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# ON THE GLOBAL DIMENSION OF THE FUNCTOR CATEGORY (mod $R$, Ab)

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In various questions in ring theory the following category turns out to be useful (cf. [3–5]). For an arbitrary ring $R$ let mod $R$ be the category of finitely presented right $R$-modules (viewed as a full subcategory of the category of all right $R$-modules) and $D(R) = (\text{mod}\, R, \text{Ab})$ the category of all additive functors from mod $R$ to the category Ab of abelian groups. $D(R)$ is a Grothendieck category with a generator and hence has a well-defined global (cohomological) dimension gl.dim $D(R)$.

For a left $R$-module $M$ the functor $-\otimes_R M$ belongs to $D(R)$ and its injective dimension in $D(R)$ is called $L$-dim $M$ and further investigated in [3, 5]. The global $L$-dimension of $R$, gl. $L$-dim $R$, is naturally defined as sup{$L$-dim $M$}, $M$ running through all left $R$-modules. gl. $L$-dim $R$ coincides with the left pure global dimension of $R$.

**Proposition 1.** *For any ring $R$ the following inequalities hold*:

$$\text{gl.}\,L\text{-dim}\,R \leq \text{gl.dim}\,D(R) \leq 2 + \text{gl.}\,L\text{-dim}\,R.$$

**Proof.** The first inequality is trivial. To prove the second inequality consider an arbitrary functor $T \in D(R)$ and let

$$0 \to T \to I_0 \to I_1 \to C_1 \to 0$$

be an exact sequence where $I_0$ and $I_1$ are injectives in $D(R)$. The injective objects in $D(R)$ are the functors $-\otimes_R M$, $M$ running through the algebraically compact (= pure-injective) left $R$-modules [3, 8]. By a diagram chasing diagram $C_1$ is seen to be a right exact functor and hence of the form $-\otimes_R A$ for some left $R$-module $A$. Since the injective dimension of $-\otimes_R A$ is $L$-dim $A \leq$ gl.$L$-dim $R$ it follows that inj.dim$_{D(R)} T \leq 2 +$ gl.$L$-dim $R$ and hence gl.dim $D(R) \leq 2 +$ gl.$L$-dim $R$.

In general, examples show that the above inequalities cannot be improved. By results in [3, 7] Proposition 1 implies that gl.dim $D(R) \leq n+3$ if $R$ is a ring of cardinality $\aleph_n$.

If $R$ is von Neumann regular any exact sequence is pure and any functor

$\in D(R)$ has the form $-\otimes_R M$ for some left module $M$; hence gl.dim $D(R)$ = gl.$L$-dim $R$ = l.gl.dim $R$.

We can now easily exhibit the rings $R$ of small gl.dim $D(R)$.

**Proposition 2.** *Let R be an arbitrary ring. Then* gl.dim $D(R) = 0$ *if and only if R is semi-simple Artinian, and* gl.dim $D(R) = 1$ *if and only if R is a left hereditary von Neumann regular ring which is not Artinian.*

**Proof.** In view of the preceding remarks it suffices to show that $R$ is von Neumann regular if gl.dim $D(R) \leq 1$ In that case for any $T \in D(R)$ there is an exact sequence

$$0 \to T \to I_0 \to I_1 \to 0$$

with injective (and hence right exact) $I_0$ and $I_1$. Thus $T$ is right exact. Applying this to the functor $\mathrm{Hom}_R(N, -)$ where $N = R/aR$, $a \in R$, we conclude that the sequence $R \to R/aR$ splits for any $a \in R$ and consequently $R$ is von Neumann regular.

If $R$ is a ring not of the type described in Proposition 2 gl.dim $D(R) \geq 2$. If $R$ is right Artinian it follows from [5] that gl.dim $D(R) = 2 +$ gl.$L$-dim $R$ and hence gl.dim $D(R) = 0$ when $R$ is semi-simple and gl.dim $D(R) = 2$ exactly when $R$ is a non-semi-simple ring for which any left $R$-module is a direct sum of finitely presented modules. In the general situation it is an open problem to classify the rings for which gl.dim $D(R) = 2$.

In this paper we shall prove that a commutative Noetherian ring $R$ is an Artinian principal ideal ring if gl.dim $D(R) \leq 2$. (The converse is obvious since any module over a commutative Artinian principal ideal ring is a direct sum of finitely presented modules.)

We first bring some results concerning algebraically compact modules in a slightly more general setting which might be of some independent interest.

Let $R$ be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and $M$ any (not necessarily finitely generated) $R$-module. Call $M$ cotorsion if $\mathrm{Ext}^1_R(P, M) = 0$ for any flat $R$-module $P$. If $M$ is algebraically compact (= pure-injective) any short exact sequence of the form

$$0 \to M \to A \to P \to 0$$

with flat $P$ splits. Hence any algebraically compact module is cotorsion. By an argument from [6, §8] it is easily seen that the canonical mapping $M \to \tilde{M} = \varprojlim_t M/\mathfrak{m}^t M$ is surjective for any cotorsion module. Thus we have the following hierarchy: $M$ algebraically compact $\Rightarrow$ $M$ cotorsion $\Rightarrow$ $M \to \varprojlim M/\mathfrak{m}^t M$ is surjective. Under certain additional assumptions the arrow can be reversed.

**Proposition 3.** *Let R be a commutative local Noetherian ring with maximal ideal*

*$\mathfrak{m}$ and $M$ an $R$-module which is separated in the $\mathfrak{m}$-adic topology (i.e. $\bigcap_t \mathfrak{m}^t M = 0$). Then $M$ is cotorsion if and only if $M$ is complete in the $\mathfrak{m}$-adic topology.*

**Proof.** It suffices to show that $M$ complete implies $M$ is cotorsion. We have to prove that any exact sequence

$$(\dagger) \qquad 0 \to M \xrightarrow{f} A \xrightarrow{g} P \to 0$$

splits whenever $P$ is flat. Since $P$ is flat, for any $t \geq 0$ the sequence (with the obvious mappings)

$$(\dagger\dagger) \qquad 0 \to M/\mathfrak{m}^t M \xrightarrow{f_t} A/\mathfrak{m}^t A \xrightarrow{g_t} P/\mathfrak{m}^t P \to 0$$

is exact. $P/\mathfrak{m}^t P$ is a flat $(R/\mathfrak{m}^t)$-module and hence $(R/\mathfrak{m}^t)$-projective, $R/\mathfrak{m}^t$ being Artinian. Thus ($\dagger\dagger$) splits for all $t$ and it is readily checked that there exist right inverses $h_t$ of $g_t$ commuting with the canonical mappings $P/\mathfrak{m}^{t+1} P \to P/\mathfrak{m}^t P$ and $A/\mathfrak{m}^{t+1} A \to A/\mathfrak{m}^t A$, $t \geq 0$. Therefore the sequence

$$0 \to \tilde{M} \to \tilde{A} \to \tilde{P} \to 0$$

is split exact (where generally $\tilde{N}$ denotes the completion $\varprojlim N/\mathfrak{m}^t N$). Since $M \simeq \tilde{M}$ this implies that ($\dagger$) splits.

**Proposition 4.** *Let $R$ be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and $P$ a flat $R$-module. The $\mathfrak{m}$-adic completion $\tilde{P}$ is a direct summand of a direct product of copies of the $\mathfrak{m}$-adic completion $\tilde{R}$ of $R$ and hence an algebraically compact flat $R$-module. In particular, a flat module which is separated in the $\mathfrak{m}$-adic topology is algebraically compact if and only if it is complete in this topology.*

**Proof.** For $t \geq 0$, $P/\mathfrak{m}^t P$ is a free $(R/\mathfrak{m}^t)$-module. By Nakayama's lemma it is easy to see that there exists a basis $e_{\alpha,t}$, $\alpha \in I$, for $P/\mathfrak{m}^t P$ such that $e_{\alpha,t} = \kappa_{t+1} e_{\alpha,t+1}$, $\alpha \in I$, $t \geq 0$, where $\kappa_{t+1}$ denotes the canonical mapping $P/\mathfrak{m}^{t+1} P \to P/\mathfrak{m}^t P$.

Let $i_t$ be the natural injection of $P/\mathfrak{m}^t P = \Sigma_\alpha \oplus (R/\mathfrak{m}^t) e_{\alpha,t}$ into the product $(R/\mathfrak{m}^t)^I$. Since the corresponding cokernel is projective, $i_t$ splits, and, as above, there exist left inverses commuting with $\{\kappa_t\}$ and the canonical homomorphisms $(R/\mathfrak{m}^{t+1})^I \to (R/\mathfrak{m}^t)^I$. Hence the injection

$$\tilde{P} = \varprojlim P/\mathfrak{m}^t P \to \varprojlim (R/\mathfrak{m}^t R)^I = \tilde{R}^I$$

splits and $\tilde{P}$ is a direct summand of $\tilde{R}^I$.

**Note.** If $R$ moreover is assumed to be one-dimensional it can be proved that a flat $R$-module $P$ is algebraically compact if and only if the canonical mapping $P \to \varprojlim P/\mathfrak{m}^t P$ is surjective.

We are now able to prove

**Theorem.** *Let R be a commutative Noetherian ring. Then* gl.dim $D(R) \leq 2$ *if and only if R is an Artinian principal ideal ring.*

**Proof.** In view of the earlier remarks it suffices to prove the "only if" part. Assume $R$ were not Artinian. We shall then obtain a contradiction by constructing an exact sequence of $R$-modules.

$$(*) \qquad A \underset{f}{\to} B \underset{g}{\to} C \to 0$$

where $A$ and $B$ are algebraically compact, but $C$ is not algebraically compact. In fact, $(*)$ gives rise to an exact sequence of functors in $D(R)$

$$0 \to \mathrm{Ker}(f \otimes -) \to (A \otimes -) \to (B \otimes -) \to (C \otimes -) \to 0.$$

Since $(A \otimes -)$ and $(B \otimes -)$ are injective objects in $D(R)$ while $(C \otimes -)$ is not, we get $\mathrm{inj.dim}_{D(R)} \mathrm{Ker}(f \otimes -) > 2$ and hence gl.dim $D(R) > 2$.

To construct the sequence $(*)$ we observe that for any ideal $\mathfrak{A}$ of $R$ any $(R/\mathfrak{A})$-module is algebraically compact if and only if it is algebraically compact qua $R$-module. Similarly, if $S$ is a multiplicatively closed system of a domain $R$, an $R_S$-module is algebraically compact if and only if it is algebraically compact as an $R$-module.

By our assumption $R$ is a non-Artinian Noetherian ring. Let $P$ be a submaximal prime ideal of $R$. By replacing $R$ by $R/P$ and passing to a suitable ring of fractions we may assume that $R$ is a one-dimensional local domain with maximal ideal $\mathfrak{m}$, say. Let $\pi$ be a non-invertible element of $R$ and let $\tilde{R}$ be the $\mathfrak{m}$-adic completion of $R$.

We consider the submodule $A$ of $\tilde{R}^N$ consisting of all sequences tending to zero in the $\mathfrak{m}$-adic topology. $A$ is the $\mathfrak{m}$-adic completion of $R^{(N)}$ and hence by Proposition 4 a flat algebraically compact module. The mapping $\varphi$

$$\varphi(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \ldots) = \left(\pi\tilde{r}_1 + \sum_{i=2}^{\infty} r_i, -\tilde{r}_2\pi, -\tilde{r}_3\pi^2, \ldots\right)$$

is a well-defined monomorphism of $A$ into $A$. From the exact sequence

$$(**) \qquad 0 \to A \xrightarrow{\varphi} A \to C \to 0$$

where $C = \mathrm{Coker}\,\varphi$ we derive the exact sequence

$$\mathrm{Hom}(Q, A) \to \mathrm{Hom}(Q, C) \to \mathrm{Ext}^1(Q, A),$$

$Q$ denoting the quotient field of $R$. Obviously $\mathrm{Hom}(Q, A) = 0$ and since $A$ in particular is a cotorsion module we get $\mathrm{Ext}^1(Q, A) = 0$, and hence $\mathrm{Hom}(Q, C) = 0$. Thus $C$ contains no non-zero divisible submodule.

On the other hand $C$ contains a non-zero element $c$ represented by $(1, 0, 0, \ldots)$ which is divisible by any power of $\pi$. If $C$ were algebraically compact the equations

$$c = \pi x_1$$

$$x_n = \pi x_{n+1}; \quad n = 1, 2, 3, \ldots$$

would have a solution in $C$ and the submodule generated by $\{x_n\}$, $n \in N$ would be a non-zero divisible submodule of $C$. (Here we use that $Q = R[1/\pi]$, $R$ being one-dimensional.) Therefore $C$ is not algebraically compact and $(**)$ yields an exact sequence of the desired form.

Consequently $R$ must be Artinian and hence a finite product of local rings. To finish the proof of the theorem we show that a local Artinian ring $R$ is a principal ideal ring ("uniserial") if $\text{gl.dim}\, D(R) \leq 2$.

For any $R$-module $M$ there is an exact sequence

$$F_1 \to F_0 \to M \to 0$$

where $F_0$ and $F_1$ are free modules. By Proposition 4 $F_0$ and $F_1$ are algebraically compact. By reversing the argument at the beginning of the theorem it follows that the (arbitrary) $R$-module $M$ is algebraically compact. It is well known that this implies that $R$ is a principal ideal ring.

**Corollary 1.** *Let $R$ be an arbitrary Dedekind domain which is not a field. Then* $\text{gl.dim}\, D(R) = 3$.

**Proof.** We only have to prove that $\text{gl.dim}\, D(R) \leq 3$. By an obvious generalization of Kulikovs theorem [1, theorem 18.1] to Dedekind domains any submodule of a direct sum of finitely generated $R$-modules is again a direct sum of finitely generated $R$-modules. This implies that any $R$-module has a pure projective resolution of length 1, hence the pure global dimension ($= \text{gl.}L\text{-dim}\, R$) $\leq 1$ and consequently $\text{gl.dim}\, D(R) \leq 2 + \text{gl.}L\text{-dim}\, R \leq 3$.

**Corollary 2.** *Let $R$ be an arbitrary countable commutative Noetherian ring which is not an Artinian principal ideal ring. Then* $\text{gl.dim}\, D(R) = 3$.

**Proof.** By [3] $L\text{-dim}\, M \leq 1$ for any $R$-module $M$, hence $\text{gl.}L\text{-dim}\, R \leq 1$ and thus $\text{gl.dim}\, D(R) \leq 3$. The inverse inequality follows from the theorem.

**Remark.** A modification of the proof of the theorem yields for a commutative Noetherian ring $R$ that $\text{gl.dim}\, D(R) \leq 2$ if and only if any linearly compact $R$-module is a direct product of finitely generated $R$-modules.

Apart from the above cases it seems rather difficult to determine $\text{gl.dim}\, D(R)$ even for commutative Noetherian domains. By taking polynomial rings over "big" fields one can obtain Noetherian domains of arbitrarily large $\text{gl.dim}\, D(R)$. However, the bounds one gets there are probably not precise.

We conclude with some remarks concerning the case $R = K[X, Y]$ where $K$ is a field. When $K$ is finite or countable $\text{gl.dim}\, D(R) = 3$; otherwise we can only say gl.dim $D(R) \geq 3$, and it is not even known whether gl.dim $D(R)$ will be finite. However, we can compute the injective dimension ($= L$-dim $R$) of the identity functor in $D(R)$ (or, strictly speaking the forgetful functor from $R$-modules to abelian groups). If $|K| \leq \aleph_0$ we have $L$-dim$R = 1$; if $|K| > \aleph_0$ we have $L$-dim$R = 2$. By [3, 5] $L$-dim$R \leq 2$, so it remains to be shown that $L$-dim$R > 1$ in the uncountable case. This follows from the slightly more general

**Proposition 5.** *Let $A$ be a Noetherian ring containing an uncountable field $K$. Then $L$-dim$A[X, Y] \geq 2$.*

**Proof.** $R = A[X, Y]$ is Noetherian and its pure injective envelope is $\prod_{\mathfrak{m}} \tilde{R}_{\mathfrak{m}}$ where $\mathfrak{m}$ runs through the maximal ideals of $R$ and $\tilde{R}_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-adic completion of $R$. In the pure exact sequence

$$0 \to R \xrightarrow{i} \prod_{\mathfrak{m}} \tilde{R}_{\mathfrak{m}} \xrightarrow{\kappa} C = \Big(\prod_{\mathfrak{m}} \tilde{R}_{\mathfrak{m}}\Big)\Big/ R \to 0$$

(where $i$ is the diagonal map) we thus have to prove that $C$ is not algebraically compact.

Let $\mathfrak{m}'$ be some fixed maximal ideal of $R$ containing $X$ and $Y$. Then $\tilde{R}_{\mathfrak{m}'} = \tilde{A}_{\mathfrak{m}' \cap A}[[X, Y]]$ where $\tilde{A}_{\mathfrak{m}' \cap A}$ is the $(\mathfrak{m}' \cap A)$-adic completion of $A$. In particular $\tilde{R}_{\mathfrak{m}'} \geq K[[X, Y]]$. If $\alpha_j$, $1 \leq j \leq t < \infty$ are elements in $K$ and $f_j$, $1 \leq j \leq t$ are power series in $K[[Y]]$, a straightforward computation shows that the following system of linear equations

$$\Xi = (X - \alpha_j Y)Z_j + \kappa \bar{f}_j, \qquad 1 \leq j \leq t,$$

has a solution $\Xi$, $Z_j$ $(1 \leq j \leq t)$ in $C$. Here $\bar{f}_j$ denotes the element in $\prod_{\mathfrak{m}} \tilde{R}_{\mathfrak{m}}$ all of whose components are zero except the $\mathfrak{m}'$-component which is $f_j$.

Consider two power series $f(Y) = \sum_{n \geq 0} a_n Y^n$ and $g(Y) = \sum_{n \geq 0} b_n Y^n$ in $K[[Y]]$ such that $a_n \neq b_n$ for infinitely many $n$.

Further let $I = \{\alpha\}$ and $J = \{\beta\}$ be two uncountable disjoint subsets of $K$. By the preceding remark any finite subset of the following system of linear equations

$$(***) \qquad \begin{aligned} \Xi &= (X - \alpha Y)Z_\alpha + \kappa \bar{f}, \qquad \alpha \in I, \\ \Xi &= (X - \beta Y)Z_\beta + \kappa \bar{g}, \qquad \beta \in J, \end{aligned}$$

has a solution in $C$.

Assume the total system $(***)$ had a solution in $C$. Let $\xi$ be the $\mathfrak{m}'$-component of a representative for $\Xi$ in $\prod_{\mathfrak{m}} \tilde{R}_{\mathfrak{m}}$. $\xi$ can be written in the form $\xi = \xi(X, Y) = \sum_{n \geq 0} h_n(X, Y)$, where $h_n(X, Y) = \sum_{\mu+\nu=n} c_{\mu\nu} X^\mu Y^\nu \in \tilde{A}_{\mathfrak{m}' \cap R}[X, Y]$. For any $\alpha \in I$ $(***)$ implies that

$$\xi(\alpha Y, Y) = \sum_{n \geq 0} h_n(\alpha Y, Y) = f(Y) + r_\alpha(Y)$$

where $r_\alpha(Y)$ is a polynomial in $A[Y]$, depending on $\alpha$. Since $I$ is uncountable there exists an integer $n_1$ such that the degree of $r_\alpha(Y)$ is $\leq n_1$ for infinitely many $\alpha \in I$. Therefore $h_n(\alpha, 1) = a_n$, $n > n_1$ for infinitely many $\alpha \in I$. Similarly, there is an integer $n_2$ such that $h_n(\beta, 1) = b_n$, $n > n_2$, for infinitely many $\beta \in J$. From this we infer that $a_n = b_n$ for $n$ max$(n_1, n_2)$ contradicting our assumption about $f(Y)$ and $g(Y)$. Hence (**) is not solvable in $C$ and $C$ is not algebraically compact.

**Remark.** In [2] a corresponding local result is proved. However, the two results have no direct connection since, in general, the $L$-dimension may increase strictly by localization!

## References


[1] L. Fuchs, Infinite Abelian Groups (Academic Press, New York, 1970).

[2] L. Gruson, Dimension homologique des modules plats sur un anneau commutatif noetherien, Symposia Mathematica XI (1973) 243–254.

[3] L. Gruson and C.U. Jensen, Modules algebriquement compacts et foncteurs $\varprojlim^{(i)}$, C.R. Acad. Sci. Paris Ser. A-B 276 (1973) 1651–1653.

[4] L. Gruson and C.U. Jensen, Deux applications de la notion de $L$-dimension, C.R. Acad. Sci. Paris Ser. A-B 282 (1976) 23–24.

[5] L. Gruson and C.U. Jensen, $L$-dimension of rings and modules (in preparation).

[6] C.U. Jensen, Les Foncteurs Derives de $\varprojlim$ et leurs Applications en Theorie des Modules, Lecture Notes in Mathematics 254 (Springer, 1972).

[7] D. Simson, On pure global dimension of locally finitely presented Grothendieck categories (to appear).

[8] R.B. Warfield, Purity and algebraic compactness for modules, Pac. J. Math. 28 (1969) 699–719.