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On orbit configuration spaces associated to the Gaussian integers: Homotopy and homology groups

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Abstract

The purpose of this article is to analyze several Lie algebras associated to "orbit configuration spaces" obtained from the standard integral lattice $\mathbb{Z} + i\mathbb{Z}$ in the complex numbers. The Lie algebra obtained from the descending central series for the associated fundamental group is shown to be isomorphic, up to a regrading, to the Lie algebra obtained from the higher homotopy groups of "higher dimensional arrangements" modulo torsion. The resulting Lie algebras are similar to those studied by T. Kohno associated to elliptic KZ systems [Topology Appl. 78 (1997) 79–94]. A question about the generality of this behavior is posed. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this article is to compare properties of certain Lie algebras obtained from (i) the descending central series for the fundamental group of certain choices of $K(\pi, 1)$ spaces together with (ii) Lie algebras obtained from the classical higher homotopy groups of related loop spaces.

The evidence here and elsewhere suggests that in the special case of certain hyperplane arrangements, the Lie algebras encountered for the groups π, and those for the higher

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homotopy groups of "higher dimensional analogues" of hyperplane arrangements modulo torsion are isomorphic up to a regrading of the underlying Lie algebras. One "mantra" is that Vassiliev invariants for pure braids are determined by the higher homotopy groups of certain configuration spaces modulo torsion. This "mantra" is satisfied for certain other examples that are described next.

A more precise version of this conjecture is stated below for which several examples are already known. Namely, the homotopy groups of the loop space of the configuration space of k points in an even dimensional Euclidean space $\mathbb{R}^{2m+2}$, modulo torsion, admits the structure of a graded Lie algebra induced by the classical Samelson product. Apart from regrading, that Lie algebra is isomorphic to $E_0^*(P_k)$, the Lie algebra obtained from the descending central series for the pure k-stranded braid group that is "universal" for the Yang–Baxter–Lie relations [3,5]. Appealing to work of Kohno [8–10], these homotopy groups yield Vassiliev invariants for braids. By work of the second author [11], and Cohen [1], there are analogous results for certain choices of "orbit configuration spaces" of points in $\mathbb{C}^n - \{0\}$ that are discussed in this article.

The main results here arise from choices of orbit configuration spaces as introduced by the second author in [11]. Given a manifold M on which the group G acts properly discontinuously, let $F_G(M,k)$ be the orbit configuration space

$$F_G(M,k) = \left\{(m_1,\dots,m_k) \in M^k \mid Gm_i \cap Gm_j = \emptyset \text{ if } i \neq j\right\}.$$

Two examples of spaces M and groups G arise by considering (i) a parametrized lattice acting on the complex numbers $\mathbb{C}$ so that the orbit space is an elliptic curve, and (ii) a discrete group acting properly discontinuously on the upper half-plane H so that the orbit space is a complex curve. The orbit configuration spaces associated to the action of the standard integral lattice on the complex numbers is considered in this article.

Given a family of linear equations over the reals, $\Theta(k) = \{\sum_{1\leqslant i\leqslant k} a_i x_i\}$, in k vector variables $x_1,\dots,x_k$, let $X(k,\mathbb{R}^n) = (\mathbb{R}^n)^k - V(\Theta(k))$ denote the complement of the variety determined by $\Theta(k)$, $V(\Theta(k))$. Furthermore, assume that there are fibrations $X(k,\mathbb{R}^n) \to X(k-1,\mathbb{R}^n)$ having sections with fibre given by $\mathbb{R}^n - S_k$ where S_k is a discrete set. Consider the graded Lie algebra $E_0^*(\pi_1(X(k,\mathbb{R}^2)))_q$. Comparing answers for several known examples, suggests that $E_0^*(\pi_1(X(k,\mathbb{R}^2)))$ is isomorphic to the Lie algebra of primitive elements in the homology of $\Omega X(k,\mathbb{R}^{2q+2})$ for $q > 0$. Furthermore, it is an easy exercise that the module of primitives is isomorphic to the Lie algebra $\pi_*(X(k,\mathbb{R}^{2q+2}))$ modulo torsion for $q > 0$.

There is an associated natural question that is stated roughly in the following paragraphs. Start with a family of functors indexed by natural numbers, from Euclidean spaces with morphisms restricted to isometric embeddings, to topological spaces, where

(1) the kth functor is denoted $X(\mathbb{R}^n,k)$,
(2) $X(\mathbb{R}^n,1) = \mathbb{R}^n$,
(3) there are natural transformations $X(\mathbb{R}^n,k) \to X(\mathbb{R}^n,k-1)$ which are fibrations with fibre $\mathbb{R}^n - S_k$,
(4) S_k is a discrete subspace of $\mathbb{R}^n$ of fixed cardinality depending on k, and
(5) each fibration $X(\mathbb{R}^n,k) \to X(\mathbb{R}^n,k-1)$ admits a section.

An obvious example of such a family is given by $X(\mathbb{R}^n, k) = F(\mathbb{R}^n, k)$, the ordinary configuration spaces.

Next consider the Lie algebra obtained from the descending central series for the group G. For each strictly positive integer q, there is a canonical (and trivially defined) graded Lie algebra $E_0^*(G)_q$ attached to the one obtained from the descending central series for G, and which is defined as follows.

(1) Fix a strictly positive integer q.
(2) Let $\Gamma^n(G)$ denote the nth stage of the descending central series for G.
(3) $E_0^{2nq}(G)_q = \Gamma^n(G)/\Gamma^{n+1}(G)$,
(4) $E_0^i(G)_q = \{0\}$, if $i \not\equiv 0 \bmod 2q$, and
(5) the Lie bracket is induced by that for the associated graded for the $\Gamma^n(G)$.

It seems likely that for many choices of $X(\mathbb{R}^{2q+2}, k)$ the Lie algebra associated to $G = \pi_1(X(\mathbb{R}^2, k))$, satisfies the condition that it also gives the Lie algebra of primitive elements in the homology of certain loop spaces: $E_0^*(G)_q$ for $q > 0$ is isomorphic to the Lie algebra of primitive elements in the homology for the loop space $\Omega X(\mathbb{R}^{2q+2}, k)$ with $q > 0$. This last Lie algebra for the module of primitives is a graded free abelian group that is isomorphic to the Lie algebra given by the homotopy groups of $\Omega X(\mathbb{R}^{2q+2}, k)$ modulo torsion.

The results of the next theorem concern special cases of fibrations with cross-sections. After looping, the total space of such fibrations are always homotopy equivalent to a product. However, this product decomposition may be multiplicatively "twisted". The main input of this article is to analyze these extensions in special cases. This picture is reflected in the following theorem.

Theorem 1. *Assume that $n \geqslant 3$. Let $X(\mathbb{R}^n, k) \to X(\mathbb{R}^n, k-1)$ be a fibration which satisfies the following properties*:

(1) *the fibre of $X(\mathbb{R}^n, k) \to X(\mathbb{R}^n, k-1)$ is $\mathbb{R}^n - S_k$ where S_k is a discrete subspace of $\mathbb{R}^n$ of fixed cardinality depending on k,*
(2) *each fibration $X(\mathbb{R}^n, k) \to X(\mathbb{R}^n, k-1)$ admits a cross-section, and*
(3) $X(\mathbb{R}^n, 1) = \mathbb{R}^n$.

Then

(1) *There is a homotopy equivalence $\Omega X(\mathbb{R}^n, k) \xrightarrow{\simeq} \prod_{1 \leqslant i \leqslant k-1} \Omega(\mathbb{R}^n - S_i)$.*
(2) *The homology of $\Omega X(\mathbb{R}^n, k)$ is torsion free, and is isomorphic to $\bigotimes_{1 \leqslant i \leqslant k-1} H_*(\Omega(\mathbb{R}^n - S_i))$ as a coalgebra.*
(3) *The module of primitives in the integer homology of $\Omega X(\mathbb{R}^n, k)$ is isomorphic to $\pi_*(\Omega X(\mathbb{R}^n, k))$ modulo torsion as a Lie algebra.*

The proof of this theorem is an exercise that is given at the end of this article. The main purpose of this article is to study one example related to the theorem above given by the space $F_G(M, k)$ in the case when $M = \mathbb{C}$, the complex numbers, and G is the integral lattice $\mathcal{L} = \mathbb{Z} + \mathrm{i}\mathbb{Z}$, acting by translation on $\mathbb{C}$. One of the consequences of the theorem below is that the Lie algebra obtained from the fundamental group of the associated orbit configuration space also gives the Lie algebra obtained form the higher homotopy groups of the "higher dimensional analogues" of this arrangement.

Theorem 2. *Let $F_{\mathcal{L}}(\mathbb{C},k)$ be defined as above.*

(1) *The symmetric group Σ_k acts on $F_{\mathcal{L}}(\mathbb{C},k)$ and the orbit space $F_{\mathcal{L}}(\mathbb{C},k)/\Sigma_k$ is homeomorphic to the subspace of monic polynomials of degree k, $p(z)\in\mathbb{C}[z]$, with the property that the difference of any two roots of $p(z)$, α_i, α_j, lies outside of the Gaussian integers.*

(2) *It is the complement in $\mathbb{C}^k$ of the infinite (affine) hyperplane arrangement*

$$\mathcal{A}=\{H^{\sigma}_{i,j}\mid 1\leqslant j<i\leqslant k,\ \sigma\in\mathcal{L}\}$$

where $H^{\sigma}_{i,j}=\ker(z_i-z_j-\sigma)$.

(3) *It is an $\mathcal{L}^k$-cover of the ordinary configuration space of k points in the torus $T=S^1\times S^1$. This is a special case of the results in* [11], *which gives the existence of a principal bundle*

$$\mathcal{L}^k\to F_{\mathcal{L}}(\mathbb{C},k)\to F(T,k).$$

(4) *The space $F_{\mathcal{L}}(\mathbb{C},k)$ is a $K(\pi,1)$.*

(5) *The fibration $F_{\mathcal{L}}(\mathbb{C},k)\to F_{\mathcal{L}}(\mathbb{C},k-1)$ has* (i) *trivial local coefficients in homology, and* (ii) *a cross-section.*

(6) *Thus the Lie algebra given by the associated graded for the descending central series of $\pi_1(F_{\mathcal{L}}(\mathbb{C},k))$ is additively isomorphic to the direct sum $\bigoplus_{1<i\leqslant k}L[i]$ where $L[i]$ is the free Lie algebra generated by elements $B^{\sigma}_{i,j}$ for fixed i with $1\leqslant j<i\leqslant k$, and σ runs over the elements of the lattice $\mathcal{L}$.*

(7) *The relations are*

$$[B^{\sigma}_{i,j},B^{\tau}_{s,t}]=0\quad \text{if } \{i,j\}\cap\{s,t\}=\emptyset.$$

Otherwise,

$$[B^{\sigma}_{i,j},B^{\tau}_{\ell,i}]=[B^{\tau}_{\ell,i},B^{\tau+\sigma}_{\ell,j}],$$

$$[B^{\sigma}_{i,j},B^{\tau}_{\ell,j}]=[B^{\tau}_{\ell,j},B^{\tau-\sigma}_{\ell,i}].$$

(8) *The integral homology of $F_{\mathcal{L}}(\mathbb{C},k)$ is additively given by*

$$H_*F_{\mathcal{L}}(\mathbb{C},k)\cong H_*(C_1)\otimes H_*(C_2)\otimes\cdots\otimes H_*(C_{k-1})$$

where C_i is the infinite bouquet of circles $\bigvee_{|Q_i^{\mathcal{L}}|}S^1$ and $Q_i^{\mathcal{L}}$ as defined in the beginning of the next section.

Remark. Work of Kohno, as well as work of Falk and Randell [6,8–10] together Theorem 2(5) give the additive structure for the Lie algebra above.

Consider the "orbit configuration space" $F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k)$ where $\mathcal{L}$ operates diagonally on $\mathbb{C} \times \mathbb{C}^q$, by translation on $\mathbb{C}$ and trivially on $\mathbb{C}^q$.

Theorem 3. *Assume that* $q \geqslant 1$.

(1) *The loop space* $\Omega F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k)$ *is homotopy equivalent to the product*

$$\prod_{1 \leqslant i \leqslant k-1} \Omega\left(\mathbb{C} \times \mathbb{C}^q - Q_i^{\mathcal{L}}\right)$$

(*although this product decomposition is not multiplicative*).

(2) *The integral homology of* $\Omega F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k)$ *is isomorphic to*

$$\bigotimes_{1 \leqslant i \leqslant k-1} H_*\left(\Omega\left(\mathbb{C} \times \mathbb{C}^q - Q_i^{\mathcal{L}}\right)\right)$$

as a coalgebra.

(3) *The Lie algebra of primitives is isomorphic to the Lie algebra given by* $\pi_*(\Omega F_{\mathcal{L}}(\mathbb{C}^q, k))/$*torsion.*

(4) *The Lie algebra of of primitive elements in the homology of* $\Omega F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k)$ *is a direct sum of free* (*graded*) *Lie algebras* $\bigoplus_{1<i\leqslant k} L[i]$ *where* $L[i]$ *is the free graded Lie algebra generated by elements* $B_{i,j}^{\sigma}$ *of degree* $2q$ *for fixed* i *with* $1 \leqslant j < i \leqslant k$, *and* σ *runs over the elements of the lattice* $\mathcal{L}$. *The relations are*

$$\left[B_{i,j}^{\sigma}, B_{s,t}^{\tau}\right] = 0 \quad \text{if } \{i, j\} \cap \{s, t\} = \emptyset.$$

Otherwise,

$$\left[B_{i,j}^{\sigma}, B_{\ell,i}^{\tau}\right] = \left[B_{\ell,i}^{\tau}, B_{\ell,j}^{\tau+\sigma}\right],$$

$$\left[B_{i,j}^{\sigma}, B_{\ell,j}^{\tau}\right] = \left[B_{\ell,j}^{\tau}, B_{\ell,i}^{\tau-\sigma}\right].$$

(5) *The Lie algebras* $\pi_*(\Omega F_{\mathcal{L}}(\mathbb{C}^q, k))/$*torsion, and* $E_0^*(\pi_1 F_{\mathcal{L}}(\mathbb{C}, k))_q$ *are isomorphic as Lie algebras.*

2. Proof of Theorem 2

A useful result is the following analogue of the Fadell and Neuwirth fibrations for ordinary configuration spaces. For any natural number ℓ, let $Q_\ell^G \subset M$ be the union of ℓ distinct orbits $Gm_1, \ldots, Gm_\ell$.

Lemma 4. *Let* M *be a manifold with a properly discontinuous action of a group* G *and such that the orbit space* M/G *is again a manifold. Then for* $\ell \leqslant k$, *the projection* $p : F_G(M, k) \to F_G(M, \ell)$ *onto the first* ℓ *coordinates is a locally trivial bundle, with fibre* $F_G(M - Q_\ell^G, k - \ell)$.

Remark. The extra hypothesis on the action are needed for the case when G is not finite. Notice that they are trivially satisfied when G is a finite group acting freely on M.

Let $q_1, \ldots, q_k \in \mathbb{C}$ be the points: $q_1 = 0$ and $q_i = q_{i-1} + 1/2^i$ for $i = 2, \ldots, k$. Notice these points lie in different orbits and let $Q_\ell^{\mathcal{L}} = \bigcup_{i=1}^{\ell}(q_i + \mathcal{L})$ the union of the first ℓ of them. Now an iterated application of Lemma 4 yields a sequence of fibrations:

$$\begin{array}{ccccc} F_{\mathcal{L}}(\mathbb{C}, k) & \longleftarrow & F_{\mathcal{L}}\big(\mathbb{C} - Q_1^{\mathcal{L}}, k-1\big) & \longleftarrow \cdots \longleftarrow & F_{\mathcal{L}}\big(\mathbb{C} - Q_{k-1}^{\mathcal{L}}, 1\big) \\ \downarrow & & s_1 \uparrow\downarrow & & s_{k-1} \uparrow\downarrow \approx \\ \mathbb{C} & & \mathbb{C} - Q_1^{\mathcal{L}} & & \mathbb{C} - Q_{k-1}^{\mathcal{L}} \end{array} \tag{1}$$

(where the horizontal maps are the inclusions of the fibers), and sections s_i, defined as follows. For $z \in \mathbb{C}$ set $d(z) = \min\{|z - \sigma| \mid \sigma \in \mathcal{L}\}$ and notice that d is a continuous function of z. Let $y_1, \ldots, y_{k-i-1}$ be $k - i - 1$ distinct points in the boundary of the disc of radius $\frac{1}{16}$ and centered at the origin. Let

$$s_i : \mathbb{C} - Q_i^{\mathcal{L}} \to F_{\mathcal{L}}\big(\mathbb{C} - Q_i^{\mathcal{L}}, k - i\big) \tag{2}$$

be given by:

$$s_i(z) = \begin{cases} \big(z, 8d(z)y_1, \ldots, 8d(z)y_{k-i-1}\big) & \text{if } d(z) \leqslant \frac{1}{8}, \\ (z, y_1, \ldots, y_{k-i-1}) & \text{if } d(z) \geqslant \frac{1}{8}. \end{cases}$$

The local coefficient system for these fibrations is trivial in integral homology. (See Lemma 8 below.) Since the homology of the fibre is concentrated in degree one, and the fibration has a cross-section, the Serre spectral sequence for these fibrations collapses. The next proposition follows.

The next proposition gives the additive structure for the integral homology of $F_{\mathcal{L}}(\mathbb{C}, k)$ as stated in Theorem 2(7).

Proposition 5. *The integral homology of $F_{\mathcal{L}}(\mathbb{C}, k)$ is additively given by*

$$H_*F_{\mathcal{L}}(\mathbb{C}, k) \cong H_*(C_1) \otimes H_*(C_2) \otimes \cdots \otimes H_*(C_{k-1}),$$

where C_i is the infinite bouquet of circles $\bigvee_{|Q_i^{\mathcal{L}}|} S^1$.

In particular, the homology of $F_{\mathcal{L}}(\mathbb{C}, k)$ is not of finite type and one must proceed with certain care when dualizing. However, something can be said about cohomology.

Proposition 6. *The integral cohomology of $F_{\mathcal{L}}(\mathbb{C}, k)$ is additively given by*

$$H^*F_{\mathcal{L}}(\mathbb{C}, k) \cong H^*(C_1) \otimes H^*(C_2) \otimes \cdots \otimes H^*(C_{k-1})$$

Moreover, for every $i = 2, \ldots k$, there is a choice of cohomology classes in $H^*F_{\mathcal{L}}(\mathbb{C}, k)$ $\{A_{i,1}^{\sigma_1}, A_{i,2}^{\sigma_2}, \ldots, A_{i,i-1}^{\sigma_{i-1}} \mid \sigma_j \in \mathcal{L}\}$, which are in a 1–1 correspondence with the generators in $H_1(C_{i-1})$, satisfying the following relations:

(a) $A_{i,j}^{\mu} A_{i,j}^{\nu} = 0$ *for all* $\mu, \nu \in \mathcal{L}$.
(b) $A_{i,\ell}^{\mu} A_{i,j}^{\nu} = A_{j,\ell}^{\mu-\nu}(A_{i,j}^{\nu} - A_{i,\ell}^{\mu})$ *if* $1 \leqslant \ell < j < i \leqslant k$ *and* $\mu, \nu \in \mathcal{L}$.

Remark. Perhaps a better way to say this is that $H^*F_{\mathcal{L}}(\mathbb{C}, k)$ contains a subalgebra isomorphic to the associative, commutative graded algebra $\mathcal{A}^*$, generated by the following collection of degree one elements, $\{A^{\sigma}_{i,j}\}^{\sigma\in\mathcal{L}}_{i>j}$, subject to relations (a) and (b). Moreover, the duals to the standard generators in $H_*F_{\mathcal{L}}(\mathbb{C},k)$ realize the inclusion of algebras $\mathcal{A}^*\subset H^*F_{\mathcal{L}}(\mathbb{C},k)$.

Remark. Notice that in the case $\mu=\nu=0$, relations (a) and (b) recover the relations among the standard generators in $H^*F(\mathbb{C},k)$ (see for example [2]). Roughly speaking, the spaces $F(\mathbb{C},k)$ and $F_{\mathcal{L}}(\mathbb{C},k)$ have *locally* the same cohomology.

Proof. Begin by identifying the classes $A^{\sigma}_{i,j}$. In the case $k=2$, the projection onto the first coordinate gives a fibration: $(\mathbb{C}-Q^{\mathcal{L}}_1)\to F_{\mathcal{L}}(\mathbb{C},k)\to\mathbb{C}$ for which the inclusion of the fiber over $q_1=0$, is a homotopy equivalence. Therefore:

$$F_{\mathcal{L}}(\mathbb{C},2)\simeq\bigvee_{\mathcal{L}}S^1.$$

For every $\sigma\in\mathcal{L}$ let $p_\sigma:F_{\mathcal{L}}(\mathbb{C},2)\to S^1$ be the map

$$p_\sigma(x,y)=\frac{y-x-\sigma}{|y-x-\sigma|}$$

and put $A_\sigma=p_\sigma{}^*(\iota)$ where ι is a fixed generator of $H^1(S^1)=\mathbb{Z}$. Then $\{A_\sigma\}_{\sigma\in\mathcal{L}}$ is a linearly independent family in $H^1F_{\mathcal{L}}(\mathbb{C},2)$, where each A_σ corresponds to one of the circles in the bouquet.

Finally, let $\pi_{i,j}:F_{\mathcal{L}}(\mathbb{C},k)\to F_{\mathcal{L}}(\mathbb{C},2)$ be the projections given by $\pi_{i,j}(x_1,\ldots,x_k)=(x_j,x_i)$ for $j<i$. Define $A^{\sigma}_{i,j}=\pi_{i,j}{}^*(A_\sigma)$ for $\sigma\in\mathcal{L}$. Thus $A^{\sigma}_{i,j}A^{\tau}_{i,j}=0$ for all σ, τ as $A_\sigma A_\tau=0$ in $H^*F_{\mathcal{L}}(\mathbb{C},2)\cong H^*(\bigvee_{\mathcal{L}}S^1)$.

To check relation (b), it suffices to check the case when $i=3$, $j=2$, $\ell=1$ by applying the projections $\pi_{i,j,\ell}:F_{\mathcal{L}}(\mathbb{C},k)\to F_{\mathcal{L}}(\mathbb{C},3)$, together with naturality. The cohomology classes $A^{\sigma}_{2,1}$, $A^{\sigma}_{3,1}$,$A^{\sigma}_{3,2}$ are the duals to the homology classes represented by the maps: $\varphi^{\sigma}_{i,j}:S^1\to F_{\mathcal{L}}(\mathbb{C},3)$

$$\begin{aligned}\varphi^{\sigma}_{2,1}(z)&=\big(q_1,q_1+\sigma+\tfrac{1}{8}z,q_3\big),\\ \varphi^{\sigma}_{3,1}(z)&=\big(q_1,q_2,q_1+\sigma+\tfrac{1}{8}z\big),\\ \varphi^{\sigma}_{3,2}(z)&=\big(q_1,q_2,q_2+\sigma+\tfrac{1}{16}z\big)\end{aligned}$$

which generate $H_1(C_1)$ and $H_1(C_2)$. From the sequence of fibrations (1) for $k=3$, it follows that $H^*F_{\mathcal{L}}(\mathbb{C},3)\cong H^*(C_1)\otimes H^*(C_2)$ and therefore

$$A^{\mu}_{i,j}A^{\nu}_{s,t}=\sum_{\sigma,\tau}\alpha_{\sigma,\tau}A^{\sigma}_{2,1}A^{\tau}_{3,1}+\sum_{\sigma,\tau}\beta_{\sigma,\tau}A^{\sigma}_{2,1}A^{\tau}_{3,2},$$

where $\alpha_{\sigma,\tau}$ and $\beta_{\sigma,\tau}$ are integers to be determined. For this purpose there are maps which detect the $A^{\sigma}_{2,1}A^{\tau}_{3,1}$ and $A^{\sigma}_{2,1}A^{\tau}_{3,2}$ and then study their effect on $A^{\mu}_{i,j}A^{\nu}_{s,t}$. For fixed $\sigma,\tau\in\mathcal{L}$, let $F_1,F_2:S^1\times S^1\to F_{\mathcal{L}}(\mathbb{C},3)$ be the maps

$$\begin{aligned}F_1(z,w)&=\big(q_1,q_1+\sigma+\tfrac{1}{8}z,q_1+\tau+\tfrac{1}{16}w\big),\\ F_2(z,w)&=\big(q_1,q_1+\sigma+\tfrac{1}{8}z,q_1+\sigma+\tau+\tfrac{1}{8}z+\tfrac{1}{16}w\big).\end{aligned}$$

It is immediate to verify their effect in cohomology:

$$(F_1)^*\big(A^{\lambda}_{2,1}\big) = \begin{cases} \iota \otimes 1, & \text{if } \lambda = \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_1)^*\big(A^{\lambda}_{3,1}\big) = \begin{cases} 1 \otimes \iota, & \text{if } \lambda = \tau, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_1)^*\big(A^{\lambda}_{3,2}\big) = \begin{cases} \iota \otimes 1, & \text{if } \lambda = \tau - \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_2)^*\big(A^{\lambda}_{2,1}\big) = \begin{cases} \iota \otimes 1, & \text{if } \lambda = \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_2)^*\big(A^{\lambda}_{3,1}\big) = \begin{cases} \iota \otimes 1, & \text{if } \lambda = \sigma + \tau, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_2)^*\big(A^{\lambda}_{3,2}\big) = \begin{cases} 1 \otimes \iota, & \text{if } \lambda = \tau, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the homomorphisms F_1^* and F_2^* detect the basic products $A^{\sigma}_{2,1}A^{\tau}_{3,1}$ and $A^{\sigma}_{2,1}A^{\tau}_{3,2}$, respectively. Since they are also ring homomorphisms, we have

$$(F_1)^*\big(A^{\mu}_{3,1}A^{\nu}_{3,2}\big) = \begin{cases} -\iota \otimes \iota, & \text{if } \mu = \tau,\ \nu = \tau - \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$$(F_2)^*\big(A^{\mu}_{3,1}A^{\nu}_{3,2}\big) = \begin{cases} \iota \otimes \iota, & \text{if } \mu = \sigma + \tau,\ \nu = \tau. \\ 0, & \text{otherwise} \end{cases}$$

and therefore we get: $A^{\mu}_{3,1}A^{\nu}_{3,2} = -A^{\mu-\nu}_{2,1}A^{\mu}_{3,1} + A^{\mu-\nu}_{2,1}A^{\nu}_{3,2}$. □

Lemma 7. *Let $p : E \to B$ be a locally trivial bundle with B path-connected. Let $x, y \in B$ and let F_x and F_y be the corresponding fibers. Then $\pi_1(B, x)$ acts trivially on $H_*(F_x)$ if and only if $\pi_1(B, y)$ acts trivially on $H_*(F_y)$.*

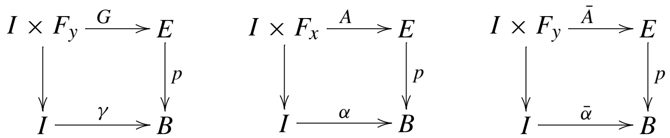
Proof. Assume that $\pi_1(B, y)$ acts trivially on $H_*(F_y)$. Let $\alpha : I \to B$ be a path from x to y and let

$$\begin{aligned} \widehat{\alpha} : \pi_1(B, y) &\to \pi_1(B, x) \\ [\gamma] &\mapsto [\alpha] * [\gamma] * [\alpha]^{-1} \end{aligned}$$

be the induced isomorphism of fundamental groups. We prove that $\widehat{\alpha}[\gamma]$ acts trivially on $H_*(F_x)$. Let:

- G be a lifting for γ,
- A be a lifting for α, and
- $\bar{A}$ be a lifting for $\bar{\alpha}$, where $\bar{\alpha}(t) = \alpha(1 - t)$.

This is, G, A and $\bar{A}$ are maps such that the following diagrams commute:



To construct a lifting for $\alpha * \gamma * \bar{\alpha}$, define $H : I \times F_x \to E$ by

$$H(t,e) = \begin{cases} A(3t,e) & \text{if } 0 \leqslant t \leqslant \frac{1}{3}, \\ G\big(3t-1, A(1,e)\big) & \text{if } \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ \bar{A}\big(3t-2, G\big(1, A(1,e)\big)\big) & \text{if } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Notice that in this case, the right end of the homotopy $H_1 = H\,|_{t=1}$ is a continuous mapping $F_x \to F_x$ given by $H_1(e) = \bar{A}(1, G(1, A(1,e)))$, that is to say $H_1 = \bar{A}_1 \circ G_1 \circ A_1$ and this composite induces a commutative diagram of homology groups:

$$\begin{array}{ccc} H_*(F_x) & \xrightarrow{(H_1)_*} & H_*(F_x) \\ {\scriptstyle (A_1)_*}\big\downarrow & & \big\uparrow{\scriptstyle (\bar{A}_1)_*} \\ H_*(F_y) & \xrightarrow{(G_1)_*} & H_*(F_y) \end{array}$$

where $(G_1)_* = 1_{H_*(F_y)}$ since $[\gamma]$ acts trivially on $H_*(F_y)$. Thus, $(H_1)_* = (\bar{A}_1)_* \circ (A_1)_*$. But this is precisely the isomorphism of $H_*(F_x)$ induced by the trivial loop $\alpha * \bar{\alpha}$ and then $(H_1)_*$ must be the identity. Therefore, $\widehat{\alpha}[\gamma]$ acts trivially on $H_*(F_x)$. □

Remark. The same proof works if using cohomology instead of homology. The way in which Lemma 7 is actually used is the following. Consider the fiber bundle given by projection onto the first coordinate $\pi : F_{\mathcal{L}}(\mathbb{C} - Q_r^{\mathcal{L}}, k-r) \to (\mathbb{C} - Q_r^{\mathcal{L}})$. Let $z_0 \in \mathbb{C} - Q_r^{\mathcal{L}}$ be a base point and then

$$\pi_1\big(\mathbb{C} - Q_r^{\mathcal{L}}, z_0\big) \cong F\big[x_1^{\sigma_1}, \ldots, x_r^{\sigma_r} \mid \sigma_i \in \mathcal{L}\big].$$

One uses that for each generator $x_i^{\sigma_i}$, there exist a point $z_{i,\sigma_i} \in \mathbb{C} - Q_r^{\mathcal{L}}$ (namely take $z_{i,\sigma_i} = q_{r+1} + \sigma_i$) for which is relatively easy to prove that the action of the corresponding generator in $\pi_1(\mathbb{C} - Q_r^{\mathcal{L}}, z_{i,\sigma_i})$ is trivial on the homology of the fiber over z_{i,σ_i}. Since $\mathbb{C} - Q_r^{\mathcal{L}}$ is path connected, Lemma 7 implies that $x_i^{\sigma_i}$ acts trivially on the homology of the fiber over z_0.

Lemma 8. *The fibration $\pi : F_{\mathcal{L}}(\mathbb{C} - Q_r^{\mathcal{L}}, k-r) \to \mathbb{C} - Q_r^{\mathcal{L}}$ has trivial local coefficients.*

Proof. The proof is by downward induction on r and it is the equivariant analogue to the proof given in [4]. For $r = k-1$ the result is clear. Assume the result for $r+1$. Consider the fibration $\pi : F_{\mathcal{L}}(\mathbb{C} - Q_r^{\mathcal{L}}, k-r) \to \mathbb{C} - Q_r^{\mathcal{L}}$ with fibre $F_{\mathcal{L}}(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1)$. Define a function

$$\rho_i : I \times \big(\mathbb{C} - Q_r^{\mathcal{L}}\big) \to \big(\mathbb{C} - Q_r^{\mathcal{L}}\big)$$

as follows. Let D_i be a disc containing the points q_i and q_{r+1} as in [4, p. 252], with two additional conditions:

(i) D_i must be entirely contained in the interior of a unit square, so that $D_i \cap (D_i + \sigma) = \emptyset$ for $\sigma \neq 0$.

(ii) $D_i \cap B(0, \frac{1}{8}) = \emptyset$ for $i \neq 1$, and $D_1 \cap B(q_2, \frac{1}{8}) = \emptyset$,

where $B(z, r)$ denotes the closed ball in $\mathbb{C}$ with center z and radius r (see Fig. 1). Since $q_i = 0 + \frac{1}{4} + \cdots + \frac{1}{2^i}$, this is not difficult to achieve. The function ρ_i does the following:

- Inside D_i rotates the 2-disc with center q_i contained within the 2-annulus by an angle of $2\pi t$ at time t, fixes the boundary of D_i and appropriately "twists" the 2-annulus, at time t to insure the continuity of ρ_i.
- Twists all discs $(D_i + \sigma)$, for $\sigma \in \mathcal{L}$, in such a way that the resulting function is $\mathcal{L}$-equivariant, namely

$$\rho_i(t, z) = \rho_i(t, z - \sigma) + \sigma \quad \forall z \in (D_i + \sigma).$$

- Fixes the complement of the union of all discs $\mathbb{C} - \bigcup_{\sigma \in \mathcal{L}} (D_i + \sigma)$.

Recall that $\pi_1(\mathbb{C} - Q_r^{\mathcal{L}})$ is isomorphic to the free group $F[x_i^\sigma \mid 1 \leqslant i \leqslant r,\ \sigma \in \mathcal{L}]$. Using the function ρ_i, we prove that all generators x_i^σ act trivially on $H_* F_{\mathcal{L}}(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k - r - 1)$. More precisely, it will follow that the generator corresponding to the point $q_i + \sigma$ acts trivially on the homology of the fiber over $q_{r+1} + \sigma$. By Lemma 7, this is enough to insure the triviality of the local coefficient system.

Define $\gamma_{i,\sigma}(t) = \rho_i(t, q_{r+1} + \sigma)$. Then $\gamma_{i,\sigma} : I \to (\mathbb{C} - Q_r^{\mathcal{L}})$ is a generator of the group $\pi_1(\mathbb{C} - Q_r^{\mathcal{L}}, q_{r+1} + \sigma)$ which corresponds to the point $q_i + \sigma$. Define a "lift"

$$H_{i,\sigma} : I \times F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k - r - 1\big) \to F_{\mathcal{L}}\big(\mathbb{C} - Q_r^{\mathcal{L}}, k - r\big)$$

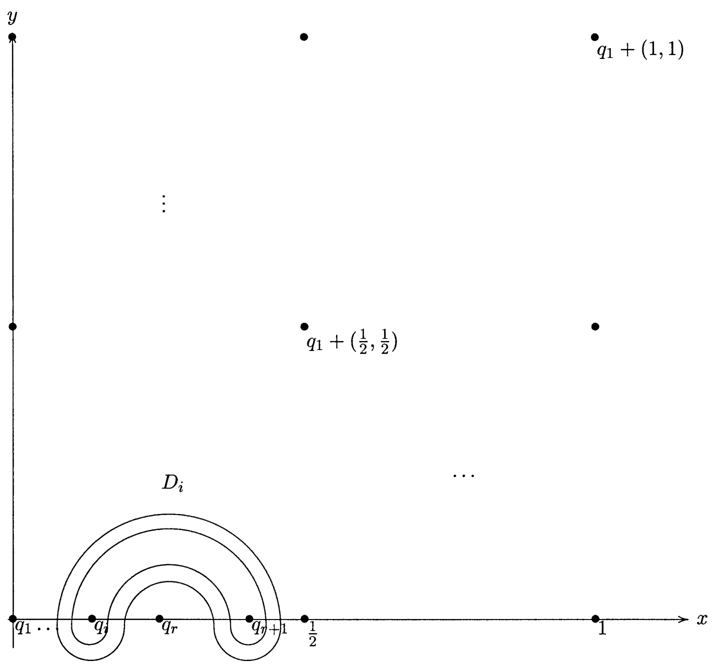


Fig. 1.

of $\gamma_{i,\sigma}$ by:

$$H_{i,\sigma}(t; z_1, z_2, \ldots, z_{k-r-1}) = \big(\rho_i(t, q_{r+1} + \sigma), \rho_i(t, z_1), \ldots, \rho_i(t, z_{k-r-1})\big). \tag{3}$$

Notice that the right side of (3) lies on $F_{\mathcal{L}}(\mathbb{C} - Q_r^{\mathcal{L}}, k-r)$ since $z_j \not\equiv q_{r+1} \bmod \mathcal{L}$, $z_j \not\equiv z_\ell \bmod \mathcal{L}$ and ρ_i is an $\mathcal{L}$-*equivariant* isotopy. Then we have a commutative diagram:

$$\begin{array}{ccc} I \times F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1\big) & \xrightarrow{H_{i,\sigma}} & F_{\mathcal{L}}\big(\mathbb{C} - Q_r^{\mathcal{L}}, k-r\big) \\ \downarrow & & \downarrow \\ I & \xrightarrow{\gamma_{i,\sigma}} & \mathbb{C} - Q_r^{\mathcal{L}} \end{array}$$

By definition, the local coefficient system is trivial if the map

$$(H_{i,\sigma})_1 : F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1\big) \to F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1\big)$$

is the identity in homology. Let $B_{i,j}^{\sigma}$ be the homology dual of $A_{i,j}^{\sigma}$. Recall that

$$H_* F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1\big) \cong \bigotimes_{j=2}^{k-r} \big\langle B_{r+j,1}^{\sigma_1}, \ldots, B_{r+j,r+j-1}^{\sigma_{r+j-1}}\big\rangle \tag{4}$$

so one must check the effect of $(H_{i,\sigma})_{1*}$ on the $B_{r+j,\ell}^{\sigma}$'s. For a fixed j, $1 \leqslant j \leqslant k-r-1$, there are sections

$$F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+j}^{\mathcal{L}}, k-r-j\big) \underset{s_{r+j}}{\rightleftarrows} \mathbb{C} - Q_{r+j}^{\mathcal{L}}$$

such that the classes $B_{r+j+1,\ell}^{\sigma}$ for $\ell = 1, \ldots, (r+j)$ and $\sigma \in \mathcal{L}$, are the images in homology of the generators $H_1(\mathbb{C} - Q_{r+j}^{\mathcal{L}})$ under s_{r+j}, followed by the morphism induced by the inclusion in $F_{\mathcal{L}}(\mathbb{C} - Q_{r+1}^{\mathcal{L}}, k-r-1)$. So, to prove that $[\gamma_{i,\sigma}]$ acts trivially on the jth factor of (4), notice that the map $(H_{i,\sigma})_1$ induces maps of fibrations

$$\begin{array}{ccc} F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+j}^{\mathcal{L}}, k-r-j\big) & \xrightarrow{\widehat{H}_{i,\sigma}} & F_{\mathcal{L}}\big(\mathbb{C} - Q_{r+j}^{\mathcal{L}}, k-r-j\big) \\ s_{r+j}\uparrow\downarrow & & s_{r+j}\uparrow\downarrow \\ \mathbb{C} - Q_{r+j}^{\mathcal{L}} & \xrightarrow{\overline{H}_{i,\sigma}} & \mathbb{C} - Q_{r+j}^{\mathcal{L}} \end{array}$$

which are compatible with the sections, in the sense that $\widehat{H}_{i,\sigma} \circ s_{r+j} = s_{r+j} \circ \overline{H}_{i,\sigma}$

For $i = 2, \ldots, r$, the sections given by (2) suffice. For $i = 1$, there is a straightforward modification of s_1 that uses $B(q_2, \frac{1}{8})$ instead of $B(0, \frac{1}{8})$. Then it will be enough to show that

$$\big(\overline{H}_{i,\sigma}\big)_* : H_1\big(\mathbb{C} - Q_{r+j-1}^{\mathcal{L}}\big) \to H_1\big(\mathbb{C} - Q_{r+j-1}^{\mathcal{L}}\big)$$

is the identity morphism and this is clear, since at the level of fundamental groups

$$\big(\overline{H}_{i,\sigma}\big)_\sharp : \pi_1\big(\mathbb{C} - Q_{r+j}^{\mathcal{L}}\big) \to \pi_1\big(\mathbb{C} - Q_{r+j}^{\mathcal{L}}\big)$$

the induced homomorphism is given by either: the identity, conjugation by a fixed element in π_1 or multiplication by a certain commutator, see [4, p. 254.] □

3. Proof of Theorem 1

Recall that a multiplicative fibration with section is homotopy equivalent to a product. Thus $\Omega X(\mathbb{R}^n, k)$ is homotopy equivalent to $\Omega X(\mathbb{R}^n, k-1) \times \Omega(\mathbb{R}^n - S_{k-1})$, and the first part of the theorem follows by induction.

The second part of the theorem follows from the *Künneth* theorem, and part 1 of the theorem.

Since $\mathbb{R}^n - S_{k-1}$ has the homotopy type of a (possibly infinite) bouquet of $(n-1)$-spheres, the homology of its loop space follows from the Bott–Samelson theorem. In this case, it is well-known that there are isomorphisms of Lie algebras

$$\pi_*\bigl(\Omega\bigl(\mathbb{R}^n - S_{k-1}\bigr)\bigr)/\mathit{torsion} \to \mathit{Prim}\, H_*\bigl(\Omega\bigl(\mathbb{R}^n - S_{k-1}\bigr)\bigr). \tag{5}$$

Furthermore, the existence of sections implies that the Hurewicz homomorphism

$$\pi_*\bigl(\Omega X\bigl(\mathbb{R}^n, k\bigr)\bigr) \to \mathit{Prim}\, H_*\bigl(\Omega X\bigl(\mathbb{R}^n, k\bigr)\bigr)$$

is a surjection. Since the map (5) is an injection, the theorem follows.

4. Proof of Theorem 3

Consider the fibration with section $F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k) \to F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k-1)$. The fibre of this map is $\mathbb{C} \times \mathbb{R}^{2q} - Q^{\mathcal{L}}_{k-1}$. By Theorem 1, the conclusions of Theorem 3 all follow except possibly the last two which state the precise extension of Lie algebras.

To finish, it suffices to prove parts (4), and (5) of the theorem by a direct comparison of the two Lie algebras $\pi_*(\Omega F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, k))/\mathit{torsion}$, and $E_0^*(\pi_1 F_{\mathcal{L}}(\mathbb{C}, k))_q$. The computation of the Lie algebra attached to the descending central series, is just an explicit a calculation that involves the local coefficient system and the generators constructed above.

Thus define maps $F_i : S^{2q+1} \times S^{2q+1} \to F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, 3)$ analogous to those in the proof of the relations for the cohomology of $F_{\mathcal{L}}(\mathbb{C}, k)$ by the formulas:

$$\begin{aligned} F_1(z, w) &= \bigl(q_1, q_1 + \sigma + \tfrac{1}{8}z, q_1 + \tau + \tfrac{1}{16}w\bigr), \\ F_2(z, w) &= \bigl(q_1, q_1 + \sigma + \tfrac{1}{8}z, q_1 + \sigma + \tau + \tfrac{1}{8}z + \tfrac{1}{16}w\bigr). \end{aligned}$$

Consider the loopings of these maps

$$\Omega(F_i) : \Omega\bigl(S^{2q+1} \times S^{2q+1}\bigr) \to \Omega\bigl(F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, 3)\bigr).$$

Notice that the fundamental cycles in degree $2q$ for the integer homology of $\Omega(S^{2q+1} \times S^{2q+1})$ commute. Thus it suffices to calculate the image of the fundamental cycles in the homology of $\Omega(F_{\mathcal{L}}(\mathbb{C} \times \mathbb{C}^q, 3))$. This gives the precise relations as stated in parts (4)–(5) of Theorem 3 which follows at once.

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