mathbb{N}. \tag{1.2}$$

We will focus on the standardized Poisson process, restricted to the unit cube $[0, 1]^d$,

$$Z_n = \{Z_n(t) = n^{-1/2}(N_n(t) - nF(t)), \qquad t \in [0, 1]^d\}. \tag{1.3}$$

In this note we will exploit the fact that, conditional on $N_n(1, \ldots, 1) = n$, the processes Z_n in (1.3) are equal in law to the processes U_n in (1.1).

The half-open rectangles $(s_1, t_1] \times \cdots \times (s_d, t_d]$ will be written as $R = R(s, t)$ and by

$$\mathscr{R} = \{R(s, t): R(s, t) \subset [0, 1]^d\}, \tag{1.4}$$

we denote the class of all such rectangles contained in the unit cube. Given any (random) function Λ: $[0, 1]^d \to \mathbb{R}$ and an arbitrary rectangle $R = R(s, t) \in \mathscr{R}$ we write

$$\Lambda\{R\} = \Lambda\{R(s, t)\} = \Delta_s^t \Lambda, \tag{1.5}$$

where Δ_s^t is the usual difference operator. The modulus of continuity that will be considered here is based on the partition $\mathscr{P}_m \subset \mathscr{R}$ of the unit cube into the squares of equal size

$$R = R_{k(1), \ldots, k(d)} = \left(\frac{k(1) - 1}{m}, \frac{k(1)}{m}\right] \times \cdots \times \left(\frac{k(d) - 1}{m}, \frac{k(d)}{m}\right], \quad m \in \mathbb{N}, \tag{1.6}$$

and is for the arbitrary function Λ defined as

$$M(\Lambda; m) = \max_{R \in \mathscr{P}_m} \sup_{s, t \in R} |\Lambda(s) - \Lambda(t)|. \tag{1.7}$$

This modulus of continuity is obviously of the same order of magnitude as the official modulus of continuity.

THEOREM 1.1. *Let $\delta \in (0, \frac{1}{2})$ be arbitrary. For integers $m \geqslant 2$ and $\nu \in \mathbb{N}$*

there exist numbers $0 < C_1 = C_1(d, \nu, \delta) \leqslant C_2 = C_2(d, \nu, \delta) < \infty$ *such that for each* $F \in \mathscr{F}$ *we have*

$$C_1\left(\frac{\log m}{m}\right)^{\nu/2} \leqslant \mathscr{E}M^{\nu}(Z_n; m) \leqslant C_2\left(\frac{\log m}{m}\right)^{\nu/2}, \tag{1.8}$$

for all $n \geqslant m^{1+2\delta}$. *The same result holds true for* $\mathscr{E}M^{\nu}(U_n; m)$.

2. Proof of the Theorem

First, $M(\Lambda; m)$ will be bounded above by a (random) variable that is easier to deal with in the Poisson case. For this purpose we introduce the cylinder sets

$$C_{j,k} = (0, 1]^{j-1} \times \left(\frac{k-1}{m}, \frac{k}{m}\right] \times (0, 1]^{d-j}, \tag{2.1}$$

and the (random) variables

$$W_{j,k}(\Lambda) = \sup_{S \subset C_{j,k}} |\Lambda\{S\}|, \tag{2.2}$$

where $S \in \mathscr{R}$ and the notation (1.5) is used. Let us now observe that

$$\sup_{s,t \in R_{k(1),\dots,k(d)}} |\Lambda(s) - \Lambda(t)| \leqslant \sum_{j=1}^{d} W_{j,k(j)}(\Lambda), \tag{2.3}$$

and consequently

$$\begin{aligned} M(\Lambda; m) &\leqslant \max_{k(1),\dots,k(d)} \sum_{j=1}^{d} W_{j,k(j)}(\Lambda) \\ &\leqslant \sum_{j=1}^{d} \max_{k(1),\dots,k(d)} W_{j,k(j)}(\Lambda) \\ &\leqslant \sum_{j=1}^{d} \max_{1 \leqslant k \leqslant m} W_{j,k}(\Lambda). \end{aligned} \tag{2.4}$$

Before we start with the actual proof let us make the convention that $a, b, c_1, \dots, c_{11} \in (0, \infty)$ will denote numbers that may depend on d, ν, or δ, but *not* on m, n, nor F.

Proof for the Poisson Process. For fixed j the $C_{j,k}$ are disjoint for different k. Since the Poisson process has the property that

$$Z_n\{R_1\} \text{ and } Z_n\{R_2\} \text{ are stochastically independent} \tag{2.5}$$

for rectangles $R_1, R_2 \in \mathscr{R}$ with $R_1 \cap R_2 = \varnothing$, it follows that the random variables

$$W_{j,1}(Z_n),\ldots, W_{j,m}(Z_n) \text{ are mutually independent} \tag{2.6}$$

for any $j \in \{1,\ldots, d\}$.

Since we have

$$\mathscr{E}M^{\nu}(Z_n; m) = \int_0^{\infty} P(M(Z_n; m) > \lambda^{1/\nu})\, d\lambda, \tag{2.7}$$

we need to find an upper and lower bound of the same order for the exceedance probability in (2.7). Let us note that

$$\begin{aligned}
1 - \prod_{k=1}^{m} &\{1 - P(Z_n\{C_{j,k}\} \geqslant \lambda^{1/\nu})\} \\
&\leqslant P(\max_{1 \leqslant k \leqslant m} |Z_n\{C_{j,k}\}| \geqslant \lambda^{1/\nu}) \leqslant P(M(Z_n; m) \geqslant \lambda^{1/\nu}) \\
&\leqslant P\left(\sum_{j=1}^{d} \max_{1 \leqslant k \leqslant m} W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}\right) \\
&\leqslant \sum_{j=1}^{d} P(\max_{1 \leqslant k \leqslant m} W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}/d) \\
&= \sum_{j=1}^{d} \left(1 - \prod_{k=1}^{m} \{1 - P(W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}/d)\}\right);
\end{aligned} \tag{2.8}$$

for the lower bound we may take any $j \in \{1,\ldots, d\}$.

Let $\psi : [0, \infty) \to (0, \infty)$ be the decreasing function given by

$$\psi(\lambda) = 2\lambda^{-2} \int_0^{\lambda} \log(1 + x)\, dx, \qquad \lambda > 0; \psi(0) = 1. \tag{2.9}$$

It has been shown in Shorack and Wellner [7] that

$$\psi(\lambda) \geqslant 1/(1 + \lambda/3), \qquad \lambda \geqslant 0. \tag{2.10}$$

For any $R \in \mathscr{R}$ and F as in (1.1)–(1.3) let us moreover introduce the function

$$\Psi_{F\{R\}}(\lambda) = \begin{cases} 1 & 0 \leqslant \lambda < c_1 F^{1/2}\{R\} \\ \exp\left(\dfrac{-a\lambda^2}{F\{R\}}\right), & c_1 F^{1/2}\{R\} \leqslant \lambda < c_2 F^{1/2-\delta}\{R\} \\ \exp\left(\dfrac{-b\lambda}{F^{1/2}\{R\}}\right), & c_2 F^{1/2-\delta}\{R\} \leqslant \lambda, \end{cases} \tag{2.11}$$

where $\delta \in (0, \frac{1}{2})$ is arbitrary but fixed.

The following inequalities are the key part of the proof. Provided that $n \geqslant F^{-1-2\delta}\{R\}$, *there exist* $a, b, c_1, c_2 \in (0, \infty)$ *such that*

$$P\left(\sup_{S \subset R} |Z_n\{S\}| \geqslant \lambda\right) \leqslant \Psi_{F\{R\}}(\lambda), \; \lambda \geqslant 0, \tag{2.12}$$

and there exist possibly different $a, b, c_1, c_2 \in (0, \infty)$ *such that*

$$P(Z_n\{R\} \geqslant \lambda) \geqslant \Psi_{F\{R\}}(\lambda), \qquad c_1 F^{1/2}\{R\} \leqslant \lambda < c_2 F^{1/2-\delta}\{R\}. \tag{2.13}$$

It is clear from the proof of Theorem 1.1 in Ruymgaart and Wellner [6] that for the left-hand side in (2.12) we have the upper bound

$$2^{2d+3} \exp\left(\frac{-\lambda^2}{32F\{R\}} \psi\left(\frac{\lambda}{4F\{R\}\, n^{1/2}}\right)\right), \qquad \lambda \geqslant (8F\{R\})^{1/2}. \tag{2.14}$$

Using (2.10) it follows that

$$\begin{aligned} &\frac{\lambda^2}{32F\{R\}} \psi\left(\frac{\lambda}{4F\{R\}\, n^{1/2}}\right) \\ &\qquad \geqslant \frac{\lambda^2}{32F\{R\}} \frac{F^{1/2}\{R\}}{F^{1/2}\{R\} + \lambda/(12(F\{R\}\, n)^{1/2})} \\ &\qquad \geqslant \frac{\lambda^2}{32F^{1/2}\{R\}} \frac{1}{\lambda(1+1/12)} \geqslant \frac{\lambda}{35F^{1/2}\{R\}}, \end{aligned} \tag{2.15}$$

provided we choose $\lambda \geqslant F^{1/2}\{R\}$ and $n \geqslant 1/F\{R\}$. Hence for $\lambda \geqslant c_2 F^{1/2-\delta}\{R\}$ we get the upper bound

$$2^{2d+3} \exp\left(\frac{-\lambda}{35F^{1/2}\{R\}}\right) \leqslant \exp\left(\frac{-\lambda}{70F^{1/2}\{R\}}\right). \tag{2.16}$$

Assuming that $\lambda < c_2 F^{1/2-\delta}\{R\}$ and again using (2.10) we see that

$$\begin{aligned} &\frac{\lambda^2}{32F\{R\}} \psi\left(\frac{\lambda}{4F\{R\}\, n^{1/2}}\right) \\ &\quad \geqslant \frac{\lambda^2}{32F\{R\}} \psi\left(\frac{\lambda}{4F^{1/2-\delta}\{R\}}\right) \\ &\quad \geqslant \frac{\lambda^2}{32F\{R\}} \psi\left(\frac{c_2}{4}\right) \geqslant \frac{\lambda^2}{32F\{R\}} \frac{1}{1+c_2/12} \geqslant \frac{\lambda^2}{(32+3c_2)\, F\{R\}}. \end{aligned} \tag{2.17}$$

A computation similar to the one in (2.16) yields for the probability the upper bound in the middle of (2.11), provided $c_1 F^{1/2}\{R\} \leqslant \lambda < c_2 F^{1/2-\delta}\{R\}$.

Let us now turn to the lower bound in (2.13). First observe that $N_n\{R\}$ is a Poisson($nF\{R\}$) random variable. Without loss of generality we may assume that both $nF\{R\}$ and $n^{1/2}\lambda$ are integers. Applying Stirling's formula we obtain

$$\begin{aligned} &P(Z_n\{R\} \geqslant \lambda) \\ &\quad \geqslant \sum_{k=0}^{n^{1/2}\lambda-1} P(N_n\{R\} = nF\{R\} + n^{1/2}\lambda + k) \\ &\quad = \sum_{k=0}^{n^{1/2}\lambda-1} \exp(-nF\{R\}) \frac{(nF\{R\})^{nF\{R\}+n^{1/2}\lambda+k}}{(nF\{R\}+n^{1/2}\lambda+k)!} \\ &\quad \geqslant \left(\frac{1}{2\pi}\right)^{1/2} \sum_{k=0}^{n^{1/2}\lambda-1} \left(\frac{1}{nF\{R\}+n^{1/2}\lambda+k}\right)^{1/2} \exp\left(\frac{-(n^{1/2}\lambda+k)^2}{2nF\{R\}}\right). \qquad (2.18) \end{aligned}$$

It is clear that for λ in the indicated range and for $n \geqslant F^{-1-2\delta}\{R\}$ the last expression is bounded below by

$$\begin{aligned} &c_3 \sum_{k=0}^{n^{1/2}\lambda-1} (nF\{R\})^{-1/2} \exp\left(\frac{-c_4\lambda^2}{F\{R\}}\right) \\ &\qquad = c_3 F^{-1/2}\{R\}\, \lambda \exp\left(\frac{-c_4\lambda^2}{F\{R\}}\right) \geqslant \exp\left(\frac{-a\lambda^2}{F\{R\}}\right). \qquad (2.19) \end{aligned}$$

Hence (2.13) is proved.

Applying (2.12) to $W_{j,k}(Z_n)$ and noting that $F(C_{j,k}) = 1/m$ for all indices, it follows that

$$P(W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}/d) \leqslant \Psi_{1/m}(\lambda^{1/\nu}/d), \qquad \lambda \geqslant 0. \qquad (2.20)$$

Combining (2.7), (2.8), and (2.20) we see that the order of magnitude of an upper bound of the νth moment is determined by

$$\int_0^\infty (1 - \{1 - \Psi_{1/m}(\lambda^{1/\nu}/d)\}^m)\, d\lambda = \int_{I(1)} + \int_{I(2)} + \int_{I(3)}, \qquad (2.21)$$

where the intervals of integration are

$$\begin{aligned} I(1) &= [0, d^\nu c_1^\nu m^{-\nu/2}), \\ I(2) &= [d^\nu c_1^\nu m^{-\nu/2}, d^\nu c_2^\nu m^{(-1/2+\delta)\nu}), \\ I(3) &= [d^\nu c_2^\nu m^{(-1/2+\delta)\nu}, \infty). \end{aligned}$$

For the first and the third integral on the right in (2.21) we easily find the upper bounds (see also (2.11))

$$\int_{I(1)} \leqslant d^{\nu} c_1^{\nu} m^{-\nu/2}, \tag{2.22}$$

$$\int_{I(3)} = \int_{I(3)} (1 - \{1 - \exp(-b\lambda^{1/\nu} m^{1/2})\}^m)\, d\lambda$$
$$\leqslant c_5 (m^{-1} \log m)^{\nu/2}. \tag{2.23}$$

For the second integral, which is slightly more complicated, we need the inequality

$$\int_1^{\infty} (1 - \{1 - (1/m)^x\}^m)\, x^{\nu/2 - 1}\, dx \leqslant c_6, \tag{2.24}$$

for $m, \nu \in \mathbb{N}$, $m \geqslant 2$. To see this note that for fixed $x \geqslant 1$ the function $\{1 - (1/m)^x\}^m$ is non-decreasing in m for $m \geqslant 2$. Hence we obtain

$$\int_1^{\infty} (1 - \{1 - (1/m)^x\}^m)\, x^{\nu/2 - 1}\, dx$$
$$\leqslant \int_1^{\infty} (1 - \{1 - (\tfrac{1}{2})^x\}^2)\, x^{\nu/2 - 1}\, dx$$
$$\leqslant 2 \int_1^{\infty} (\tfrac{1}{2})^x\, x^{\nu/2 - 1}\, dx = 2(1/\log 2)^{\nu/2}\ \Gamma(\tfrac{1}{2}\nu) = c_6.$$

First, let us note that by substituting $am\lambda^{2/\nu}/d^2 = y^{2/\nu}$ we find

$$\int_{I(2)} \leqslant \int_0^{\infty} (1 - \{1 - \exp(-am\lambda^{2/\nu}/d^2)\}^m)\, d\lambda$$
$$= d^{\nu} a^{-\nu/2} m^{-\nu/2} \int_0^{\infty} (1 - \{1 - \exp(-y^{2/\nu})\}^m)\, dy$$
$$= d^{\nu} a^{-\nu/2} m^{-\nu/2} \left(\int_{I(2,1)} + \int_{I(2,2)} \right), \tag{2.25}$$

where the intervals of integration are

$$I(2, 1) = [0, (\log m)^{\nu/2}),\ I(2, 2) = [(\log m)^{\nu/2}, \infty).$$

It is immediate that

$$\int_{I(2,1)} \leqslant (\log m)^{\nu/2}. \tag{2.26}$$

By substituting $y^{2/\nu} = x \log m$ and using (2.24) we see that

$$\int_{I(2,2)} = \tfrac{1}{2}\nu (\log m)^{\nu/2} \int_1^{\infty} (1 - \{1 - (1/m)^x\}^m)\, x^{\nu/2 - 1}\, dx$$
$$\leqslant c_7 (\log m)^{\nu/2}. \tag{2.27}$$

Combination of (2.25)–(2.27) yields

$$\int_{I(2)} \leqslant c_8 (m^{-1} \log m)^{\nu/2}. \tag{2.28}$$

The upper bound of the theorem is immediate from (2.21), (2.22), (2.23), and (2.28), so that it remains to prove the lower bound. In this case we apply (2.13) to $Z_n\{C_{j,k}\}$. Combination of (2.7) and (2.8) now yields

$$\mathscr{E} M^{\nu}(Z_n; m) \geqslant \int_{I(4)} (1 - \{1 - \Psi_{1/m}(\lambda^{1/\nu})\}^m)\, d\lambda, \tag{2.29}$$

where

$$I(4) = [c_1^{\nu} m^{-\nu/2}, c_2^{\nu} m^{(-1/2 + \delta)\nu}).$$

Similar to (2.25) let us make the substitution $am\lambda^{2/\nu} = x^{2/\nu}$, then we find

$$\int_{I(4)} = a^{-\nu/2} m^{-\nu/2} \int_{I(4,1)} (1 - \{1 - \exp(-x^{2/\nu})\}^m)\, dx, \tag{2.30}$$

with the interval

$$I(4, 1) = [a^{\nu/2} c_1^{\nu}, a^{\nu/2} c_2^{\nu} m^{\delta\nu}).$$

Since there exists $m_0 = m_0(a, c_2, \nu, \delta)$ such that

$$I(4, 1) \supset I(4, 2) = [a^{\nu/2} c_1^{\nu}, (\log m)^{\nu/2}) \qquad \text{for} \quad m \geqslant m_0, \tag{2.31}$$

it follows that

$$\int_{I(4)} \geqslant a^{-\nu/2} m^{-\nu/2} \int_{I(4,2)} (1 - \{1 - 1/m\}^m)\, dx$$
$$\geqslant a^{-\nu/2} m^{-\nu/2} \int_{I(4,2)} (1 - 1/e)\, dx$$
$$\geqslant c_9 (m^{-1} \log m)^{\nu/2}, \tag{2.32}$$

provided only that $m \geqslant m_0$. This proves the lower bound. The proof for the Poisson case is now completed.

Proof for the Empirical Process. To a large extent the method can be reduced to that of the previous case. Hence we may restrict ourselves to an outline. The following inequality relates the upper bound almost immediately to the upper bound for the Poisson process. Given the (random) function Λ and a rectangle $R \in \mathscr{R}$, by Λ_R we denote the corresponding set function restricted to subrectangles of R. Let $\mathscr{L}_R$ be the class of all such set functions and let $\phi\colon \mathscr{L}_R \to \mathbb{R}$. *Provided that* $F\{R\} \leqslant \frac{1}{2}$ *we have*

$$P(\phi(U_{n,R}) \in A) \leqslant 2P(\phi(Z_{n,R}) \in A), \qquad A \in \mathscr{B}, \tag{2.33}$$

under the assumption that the sets are measurable. For a proof see Ruymgaart and Wellner [6, Lemma 1.1].

Without loss of generality we take $m = 2l$, $l \in \mathbb{N}$. Proceeding as in (2.7) and (2.8) we obtain the upper bound

$$\begin{aligned} &P(M(U_{n;m}) \geqslant \lambda^{1/\nu}) \\ &\quad \leqslant \sum_{j=1}^{d} P(\max_{1 \leqslant k \leqslant m} W_{j,k}(U_n) \geqslant \lambda^{1/\nu}/d) \\ &\quad \leqslant \sum_{j=1}^{d} P(\max_{1 \leqslant k \leqslant l} W_{j,k}(U_n) \geqslant \lambda^{1/\nu}/d) \\ &\qquad + \sum_{j=1}^{d} P(\max_{l < k \leqslant m} W_{j,k}(U_n) \geqslant \lambda^{1/\nu}/d) \\ &\quad \leqslant 2 \sum_{j=1}^{d} P(\max_{1 \leqslant k \leqslant l} W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}/d) \\ &\qquad + 2 \sum_{j=1}^{d} P(\max_{l < k \leqslant m} W_{j,k}(Z_n) \geqslant \lambda^{1/\nu}/d). \end{aligned} \tag{2.34}$$

In the last step in (2.34) we apply (2.33) to the first sum with $R = C_{j,1} \cup \cdots \cup C_{j,l}$ and to the second sum with $R = C_{j,l+1} \cup \cdots \cup C_{j,m}$; in both cases we have $F\{R\} = \frac{1}{2}$. Now we are back in the Poisson case and essentially the same technique as before applies, with only a minor modification. This yields an upper bound of the same order of magnitude.

For the lower bound first observe that the random vector $(n\hat{F}_n\{C_{j,1}\}, \ldots, n\hat{F}_n\{C_{j,m}\})$ has a multinomial distribution for any j. Using a result in Mallows [3] we see that (cf. (2.8))

$$\begin{aligned} &P(M(U_n; m) \geqslant \lambda^{1/\nu}) \\ &\quad \geqslant P(\max_{1 \leqslant k \leqslant m} |U_n\{C_{j,k}\}| \geqslant \lambda^{1/\nu}) \\ &\quad = 1 - P(\max_{1 \leqslant k \leqslant m} |U_n\{C_{j,k}\}| < \lambda^{1/\nu}) \\ &\quad \geqslant 1 - \prod_{k=1}^{m} \{1 - P(U_n\{C_{j,k}\} \geqslant \lambda^{1/\nu})\}, \end{aligned} \tag{2.35}$$

for an arbitrary j. At this point we might again proceed as in the Poisson case, provided that we show that

$$P(U_n\{R\} \geqslant \lambda) \geqslant \exp\left(\frac{-a\lambda^2}{F\{R\}}\right), \tag{2.36}$$

for $c_{10}F^{1/2}\{R\} \leqslant \lambda < c_{11}F^{1/2-\delta}\{R\}$. The proof is very similar to that of (2.13), see (2.18), and will be omitted. Hence the theorem is also proved for the empirical process.

3. Some Comments

The order of magnitude of the moments appears to be *independent of both* the dimension d (apart from multiplicative constants) *and* the intensity or distribution function F (within the class $\mathscr{F}$). It should be noted that the choice of intensity function may have its influence on the actual dimension of the process. Taking, e.g., $d=2$ and $F(s, t) = s \wedge t$ for $(s, t) \in [0, 1]^2$, this intensity function corresponds to a degenerate probability measure concentrating mass 1 on the diagonal from $(0, 0)$ to $(1, 1)$. Hence the actual dimension is 1 rather than 2 in this example. This observation makes clear that independence of only one of the two, i.e., either on dimension or on intensity function, would be suspect.

Writing B for the d-parameter Brownian sheet, in our notation Orey and Pruitt [5, Theorem 2.4] requires that there exists $C(d) \in (0, \infty)$ such that

$$\limsup_{m \to \infty} \left(\frac{m}{\log m}\right)^{1/2} M(B; m) \leqslant C(d) \qquad \text{a.s.} \tag{3.1}$$

This result too is apparently independent of the dimension (apart from the constant). The order of magnitude, moreover, coincides with ours for the expectation in the case of d-parameter Poisson processes.

Without going into any detail let us just mention that computations of the kind in (2.25)–(2.28) and (2.30)–(2.32) lead to the following result: for any $m \in \mathbb{N}$ with $m \geqslant 2$ and any $x > 0$ there exists $C_j(x) \in (0, \infty)$ such that

$$C_1(x)(\log m)^x \leqslant \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} (1/k)^x \leqslant C_2(x)(\log m)^x. \tag{3.2}$$

This is an extension of the formula

$$\sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} (1/k) = \sum_{k=1}^{m} 1/k, \tag{3.3}$$

see, e.g., Gradshteyn and Ryzhik [1, Formula 0.155,4]. The formula in (3.3) can be generalized to arbitrary integer powers of $1/k$ by standard methods. The main novelty of (3.2) is that the power x need no longer be restricted to integer values.

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References

[1] Gradshteyn, I. S., and Ryzhik, I. M. (1980). *Table of Integrals, Series and Products.* Academic Press, New York.

[2] Komlós, J., Major, P., and Tusnády, G. (1975). Weak convergence and embedding. *Coll. Math. Soc. János Bolyai: Limit Theorems of Probability Theory*, pp. 149–165. North-Holland, Amsterdam.

[3] Mallows, C. L. (1968). An inequality involving multinomial probabilities. *Biometrika* **55** 422–424.

[4] Mason, D. M., Shorack, G. R., and Wellner, J. A. (1983). Strong limit theorems for oscillation moduli of the uniform empirical process. *Z. Wahrsch. Verw. Gebiete* **65** 83–97.

[5] Orey, S., and Pruitt, W. E. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probab.* **1** 138–163.

[6] Ruymgaart, F. H., and Wellner, J. A. (1984). Some properties of weighted multivariate empirical processes. *Statist. Decisions* **2** 199–223.

[7] Shorack, G. R., and Wellner, J. A. (1982). Limit theorems and inequalities for the uniform empirical process indexed by intervals. *Ann. Probab.* **10** 639–652.

[8] Silverman, B. W. (1983). Convergence of a class of empirical distribution functions of dependent random variables. *Ann. Probab.* **11** 745–751.