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The enumeration of fully commutative affine permutations

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ABSTRACT

We give a generating function for the fully commutative affine permutations enumerated by rank and Coxeter length, extending formulas due to Stembridge and Barcucci–Del Lungo–Pergola–Pinzani. For fixed rank, the length generating functions have coefficients that are periodic with period dividing the rank. In the course of proving these formulas, we obtain results that elucidate the structure of the fully commutative affine permutations.

1. Introduction

Let W be a Coxeter group. An element w of W is *fully commutative* if any reduced expression for w can be obtained from any other using only commutation relations among the generators. For example, if W is simply laced then the fully commutative elements of W are those with no $s_i s_j s_i$ factor in any reduced expression, where s_i and s_j are any noncommuting generators.

The fully commutative elements form an interesting class of Coxeter group elements with many special properties relating to smoothness of Schubert varieties [12], Kazhdan–Lusztig polynomials and μ-coefficients [6,16], Lusztig's $a(w)$-function [9,26], and decompositions into cells [25,22,18]. Some of the properties carry over to the freely braided and maximally clustered elements introduced in [19,20,23]. At the level of the Coxeter group, [28] shows that each fully commutative element w has a unique labeled partial order called the *heap* of w whose linear extensions encode all of the reduced expressions for w.

Stembridge [28] (see also [11,14]) classified the Coxeter groups having finitely many fully commutative elements. In [29], he then enumerated the total number of fully commutative elements in each of these Coxeter groups. The type A series yields the Catalan numbers, a result previously given in [5]. Barcucci et al. [3] have enumerated the fully commutative permutations by Coxeter length,

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obtaining a q-analogue of the Catalan numbers. Our main result in Theorem 3.2 is an analogue of this result for the affine symmetric group.

In a general Coxeter group, the fully commutative elements index a basis for a quotient of the corresponding Hecke algebra [14,11,17]. In type A, this quotient is the Temperley–Lieb algebra; see [30,34]. Therefore, our result can be interpreted as a graded dimension formula for the affine analogue of this algebra.

In Section 2, we introduce the necessary definitions and background information. In Section 3, we enumerate the fully commutative affine permutations by decomposing them into several subsets. The formula that we obtain turns out to involve a ratio of q-Bessel functions as described in [2] arising as the solution obtained by [8] of a certain recurrence relation on the generating function. A similar phenomenon occurred in [3], and our work can be viewed as a description of how to lift this formula to the affine case. It turns out that the only additional ingredients that we need for our formula are certain sums and products of q-binomial coefficients. In Section 4, we prove that for fixed rank, the coefficients of the length generating functions are periodic with period dividing the rank. This result gives another way to determine the generating functions by computing the finite initial sequence of coefficients until the periodicity takes over. We mention some further questions in Section 5.

2. Background

In this section, we introduce the affine symmetric group, abacus diagrams for minimal length coset representatives, and q-binomial coefficients.

2.1. The affine symmetric group

We view the symmetric group S_n as the Coxeter group of type A with generating set $S = \{s_1, \ldots, s_{n-1}\}$ and relations of the form $(s_i s_{i\pm1})^3 = 1$ together with $(s_i s_j)^2 = 1$ for $|i-j| \geq 2$ and ${s_i}^2 = 1$. We denote $\bigcup_{n\geq1} S_n$ by S_∞ and call $n(w)$ the minimal rank n of $w \in S_n \subset S_\infty$. The affine symmetric group $\widetilde{S}_n$ is also a Coxeter group; it is generated by $\widetilde{S} = S \cup \{s_0\}$ with the same relations as in the symmetric group together with ${s_0}^2 = 1$, $(s_{n-1}s_0)^3 = 1$, $(s_0 s_1)^3 = 1$, and $(s_0 s_j)^2 = 1$ for $2 \leq j \leq n-2$.

Recall that the products of generators from S or $\widetilde{S}$ with a minimal number of factors are called *reduced expressions*, and $\ell(w)$ is the length of such an expression for an (affine) permutation w. Given an (affine) permutation w, we represent reduced expressions for w in sans serif font, say $\mathsf{w} = \mathsf{w}_1\mathsf{w}_2\cdots\mathsf{w}_p$ where each $\mathsf{w}_i \in S$ or $\widetilde{S}$. We call any expression of the form $s_i s_{i\pm1} s_i$ a *short braid*, where the indices $i, i\pm1$ are taken mod n if we are working in $\widetilde{S}_n$. There is a well-known theorem of Matsumoto [24] and Tits [31], which states that any reduced expression for w can be transformed into any other by applying a sequence of relations of the form $(s_i s_{i\pm1})^3 = 1$ (where again $i, i\pm1$ are taken mod n in $\widetilde{S}_n$) together with $(s_i s_j)^2 = 1$ for $|i-j| > 1$. We say that s_i is a *left descent* for $w \in \widetilde{S}_n$ if $\ell(s_i w) < \ell(w)$ and we say that s_i is a *right descent* for $w \in \widetilde{S}_n$ if $\ell(w s_i) < \ell(w)$.

As in [28], we define an equivalence relation on the set of reduced expressions for an (affine) permutation by saying that two reduced expressions are in the same *commutativity class* if one can be obtained from the other by a sequence of *commuting moves* of the form $s_i s_j \mapsto s_j s_i$ where $|i-j| \geq 2$. If the reduced expressions for a permutation w form a single commutativity class, then we say w is *fully commutative*.

If $\mathsf{w} = \mathsf{w}_1\cdots\mathsf{w}_k$ is a reduced expression for any permutation, then following [28] we define a partial ordering on the indices $\{1, \ldots, k\}$ by the transitive closure of the relation $i \lessdot j$ if $i < j$ and w_i does not commute with w_j. We label each element i of the poset by the corresponding generator w_i. It follows quickly from the definition that if w and w' are two reduced expressions for an element w that are in the same commutativity class then the labeled posets of w and w' are isomorphic. This isomorphism class of labeled posets is called the *heap* of w, where w is a reduced expression representative for a commutativity class of w. In particular, if $w \in S_n$ is fully commutative then it has a single commutativity class, and so there is a unique heap of w. Cartier and Foata [10] were among the first to study heaps of dimers, which were generalized to other settings by Viennot [32].

We also refer to elements in the symmetric group by the *one-line notation* $w = [w_1 w_2 \cdots w_n]$, where w is the bijection mapping i to w_i. Then the generators s_i are the adjacent transpositions interchanging the entries i and $i+1$ in the one-line notation. Let $w = [w_1 \cdots w_n]$, and suppose that $p = [p_1 \cdots p_k]$ is another permutation in S_k for $k \leq n$. We say w *contains the permutation pattern p* or w *contains p as a one-line pattern* whenever there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$w_{i_a} < w_{i_b} \quad \text{if and only if} \quad p_a < p_b$$

for all $1 \leq a < b \leq k$. We call $(i_1, i_2, \ldots, i_k)$ the *pattern instance*. For example, $[\underline{5}32\underline{41}]$ contains the pattern [321] in several ways, including the underlined subsequence. If w does not contain the pattern p, we say that w *avoids p*.

The *affine symmetric group* $\widetilde{S}_n$ is realized in [7, Chapter 8] as the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i+n) = w(i) + n$ and $\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i = \binom{n+1}{2}$. We call the infinite sequence

$$(\ldots, w(-1), w(0), w(1), w(2), \ldots, w(n), w(n+1), \ldots)$$

the *complete notation* for w and

$$[w(1), w(2), \ldots, w(n)]$$

the *base window* for w. By definition, the entries of the base window determine w and its complete notation. Moreover, the entries of the base window can be any set of integers that are normalized to sum to $\binom{n+1}{2}$ and such that the entries form a permutation of the residue classes in $\mathbb{Z}/(n\mathbb{Z})$ when reduced mod n. That is, no two entries of the base window have the same residue mod n. With these considerations in mind, we will represent an affine permutation using an abacus diagram together with a finite permutation.

To describe this, observe that S_n acts on the base window by permuting the entries, which induces an action of S_n on $\mathbb{Z}$. In this action, the Coxeter generator s_i simultaneously interchanges $w(i+kn)$ with $w(i+1+kn)$ for all $k \in \mathbb{Z}$. Moreover, the affine generator s_0 interchanges all $w(kn)$ with $w(kn+1)$. Hence, S_n is a parabolic subgroup of $\widetilde{S}_n$. We form the parabolic quotient

$$\widetilde{S}_n/S_n = \{w \in \widetilde{S}_n : \ell(ws_i) > \ell(w) \text{ for all } s_i \text{ where } 1 \leq i \leq n-1\}.$$

By a standard result in the theory of Coxeter groups, this set gives a unique representative of minimal length from each coset wS_n of $\widetilde{S}_n$. For more on this construction, see [7, Section 2.4]. In our case, the base window of the minimal length coset representative of an element is obtained by ordering the entries that appear in the base window increasingly. This construction implies that, as sets, the affine symmetric group can be identified with the set $(\widetilde{S}_n/S_n) \times S_n$. The minimal length coset representative determines which entries appear in the base window, and the finite permutation orders these entries in the base window.

We say that w has a *descent* at i whenever $w(i) > w(i+1)$. Note that if w has a descent at i, then $s_{(i \bmod n)}$ is a right descent in the usual Coxeter theoretic sense that $\ell(ws_i) < \ell(w)$.

2.2. Abacus diagrams

The abacus diagrams of [21] give a combinatorial model for the minimal length coset representatives in $\widetilde{S}_n/S_n$. Other combinatorial models and references for these are given in [4].

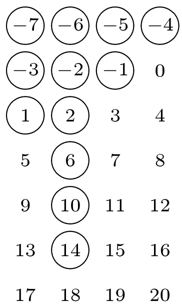
An *abacus diagram* is a diagram containing n columns labeled $1, 2, \ldots, n$, called *runners*. The horizontal rows are called *levels* and runner i contains entries labeled by $rn + i$ on each level r where $-\infty < r < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, with runner 1 in the leftmost position, increasing to runner n in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. Entries that are not circled are called *gaps*. The linear ordering of the entries given by the labels $rn + i$ is called the *reading order* of the abacus which corresponds to scanning left to right, top to bottom.

We associate an abacus to each minimal length coset representative $w \in \widetilde{S}_n/S_n$ by drawing beads down to level w_i in runner i for each $1 \leq i \leq n$ where $\{w_1, w_2, \ldots, w_n\}$ is the set of integers in the base window of w, with no two having the same residue mod n. Since the entries w_i sum to $\binom{n+1}{2}$, we call the abacus constructed in this way *balanced*. It follows from the construction that the Coxeter length of the minimal length coset representative can be determined from the abacus.

Proposition 2.1. *Let $w \in \widetilde{S}_n/S_n$ and form the abacus for w as described above. Let m_i denote the number of gaps preceding the lowest bead of runner i in reading order, for each $1 \le i \le n$. Then, the Coxeter length $\ell(w)$ is $\sum_{i=1}^{n} m_i$.*

Proof. This result is part of the folklore of the subject. One proof can be obtained by combining Propositions 3.2.5 and 3.2.8 of [4]. □

Example 2.2. The affine permutation $\widetilde{w} = [-1, -4, 14, 1]$ is identified with the pair (w^0, w) where w is the finite permutation $s_1 s_3 = [2143]$ which sorts the elements of the minimal length coset representative $w^0 = [-4, -1, 1, 14]$. Note that the entries of w^0 sum to $\binom{5}{2} = 10$. The abacus of w^0 is shown below.



From the abacus, we see that w^0 has Coxeter length $1 + 10 + 0 + 0 = 11$. For example, the ten gaps preceding the lowest bead in runner 2 are 13, 12, 11, 9, 8, 7, 5, 4, 3, and 0. Hence, $\widetilde{w}$ has length $\ell(w^0) + \ell(w) = 13$.

In this work, we are primarily concerned with the fully commutative affine permutations. Green has given a criterion for these in terms of the complete notation for w. His result is a generalization of a theorem from [5] which states that $w \in S_n$ is fully commutative if and only if w avoids [321] as a permutation pattern.

Theorem 2.3 ([15])**.** *Let $w \in \widetilde{S}_n$. Then, w is fully commutative if and only if there do not exist integers $i < j < k$ such that $w(i) > w(j) > w(k)$.*

Observe that even though the entries in the base window of a minimal length coset representative are sorted, the element may not be fully commutative by Theorem 2.3. For example, if we write the element $w^0 = [-4, -1, 1, 14]$ in complete notation

$$w^0 = (\ldots, -8, -5, -3, \mathbf{10}, -4, -1, \mathbf{1}, 14, \mathbf{0}, 3, 5, 18, \ldots)$$

we obtain a [321]-instance as indicated in boldface.

In order to more easily exploit this phenomenon, we slightly modify the construction of the abacus. Observe that the length formula in Proposition 2.1 depends only on the relative positions of the beads in the abacus, and is unchanged if we shift every bead in the abacus exactly k positions to the right in reading order. Moreover, each time we shift the beads one unit to the right, we change the sum of the entries occurring on the lowest bead in each runner by exactly n. In fact, this shifting corresponds to shifting the base window inside the complete notation. Therefore, we may define an abacus in which all of the beads are shifted so that position $n + 1$ becomes the first gap in reading order. We call such abaci *normalized*. Although the entries of the lowest beads in each runner will no longer sum to $\binom{n+1}{2}$, we can reverse the shifting to recover the balanced abacus. Hence, this process is a bijection on abaci, and we may assume from now on that our abaci are normalized.

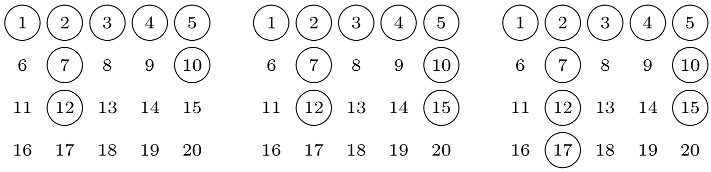


Fig. 1. The first three members—$A_1^{\mathcal{R}}$, $A_2^{\mathcal{R}}$, and $A_3^{\mathcal{R}}$—of the infinite family of abaci $\{A_i^{\mathcal{R}}\}$ with long runners $\mathcal{R} = \{2, 5\}$. These abaci correspond to certain long fully commutative elements of $\widetilde{S}_5$.

Proposition 2.4. *Let A be a normalized abacus for $w^0 \in \widetilde{S}_n/S_n$, and suppose the last bead occurs at entry i. Then, w^0 is fully commutative if and only if the lowest beads on runners of A occur only in positions that are a subset of* $\{1, 2, \ldots, n\} \cup \{i - n + 1, i - n + 2, \ldots, i\}$.

Proof. By construction, position $n + 1$ is the first gap in A, so the lowest bead on runner 1 occurs at position 1, and all of the positions $2, 3, \ldots, n$ are occupied by beads. Suppose there exists a lowest bead at position j with $n < j < i - n + 1$, and consider the complete notation for w^0 obtained by arranging the positions of the lowest beads in each runner sequentially in the base window. We obtain a [321]-instance in positions $i - n$ from the window immediately preceding the base window, j from the base window, and $n + 1$ from the window immediately succeeding the base window. Hence, w^0 is not fully commutative by Theorem 2.3.

Otherwise, there does not exist j such that $n < j < i - n + 1$. Hence, each of the entries in the base window belongs to $\{1, 2, \ldots, n\}$ in which case we say that the entry is *short*, or it belongs to $\{i - n + 1, i - n + 2, \ldots, i\}$ in which case we say that the entry is *long*. If $i \geq 2n$, then these designations are disjoint; otherwise, some entries may be both short and long. For example, in the first normalized abacus shown in Fig. 1, the entries 1, 3, and 4 are short while entries 12 and 10 are long. The corresponding complete notation is $(\ldots, 7|1, 3, 4, 10, 12|6, 8, 9, 15, 17|11, \ldots)$.

Because w^0 is constructed with an increasing base window, any entry $w^0(b)$ that is equivalent mod n to one of the short entries has the property that $w^0(c) > w^0(b)$ for all $c > b$. Therefore, the only inversions $w^0(a) > w^0(b)$ for $a < b$ in the complete notation occur between an entry $w^0(a)$ that is equivalent mod n to a long entry with an entry $w^0(b)$ that is equivalent mod n to a short entry. Hence, w^0 is [321]-avoiding, from which it follows that w^0 is fully commutative by Theorem 2.3. □

We distinguish between two types of fully commutative elements through the position of the last bead in its normalized abacus A. If the last bead occurs in a position $i > 2n$, then we call the element a *long* element. Otherwise, the last bead occurs in a position $n \leq i \leq 2n$, and we call the element a *short* element. As evidenced in Section 3, the long fully commutative elements have a nice structure that allows for an elegant enumeration; the short elements lack this structure.

2.3. q-analogues of binomial coefficients

Calculations involving q-analogues of combinatorial objects often involve q-analogues of counting functions. A few standard references on the subject are [1,13,27]. Define $(a, q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ and $(q)_n = (q, q)_n$. The *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (also called the *Gaussian polynomial*) is a q-analogue of the binomial coefficient $\binom{n}{k}$. To calculate a q-binomial coefficient directly, we use the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q^1)} = \frac{(q)_n}{(q)_k (q)_{n-k}}. \tag{2.1}$$

Just as with ordinary binomial coefficients, q-binomial coefficients have multiple combinatorial interpretations and satisfy many identities, a few of which are highlighted below.

Interpretation 1 (*[27, Proposition 1.3.17]*). *Let M be the multiset $M = \{1^k, 2^{n-k}\}$. For an ordering π of the n elements of M, the number of inversions of π, denoted* $\operatorname{inv}(\pi)$, *is the number of instances of two entries i and j such that $i < j$ and $\pi(i) > \pi(j)$. Then* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_\pi q^{\operatorname{inv}(\pi)}$.

Interpretation 2 (*[27, Proposition 1.3.19]*). *Let Λ be the set of partitions λ whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle. Then $\left[{n \atop k} \right]_q = \sum_{\lambda \in \Lambda} q^{|\lambda|}$, where $|\lambda|$ denotes the number of boxes in the diagram of λ.*

Interpretation 3. *Let $\left[{n \atop k} \right]$ be the set of subsets of $[n] = \{1, 2, \ldots, n\}$ of size k. Given $\mathcal{R} = \{r_1, \ldots, r_k\} \in \left[{n \atop k} \right]$, define $|\mathcal{R}| = \sum_{j=1}^{k} (r_j - j)$. Then $\left[{n \atop k} \right]_q = \sum_{\mathcal{R} \in \left[{n \atop k} \right]} q^{|\mathcal{R}|}$.*

Proof. There is a standard bijection between the diagram of a partition λ drawn in English notation inside an $(n-k) \times k$ rectangle and lattice paths of length n consisting of down and left steps that contain k left steps. This bijection is given by tracing the lattice path formed by the boundary of the partition λ from the upper right to the lower left corners of the bounding rectangle. We can obtain another bijection to subsets $\mathcal{R} = \{r_1, \ldots, r_k\} \in \left[{n \atop k} \right]$ by recording the index $r_j \in \{1, 2, \ldots, n\}$ of the horizontal steps of the path in $\mathcal{R}$ for each $j = 1, \ldots, k$. Then, the number of boxes of λ that are added in the column above each horizontal step is precisely $(r_j - j)$. Hence, Interpretation 3 follows from transposing the Ferrers diagrams in Interpretation 2. □

Interpretation 1 is used most frequently in this article. Interpretation 3 is used in Section 3.1 when counting long fully commutative elements.

The following identities follow directly from Eq. (2.1).

Identity 2.2. $\left[{n \atop k} \right]_q = \left[{n \atop n-k} \right]_q$.

Identity 2.3. $(1 - q^{n-k}) \left[{n \atop k} \right]_q = (1 - q^n) \left[{n-1 \atop k} \right]_q$.

3. Decomposition and enumeration of fully commutative elements

Let S_n^{FC} denote the set of fully commutative permutations in S_n. In the following result, Barcucci et al. enumerate these elements by Coxeter length.

Theorem 3.1 (*[3]*). *Let $C(x, q) = \sum_{n \geq 0} \sum_{w \in S_n^{FC}} x^n q^{\ell(w)}$. Then,*

$$C(x, q) = \frac{\sum_{n \geq 0} (-1)^n x^{n+1} q^{(n(n+3))/2} / (x, q)_{n+1} (q, q)_n}{\sum_{n \geq 0} (-1)^n x^n q^{(n(n+1))/2} / (x, q)_n (q, q)_n}.$$

This formula is a ratio of q-Bessel functions as described in [2]. It arises as the solution obtained by [8] of a recurrence relation given on the generating function. We will encounter such a recurrence in the proof of Lemma 3.12.

We enumerate the fully commutative elements $\widetilde{w} \in \widetilde{S}_n$ by identifying each as the product of its minimal length coset representative $w^0 \in \widetilde{S}_n / S_n$ and a finite permutation $w \in S_n$ as described in Section 2.2. Recall that we decompose the set of fully commutative elements into long and short elements. The elements with a short abacus structure break down into those where certain entries intertwine and those in which there is no intertwining. When we assemble these cases, we obtain our main theorem.

Theorem 3.2. *Let $\widetilde{S}_n^{FC}$ denote the set of fully commutative affine permutations in $\widetilde{S}_n$, and let $G(x, q) = \sum_{n \geq 0} \sum_{w \in \widetilde{S}_n^{FC}} x^n q^{\ell(w)}$, where $\ell(w)$ denotes the Coxeter length of w. Then,*

$$G(x, q) = \left(\sum_{n \geq 0} \frac{x^n q^n}{1 - q^n} \sum_{k=1}^{n-1} \left[{n \atop k} \right]_q^2 \right) + C(x, q) + \left(\sum_{R, L \geq 1} q^{R+L-1} \left[{L + R - 2 \atop L - 1} \right]_q S(x, q) \right),$$

where $C(x, q)$ is given by Theorem 3.1, and the component parts of $S(x, q) = S_I(x, q) + S_0(x, q) + S_1(x, q) + S_2(x, q)$ are given in Lemmas 3.8–3.10 and 3.12, respectively.

The first summand of $G(x, q)$ counts the long elements, while the remaining summands count the short elements. This theorem will be proved in Section 3.3.

3.1. Long elements

In this section, we enumerate the long elements. Recall that the last bead in the normalized abacus for these elements occurs in position $> 2n$.

Lemma 3.3. *For fixed $n \geq 0$, we have*

$$\sum_{w \in \widetilde{S}_n^{FC} \text{ such that } w \text{ is long}} q^{\ell(w)} = \frac{q^n}{1-q^n} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 .$$

Proof. Fix a long fully commutative element, and define the set of *long runners* $\mathcal{R}$ of its normalized abacus A to be the set of runners $\{r_1, \ldots, r_k\} \subset [n] \setminus \{1\}$ in which there exists a bead in position $n + r_j$ for $1 \leq j \leq k$. We will enumerate the long fully commutative elements by conditioning on $k = |\mathcal{R}|$, the size of the set of long runners of its normalized abacus. Note that by Proposition 2.4, all subsets $\mathcal{R} \subset [n] \setminus \{1\}$ are indeed the set of long runners for some fully commutative element.

For a fixed $\mathcal{R}$, there is an infinite family of abaci $\{A_i^{\mathcal{R}}\}_{i \geq 1}$, each having beads in positions $n + r_j$ for $r_j \in \mathcal{R}$, together with i additional beads that are placed sequentially in the long runners in positions larger than $2n$. See Fig. 1 for an example.

By Proposition 2.1, the Coxeter length of the minimal length coset representative $w^0 \in \widetilde{S}_n / S_n$ having $A_i^{\mathcal{R}}$ as its abacus is $i(n-k) + \sum_{j=1}^{k} (r_j - j)$. In addition, w^0 has base window $[aa \cdots abb \cdots b]$, where the $(n-k)$ numbers a are all at most n, and the k numbers b are all at least $n+2$. The finite permutations w that can be applied to this standard window may not invert any of the larger numbers (b's) without creating a [321]-pattern with $n+1$ in the window following the standard window. Similarly, none of the a's can be inverted. All that remains is to intersperse the a's and the b's, keeping track of how many transpositions are used. This contributes exactly $\begin{bmatrix} n \\ k \end{bmatrix}_q$ to the Coxeter length, by Interpretation 1 of $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Therefore, the generating function for the long fully commutative elements of $\widetilde{S}_n$ by Coxeter length is

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{i \geq 1} \sum_{\substack{\mathcal{R} \subset [n] \setminus \{1\} \\ |\mathcal{R}| = k}} q^{i(n-k) + \sum_{j=1}^{k} (r_j - j)} .$$

Taking the sum over i and incorporating a factor of q^{-k} into the summation in the exponent of q yields

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^n}{1-q^{n-k}} \sum_{\substack{\mathcal{R} \subset [n] \setminus \{1\} \\ |\mathcal{R}| = k}} q^{\sum_{j=1}^{k} (r_j - 1 - j)} .$$

Reindexing the entries r_j to be from 1 to $n-1$ instead of from 2 to n gives

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^n}{1-q^{n-k}} \sum_{\substack{\mathcal{R} \subset [n-1] \\ |\mathcal{R}| = k}} q^{\sum_{j=1}^{k} (r_j - j)} ,$$

which simplifies by Interpretation 3 of the q-binomial coefficients to

$$\sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{q^n}{1-q^{n-k}} .$$

Applying Identity 2.3 proves the desired result. $\square$

3.2. Short elements

The normalized abacus of every short fully commutative element has a particular structure. There must be a gap in position $n+1$, and for runners $2 \le i \le n$, the lowest bead is either in position i or $n+i$. In the following arguments, we will assign a status to each runner, depending on the position of the lowest bead in that runner.

Definition 3.4. An R-*entry* is a bead lying in some position $> n$. Let $n+j$ be the position of the last R-entry, or set $j = n$ if there are no R-entries; an M-*entry* is a lowest bead lying in position i where $j+1 \le i \le n$. Note that it is possible that there do not exist any M-entries. The L-*entries* are the remaining lowest beads in position i for $i \le j$. This assigns a status *left, middle*, or *right* to each entry of the base window, depending on the position of the lowest bead in the corresponding runner. We will call an abacus containing L L-entries, M M-entries, and R R-entries an $(L)(M)(R)$ *abacus*.

Example 3.5. Fig. 2 shows the three (3)(1)(2) abaci. In each case, 6 is the unique M-entry and 11 is an R-entry. In the first abacus, the L-entries are {1, 3, 4} and the R-entries are {8, 11}.

The rationale for this assignment is that in the base window of a fully commutative element, not of type $(n)(0)(0)$, neither the L-entries nor the R-entries can have a descent amongst themselves, respectively. To see this, consider the contrary where two R-entries have a descent. These two entries, along with the $n+1$ entry in the window following the standard window, form a [321]-instance. Similarly, the last R-entry in the window previous to the standard window together with two L-entries that have a descent in the standard window would form a [321]-instance.

When the normalized abacus of a short fully commutative element has no R-entries (and therefore no M-entries), the base window for its minimal length coset representative is $[12\cdots n]$. That is, the fully commutative elements of $\widetilde{S}_n$ having this abacus are in one-to-one correspondence with fully commutative elements of finite S_n. These elements have been enumerated in Theorem 3.1.

From now on, we only concern ourselves with $(L)(M)(R)$ abaci where $R > 0$. Proposition 3.6 proves that it is solely the parameters L, M, and R that determine the set of finite permutations that we can apply to the minimal length coset representative, and not the exact abacus. In Proposition 3.7 we determine the cumulative contribution to the Coxeter length of the minimal length coset representative from all $(L)(M)(R)$ abaci for fixed L, M, and R.

Proposition 3.6. *Let $w_1^0, w_2^0 \in \widetilde{S}_n/S_n$, each corresponding to an $(L)(M)(R)$ abacus for the same L, M, and R with $R > 0$. For any finite permutation $w \in S_n$, $w_1^0 w$ is fully commutative in $\widetilde{S}_n$ if and only if $w_2^0 w$ is fully commutative in $\widetilde{S}_n$.*

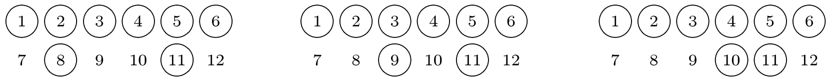
Proof. For $v \in \widetilde{S}_n$ and $i \in \mathbb{Z}$, we will say that $v(i)$ has the same left, middle, or right status as the entry $v(i \bmod n)$ of the base window, as in Definition 3.4. Observe that $(w_1^0 w)(i)$ has the same left, middle, or right status as $(w_2^0 w)(i)$ for all $1 \le i \le n$, and that the relative order of these entries is the same for $w_1^0 w$ as for $w_2^0 w$.

Next, suppose that $w_1^0 w$ has a [321]-instance with two inverted L-entries or two inverted R-entries. By construction, these entries must occur in the same window j. Then any R-entry from window $j-1$ yields a [321]-instance in $w_2^0 w$, and such an entry exists since we are assuming that $R > 0$. Similarly, if the [321]-instance has two inverted R-entries occurring in window j, then any L-entry from window $j+1$ yields a [321]-instance in $w_2^0 w$, and such an entry exists since 1 is always an L-entry in the base window. Hence, $w_2^0 w$ is not fully commutative by Theorem 2.3.

Next, suppose that $w_1^0 w$ has a [321]-instance that includes two M-entries, at least one of which lies in window j. Observe that every M-entry in window j is larger than every entry in window $j-1$, and smaller than every entry in window $j+1$. Therefore, if the [321]-instance involves two M-entries then the entire [321]-instance must occur within window j, which implies that w is not fully commutative. Thus, $w_2^0 w$ is not fully commutative.

Finally, if $w_1^0 w$ has a [321]-instance that includes one R-entry, one M-entry, and one L-entry, then all three of these entries must lie in the same window. Hence, neither w nor $w_2^0 w$ are fully commutative.

Thus, we have shown that the result holds in all cases. $\square$



$\ell([1,3,4,6,8,11]) = 4$ $\quad\ell([1,2,4,6,9,11]) = 5$ $\quad\ell([1,2,3,6,10,11]) = 6$

Fig. 2. The three (3)(1)(2) abaci and Coxeter length of their corresponding minimal length coset representatives.

Proposition 3.7. *Let L, M and $R > 0$ be fixed. Then, we have*

$$\sum_{w} q^{\ell(w)} = q^{L+R-1} \begin{bmatrix} L+R-2 \\ L-1 \end{bmatrix}_q$$

where the sum on the left is over all minimal length coset representatives w having an $(L)(M)(R)$ abacus.

Proof. Every $(L)(M)(R)$ abacus contains beads in all positions through n and in position $2n-M$ as well as gaps in position $n+1$ and all positions starting with $2n-M+1$. Depending on the positions of the $L-1$ remaining gaps (and $R-1$ remaining beads), the Coxeter length of the minimal length coset representative changes as illustrated by example in Fig. 2.

The minimal length coset representative corresponding to an $(L)(M)(R)$ abacus having beads in positions i for $n+2 \le i \le n+R$ together with a bead at position $2n-M$, and gaps in positions i for $n+R+1 \le i \le 2n-M-1$ has Coxeter length $L+R-1$. Notice that every time we move a bead from one of the positions between $n+2$ and $2n-M-1$ into a gap in the position directly to its right, the Coxeter length increases by exactly one. In essence, we are intertwining one sequence of length $L-1$ and one sequence of length $R-1$ and keeping track of the number of inversions we apply. By q-binomial Interpretation 1, the contribution to the Coxeter length of the minimal length coset representatives corresponding to the $(L)(M)(R)$ abaci is $q^{L+R-1}\begin{bmatrix} L+R-2 \\ L-1 \end{bmatrix}_q$. □

For the remaining arguments, we ignore the exact entries in the base window and simply fix both some positive number L of L-entries and some positive number R of R-entries, and then enumerate the permutations $w \in S_n$ that we can apply to a minimal length coset representative w^0 with base window of the form [L · · · LM · · · MR · · · R]. In Theorem 3.2, we sum the contributions over all possible values of L and R.

3.2.1. Short elements with intertwining

One possibility is that after $w \in S_n$ is applied to our minimal length coset representative w^0 with base window of the form [L · · · LM · · · MR · · · R], an R-entry lies to the left of an L-entry. In this case, we say that w is *intertwining*, the L-entries are *intertwining* with the R-entries, and that the interval between the leftmost R and the rightmost L inclusive is the *intertwining zone*.

Lemma 3.8. *Fix L and $R > 0$. Then, we have*

$$S_I(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = \sum_{M\ge 0} x^{L+M+R} \sum_{\rho=0}^{R-1}\sum_{\lambda=0}^{L-1}\sum_{\mu=0}^{M} q^{Q} \begin{bmatrix} M \\ \mu \end{bmatrix}_q \times \begin{bmatrix} L-\lambda-1+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} \lambda+\rho \\ \lambda \end{bmatrix}_q \begin{bmatrix} M-\mu+R-\rho-1 \\ M-\mu \end{bmatrix}_q,$$

where the sum on the left is over all $w \in S_\infty$ that are intertwining and apply to a short $(L)(M)(R)$ abacus for some M, and $Q = (\lambda+1)(\mu+1) + (\rho+1)(M-\mu+1) - 1$.

Proof. By Theorem 2.3, there are no M-entries between the leftmost R and the rightmost L, because this would create a [321]-pattern. So the M-entries only occur before the leftmost R and after the rightmost L.

$$[\overbrace{\mathrm{LM}\cdots\mathrm{LM}}^{\text{L's and M's}}\quad\overbrace{\underline{\mathrm{R}}\mathrm{L}\cdots\mathrm{R}\underline{\mathrm{L}}}^{\text{intertwining zone}}\quad\overbrace{\mathrm{MR}\cdots\mathrm{MR}}^{\text{R's and M's}}]$$

Fig. 3. The structure of the base window of an intertwined short fully commutative element. The leftmost R and the rightmost L are underlined.

$$[\overbrace{\mathrm{LM}\cdots\mathrm{M}\underline{\mathrm{L}}}^{\text{L's and M's}}\quad\overbrace{\mathrm{MM}\cdots\mathrm{MM}}^{\text{M descents may occur}}\quad\overbrace{\underline{\mathrm{R}}\mathrm{M}\cdots\mathrm{MR}}^{\text{R's and M's}}]$$

Fig. 4. The structure of the base window of a non-intertwined short fully commutative element. The rightmost L and the leftmost R are underlined.

Notice that any descent in the M-entries before the leftmost R would create a [321]-pattern when coupled with the rightmost L. Similarly, any descent in the M-entries after the rightmost L would create a [321]-pattern when coupled with the leftmost R. Therefore, the M-entries are allowed to have at most one descent, which must occur between the M-entries on either side of the intertwining zone.

From this, we know that the structure of w^0w is as follows. Some number $\lambda + 1$ of L-entries are intertwining with some number $\rho + 1$ of R-entries in the intertwining zone. The remaining L-entries are intertwining with some number μ of M-entries on the left side of the intertwining zone, and the remaining R-entries are intertwining with $(M - \mu)$ M-entries on the right side of the intertwining zone. This structure is illustrated in Fig. 3.

The contribution to the Coxeter length generating function from splitting the M-entries into two sets of size μ and $M - \mu$, and transposing as necessary in order to place the first set on the left and the right set on the right is $\begin{bmatrix} M \\ \mu \end{bmatrix}_q$ by Interpretation 1.

Once these entries have been ordered in the minimal length configuration that conforms to the structure shown in Fig. 3, we compute the Coxeter length offset Q by counting the remaining inversions among the entries in the base window. We have μ M-entries inverted with $(\lambda+1)$L-entries, and $(\rho+1)$ R-entries inverted with $(M-\mu)$ M-entries. In addition, the leftmost R is inverted with λ L-entries not including the rightmost L, and the rightmost L is inverted with ρ R-entries not including the leftmost R. Finally, the leftmost R is inverted with the rightmost L. These inversions contribute $Q = \mu(\lambda + 1) + (M - \mu)(\rho + 1) + \lambda + \rho + 1$ to the Coxeter length.

Lastly, we can intertwine the $(L - \lambda - 1)$ L-entries and μ M-entries to the left of the zone, the λ L-entries and ρ R-entries in the zone, and the $(M - \mu)$ M-entries with the $(R - \rho - 1)$ R-entries to the right of the zone. This proves the formula. □

3.2.2. Short elements without intertwining

If the L-entries and R-entries are not intertwined, there may be M-entries lying between the rightmost L and the leftmost R. There can be no descents in the M-entries to the left of the rightmost L nor to the right of the leftmost R by the same reasoning as above. However, multiple descents may now occur among the M-entries. This structure is illustrated in Fig. 4. We enumerate these short elements without intertwining by conditioning on the number of descents that occur among the M-entries. Lemmas 3.9, 3.10 and 3.12 enumerate the short elements in which there are zero, one, or two or more descents among the M-entries, respectively.

Lemma 3.9. *Fix L and R > 0. Then, we have*

$$S_0(x, q) = \sum_w x^{n(w)} q^{\ell(w)} = \sum_{M \geq 0} x^{L+M+R} \sum_{\mu=0}^{M} q^{\mu} \begin{bmatrix} L - 1 + \mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R + M - \mu \\ M - \mu \end{bmatrix}_q,$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have no descents among the M-entries, and apply to a short $(L)(M)(R)$ abacus for some M.

Proof. Let μ be the number of M-entries lying to the left of the rightmost L. Then, the μ M-entries can be intertwined with the remaining $(L - 1)$ L-entries, and the remaining $(M - \mu)$ M-entries can be intertwined with the R R-entries.

We compute the Coxeter length offset by counting the inversions among the entries in the base window in the minimal length configuration of this type. In this case, there are simply μ M-entries that are inverted with the rightmost L. Summing over all valid values of μ gives the formula. □

Lemma 3.10. *Fix L and $R > 0$. Then, we have*

$$S_1(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = \sum_{M \geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left(\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q,$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have exactly one descent among the M-entries, and apply to a short $(L)(M)(R)$ abacus for some M.

Proof. Consider such permutations having a descent at the μth M-entry. The choices for the M-entries that are not the identity permutation are enumerated by $\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1$ by Interpretation 1. Then, the M-entries to the left of the descent can be intertwined with the L-entries, and the M-entries to the right of the descent can be intertwined with the R-entries. Summing over all valid values of μ gives the formula. □

To prepare for the proof of our next result, we recall the following lemma which solves certain generating function recurrences.

Lemma 3.11 ([8, Lemma 2.3])**.** *Let $\mathcal{A}$ be the sub-algebra of the formal power series algebra $\mathbb{R}[[s,t,x,y,q]]$ formed with series S such that $S(1,t,x,y,q)$ and $S'(1,t,x,y,q)$ are well defined in $\mathbb{R}[[t,x,y,q]]$. Moreover, we abbreviate $f(s,t,x,y,q) \in \mathcal{A}$ by $f(s)$. Let $X(s,t,x,y,q)$ be a formal power series in $\mathcal{A}$. Suppose that*

$$X(s) = xe(s) + xf(s)X(1) + xg(s)X(sq)$$

where e, f, and g are in $\mathcal{A}$. Then, $X(s,t,x,y,q)$ is equal to

$$\frac{E(s) + E(1)F(s) - E(s)F(1)}{1 - F(1)}$$

where

$$E(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\cdots g(sq^{n-1})e(sq^n) \quad \text{and}$$
$$F(s) = \sum_{n \geq 0} x^{n+1} g(s)g(sq)\cdots g(sq^{n-1})f(sq^n).$$

We are now in a position to enumerate the remaining elements.

Lemma 3.12. *Fix L and $R > 0$. Then, we have*

$$S_2(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = x^{L+R} \sum_{i,j \geq 1} \begin{bmatrix} L+i \\ L \end{bmatrix}_q \begin{bmatrix} R+j \\ R \end{bmatrix}_q d_{i,j}(x,q),$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have at least two descents among the M-entries, and apply to a short $(L)(M)(R)$ abacus for some M. Here, $d_{i,j}(x,q)$ is the coefficient of $z^i s^j$ in the generating function that satisfies the functional equation

$$D(x,q,z,s) = \frac{\sum_{n \geq 0} x^{n+1} \sum_{i=1}^{n-1} \left(\begin{bmatrix} n \\ i \end{bmatrix}_q - 1 \right) z^i \big((qs) - (qs)^{n-i} \big)}{(1-qs)(1-xs)} + \frac{xqs\big(D(x,q,z,1) - D(x,q,z,qs)\big)}{(1-qs)(1-xs)}$$

and whose solution is given explicitly below.

Proof. In this proof, we use the ideas of Barcucci et al. [3], Bousquet-Mélou [8], and West [33], to investigate the structure of the permutations restricted to the M's in the base window.

For such a finite permutation $w \in S_M$, we consider the following statistics:

- $n(w)$ is the size of the element (represented by variable x),
- $\ell(w)$ is the number of inversions (represented by variable q),
- $i(w)$ is the number of entries to the left of the leftmost descent (represented by variable z), and
- $j(w)$ is the number of entries to the right of the rightmost descent (represented by variable s).

Let

$$D(x, q, z, s) = \sum_{w} x^{n(w)} q^{\ell(w)} z^{i(w)} s^{j(w)}$$

where we sum over all fully commutative permutations with at least two descents.

We require the auxiliary function $N(x, q, z, s) = \sum_w x^{n(w)} q^{\ell(w)} z^{i(w)} s^{j(w)}$ where we sum over all fully commutative permutations $w \in S_n$ with at least two descents such that removing the largest entry from the one-line notation of w results in a permutation that has only one descent. Then, the permutations counted by $N(x, q, z, s)$ are generated from fully commutative permutations w' with exactly one descent by inserting the entry $n(w')+1$ into the one-line notation of w' at some position to the right of the existing descent in order to avoid creating a [321]-instance, and this creates the second descent. If we fix the existing descent to occur at entry i, then the fully commutative permutations with exactly one descent contribute $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q - 1$ to $N(x, q, z, s)$. Let k denote the number of entries of w to the right of the position where we insert entry $n+1$. As k runs from 1 to $n-i-1$, we have that n increases by 1, the Coxeter length l increases by k, there are i entries to the left of the leftmost descent, and k entries to the right of the rightmost descent. Therefore,

$$\begin{aligned} N(x, q, z, s) &= \sum_{n\geq 0} \sum_{i=1}^{n-1} x^{n+1} \left(\begin{bmatrix} n \\ i \end{bmatrix}_q - 1 \right) z^i \sum_{k=1}^{n-i-1} q^k s^k \\ &= \sum_{n\geq 0} \sum_{i=1}^{n-1} x^{n+1} \left(\begin{bmatrix} n \\ i \end{bmatrix}_q - 1 \right) z^i \frac{(qs) - (qs)^{n-i}}{1 - qs}. \end{aligned}$$

We remark that x divides $N(x, q, z, s)$.

Next, we have

$$D(x, q, z, s) = N(x, q, z, s) + \sum_{w} \left(\sum_{k=1}^{j(w)} \left(x^{n(w)+1} q^{\ell(w)+k} z^{i(w)} s^k \right) + x^{n(w)+1} q^{\ell(w)} z^{i(w)} s^{j(w)+1} \right)$$

where the leftmost sum is over all fully commutative permutations with at least two descents. This sum counts such permutations that are obtained by inserting entry $n(w)+1$ into the one-line notation of an existing fully commutative permutation w having at least two descents, and such that $n(w)+1$ is inserted into a position to the right of the rightmost descent. The rightmost term corresponds to inserting into the rightmost position in the one-line notation, while the sum from $k = 1$ to $j(w)$ corresponds to inserting into the remaining positions in the one-line notation from right to left. This formula expresses a recursive construction of the permutations we are counting, known as the generating tree.

Hence,

$$D(x, q, z, s) = N(x, q, z, s) + \frac{xqs}{1 - qs} \big(D(x, q, z, 1) - D(x, q, z, qs) \big) + xsD(x, q, z, s),$$

and therefore,

$$D(x, q, z, s) = \frac{N(x, q, z, s)}{1 - xs} + \frac{xqs}{(1 - qs)(1 - xs)} D(x, q, z, 1) + \frac{-xqs}{(1 - qs)(1 - xs)} D(x, q, z, qs).$$

This functional equation has exactly the same form as those discussed in [8]; applying Lemma 3.11 proves that

$$D(x,q,z,s)=\frac{E(x,q,z,s)+E(x,q,z,1)F(x,q,z,s)-E(x,q,z,s)F(x,q,z,1)}{1-F(x,q,z,1)},$$

where

$$E(x,q,z,s)=\sum_{n\geq 0}x^{n+1}\frac{-qs}{(1-qs)(1-xs)}\cdots\frac{-q^n s}{(1-q^n s)(1-xq^{n-1}s)}\frac{N(x,q,z,sq^n)/x}{1-xq^n s}$$

and

$$F(x,q,z,s)=\sum_{n\geq 0}x^{n+1}\frac{-qs}{(1-qs)(1-xs)}\cdots\frac{-q^n s}{(1-q^n s)(1-xq^{n-1}s)}\frac{q^{n+1}s}{(1-q^{n+1}s)(1-xq^n s)}.$$

Condensing these formulas,

$$E(x,q,z,s)=\sum_{n\geq 0}\frac{(-1)^n(sx)^n q^{\binom{n+1}{2}}}{(qs,q)_n(xs,q)_{n+1}}N(x,q,z,sq^n)\quad\text{and}$$

$$F(x,q,z,s)=\sum_{n\geq 0}\frac{(-1)^n(sx)^{n+1} q^{\binom{n+2}{2}}}{(qs,q)_{n+1}(xs,q)_{n+1}}.$$

The coefficient $d_{i,j}(x,q)$ of $z^i s^j$ in $D(x,q,z,s)$ enumerates the permutations applied to the M's in the base window of all sizes and lengths such that there are at least two descents and the leftmost descent is after i entries and the rightmost descent is before j entries. Intertwining the L-entries with the first i M-entries and intertwining the R-entries with the last j M-entries gives the desired result. □

3.3. Proof of the main theorem

In this section, we complete the proof of our main result.

Proof of Theorem 3.2. Partition the set of fully commutative elements $\widetilde{w}$ into long elements and short elements. The long elements in $\widetilde{S}_n$ are enumerated by Lemma 3.3; we must sum over all n.

Each short element $\widetilde{w}$ has a normalized abacus of type $(L)(M)(R)$ for some L, M, and R. When this abacus is of type $(n)(0)(0)$ for some n, the base window for the corresponding minimal length coset representative is $[1\,2\,\ldots\,n]$. These elements $\widetilde{w}\in\widetilde{S}_n^{FC}$ are therefore in one-to-one correspondence with elements of S_n^{FC}. Therefore, the generating function $C(x,q)$ enumerates these elements for all n.

The elements that remain to be enumerated are short elements with normalized abacus of type $(L)(M)(R)$ for $R>0$. We enumerate these elements by grouping these elements into families based on the values of L, M, and R. Decompose each element $\widetilde{w}$ into the product of its minimal length coset representative w^0 and a finite permutation w. Proposition 3.6 proves that for two minimal length coset representatives w_1^0 and w_2^0 of the same abacus type, the set of finite permutations w that multiply to form a fully commutative element is the same. Proposition 3.7 proves that in an $(L)(M)(R)$-family of fully commutative elements, the contribution to the length from the minimal length coset representatives is $q^{L+R-1}\begin{bmatrix}L+R-2\\L-1\end{bmatrix}_q$. What remains to be determined is the generating function for the contributions of the finite permutations w.

In an $(L)(M)(R)$-family of fully commutative elements, the finite permutations w might intermingle the L entries and the R entries of the base window in which case there is at most one descent among the M entries at a prescribed position; the contribution of such w is given by S_I in Lemma 3.8. Otherwise, there is no intermingling and the finite permutations w may induce zero, one, or two or more descents among the M entries; these cases are enumerated by generating functions S_0, S_1, and S_2 in Lemmas 3.9, 3.10 and 3.12, respectively. In each of these lemmas, the values for L and R are held constant as M varies. Summing the product of the contributions of the minimal length coset representatives and the finite permutations over all possible values of L and R completes the enumeration. □

4. Numerical conclusions

Theorem 3.2 allows us to determine the length generating function $f_n(q)$ for the fully commutative elements of $\widetilde{S}_n$ as n varies. The first few series $f_n(q)$ are presented below.

$$f_3(q) = 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots$$

$$f_4(q) = 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots$$

$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots$$

$$\begin{aligned} f_6(q) &= 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} \\ &\quad + \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \cdots \end{aligned}$$

$$\begin{aligned} f_7(q) &= 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 \\ &\quad + \mathbf{490q^{10}} + \mathbf{490q^{11}} + \mathbf{490q^{12}} + \mathbf{490q^{13}} + \mathbf{490q^{14}} + \mathbf{490q^{15}} + \cdots \end{aligned}$$

$$\begin{aligned} f_8(q) &= 1 + 8q + 36q^2 + 112q^3 + 266q^4 + 504q^5 + 792q^6 + 1064q^7 + 1274q^8 + 1416q^9 \\ &\quad + 1520q^{10} + 1568q^{11} + 1602q^{12} + \mathbf{1600q^{13}} + \mathbf{1616q^{14}} + \mathbf{1600q^{15}} + \mathbf{1618q^{16}} \\ &\quad + \mathbf{1600q^{17}} + \mathbf{1616q^{18}} + \mathbf{1600q^{19}} + \mathbf{1618q^{20}} + \cdots \end{aligned}$$

$$\begin{aligned} f_9(q) &= 1 + 9q + 45q^2 + 156q^3 + 414q^4 + 882q^5 + 1563q^6 + 2367q^7 + 3159q^8 + 3831q^9 \\ &\quad + 4365q^{10} + 4770q^{11} + 5046q^{12} + 5220q^{13} + 5319q^{14} + 5370q^{15} + 5391q^{16} \\ &\quad + \mathbf{5400q^{17}} + \mathbf{5406q^{18}} + \mathbf{5400q^{19}} + \mathbf{5400q^{20}} + \mathbf{5406q^{21}} + \mathbf{5400q^{22}} + \mathbf{5400q^{23}} + \cdots \end{aligned}$$

$$\begin{aligned} f_{10}(q) &= 1 + 10q + 55q^2 + 210q^3 + 615q^4 + 1452q^5 + 2860q^6 + 4820q^7 + 7125q^8 \\ &\quad + 9470q^9 + 11622q^{10} + 13470q^{11} + 15000q^{12} + 16160q^{13} + 17030q^{14} \\ &\quad + 17602q^{15} + 18010q^{16} + 18210q^{17} + 18380q^{18} + 18410q^{19} + 18482q^{20} \\ &\quad + \mathbf{18450q^{21}} + \mathbf{18500q^{22}} + \mathbf{18450q^{23}} + \mathbf{18500q^{24}} + \mathbf{18452q^{25}} + \mathbf{18500q^{26}} \\ &\quad + \mathbf{18450q^{27}} + \mathbf{18500q^{28}} + \mathbf{18450q^{29}} + \mathbf{18502q^{30}} + \mathbf{18450q^{31}} + \mathbf{18500q^{32}} \\ &\quad + \mathbf{18450q^{33}} + \mathbf{18500q^{34}} + \mathbf{18452q^{35}} + \cdots \end{aligned}$$

$$\begin{aligned} f_{11}(q) &= 1 + 11q + 66q^2 + 275q^3 + 880q^4 + 2277q^5 + 4928q^6 + 9141q^7 + 14850q^8 \\ &\quad + 21571q^9 + 28633q^{10} + 35453q^{11} + 41690q^{12} + 47135q^{13} + 51667q^{14} \\ &\quad + 55297q^{15} + 58091q^{16} + 60159q^{17} + 61622q^{18} + 62623q^{19} + 63272q^{20} \\ &\quad + 63668q^{21} + 63910q^{22} + 64031q^{23} + 64086q^{24} + 64119q^{25} + \mathbf{64130q^{26}} \\ &\quad + \mathbf{64130q^{27}} + \mathbf{64130q^{28}} + \mathbf{64130q^{29}} + \cdots \end{aligned}$$

$$\begin{aligned} f_{12}(q) &= 1 + 12q + 78q^2 + 352q^3 + 1221q^4 + 3432q^5 + 8086q^6 + 16356q^7 + 28974q^8 \\ &\quad + 45772q^9 + 65670q^{10} + 87120q^{11} + 108690q^{12} + 129288q^{13} + 148170q^{14} \\ &\quad + 164776q^{15} + 178980q^{16} + 190680q^{17} + 200148q^{18} + 207444q^{19} + 213084q^{20} \\ &\quad + 217096q^{21} + 220098q^{22} + 222012q^{23} + 223458q^{24} + 224172q^{25} + 224814q^{26} \\ &\quad + 224992q^{27} + 225276q^{28} + 225216q^{29} + 225408q^{30} + \mathbf{225264q^{31}} + \mathbf{225420q^{32}} \\ &\quad + \mathbf{225280q^{33}} + \mathbf{225414q}^{34} + \mathbf{225264q}^{35} + \mathbf{225438q^{36}} + \mathbf{225264q^{37}} + \mathbf{225414q}^{38} \\ &\quad + \mathbf{225280q^{39}} + \mathbf{225420q^{40}} + \mathbf{225264q^{41}} + \mathbf{225432q^{42}} + \mathbf{225264q^{43}} + \mathbf{225420q}^{44} \\ &\quad + \mathbf{225280q}^{45} + \mathbf{225414q^{46}} + \mathbf{225264q^{47}} + \mathbf{225438q^{48}} \\ &\quad + \mathbf{225264q^{49}} + \mathbf{225414q}^{50} + \cdots. \end{aligned}$$

One remarkable quality of these series is their periodicity, given by the bold-faced terms. This behavior is explained by the following corollary to Lemma 3.3.

Corollary 4.1. *The coefficients a_i of $f_n(q) = \sum_{w \in \widetilde{S}_n^{FC}} q^{\ell(w)} = \sum_{i \geq 0} a_i q^i$ are periodic with period $m|n$ for sufficiently large i. When $n = p$ is prime, the period $m = 1$ and in this case there are precisely*

$$\frac{1}{p}\left(\binom{2p}{p} - 2\right)$$

fully commutative elements of length i in $\widetilde{S}_p$, when i is sufficiently large.

Proof. For a given n, the number of short fully commutative elements is finite. The formula for long elements in Lemma 3.3 is a polynomial divided by $1 - q^n$. Hence, the coefficients of this generating function satisfy $a_{i+n} = a_i$, by a fundamental result on rational generating functions.

We have a factor of $(1-q^n) = (1-q)(1+q+\cdots+q^{n-1})$ in the numerator of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and when n is prime, $(1+q+\cdots+q^{n-1})$ is irreducible. Therefore $\begin{bmatrix} n \\ k \end{bmatrix}_q$ contains a factor of $(1+q+\cdots+q^{n-1})$ for every k between 1 and $n-1$. Factoring one copy out of the sum in the expression of Lemma 3.3 and canceling with the same factor in the denominator of $\frac{q^n}{1-q^n}$ leaves a denominator of $(1-q)$.

Hence, we have that

$$P(q) := \frac{q^n}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q^2$$

is the polynomial numerator of the rational generating function $P(q)/(1-q)$ for the number of long fully commutative elements. Therefore, when i is larger than the degree of $P(q)$, the coefficient of q^i in the series expansion of $P(q)/(1-q)$ is $P(1)$. After substituting $q = 1$ and applying Vandermonde's identity,

$$P(1) = \frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}^2 = \frac{1}{n}\left(\sum_{k=0}^{n}\binom{n}{k}^2 - 2\right) = \frac{1}{n}\left(\binom{2n}{n} - 2\right),$$

as desired. □

The distinction between long and short elements allows us to enumerate the fully commutative elements efficiently. In some respects, this division is not the most natural in that the periodicity of the above series begins before there exist no more short elements. Experimentally, it appears that the periodicity begins at $1 + \lfloor (n-1)/2 \rfloor \lceil (n-1)/2 \rceil$; whereas, we can prove that the longest short element has length $2\lfloor n/2 \rfloor \lceil n/2 \rceil$.

We begin by bounding the Coxeter length of finite fully commutative permutations.

Definition 4.2. If w has a unique left descent and w has a unique right descent, then we say that w is *bi-Grassmannian.*

Lemma 4.3. *Suppose* w *is a reduced expression for* $w \in S_n^{FC}$. *Then there exists a bi-Grassmannian permutation x with reduced expression* $\mathsf{x} = \mathsf{uwv}$. *In particular,* $\ell(w) \le \lfloor n/2 \rfloor \lceil n/2 \rceil$.

Proof. Recall the coalesced heap diagram from [6, Section 3] associated to any fully commutative element w. This diagram is an embedding of the Hasse diagram of the heap poset defined in [28] into $\mathbb{Z}^2$. In this diagram, an entry of the heap poset represented by $(x, y) \in \mathbb{Z}^2$ is labeled by the Coxeter generator s_i if and only if $x = i$. Moreover, we have that a generator represented by (x, y) covers a generator represented by (x', y') in the heap poset if and only if $y = y' + 1$ and $x = x' \pm 1$. See [6, Remark 5] for details. An example of a heap diagram is shown in Fig. 5.

Next, we describe a sequence of length-increasing multiplications on the left and right that will transform w into a bi-Grassmannian permutation. An example of this construction is illustrated in Fig. 6. First, if there are any columns $1 \le i \le (n-1)$ in the heap diagram of w that do not contain an entry, then multiply on the right by s_i to add an entry to the heap diagram, and then recoalesce the heap diagram. Henceforth, we assume that every column in the heap diagram of w has at least one entry. Moreover, it follows from [6, Lemma 1] that columns 1 and $(n-1)$ of the heap diagram contain precisely one entry.

Next, consider the *ridgeline* in the heap diagram of w consisting of the points that correspond to maximal elements in the heap poset. By construction, the ridgeline can be interpreted as a lattice path

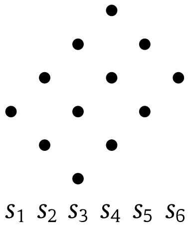


Fig. 5. The heap diagram of a bi-Grassmannian permutation $w = s_3s_2s_4s_1s_3s_5s_2s_4s_6s_3s_5s_4$.

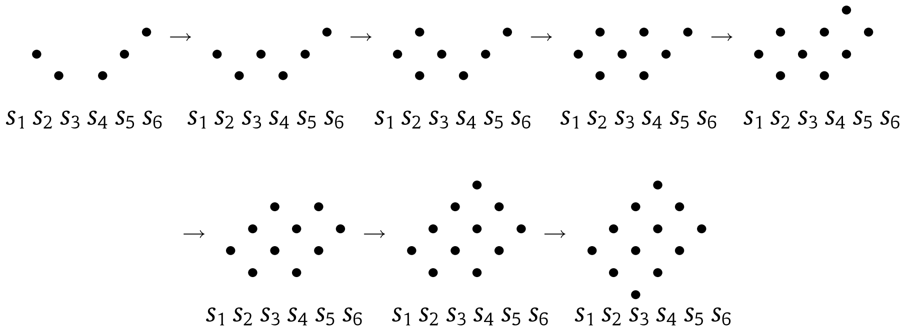


Fig. 6. The construction of a bi-Grassmannian permutation containing $w = s_2s_4s_1s_5s_6$.

consisting of *up-steps* of the form $\big((i, y), (i+1, y+1)\big)$ and *down-steps* of the form $\big((i, y), (i+1, y-1)\big)$. For each sequence of the form $\big((i, y), (i+1, y-1), (i+2, y)\big)$, we multiply on the right by an s_{i+1} generator to add a new entry to the ridgeline and transform the sequence from down–up to up–down. When we have performed these multiplications until there are no more down–up sequences along the ridgeline, our heap diagram encodes a fully commutative permutation with a unique right descent. In a completely similar fashion, we can also perform multiplications on the left to produce a heap which encodes a fully commutative permutation with a unique left descent. Hence, our transformed permutation is bi-Grassmannian.

When w is bi-Grassmannian, the heap of w forms a quadrilateral by [6, Lemma 1] as illustrated in Fig. 5. The Coxeter length of w is the number of lattice points in the quadrilateral, and this is maximized when the unique left and right descents occur as close to $n/2$ as possible. Hence, $\ell(w) \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. □

Proposition 4.4. *Let $w \in \widetilde{S}_n^{FC}$ be a short element. Then $\ell(w) \leq 2\lfloor n/2 \rfloor \lceil n/2 \rceil$. In addition, there exists a $w \in \widetilde{S}_n^{FC}$ with length $2\lfloor n/2 \rfloor \lceil n/2 \rceil$.*

Proof. Let $\widetilde{w} \in \widetilde{S}_n^{FC}$ be a short element. Then, by the parabolic decomposition, $\widetilde{w} = w^0 w$ where $w \in S_n^{FC}$ and w^0 is a minimal length coset representative with an associated $(L)(M)(R)$ abacus.

First, we determine the values of L, M, and R that give $w^0 \in \widetilde{S}_n/S_n$ of longest Coxeter length. For fixed L, M, and R, Proposition 2.1 implies that the longest Coxeter length of a minimal length coset representative having an $(L)(M)(R)$ abacus occurs when there are gaps in positions $n+1$ through $n+L$, beads in positions $n+L+1$ through $n+L+R$ and gaps in positions $n+L+R+1$ through $2n$. The length of this minimal length coset representative is LR, which is maximized when $M = 0$ and L and R are as close to $n/2$ as possible. Therefore, $\ell(w^0) \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$ and is exactly equal in the case where $w^0 = \big[1, 2, \ldots, \lfloor n/2 \rfloor, n + \lfloor n/2 \rfloor + 1, \ldots, 2n\big]$.

Considering the other factor $w \in S_n^{FC}$, it follows from Lemma 4.3 that $\ell(w) \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$. Moreover, this length is maximized when w is a bi-Grassmannian permutation with left and right descents occurring as close to $n/2$ as possible.

Adding the bounds to obtain $\ell(\widetilde{w}) = \ell(w) + \ell(w^0) \leq 2\lfloor n/2 \rfloor \lceil n/2 \rceil$ proves the result. In addition, the one-line notation for a bi-Grassmannian permutation has the form $[i+1, i+2, \dots, n, 1, 2, \dots, i]$ for some i. When $i = \lfloor n/2 \rfloor$, this bi-Grassmannian applies directly to the above affine permutation to give the fully commutative affine permutation $\left[n + \lfloor n/2 \rfloor + 1, \dots, 2n, 1, 2, \dots, \lfloor n/2 \rfloor\right]$ of length $2\lfloor n/2 \rfloor \lceil n/2 \rceil$. □

Corollary 4.1 and Proposition 4.4 give another way to compute the series $f_n(q)$, without invoking Theorem 3.2. Using a computer program, one needs simply to count the fully commutative elements of $\widetilde{S}_n$ of length up to $n + 2\lfloor n/2 \rfloor \lceil n/2 \rceil$.

5. Further questions

In this work, we have studied the length generating function for the fully commutative affine permutations. It would be interesting to explore the ramifications of the periodic structure of these elements in terms of the affine Temperley–Lieb algebra. Also, all of our work should have natural extensions to the other Coxeter groups. In fact, we know of no analogue of [3] enumerating the fully commutative elements by length for finite types beyond type A. It is a natural open problem to establish the periodicity of the length generating functions for the other affine types. It would also be interesting to determine the analogues for other types of the q-binomial coefficients and q-Bessel functions that played prominent roles in our enumerative formulas.

Finally, it remains an open problem to prove that the periodicity of the length generating function coefficients for fixed rank begins at length $1 + \lfloor (n-1)/2 \rfloor \lceil (n-1)/2 \rceil$, as indicated by the data. By examining the structure of the heap diagrams associated to the fully commutative affine permutations, we have discovered some plausible reasoning indicating this tighter bound, but a proof remains elusive.

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