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Relations between theta-functions Hardy sums Eisenstein and Lambert series in the transformation formula of $\log \eta_{g,h}(z)$

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Abstract

In this paper, by using generalized logarithms of Dedekind eta-functions, generalized logarithms of theta-functions are obtained. Applying these functions, the relations between Hardy sums and Theta-functions are found. The special cases of these relations give Berndt's Theorems 6.1–8.1 (J. Reine Angew. Math. 303/304 (1978) 332) and explicit formulae of Hardy sums. Using derivative of logarithms of the Dedekind eta-function, relations between logarithm of the theta-functions and Eisenstein series are given. Applying connection between Lambert series and generalized Dedekind sums, the relation between theta-functions and Lambert series are obtained.

MSC: 11F20; 14K25; 11M36

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1. Introduction

In 1972, Lewittes [20] generalized Eisenstein series as follows: Let $\mathbb{H}$ be a upper half-space, $\mathbb{H} = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$. For $z \in \mathbb{H}$, $s = u + iv \in \mathbb{C}$ with $u > 1$ and

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$h = (h_1, h_2),\ r = (r_1, r_2) \in \mathbb{R}^2$

$$G(z,s,r,h) = \sum_{r \neq (m,n) \in \mathbb{Z}^2} \frac{e^{2\pi i(mh_1+nh_2)}}{((m+r_1)z+n+r_2)^s}.$$

The special case $h = r = 0$ gives $G(z,s) = \sum_{0 \neq (m,n) \in \mathbb{Z}^2} (mz+n)^{-s}$ (for detail see [20,2]). Lewittes [20] proved transformation formulae for the analytic continuation of a very large class of Eisenstein series. These results give transformation formulae for a large class of functions which generalizes Dedekind eta-function. He also showed that this function has an analytic continuation over the entire s plane, exhibited explicitly by a convergent Fourier expansion.

Recall that the Dedekind eta-function $\eta(z)$ is defined as follows;

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

for $z \in \mathbb{H}$. (Here, we shall employ some of the notation in [2].) In many applications of elliptic modular functions to Number Theory the η-function, which was introduced by Dedekind in 1877, plays a central role. The product has the form $\prod_{n=1}^{\infty}(1-x^n)$ where $x = e^{2\pi i z}$. If $z \in \mathbb{H}$ then $|x| < 1$, so the product converges absolutely and is nonzero. Moreover, since the convergence is uniform on compact subsets of $\mathbb{H}$, $\eta(z)$ is analytic on $\mathbb{H}$. Dedekind sums appear in the transformation formulae of $\eta(z)$. Here we shall review some well-known classical results on $\eta(z)$. For proofs and more details, we refer to [2,10,13,18]. Dedekind [25] gave under the modular transformation an elegant functional equation which contains Dedekind sums in the following theorem.

Theorem 1. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, c > 0$ *and* $z \in \mathbb{H}$, *we have*

$$\eta(Az) = \eta\left(\frac{az+b}{cz+d}\right) = e^{\frac{\pi i(a+d)}{12c} + \pi i s(-d,c)} (-i(cz+d))^{\frac{1}{2}} \eta(z), \tag{1.1}$$

where

$$s(h,k) = \sum_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

where h is an arbitrary integer, k is a positive integer and the function $((x))$ *is defined as follows*:

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2}, & x \text{ is not an integer}, \\ 0, & \text{otherwise}, \end{cases}$$

where $[x]$ *is the largest integer* $\leqslant x$ (*for detail see* [2,25]).

The sum $s(h,k)$ in (1.1) is known Dedekind sum. For $z = x + iy$ with x, y real. For any complex number w, we choose that branch of $\log w$ with $-\pi \leqslant \arg w < \pi$. We note that Dedekind's formula is a consequence of the following equation, obtained by taking logarithms of both sides of (1.1) [2,24]

$$\log \eta(Az) = \log \eta(z) + \frac{\pi i(a+d)}{12c} - \pi i s(d,c) - \frac{\pi i}{4} + \frac{1}{2}\log(cz+d). \tag{1.2}$$

Rademacher's first paper, which was published in 1932, devoted to the eta-function and Dedekind sums [25]. Here he proves the transformation formula for $\log \eta(z)$ under modular transformations via contour integration and the functional equation of the Hurwitz zeta-function. The proof of (1.2) was given by Berndt [5] and Apostol [2]. Lewittes [20] generalized $\log \eta(z)$ as follows:

$$A(z,s,r_1,r_2) = \sum_{n>-r_1} \sum_{k=1}^{\infty} k^{s-1} e^{2\pi i k r_2 + 2\pi i k(r_1+n)z},$$

$$H(z,s,r_1,r_2) = A(z,s,r_1,r_2) + e^{\pi i s} A(z,s,-r_1,-r_2)$$

for $z \in \mathbb{H}$. If $r_1 = 0, r_2 = 0$ in the above, then we obtain immediately $A(z,s) = \sum_{n=1}^{\infty} \sigma_{s-1}(n) e^{2\pi i n z}$, where $\sigma_{s-1}(n) = \sum_{k|n} k^{s-1}$. We note that $A(z,0)$ is closely related to the Dedekind eta-function. By the definition of $\eta(z)$, we have

$$\log \eta(z) - \frac{\pi i z}{12} = \sum_{n=1}^{\infty} \log(1 - e^{2\pi i n z}) = -\sum_{n=1}^{\infty} \sigma_{-1}(n) e^{2\pi i n z} = -A(z,0).$$

Lewittes also showed transformation formulae of $A(z,s,r_1,r_2)$ and $H(z,s,r_1,r_2)$ under the modular substitutions. However, the formulae (Theorem 3, Eq. (51) in [20]) are so complicated that even in the simplest case of the Dedekind eta-function it is exceedingly difficult to deduce the usual transformation formulae in terms of Dedekind sums. Berndt [7] gave a different proof of this formulae (Theorem 3, Eq. (51) in [20]). He also proved a transformation formula under modular substitutions, which is derived for a very large class of generalized Eisentein series. This transformation formula is easily converted into a transformation formula for a large class of functions that includes and generalizes the classical Dedekind eta-function. In addition, he gave elegant transformation formulae in which Dedekind sums or various generalizations of Dedekind sums appear. These formulae include those functions studied by Dieter [13], Schoeneberg [26] and Tezeng and Miao [29]. Dieter [13] and Schoeneberg [26] have derived the result for a subset of Berndt's functions (Eqs. (21)–(22) in [5]). The results of Dieter and Schoeneberg are special cases of Berndt's function(see Eq. (22) in [5]). We shall employ some of the notation in [5,26]. Let g and h be integers, and N be a positive integer N. We define generalized Dedekind eta-function, $\eta_{g,h}(z)$ as follows:

$$\eta_{g,h}(z;N) = \alpha_{g,h}(N)e^{\pi iz\bar{B}_2(\frac{g}{N})} \prod_{m\equiv g(N),m>0} (1-\zeta_N^h q_N^m) \prod_{m\equiv -g(N),m>0} (1-\zeta_N^{-h} q_N^m)$$

for $z\in\mathbb{H}$, where $\zeta_N = e^{\frac{2\pi i}{N}}$, $q_N = e^{\frac{2\pi iz}{N}}$ and

$$\alpha_{g,h}(N) = \begin{cases} e^{\pi i\bar{B}_1(\frac{h}{N})}(1-\zeta_N^{-h}) & \text{if } g\equiv 0,\ h\not\equiv 0 \bmod N, \\ 1 & \text{otherwise.} \end{cases}$$

$\bar{B}_1$ and $\bar{B}_2$ in the formulae are Bernoulli functions:

$$\bar{B}_1(x) = x-[x]-\frac{1}{2},\ \bar{B}_2(x) = (x-[x])^2-(x-[x])+\frac{1}{6}.$$

The functions $\eta_{g,h}(z;N)$ are holomorphic for $z\in\mathbb{H}$ and depend upon g,h modulo N. Furthermore, $\eta_{g,h}(z;N) = \eta_{-g,-h}(z;N)$ for each g and h, and $\eta_{g,h}(z;N) = \eta^2(z)$ for $(g,h)\equiv(0,0) \pmod N$. $\eta_{g,h}(z;N)$ is a modular function with a multiplication of absolute value 1 for the principal congruence group $\Gamma(N)$ if $(g,h\not\equiv(0,0) \pmod N)$ (see [13,26]). In the following, we regard these functions from the point of view of the modular group Γ (see [18] about modular group and subgroups of modular group). Tzeng and Miao [29] showed that $\eta_{g,h}(z;N)$ is an almost automorphic function for the whole modular group Γ if $(g,h)\not\equiv(0,0) \pmod N$. Some examples and some relations between Theta functions $\vartheta_{\mu,\nu}$ and Dedekind eta-function were given by Tzeng and Miao [29]. Generalized transformation formulae for $\log\eta_{g,h}(z;N)$ is defined as follows:

Let $(g_1,h_1) = (g,h)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and

$$b_{g,h}(N) = \begin{cases} 1, & g\equiv h\equiv 0 \pmod N, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\log\eta_{g,h}(Az;N) = \log\eta_{g_1,h_1}(z;N) + \frac{\pi ia}{c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{\pi id}{c}\bar{B}_2\left(\frac{g_1}{N}\right) - 2\pi is_{g,h}(d,c;N) - b_{g,h}(N)\left(\frac{\pi i}{2}-\log(cz+d)\right), \tag{1.3}$$

where $s_{g,h}(d,c;N)$ is generalized Dedekind sum, which is defined as [26]

$$s_{g,h}(d,c;N) = \sum_{\mu \bmod c} \left(\left(\frac{g+\mu N}{cN}\right)\right)\left(\left(\frac{g_1+d\mu N}{cN}\right)\right).$$

Remark 1. Let $0 \leqslant g, h < N$. $\log \eta_{0,0}(z; N) = 2\log \eta(z)$, and (1.3) is reduced to (1.2) and $s_{0,0}(d, c, 1) = s(h, k)$. Eq. (1.3) is proved in different ways. Generalized transformation formula for $\log \eta_{g,h}(z)$ is proved by Dieter [13] and Schoeneberg [26], and Berndt [5–10] proved a transformation formula for a fairly broad class of analytic Eisenstein series.

The most fundamental property of Dedekind sums is the remarkable reciprocity law: If $h, k > 0$ and $(h, k) = 1$, then

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right).$$

This formula was first proved by Dedekind [25] using the transformation formulae of $\log \eta(z)$. There now exist many proofs of this formula, and several of these can be found in a monograph [25] on Dedekind sums, written by Rademacher and completed by Grosswald after Rademacher's death. Rademacher found five original proofs of this formula. An elementary proof of this formula, as well as a generalization is given by Berndt [6,8,9]. An elementary proofs appear in a paper [14], which generalized this formula. Another elegant proof was given by Apostol [1], which generalized this formula by the pth Bernoulli function, $\bar{B}_p(x)$. Here we shall review some of the well-known classical results on Generalized Dedekind sums and $\bar{B}_p(x)$. For proof and more details, we refer to Apostol's paper [2]. Generalized Dedekind sums $s(h, k; p)$ are defined as follows [2]:

$$s(h,k;p) = \sum_{a \bmod k} \frac{a}{k} \bar{B}_p\left(\frac{ah}{k}\right),$$

where $h, k \in \mathbb{Z}$, $(h, k) = 1$, $\bar{B}_p(x)$ is the pth Bernoulli function. For odd p, the sums $s(h, k; p)$ have reciprocity law. Apostol [2] proved the reciprocity law of these sums. When $p = 1$, the sums $s(h, k; 1)$ are known as Dedekind sums, $s(h, k)$. Also the sums $s(h, k; p)$ are related to the Lambert series, $G_p(x)$, which are defined as follows:

$$G_p(x) = \sum_{n=1}^{\infty} n^{-p} \frac{x^n}{1 - x^n} = \sum_{m,n=1}^{\infty} n^{-p} x^{mn},$$

where $p \geqslant 1$. These functions are regular for $|x| < 1$ and have the unit circle as a natural boundary, each rational point of the circle being a singular point. The special case $p = 1$ gives $G_1(x) = -\log \prod_{m=1}^{\infty}(1 - x^m)$. Thus, $\log \eta(z)$ is the same as $\frac{\pi i z}{12} - G_1(e^{2\pi i z})$. Using a technique developed by Rademacher, transformation formulae relating $G_p(e^{2\pi i z})$ to $G_p(e^{2\pi i z'})$ are obtained for odd p, where $z' = \frac{az+b}{cz+d}$ is a modular substitution [2]. The sums $s(h, k; p)$ appear in these formulae. The sums $s(h, k; p)$ are expressible as infinite series related to certain Lambert series. In the last section we shall establish new relations connection between the sums $s(h, k; p)$ and the Lambert series $G_p(e^{2\pi i h/k})$ and theta-functions.

The classical theta-functions, $\vartheta_n(0,q)$ $(n=2,3,4)$ are defined as follows [24,30]:

$$\vartheta_2(0,q) = 2q^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2, \quad \vartheta_3(0,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,$$

$$\vartheta_4(0,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2.$$

In the remainder of our work, we shall denote $\vartheta_2(0,q)$, $\vartheta_3(0,q)$ and $\vartheta_4(0,q)$ as $\vartheta_2(z)$, $\vartheta_3(z)$ and $\vartheta_4(z)$, respectively, where $q=e^{\pi i z}$. The relations between theta-functions and Dedekind eta-function are defined as

$$\vartheta_2(z) = \frac{2\eta^2(2z)}{\eta(z)}, \quad \vartheta_3(z) = \frac{\eta^5(z)}{\eta^2(2z)\eta^2(\frac{z}{2})}, \quad \vartheta_4(z) = \frac{\eta^2(\frac{z}{2})}{\eta(z)}.$$

The above relations and the others are studied by Rademacher [24]. Further information about these functions can be found in Barner [4], Knopp [17], Peterson [21], and Köhler [19]. Raab [23] gave the relation between generalized theta-functions and Dedekind eta-functions in a different way. He also defined quadratic theta-functions.

The following relations are obtained by taking logarithms of both sides of above relations:

$$\log\vartheta_2(z) = \log 2 + 2\log\eta(2z) - \log\eta(z), \tag{1.4}$$

$$\log\vartheta_3(z) = 5\log\eta(z) - 2\log\eta(2z) - 2\log\eta\left(\frac{z}{2}\right), \tag{1.5}$$

$$\log\vartheta_4(z) = 2\log\eta\left(\frac{z}{2}\right) - \log\eta(z). \tag{1.6}$$

In contrast to $\log\eta(z)$, the logarithms of the classical theta-functions have scarcely been studied. (Here we use the notation of Whittaker and Watson [30] and Berndt [10] for the theta-functions.) Berndt [10] and Goldberg [15] derived the transformation formulae for $\log\vartheta_n(z)$ $(n=2,3,4)$. There are 9 distinct transformation formulae depending on parties of certain coefficients $a,b,c,$ and d in modular transformation $(az+b)/(cz+d)$. Arising in the transformation formulae are six different arithmetic sums, which are thus similar to Dedekind sum and well-known Hardy sums or Berndt's arithmetic sums. For $h,k\in\mathbb{Z}$ with $k>0$, these 6 sums are defined as follows:

$$S(h,k) = \sum_{j=1}^{k-1}(-1)^{j+1+[\frac{hj}{k}]}, \quad s_1(h,k) = \sum_{j=1}^{k}(-1)^{[\frac{hj}{k}]}\left(\left(\frac{j}{k}\right)\right),$$

$$s_2(h,k) = \sum_{j=1}^{k} (-1)^j \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right), \quad s_3(h,k) = \sum_{j=1}^{k} (-1)^j \left(\left(\frac{hj}{k}\right)\right),$$

$$s_4(h,k) = \sum_{j=1}^{k-1} (-1)^{[\frac{hj}{k}]}, \quad s_5(h,k) = \sum_{j=1}^{k} (-1)^{j+[\frac{hj}{k}]} \left(\left(\frac{j}{k}\right)\right).$$

Rademacher [24] studied $\log \vartheta_n(z),\ n = 2,3,4$. However, his approach was via the Dedekind eta-function, so the sums defined above were not discerned by Rademacher. Some of these sums are mentioned in a paper of Hardy [16], where reciprocity theorems are stated without proof. However, Hardy did not observe the connections between his sums and theta-functions. Hardy studied on the theory of $r_s(n)$, the number of representations of n as the sum of s squares, the sums $S(h,k)$ and $s_n(h,k),\ n = 1,2,3,4,5$ arouse this theory. Hardy gave formulae $r_s(n)$, for $5 \leqslant n \leqslant 8$ and asymptotic formulae for $s > 8$, which can be found in [13]. Employing the sums mentioned above, Goldberg [15] has shown that a substantial simplification in Hardy's proof can be effected. These sums also arise in the study of the Fourier coefficients of the reciprocals of $\vartheta_n(0,q),\ n = 2,3,4$ [12]. Berndt and Goldberg [12] found analytic properties of Hardy sums. They established infinite trigonometric series representations for Hardy sums. They also evaluated certain nonabsolutely convergent double series in terms of these sums. The most important of Hardy sums is the following reciprocity theorem due to Sitaramachandrarao [28] and Berndt [10].

Theorem 2. *Let h and k be coprime positive integers. If $h+k$ is odd, then,*

$$S(h,k) + S(k,h) = 1 \tag{1.7}$$

if h is even, then

$$s_1(h,k) - s_2(k,h) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{hk} + \frac{k}{h}\right) \tag{1.8}$$

if k is odd, then

$$2s_3(h,k) - s_4(k,h) = 1 - \frac{h}{k} \tag{1.9}$$

if h is even, then

$$s_5(h,k) + s_5(k,h) = \frac{1}{2} - \frac{1}{2hk}. \tag{1.10}$$

The reciprocity theorems (1.7)–(1.10) appear in Hardy's list, respectively, as Eqs. (viii)–(ix) [16]. By using $\log \vartheta_n(0,q),\ n = 2,3,4$, Berndt [10] proved (1.7)–(1.9) in a different way. Goldberg [15] deduced (1.10) from Berndt's transformation formulae [10]. For other proofs which do not depend on transformation theory, we refer to Apostol and Vu [3], Berndt and Goldberg [12], Berndt and Dieter [11], and Sitaramachandrarao [28]. The author [27] deduced (1.9) from three term relations defined for polynomials. Pettet and Sitaramachandrarao [22] gave new relations related to Hardy sums and three terms relations for Hardy sums. It may be noted that Sitaramachandrarao [28] expressed reciprocity theorem, by using elementary arguments. Each of the Hardy sums is explicitly deduced in terms of the sum $s(h,k)$ to Theorem 2 from reciprocity law of the Dedekind sums.

Theorem 3. *Let* $(h,k)=1$. *If* $h+k$ *is* odd, *then*

$$S(h,k)=8s(h,2k)+8s(2h,k)-20s(h,k) \tag{1.11}$$

if h *is* even, *then*

$$s_1(h,k)=2s(h,k)-4s(h,2k) \tag{1.12}$$

if k *is* even, *then*

$$s_2(h,k)=-s(h,k)+2s(2h,k) \tag{1.13}$$

if k *is* odd, *then*

$$s_3(h,k)=2s(h,k)-4s(2h,k) \tag{1.14}$$

if h *is* odd, *then*

$$s_4(h,k)=-4s(h,k)+8s(h,2k) \tag{1.15}$$

if $h+k$ *is* even, *then*

$$s_5(h,k)=-10s(h,k)+4s(2h,k)+4s(h,2k). \tag{1.16}$$

Each one of $S(h,k)(h+k\ \textit{even})$, $s_1(h,k)(h\ \textit{odd})$, $s_2(h,k)(k\ \textit{odd})$, $s_3(h,k)(k\ \textit{even})$, $s_4(h,k)(h\ \textit{even})$ *and* $s_5(h,k)(h+k\ \textit{odd})$ *is zero.*

The proof of this theorem is given by Sitaramachandrarao [28]. The relations between Hardy sums and $\log\vartheta_n(z)$ $(n=2,3,4)$ are given in theorems below.

Theorem 4. Let $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\Gamma_0(2)$, $c>0$. *If* c *is even and* $(c,d)=1$, *then*

$$\log\vartheta_2(Az)=\log\vartheta_2(z)+\frac{1}{2}\log(cz+d)-\frac{\pi i}{4}+\pi i\left(\frac{a+d}{4c}\right)-\pi i s_2(d,c). \tag{1.17}$$

Theorem 5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(2),\ c > 0$. *If* d *is odd and* $(c, d) = 1$, *then*

$$\log \vartheta_4(Az) = \log \vartheta_4(z) + \frac{1}{2}\log(cz + d) - \frac{\pi i}{4} - \frac{\pi i}{4} s_4(d, c). \tag{1.18}$$

The proofs of Theorems 4 and 5 have been given by Berndt [10] using analytic continuation of $H(z, s, r, h)$ and residue calculus. It is the first purpose of the present paper to prove and generalize Theorems 4 and 5. Our proofs will be quite different from those of Berndt [10]. We will also generalize these theorems by using the relations between Hardy sums and generalized transformation formulae of logarithms of the classical theta-functions, which will be explained in Section 2. We will deduce some new results, which are generalized Theorem 3, as well. Our second aim is to give the relations between $\frac{d}{dz}\eta_{g,h}(z)$ and Eisenstein series, which will be given in Section 3. Our third aim is to prove the relation between generalized Dedekind sums, Hardy sums and Lambert series, which will be explained in the last section.

2. Main theorems on generalized theta-functions and Hardy sums

In this section, we shall define some new functions, which are related to generalized Dedekind eta-function. These functions will give us further insights into the nature of (1.4)–(1.6). We will need these function throughout the paper. Here, we will use the notations of Berndt [5,10] and Schoeneberg [26]. By using (1.3) in (1.4), (1.5) and (1.6), we have

$$f_{2;g,h}(z, N) = \log 2 + 2\log \eta_{g,h}(2z, N) - \log \eta_{g,h}(z, N), \tag{2.1}$$

$$f_{3;g,h}(z, N) = 5\log \eta_{g,h}(z, N) - 2\log \eta_{g,h}\left(\frac{z}{2}, N\right) - 2\log \eta_{g,h}(2z, N), \tag{2.2}$$

$$f_{4;g,h}(z, N) = 2\log \eta_{g,h}\left(\frac{z}{2}, N\right) - \log \eta_{g,h}(z, N). \tag{2.3}$$

By using these functions we shall give several new theorems, which generalize Theorems 3–5. Furthermore, note that

$$f_{n;0,0}(z, N) = 2\log \vartheta_n(z), \quad n = 2, 3, 4$$

and (2.1), (2.2) and (2.3) reduces to (1.4), (1.5) and (1.6), respectively.

Proof of Theorem 4. The proof of this theorem is given by Berndt, (Theorem 6.1) [10]. We give a different proof of this theorem.

Putting $z = \frac{\tau-d}{c}$, and $2z = \frac{2(\tau-d)}{c}$, and choosing $A = \begin{bmatrix} 2a & \frac{4ad-1}{c} \\ c & 2d \end{bmatrix}$, we can rewrite (1.2) and we obtain

$$\log \eta(2z) = \log \eta\left(\frac{2\tau - 2d}{c}\right) = \log \eta\left(\frac{2a - \frac{1}{2\tau}}{c}\right) - \pi i\left(\frac{a+d}{6c} - \frac{1}{4}\right) + \pi i s(2d, c) - \frac{1}{2}\log \tau. \tag{2.4}$$

By using (2.4) and (1.2) in (1.4), we deduce that

$$\log \vartheta_2(z) = \log \vartheta_2(Az) - \frac{1}{2}\log(cz+d) + \frac{\pi i}{4} - \pi i\left(\frac{a+d}{4c}\right) + \pi i(-s(d,c) + 2s(2d,c)). \tag{2.5}$$

Hence, we assume that c is even and we apply (1.13) to (2.5) and we obtain (1.17). The proof is complete. $\square$

Now by using (1.3) and (2.1), we obtain generalized Theorem 4 as follows:

Theorem 6. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(2)$, $c > 0$. If c is even and $(c,d) = 1$, then*

$$f_{2;g,h}(Az, N) = f_{2;g_1,h_1}(z, N) + \frac{\pi i a}{c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{\pi i d}{c}\left(\bar{B}_2\left(\frac{g_3}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right) - 2\pi i T_{2;g,h}(d, c; N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz+d)\right), \tag{2.6}$$

where

$$(g_3, h_3) = (g, h)\begin{bmatrix} 2a & \frac{4ad-1}{c} \\ c & 2d \end{bmatrix}$$

and

$$T_{2;g,h}(d, c; N) = 2s_{g_3,h_3}(2d, c; N) - s_{g_1,h_1}(d, c; N).$$

Proof. Putting $\tau = cz + d$, and hence $2z = \frac{2\tau - 2d}{c}$ and choosing $A = \begin{bmatrix} 2a & \frac{4ad-1}{c} \\ c & 2d \end{bmatrix}$ we can rewrite (1.3), and we obtain

$$\log \eta_{g,h}\left(\frac{2a-\dfrac{1}{2z}}{c};N\right) = \log \eta_{g_3,h_3}\left(\frac{2z-2d}{c};N\right) + \frac{2\pi ia}{c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{2\pi id}{c}\bar{B}_2\left(\frac{g_3}{N}\right)$$
$$- 2\pi is_{g_3,h_3}(2d,c;N) - b_{g,h}(N)(\frac{\pi i}{2} - \log(\tau)). \tag{2.7}$$

Multiplying (2.7) by 2 and subtracting (1.3), using (2.1), we deduce that

$$f_{2;g,h}(Az,N) = f_{2;g_1,h_1}(z,N) + \frac{\pi ia}{c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{\pi id}{c}\left(\bar{B}_2\left(\frac{g_3}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right)$$
$$- \pi i(-4s_{g_3,h_3}(2d,c;N) + 2s_{g_1,h_1}(d,c;N))$$
$$- b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz+d)\right). \tag{2.8}$$

Hence, we assume that c is even. By using (1.13) in (2.8) we get (2.6). The proof is complete. $\square$

Remark 2. Theorems 3 and 4 are special cases of Theorem 6. If we get $g = h \equiv 0(N)$ then $f_{2,0,0}(Az;N)$ is deduced to Theorem 4 (Berndt's Theorem 6.1 [10]) and $T_{2;0,0}(d,c;N)$ is deduced to reciprocity theorem for Hardy sums (1.13) (Sitaramachandrarao's Theorem 5.1 [28]).

Proof of Theorem 5. The proof of this theorem is given by Berndt, (Theorem 8.1 [10]). We give a different proof of this theorem. Putting $z = \frac{\tau-d}{c}$, and $\frac{z}{2} = \frac{\tau-d}{2c}$, and choosing $A = \begin{bmatrix} a & \frac{ad-1}{2c} \\ 2c & d \end{bmatrix}$, we can rewrite (1.2) and we have

$$\log \eta\left(\frac{z}{2}\right) = \log \eta\left(\frac{\tau-d}{2c}\right) = \log \eta\left(\frac{a-\frac{1}{\tau}}{2c}\right) - \pi i\left(\frac{a+d}{24c} - \frac{1}{4}\right)$$
$$+ \pi is(d,2c) - \frac{1}{2}\log \tau. \tag{2.9}$$

Hence, by using (1.2) and (2.9) in (1.6), we deduce that

$$\log \vartheta_4(z) = \log \vartheta_4(Az) - \frac{1}{2}\log(cz+d) + \frac{\pi i}{4} + \frac{\pi i}{4}(-4s(d,c) + 8s(d,2c)). \tag{2.10}$$

Here, we assume that d is odd and we apply (1.15) to (2.10), thus we obtain (1.18). The proof is complete. $\square$

Now by using (1.3) and (2.3), we shall generalize Theorem 5 as follows:

Theorem 7. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(2),\ c > 0$. *If d is odd and* $(c, d) = 1$, *then*

$$f_{4;g,h}(Az, N) = f_{4;g_1,h_1}(z, N) + \frac{\pi i d}{c}\left(\bar{B}_2\left(\frac{g_2}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right) - \frac{\pi i}{2} T_{4;g,h}(d, c; N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz + d)\right), \tag{2.11}$$

where

$$(g_2, h_2) = (g, h)\begin{bmatrix} a & \dfrac{ad-1}{2c} \\ 2c & d \end{bmatrix}$$

and

$$T_{4;g,h}(d, c; N) = 8s_{g_2,h_2}(d, 2c; N) - 4s_{g_1,h_1}(d, c; N).$$

Proof. Putting $\tau = cz + d$ and hence $\frac{z}{2} = \frac{\tau - d}{2c}$ and choosing $A = \begin{bmatrix} a & \frac{ad-1}{2c} \\ 2c & d \end{bmatrix}$ we can rewrite (1.3), and we obtain

$$\log \eta_{g,h}\left(\frac{a - \frac{1}{z}}{2c}; N\right) = \log \eta_{g_2,h_2}\left(\frac{z - d}{2c}; N\right) + \frac{\pi i a}{2c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{\pi i d}{2c}\bar{B}_2\left(\frac{g_2}{N}\right) - 2\pi i s_{g_2,h_2}(d, 2c; N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(\tau)\right). \tag{2.12}$$

Multiplying (2.12) by 2 and subtracting (1.3), and using (2.3), we deduce that

$$f_{4;g,h}(Az, N) = f_{4;g_1,h_1}(z, N) + \frac{\pi i d}{c}\left(\bar{B}_2\left(\frac{g_2}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right) - \pi i(4s_{g_2,h_2}(d, 2c; N) - 2s_{g_1,h_1}(d, c; N)) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz + d)\right). \tag{2.13}$$

Hence, we assume that d is odd. Applying (1.15) to (2.13) we obtain (2.11). The proof is complete. □

Remark 3. Theorems 3 and 5 are special cases of Theorem 7. If we get $g = h \equiv 0 \pmod N$ then $f_{4,0,0}(Az; N)$ is deduced to Theorem 5 (Berndt's Theorem 8.1 [10]) and $T_{4;0,0}(d, c; N)$ is deduced to reciprocity theorem for Hardy sums (1.15) (Sitaramachandrarao's Theorem 5.1 [28]).

Theorem 8. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_\theta$, $c > 0$. *If* $d + c$ *is even and* $(c, d) = 1$, *then*

$$f_{5;g,h}(Az, N) = f_{5;g_1,h_1}(z, N) + \frac{\pi i d}{c}\left(5\bar{B}_2\left(\frac{g_2}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right) - 4\bar{B}_2\left(\frac{g_3}{N}\right)\right) + \pi i T_{5;g,h}(d, c; N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz + d)\right), \tag{2.14}$$

where

$$T_{5;g,h}(d, c; N) = 4s_{g_2,h_2}(d, 2c; N) + 4s_{g_3,h_3}(2d, c; N) - 10s_{g_1,h_1}(d, c; N).$$

Proof. The proof of this theorem is similar to above theorems. By using (1.3), (2.7), (2.12) and (1.16) in (2.2), we get (2.14). The proof is complete. □

Here,

$$T_{5;0,0}(d, c; N) = 4s_{0,0}(d, 2c; N) + 4s_{0,0}(2d, c; N) - 10s_{0,0}(d, c; N)$$

is equal to Eq. (1.16).

Corollary 1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^0(2)$, $c > 0$. *If* d *is even and* $(c, d) = 1$, *then*

$$f_{4;g,h}(Az, N) = f_{4;g_1,h_1}(z, N) + \frac{\pi i d}{c}\left(\bar{B}_2\left(\frac{g_2}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right) - \frac{\pi i}{2} T_{1;g,h}(d, c; N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz + d)\right),$$

where

$$T_{1;g,h}(d, c; N) = -4s_{g_2,h_2}(d, 2c; N) + 2s_{g_1,h_1}(d, c; N).$$

The proof of Corollary 1 is similar to Theorem 7. Here,

$$T_{1;0,0}(d, c; N) = -4s_{0,0}(d, 2c; N) + 2s_{0,0}(d, c; N)$$

is equal to Eq. (1.12).

Corollary 2. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(2),\ c > 0$. *If* c *is odd and* $(c,d)=1$, *then*

$$\begin{aligned} f_{2;g,h}(Az,N) = & f_{2;g_1,h_1}(z,N) + \frac{\pi i a}{c}\bar{B}_2\left(\frac{g}{N}\right) + \frac{\pi i d}{c}\left(\bar{B}_2\left(\frac{g_3}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right)\right) \\ & - 2\pi i T_{2;g,h}(d,c;N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz+d)\right), \end{aligned}$$

where

$$T_{3;g,h}(d,c;N) = -4s_{g_3,h_3}(2d,c;N) + 2s_{g_1,h_1}(d,c;N).$$

The proof of Corollary 2 is similar to Theorem 5. Here,

$$T_{3;0,0}(d,c;N) = -4s_{0,0}(2d,c;N) - 2s_{0,0}(d,c;N)$$

is equal to (1.14).

Corollary 3. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_\theta,\ c > 0$. *If* $d + c$ *is odd and* $(c,d)=1$, *then*

$$\begin{aligned} f_{5;g,h}(Az,N) = & f_{5;g_1,h_1}(z,N) + \frac{\pi i d}{c}\left(5\bar{B}_2\left(\frac{g_2}{N}\right) - \bar{B}_2\left(\frac{g_1}{N}\right) - 4\bar{B}_2\left(\frac{g_3}{N}\right)\right) \\ & + \frac{\pi i}{2}T_{g,h}(d,c;N) - b_{g,h}(N)\left(\frac{\pi i}{2} - \log(cz+d)\right), \end{aligned}$$

where

$$T_{g,h}(d,c;N) = 8s_{g_2,h_2}(d,2c;N) + 8s_{g_3,h_3}(2d,c;N) - 20s_{g_1,h_1}(d,c;N).$$

The proof of Corollary 3 is similar to Theorem 8. Putting $g = h = 0$,

$$T_{0,0}(d,c;N) = 8s_{0,0}(d,2c;N) + 8s_{0,0}(2d,c;N) - 20s_{0,0}(d,c;N)$$

is equal to (1.11).

3. Main theorems on the Eisenstein series and generalized Dedekind eta-functions

In this section, we shall review some of the well-known basic result on Eisenstein series. For proofs and more details we refer to Apostol [2], Berndt [1,5,10] and Lewittes [20]. We will use the following properties of Eisenstein series as follows:

For x and a both real numbers and $\mathrm{Re}\, s > 1$, let [10]

$$\psi(s, x, a) = \sum_{y+a>0} e^{2\pi iyx}(y+a)^{-s}.$$

The function $\psi(s, x, a)$ has an analytic continuation into the entire complex plane. This continuation is analytic everywhere except for a possible simple pole at $s = 1$.

For $z \in \mathbb{H}$ the Eisenstein series $G(z, 2)$ is defined as follows:

$$G(z, 2) = 2\zeta(2) + 2(2\pi i)^2 \sum_{a=1}^{\infty} \sigma(n) e^{2\pi inz},$$

where $\sigma(n) = \sum_{d|n} d$, and $\zeta(z)$ denotes Riemann zeta-function [2].

Lemma 1. *If $z \in \mathbb{H}$, then*

$$G(z, 2) = 2\zeta(2) + \sum_{0 \neq m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left(\frac{1}{mz+n}\right)^2.$$

Proof. Now we give the following equation (Equation (12) of Chapter 1 [1]):

$$-(2\pi i)^2 \sum_{r=1}^{\infty} re^{2\pi irz} = -\frac{1}{z^2} - \sum_{0 \neq m \in \mathbb{Z}} \left(\frac{1}{z+m}\right)^2. \tag{3.1}$$

Replacing z by nz, where $n > 0$, and sum over all $n > 0$, and by using Riemann zeta-function in (3.1), we have the desired result. $\square$

Apostol [2] raised the following question concerning the Dedekind eta-function and $G(z, 2)$ function: let $z \in \mathbb{H}$. We have

$$G(z, 2) = -4\pi i \frac{d}{dz} \log \eta(z). \tag{3.2}$$

Now, by using differentiating logarithm of generalized Dedekind eta-function, we find generalized (3.2) as follows:

Lemma 2. *Let g and h be integers and N be positive integer and $z \in \mathbb{H}$. Then*

$$\begin{aligned} &-4\pi i \frac{d}{dz} \log \eta_{g,h}(z, N) \\ &\quad = 4\pi^2 \bar{B}_2\left(\frac{g}{N}\right) + 2N\left(\psi\left(2, \frac{-h}{N}, 0\right) + \psi\left(2, \frac{h}{N}, 0\right)\right) \\ &\qquad + \frac{1}{N}\left(G\left(\frac{z}{N}, 2, 0, \left(\frac{h}{N}, 0\right)\right) + G\left(\frac{z}{N}, 2, 0, \left(\frac{-h}{N}, 0\right)\right)\right), \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&-4\pi i\frac{d}{dz}\log\eta_{g,h}(2z,N)\\
&= 8\pi^2\bar{B}_2\left(\frac{g}{N}\right)+N\left(\psi\left(2,\frac{-h}{N},0\right)+\psi\left(2,\frac{h}{N},0\right)\right)\\
&\quad+\frac{2}{N}\left(G\left(\frac{2z}{N},2,0,\left(\frac{h}{N},0\right)\right)+G\left(\frac{2z}{N},2,0,\left(\frac{-h}{N},0\right)\right)\right),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&-4\pi i\frac{d}{dz}\log\eta_{g,h}\left(\frac{z}{2},N\right)\\
&= 2\pi^2\bar{B}_2\left(\frac{g}{N}\right)+4N\left(\psi\left(2,\frac{-h}{N},0\right)+\psi\left(2,\frac{h}{N},0\right)\right)\\
&\quad+\frac{1}{2N}\left(G\left(\frac{z}{2N},2,0,\left(\frac{h}{N},0\right)\right)+G\left(\frac{z}{2N},2,0,\left(\frac{-h}{N},0\right)\right)\right).
\end{aligned}
\tag{3.5}
$$

Proof. We prove (3.3). By differentiating $\log\eta_{g,h}(z,N)$ and through some calculations, we get

$$
\begin{aligned}
\frac{d}{dz}\log\eta_{g,h}(z,N) &= \pi i\bar{B}_2\left(\frac{g}{N}\right)+\frac{2\pi i}{N}\sum_{m=1,m\equiv g(N)}^{\infty} m\frac{e^{2\pi i(h+zm)/N}}{1-e^{2\pi i(h+zm)/N}}\\
&\quad+\frac{2\pi i}{N}\sum_{m=1,m\equiv -g(N)}^{\infty} m\frac{e^{2\pi i(-h+zm)/N}}{1-e^{2\pi i(-h+zm)/N}}
\end{aligned}
\tag{3.6}
$$

by the well-known relation $\sum_{v\in\mathbb{N}} e^{-nv}=\frac{e^n}{e^n-1}$, using (3.1) and the above relation in (3.6), then the desired result is obtained. □

Remark 4. The proofs of (3.4) and (3.5) follow precisely along the same lines as the proof of (3.3), and so we omit them. Eq. (3.2) is a special case of Lemma 2. If we get $g=h\equiv 0(N)$ then $\frac{d}{dz}\log\eta_{0,0}(z)$ is reduced to $\frac{d}{dz}(2\log\eta(z))$. By using the above lemma, we find relations between theta-functions and Eisenstein series. These relations are given as follows:

Theorem 9. *Let g and h be integers and N be positive integer and $z\in\mathbb{H}$. Then*

$$
-4\pi i\frac{d}{dz}f_{2;g,h}(z,N)=12\pi_2^2\bar{B}\left(\frac{g}{N}\right)+\frac{1}{N}(E_1(z)+E_2(z)),
$$

where

$$
E_1(z)=4G\left(\frac{2z}{N},2,0,\left(\frac{h}{N},0\right)\right)-G\left(\frac{z}{N},2,0,\left(\frac{h}{N},0\right)\right)
$$

and

$$E_2(z) = 4G\left(\frac{2z}{N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right) - G\left(\frac{z}{N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right)$$

and $G(z,s,r,h)$ *is Eisenstein series* $\bar{B}_2(x)$ *is Bernoulli function.*

Theorem 10. *Let* g *and* h *be integers and* N *be positive integer and* $z \in \mathbb{H}$. *Then*

$$-4\pi i \frac{d}{dz} f_{3;g,h}(z,N) = \frac{1}{N}(F_1(z) + F_2(z)),$$

where

$$F_1(z) = 5G\left(\frac{z}{N}, 2, 0, \left(\frac{h}{N}, 0\right)\right) - 2G\left(\frac{2z}{N}, 2, 0, \left(\frac{h}{N}, 0\right)\right) - G\left(\frac{z}{2N}, 2, 0, \left(\frac{h}{N}, 0\right)\right)$$

and

$$F_2(z) = 5G\left(\frac{z}{N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right) - 2G\left(\frac{2z}{N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right) - G\left(\frac{z}{2N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right)$$

and $G(z,s,r,h)$ *is Eisenstein series.*

Theorem 11. *Let* g *and* h *be integers and* N *be positive integer and* $z \in \mathbb{H}$. *Then*

$$-4\pi i \frac{d}{dz} f_{4;g,h}(z,N) = 6NY(z) + \frac{1}{N}(T_1(z) + T_2(z)),$$

where

$$T_1(z) = G\left(\frac{z}{2N}, 2, 0, \left(\frac{h}{N}, 0\right)\right) - G\left(\frac{z}{N}, 2, 0, \left(\frac{h}{N}, 0\right)\right),$$

$$T_2(z) = G\left(\frac{z}{2N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right) - G\left(\frac{z}{N}, 2, 0, \left(-\frac{h}{N}, 0\right)\right)$$

and

$$Y(z) = \psi\left(2, \frac{h}{N}, 0\right) + \psi\left(2, -\frac{h}{N}, 0\right)$$

and $G(z,s,r,h)$ *is Eisenstein series.*

Proof of Theorem 9. Differentiating (2.1) and multiplying both sides by $-4\pi i$, we find

$$-4\pi i\frac{d}{dz}f_{2;g,h}(z,N)=-8\pi i\frac{d}{dz}\eta_{g,h}(2z,N)+4\pi i\frac{d}{dz}\eta_{g,h}(z,N).$$

Using (3.3), (3.4) and evaluating $G(z,s,r,h)$ and $\psi(s,x,a)$ in the above, we find that

$$\begin{aligned}-4\pi i\frac{d}{dz}f_{2;g,h}(z,N)&=12\pi_2^2\bar{B}\left(\frac{g}{N}\right)+\frac{1}{N}\left(4G\left(\frac{2z}{N},2,0,\left(\frac{h}{N},0\right)\right)\right.\\&\quad-\left.G\left(\frac{z}{N},2,0,\left(\frac{h}{N},0\right)\right)\right)+\frac{1}{N}\left(4G\left(\frac{2z}{N},2,0,\left(-\frac{h}{N},0\right)\right)\right.\\&\quad-\left.G\left(\frac{z}{N},2,0,\left(-\frac{h}{N},0\right)\right)\right).\end{aligned}$$

Thus, we obtain the desired result. □

Remark 5. The proof of Theorems 10 and 11 follow precisely along the same lines as the proof of Theorem 9, and so we omit them. By these theorems, the connection between $f_{n;g,h}(z,N),\ n=2,3,4$ and Eisenstein series are found. Eqs. (1.4)–(1.5) are special cases of these theorems. For $g=h\equiv 0(N)$ we get $\frac{d}{dz}\log\eta_{0,0}(z)$ is reduced to $\frac{d}{dz}(2\log\eta(z))$ and $\frac{d}{dz}f_{n;0,0}(z,N)$ are reduced to $\frac{d}{dz}(2\vartheta_n(z)),\ n=2,3,4$.

4. Theorems on Lambert series and generalized Dedekind sums

We review some notation from Apostol [1] and Berndt [5,8] papers. We will define Dedekind sums, $A(z,s,r,h)$ and Lambert series and discuss some of the fundamental properties of these functions which are needed in the following theorems:

$$s(a,b;p)=\frac{p!}{(2\pi i)^p}\sum_{k=1,k\not\equiv 0(b)}^{\infty}k^{-p}\left(\frac{e^{2\pi ika/b}}{1-e^{2\pi ika/b}}-\frac{e^{-2\pi ika/b}}{1-e^{-2\pi ika/b}}\right),\tag{4.1}$$

where p is fixed odd integer $\geqslant 1$, and $(a,b)=1$ (in Theorem 4 Eq. (4.11) in [2]). Berndt [8] has established representation $s(a,b)$ by the $\cot\pi z$ function as follows:

$$s(a,b)=\sum_{k=1,k\not\equiv 0(b)}^{\infty}\frac{\cot(\frac{\pi ak}{b})}{k},\tag{4.2}$$

where $(a,b)=1$ (see [14,8] for detail).

We define new relation as follows:

$$\frac{e^{2\pi iz}}{1-e^{2\pi iz}}=i\cot\pi z+\frac{e^{-2\pi iz}}{1-e^{-2\pi iz}}.\tag{4.3}$$

In Theorem 2 in [5], put $r_1 = r_2 = 0$ and $s = -m$, where $m > 0$ is even, we get

$$A(z, -m) = \sum_{k,n=1}^{\infty} k^{-m-1} e^{2\pi iknz} = \sum_{k=1}^{\infty} k^{-m-1} \frac{e^{2\pi ikz}}{1 - e^{2\pi ikz}}, \tag{4.4}$$

which is a Lambert series in the variable $e^{2\pi iz}$. By using (4.3) in (4.4), we get

$$A(z, -m) = 2i \sum_{k=1}^{\infty} k^{-m-1} \cot(\pi kz) + 2G_{m+1}(e^{-2\pi iz}).$$

Putting $z = \frac{a}{b}$, $k \not\equiv 0(b)$, $k = 1, 2, 3, \ldots, \infty$ in the above, we arrive at the following:

$$A\left(\frac{a}{b}, -m\right) = 2i \sum_{k=1, k\not\equiv 0(b)}^{\infty} k^{-m-1} \cot\left(\frac{\pi k\, a}{b}\right) + 2G_{m+1}\left(e^{-2\pi i\frac{a}{b}}\right).$$

By using (4.1) in the above, we arrive at the following lemma:

Lemma 3. *Let* $(a, b) = 1$. *For even* $m > 0$, *we have*

$$A\left(\frac{a}{b}, -m\right) = -\frac{\pi(4\pi i)^m}{(m+1)!} s(a, b, m+1) + 2 \sum_{k=1, k\not\equiv 0(b)}^{\infty} k^{-m-1} \frac{e^{-2\pi ika/b}}{1 - e^{-2\pi ika/b}}.$$

Using the definition of $A(z, 0, 0)$, Lambert series, (4.3) and (4.2), it is easy to show that

$$A\left(\frac{a}{b}, 0\right) = -\pi s(a, b) + 2 \sum_{k=1, k\not\equiv 0(b)}^{\infty} k^{-1} \frac{e^{-2\pi ika/b}}{1 - e^{-2\pi ika/b}}.$$

Recalling the following equation:

$$A(z, 0, 0, 0) = \frac{\pi iz}{6} - 2\log \eta(z) \tag{4.5}$$

and by using (4.5) in the above equation, we obtain

$$\sum_{k=1, k\not\equiv 0(b)}^{\infty} k^{-1} \frac{e^{-2\pi ika/b}}{1 - e^{-2\pi ika/b}} = \frac{\pi ia}{12b} - 2\log \eta\left(\frac{a}{b}\right) - \frac{\pi}{2} s(a, b).$$

Now, putting $A(z, -m)$ $(m \geqslant 0$, is even) in (1.4)–(1.6) and through some calculations and by using Theorem 3, we find the relations between theta-function, Hardy sums and Lambert series as follows:

Theorem 12. *Let a and b denote relatively prime integers with $b > 0$. If b is even, then*

$$\log \vartheta_2\left(\frac{a}{b}\right) = \log 2 + \frac{\pi i a}{3b} + 2\pi s_2(a,b) + L_2\left(\frac{a}{b}\right)$$

if b is odd, then

$$\log \vartheta_2\left(\frac{a}{b}\right) = \log 2 + \frac{\pi i a}{3b} - \pi s_3(a,b) + L_2\left(\frac{a}{b}\right),$$

where

$$L_2\left(\frac{a}{b}\right) = G_1(e^{-2\pi i a/b}) - 2G_1(e^{-4\pi i a/b})$$

if $a+b$ is even, then

$$\log \vartheta_3\left(\frac{a}{b}\right) = -\pi s_5(a,b) + L_3\left(\frac{a}{b}\right)$$

if $a+b$ is odd, then

$$\log \vartheta_3\left(\frac{a}{b}\right) = -\frac{\pi}{2} S(a,b) + L_3\left(\frac{a}{b}\right),$$

where

$$L_3\left(\frac{a}{b}\right) = 2G_1(e^{-\pi i a/b}) + 2G_1(e^{-4\pi i a/b}) - 5G_1(e^{-2\pi i a/b})$$

and if a is odd, then

$$\log \vartheta_4\left(\frac{a}{b}\right) = \frac{\pi}{4} s_4(a,b) + L_4\left(\frac{a}{b}\right)$$

and if a is even, then

$$\log \vartheta_4\left(\frac{a}{b}\right) = -\frac{\pi}{2} s_1(a,b) + L_4\left(\frac{a}{b}\right),$$

where

$$L_4\left(\frac{a}{b}\right) = G_1(e^{-2\pi i a/b}) - 2G_1(e^{-\pi i a/b}).$$

Now using (1.4)–(1.6) and (4.5), we define following functions as follows:

$$F(2,z,m) = \log 2 + \frac{\pi i z}{4} + A(z,-m) - 2A(2z,-m),$$

$$F(3,z,m)=2A\left(\frac{z}{2},-m\right)+2A(2z,-m)-5A(z,-m),$$

$$F(4,z,m)=A(z,-m)-2A\left(\frac{z}{2},-m\right).$$

We will use these functions in the theorems below.

Theorem 13. *Let a and b denote relatively prime integers with $b>0$ and m be even integer, with $m\geqslant 0$. If b is even, then*

$$F\left(2,\frac{a}{b},m\right)=\log 2+\frac{\pi ia}{4b}+\frac{i(2\pi i)^{m+1}}{(m+1)!}s_{2,m+1}(a,b)+L_{2,m+1}\left(\frac{a}{b}\right) \tag{4.6}$$

if b is odd, then

$$F\left(2,\frac{a}{b},m\right)=\log 2+\frac{\pi ia}{4b}+\frac{i(2\pi i)^{m+1}}{2(m+1)!}s_{3,m+1}(a,b)+L_{2,m+1}\left(\frac{a}{b}\right)$$

where

$$L_{2,m+1}\left(\frac{a}{b}\right)=G_{m+1}(e^{-2\pi iz})-2G_{m+1}(e^{-4\pi iz}),$$

if $a+b$ is even, then

$$F\left(3,\frac{a}{b},m\right)=\frac{\pi ia}{12b}-\frac{i(2\pi i)^{m+1}}{2(m+1)!}s_{5,m+1}(a,b)+L_{3,m+1}\left(\frac{a}{b}\right)$$

if $a+b$ is odd, then

$$F\left(3,\frac{a}{b},m\right)=\frac{\pi ia}{12b}-\frac{i(2\pi i)^{m+1}}{4(m+1)!}S_{m+1}(a,b)+L_{3,m+1}\left(\frac{a}{b}\right)$$

where

$$L_{3,m+1}\left(\frac{a}{b}\right)=2G_{m+1}(e^{-\pi ia/b})+2G_{m+1}(e^{-4\pi ia/b})-5G_{m+1}(e^{-2\pi ia/b})$$

if a is odd, then

$$F\left(4,\frac{a}{b},m\right)=-\frac{i(2\pi i)^{m+1}}{4(m+1)!}s_{4,m+1}(a,b)+L_{4,m+1}\left(\frac{a}{b}\right)$$

if a is even, then

$$F\left(4,\frac{a}{b},m\right)=-\frac{i(2\pi i)^{m+1}}{2(m+1)!}s_{1,m+1}(a,b)+L_{4,m+1}\left(\frac{a}{b}\right),$$

where

$$L_{4,m+1}\left(\frac{a}{b}\right) = G_{m+1}(e^{-\pi ia/b}) - 2G_{m+1}(e^{-2\pi ia/b}).$$

Proof of Theorem 13. We prove (4.6). If we employ in (4.4) the well-known Lambert series in the variables $e^{2\pi iz}$, $e^{4\pi iz}$, we find that

$$F(2,z,m) = \log 2 + \frac{\pi iz}{4} - 2\sum_{k=1}^{\infty} k^{-m-1}\frac{e^{4\pi ikz}}{1-e^{4\pi ikz}} + \sum_{k=1}^{\infty} k^{-m-1}\frac{e^{2\pi ikz}}{1-e^{2\pi ikz}}.$$

Let a and b be integers and $(a,b)=1$. Putting $z=a/b$ and substituting (4.1), (4.2), (4.3) into the above, we get

$$\begin{aligned} F\left(2,\frac{a}{b},m\right) &= \log 2 + \frac{\pi ia}{4b} - \frac{(2\pi i)^{m+1}}{(m+1)!} i(-s_{m+1}(a,b) + 2s_{m+1}(2a,b)) \\ &\quad + G_{m+1}(e^{-2\pi iz}) - 2G_{m+1}(e^{-4\pi iz}). \end{aligned}$$

If b is even, by using (1.13) in the above, we establish the desired result immediately. $\square$

Remark 6. Theorems 12 and 3 (Explicit formulae, cf. Theorem 5.1 in [28]) are special cases of Theorem 13. If we get $m=0$, then Theorem 13 is deduced to Theorem 12. $S_{m+1}(a,b)$ and $s_{x,m+1}(a,b)$, $1\leqslant x\leqslant 5$ are deduced to $S(a,b)$ and $s_x(a,b)$, $1\leqslant x\leqslant 5$. The proof of Theorem 12 and the other relations of Theorem 3 follow along the same lines as the proof of (4.6), and so we omit them.

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