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Strong convergence theorems for a new iterative method of k-strictly pseudo-contractive mappings in Hilbert spaces

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ABSTRACT

In this paper, we introduce a new iterative method of a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$ and prove that the sequence $\{x_n\}$ converges strongly to a fixed point of T, which solves a variational inequality related to the linear operator A. Our results have extended and improved the corresponding results of Y.J. Cho, S.M. Kang and X. Qin [Some results on k-strictly pseudo-contractive mappings in Hilbert spaces, Nonlinear Anal. 70 (2008) 1956–1964], and many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Note that the class of k-strictly pseudo-contractive includes strictly the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

This is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [1–3]).

It is clear that, in a real Hilbert space H, (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

The mapping T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

T is strongly pseudo-contractive if and only if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.5}$$

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In 2002, Xu [4] studied the following iterative process by the viscosity approximation defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, and then proved that the sequence $\{x_n\}$ converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality:

$$\langle (I-f)q, p-q\rangle \leq 0, \quad \forall p \in F(T). \tag{1.7}$$

Very recently, Marino and Xu [5] introduced and considered the following iterative algorithm:

$$\begin{cases} x_0 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)Tx_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions and A is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Then they proved that the sequence $\{x_n\}$ converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality:

$$\langle (A-\gamma f)q, q-x\rangle \leq 0, \quad \forall x \in F(T). \tag{1.9}$$

Moreover, Cho, Kang and Qin [6] extended and improved the result of Marino and Xu [5] (see also [2,7–14]) and introduced a general iterative algorithm:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)P_K Sx_n, \quad \forall n \geq 1 \end{cases} \tag{1.10}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$, $\{\alpha_n\}$ of parameters satisfies appropriate conditions, and A is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Then, they proved the strong convergence theorems for T being a k-strictly pseudo-contractive mapping in Hilbert spaces.

In this paper, motivated by Cho et al. [6], we introduce a new iterative scheme generated by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C Sx_n, \end{cases} \tag{1.11}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$ and $T : C \to H$ is a k-strictly pseudo-contractive mapping, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. We will prove in Section 3 that if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.11) converges strongly to the solution of variational inequality (1.9).

2. Preliminary

In this section, we collect some lemmas which will be used in the proof for the main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$
(ii) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$
(iii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$
(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1]$.

Lemma 2.2 ([12]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property*

$$a_{n+1} \leq (1-\gamma_n)a_n + b_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ be a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([15]). *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.4 ([5]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$,*

$$\langle x-y, (A-\gamma f)x - A(A-\gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x-y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.5 ([5]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.6 ([16]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.8 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.9 ([14]). *Let H be a Hilbert space, and C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.10 ([5]). *Let H be a Hilbert space, and C be a nonempty closed convex subset of H. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $T : C \to C$ be a nonexpansive mapping with fixed point x_t of contraction $C \ni x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then $\{x_t\}$ converges strongly to fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following variational inequality:*

$$\langle(\gamma f - A)\widetilde{x}, z - \widetilde{x}\rangle \leq 0, \quad \forall z \in F(T).$$

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,
(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $s = (a_0, a_1, \ldots) \in l^\infty$.

Lemma 2.11 ([13]). *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \subset l^\infty$ satisfying the condition $\mu_n(a_n) \leq a$ for all Banach limits μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

Lemma 2.12 ([17]). *Let H be a Hilbert space, and C be a nonempty closed convex subset of H. For any integer $N \geq 1$, assume that, for each $1 \leq i \leq N$, $T_i : C \to H$ be k_i-strictly pseudo-contractive mappings for some $0 \leq k_i < 1$. Assume that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a non-self-k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 2.13 ([17]). *Let $\{T_i\}_{i=1}^N$ and $\{\eta_i\}_{i=1}^N$ be given as in Lemma 2.12. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^\infty F(T_i)$.*

3. Main results

In this section, first we show that a mapping $S : C \to H$ defined by $Sx = kx + (1-k)Tx$ is a nonexpansive mapping, where C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ is a k-strictly pseudo contractive mapping with a fixed point for some $0 \leq k < 1$. Let $x, y \in C$; then from Lemma 2.1(iv) we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|kx + (1-k)Tx - (ky + (1-k)Ty)\|^2 \\
&= \|k(x-y) + (1-k)(Tx - Ty)\|^2 \\
&= k\|x-y\|^2 + (1-k)\|Tx - Ty\|^2 - k(1-k)\|(x-y)x - (Tx - Ty)\|^2 \\
&= k\|x-y\|^2 + (1-k)(\|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2) - k(1-k)\|(x-y)x - (Tx-Ty)\|^2 \\
&= \|x-y\|^2 + (1-k)k(\|(I-T)x - (I-T)y\|^2) - k(1-k)\|(I-T)x - (I-T)y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Hence $\|Sx - Sy\| \leq \|x - y\|$. Then S is a nonexpansive mapping and we have that $P_C S$ is also nonexpansive, where P_C is a metrics projection on C. For any $j \in \mathbb{N}$, define a mapping $S_j : C \to C$ by $S_j x = \frac{1}{j}\gamma f(x) + (I - \frac{1}{j}A)P_C Sx$. Let us show that S_j is a contraction: let $x, y \in C$; we have

$$\begin{aligned}\|S_jx - S_jy\| &= \left\| \frac{1}{j}\gamma f(x) + \left(I - \frac{1}{j}A\right) P_C Sx - \left(\frac{1}{j}\gamma f(y) + \left(I - \frac{1}{j}A\right) P_C Sy\right)\right\| \\ &\le \frac{1}{j}\gamma\alpha\|x-y\| + \left(1 - \frac{1}{j}\overline{\gamma}\right)\|P_CSx - P_CSy\| \\ &\le \frac{1}{j}\gamma\alpha\|x-y\| + \left(1 - \frac{1}{j}\overline{\gamma}\right)\|x-y\| \\ &\le \left(1 - \frac{1}{j}(\overline{\gamma} - \gamma\alpha)\right)(\|x-y\|).\end{aligned}$$

Hence, S_j is a contraction. By Banach's contraction principle there exists a unique fixed point $u_j \in C$ such that

$$u_j = \frac{1}{j}\gamma f(u_j) + \left(1 - \frac{1}{j}A\right) P_C S u_j. \tag{3.1}$$

Next, we prove the main results.

Theorem 3.1. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and let $T : C \to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0 \le k < 1$. Let A be a strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n, \end{cases} \tag{3.2}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9).

Proof. Note that from the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \le (1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear operator on H,

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \alpha_n\|A\| \\ &\ge 0;\end{aligned}$$

that is to say, $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle((1-\beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\le 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(T)$; we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(P_C S x_n - p)\| \\ &\le \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + \|((1-\beta_n)I - \alpha_n A)\|\|P_C S x_n - p\| \\ &\le \alpha_n\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &= (1 - \alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}.\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{(\overline{\gamma} - \gamma\alpha)}\right\}, \quad n \ge 0,$$

and hence $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_C Sx_n\}$ are bounded. From (3.1), we have, for any $n, j \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - P_C Su_j\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C Sx_n - P_C Su_j\| \\
&= \|\alpha_n(\gamma f(x_n) - AP_C Su_j) + \beta_n(x_n - P_C Su_j) + ((1-\beta_n)I - \alpha_n A)(P_C Sx_n - P_C Su_j)\| \\
&\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n - \alpha_n\overline{\gamma})\|P_C Sx_n - P_C Su_j\| \\
&\le \alpha_n\|\gamma f(x_n) - AP_C Su_j\| + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n - \alpha_n\overline{\gamma})\|x_n - u_j\| \\
&= \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n - u_j\|) + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\| \\
&= \delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|
\end{aligned}$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_C Su_j\| - \overline{\gamma}\|x_n - u_j\|)$, and from $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned}
\|x_{n+1} - P_C Su_j\|^2 &= (\delta_n + \beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)^2 \\
&= (\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)^2 + 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \\
&= \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_C Su_j\|\|x_n - u_j\| + \sigma_n
\end{aligned}$$

where $\sigma_n = 2(\beta_n\|x_n - P_C Su_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \to 0$ as $n \to \infty$, and hence

$$\begin{aligned}
\|x_{n+1} - P_C Su_j\|^2 &\le \beta_n^2\|x_n - P_C Su_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + \beta_n(1-\beta_n)(\|x_n - P_C Su_j\|^2 + \|x_n - u_j\|^2) + \sigma_n \\
&= \beta_n\|x_n - P_C Su_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n.
\end{aligned}$$

For any Banach limit μ and $\beta_n \to 0$, we have

$$\mu_n\|x_n - P_C Su_j\|^2 = \mu_n\|x_{n+1} - P_C Su_j\|^2 \le \mu_n\|x_n - u_j\|^2. \tag{3.3}$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n) + (1 - \frac{1}{j})(P_C Su_j - x_n)$; thus we have

$$\left(1 - \frac{1}{j}\right)(x_n - P_C Su_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n).$$

It follows from Lemma 2.1(ii) that

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - P_C Su_j\|^2 &= \left\|(x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_C Su_j - x_n)\right\|^2 \\
&\ge \|x_n - u_j\|^2 + \frac{2}{j}\langle(\gamma f(u_j) + (I-A)P_C Su_j - x_n), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j - (x_n - u_j), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle - \frac{2}{j}\langle x_n - u_j, x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle - \frac{2}{j}\|x_n - u_j\|^2 \\
&= \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle.
\end{aligned} \tag{3.4}$$

So, by (3.3) and (3.4), we have

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - u_j\|^2 &\ge \left(1 - \frac{1}{j}\right)^2\|P_C Su_j - x_n\|^2 \\
&\ge \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle
\end{aligned}$$

and hence

$$\frac{1}{j^2}\|x_n - u_j\|^2 \ge \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_C Su_j - u_j, x_n - u_j\rangle.$$

This implies that

$$\frac{2}{j}\mu_n\|x_n-u_j\|^2 \geq \mu_n\langle \gamma f(u_j)+(I-A)P_CSu_j-u_j, x_n-u_j\rangle.$$

From Lemmas 2.8 and 2.10, $u_j \to p \in F(T)=F(P_CS)$ as $j\to\infty$, we get

$$\mu_n\langle \gamma f(p)-Ap, x_n-p\rangle \leq 0, \tag{3.5}$$

where p is the solution of variational inequality (1.9). Since $\{x_n\}$, $\{f(x_n)\}$ and $\{P_CSx_n\}$ are bounded, we choose

$$M=\sup\{\|f(x_n)\|+\|x_n\|+\|P_CSx_n\|+\|AP_CSx_n\| : n\in\mathbb{N}\}.$$

On the other hand,

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1})+\beta_{n+1}x_{n+1}+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}\\ &\quad -(\alpha_{n+1}\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n)\|\\ &= \|\alpha_{n+1}\gamma f(x_{n+1})-\alpha_{n+1}\gamma f(x_n)+\alpha_{n+1}\gamma f(x_n)-\alpha_n\gamma f(x_n)+\beta_{n+1}x_{n+1}-\beta_{n+1}x_n\\ &\quad +\beta_{n+1}x_n-\beta_nx_n+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}-((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n\\ &\quad +((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n-((1-\beta_n)I-\alpha_nA)P_CSx_n\|\\ &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\ &\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|P_CSx_{n+1}-P_CSx_n\|\\ &\quad +\|((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_nA)\|\|P_CSx_n\|\\ &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)\|+\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|x_n\|\\ &\quad +(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|P_CSx_n\|+|\alpha_{n+1}-\alpha_n|\|AP_CSx_n\|\\ &\leq (1-\alpha_{n+1}(\overline{\gamma}-\gamma\alpha))\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\gamma M+|\beta_{n+1}-\beta_n|M\\ &\quad +|\beta_{n+1}-\beta_n|M+|\alpha_{n+1}-\alpha_n|M.\end{aligned}$$

From (ii), (iii) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Next, we show that $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$. We consider

$$\begin{aligned}\|x_n-P_CSx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-P_CSx_n\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n-Ap)\|+\beta_n\|x_n-P_CSx_n\|.\end{aligned}$$

From $\alpha_n\to 0$, $\beta_n\to 0$ and (3.6), it follows that $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$.

Next, we show that

$$\limsup_{n\to\infty}\langle \gamma f(p)-Ap, x_n-p\rangle \leq 0,$$

where $p\in F(T)$, where p is the solution of variational inequality (1.9). From (3.6), we have

$$\limsup_{n\to\infty}|\langle \gamma f(p)-Ap, x_{n+1}-p\rangle-\langle \gamma f(p)-Ap, x_n-p\rangle|=0. \tag{3.7}$$

Hence it follows from (3.5) and (3.7) and Lemma 2.11 that

$$\limsup_{n\to\infty}\langle \gamma f(p)-Ap, x_n-p\rangle \leq 0, \tag{3.8}$$

and from $\lim_{n\to\infty}\|x_n-P_CSx_n\|=0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(p)-Ap, P_CSx_n-p\rangle &= \limsup_{n\to\infty}\langle \gamma f(p)-Ap, (P_CSx_n-x_n)+(x_n-p)\rangle\\ &= \limsup_{\to\infty}\langle \gamma f(p)-Ap, x_n-p\rangle \leq 0.\end{aligned} \tag{3.9}$$

Finally, we prove that $x_n\to p$ as $n\to\infty$. We note that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n-p\|^2\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p)\|^2\\ &= \|\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p)\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\ &\quad +2\langle\beta_n(x_n-p)+((1-\beta_n)I-\alpha_nA)(P_CSx_n-p), \alpha_n(\gamma f(x_n)-Ap)\rangle\\ &\leq (\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|P_CSx_n-p\|)^2+2\beta_n\alpha_n\langle x_n-p, (\gamma f(x_n)-Ap)\rangle+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\end{aligned}$$

$$
\begin{aligned}
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle-2\alpha_n^2\langle A(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle\\
&\le(\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|)^2+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle-2\alpha_n^2\langle A(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\le(1-\alpha_n\overline{\gamma})^2\|x_n-p\|^2+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_n^2M\\
&=(1-2(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|^2+(\alpha_n\overline{\gamma})^2M+2\beta_n\alpha_n\alpha\gamma\|x_n-p\|^2+2\beta_n\alpha_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\alpha_n\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_n^2M\\
&=(1-2(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\langle x_n-p,(\gamma f(p)-Ap)\rangle\\
&+2(1-\beta_n)\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_nM+\alpha_n\overline{\gamma}^2M]\\
&=:(1-\gamma_n)\|x_n-p\|^2+b_n
\end{aligned}
$$

where $\gamma_n=2(\overline{\gamma}-\gamma\alpha)\alpha_n$ and $b_n=\alpha_n[2\beta_n\langle x_n-p,(\gamma f(p)-Ap)\rangle+2(1-\beta_n)\langle(P_CSx_n-p),(\gamma f(x_n)-Ap)\rangle+3\alpha_nM+\alpha_n\overline{\gamma}^2M]$. From $\sum_{n=1}^{\infty}\alpha_n=\infty$, (3.8) and (3.9), we have $\Sigma_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\frac{b_n}{\gamma_n}\le 0$. By Lemma 2.2, we have that the sequence $\{x_n\}$ converges strongly to a fixed point p of T, which is the solution of variational inequality (1.9). This completes the proof. □

If $\beta_n\equiv 0$, in Theorem 3.1, we obtain the following corollary.

Corollary 3.2 ([6]). *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C\pm C\subset C$, and let $T:C\to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0\le k<1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma}>0$ and $f:C\to C$ be a contraction with the contractive constant $(0<\alpha<1)$ such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}x_1\in C,\\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)P_CSx_n,\end{cases}\tag{3.10}
$$

where $S:C\to H$ is a mapping defined by $Sx=kx+(1-k)Tx$. If the control sequence $\{\alpha_n\}\subset(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n=0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9). □

Theorem 3.3. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C\pm C\subset C$, and $T:C\to H$ be a k-strictly pseudo-contractive mapping with a fixed point for some $0\le k<1$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma}>0$ and $f:C\to C$ be a contraction with the contractive constant $(0<\alpha<1)$ such that $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}x_1\in C,\\ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)P_CSx_n,\end{cases}\tag{3.11}
$$

where $S:C\to H$ is a mapping defined by $Sx=kx+(1-k)Tx$. If the control sequence $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n=0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$,
(iv) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

Then $\{x_n\}$ converges strongly to a fixed point p of T, which solves the following solution of variational inequality (1.9).

Proof. In the proof of Theorem 3.1, we have that $\{x_n\}$ is bounded. We also obtain that $\{f(x_n)\}$ and $\{P_CSx_n\}$ are bounded. Next, we show that $\|x_{n+1}-x_n\|\to 0$. Define the sequence $z_n=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)P_CSx_n}{1-\beta_n}$, such that $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$, $n\ge 0$. Observe that from the definition of z_n we obtain

$$
\begin{aligned}
z_{n+1}-z_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)P_CSx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}\gamma f(x_n)}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)}{1-\beta_n}\\
&\quad+\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_{n+1}}{1-\beta_{n+1}}-\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n}{1-\beta_{n+1}}+\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)P_CSx_n}{1-\beta_{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}}+\frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_{n+1}}-\frac{((1-\beta_n)I-\alpha_n A)P_C Sx_n}{1-\beta_n}\\
=\;&\frac{\alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))}{1-\beta_{n+1}}+(\alpha_{n+1}-\alpha_n)\frac{(\gamma f(x_{n+1}))}{1-\beta_{n+1}}+\frac{((1-\beta_{n+1})I-\alpha_{n+1}A)}{1-\beta_{n+1}}(P_C Sx_{n+1}-P_C Sx_n)\\
&+\frac{[((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_n A)]}{1-\beta_{n+1}}(P_C Sx_n)\\
&+((1-\beta_n)I-\alpha_n A)\left(\frac{1}{1-\beta_{n+1}}-\frac{1}{1-\beta_n}\right)(P_C Sx_n).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|z_{n+1}-z_n\|\le\;&\frac{\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|}{1-\beta_{n+1}}+|\alpha_{n+1}-\alpha_n|\frac{\|\gamma f(x_{n+1})\|}{1-\beta_{n+1}}+\frac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|P_C Sx_{n+1}-P_C Sx_n\|\\
&+\frac{\|((1-\beta_{n+1})I-\alpha_{n+1}A)-((1-\beta_n)I-\alpha_n A)\|}{1-\beta_{n+1}}\|P_C Sx_n\|+((1-\beta_n-\alpha_n\overline{\gamma})|\frac{1}{1-\beta_{n+1}}\\
&-\frac{1}{1-\beta_n}|\|P_C Sx_n\|)\le\frac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M+\frac{(1-\beta_{n+1}-\alpha_{n+1}\overline{\gamma})}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&+\frac{[|\beta_{n+1}-\beta_n|+|\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}\|AP_C Sx_n\|+\left((1-\beta_n-\alpha_n\overline{\gamma})\left|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}\right.\|P_C Sx_n\|\right)\\
=\;&\frac{\alpha_{n+1}\gamma\alpha}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M+\|x_{n+1}-x_n\|-\frac{\alpha_{n+1}\overline{\gamma}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\
&+\frac{[|\beta_{n+1}-\beta_n|+|\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}M+\left((1-\beta_n-\alpha_n\overline{\gamma})|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M\right)
\end{aligned}
$$

where $M=\sup\{\|f(x_n)\|+\|P_C Sx_n\|+\|AP_C Sx_n\|+\|x_{n+1}-x_n\|:n\in\mathbb{N}\}$. It follows that

$$
\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\le\;&\left|\frac{|\alpha_{n+1}-\alpha_n|}{1-\beta_{n+1}}\gamma M+\frac{[|\beta_{n+1}-\beta_n|+|\alpha_{n+1}-\alpha_n|\overline{\gamma}]}{1-\beta_{n+1}}M\right.\\
&+\left((1-\beta_n-\alpha_n\overline{\gamma})\left|\frac{|\beta_{n+1}-\beta_n|}{(1-\beta_{n+1})(1-\beta_n)}M\right.\right).
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$, we have

$$
\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.12}
$$

From $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, (3.12) and Lemma 2.6, we have

$$
\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.13}
$$

We consider

$$
\begin{aligned}
\|x_{n+1}-x_n\|&=\|(1-\beta_n)z_n-\beta_n x_n-x_n\|\\
&=(1-\beta_n)\|z_n-x_n\|
\end{aligned}
$$

then

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0.
$$

Next, we show that $\lim_{n\to\infty}\|x_n-P_C Sx_n\|=0$. We note that

$$
\begin{aligned}
\|x_n-P_C Sx_n\|&\le\|x_n-x_{n+1}\|+\|x_{n+1}-P_C Sx_n\|\\
&\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AP_C Sx_n\|+\beta_n\|x_n-P_C Sx_n\|,
\end{aligned} \tag{3.14}
$$

and hence

$$
(1-\beta_n)\|x_n-P_C Sx_n\|\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AP_C Sx_n\|.
$$

From $\alpha_n\to 0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, it follows that $\lim_{n\to\infty}\|x_n-P_C Sx_n\|=0$. From (3.1), we have, for any $n,j\in\mathbb{N}$,

$$
\begin{aligned}
\|x_{n+1}-P_C Su_j\|&=\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)P_C Sx_n-P_C Su_j\|\\
&=\|\alpha_n(\gamma f(x_n)-AP_C Su_j)+\beta_n(x_n-P_C Su_j)+((1-\beta_n)I-\alpha_n A)(P_C Sx_n-P_C Su_j)\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_n\|\gamma f(x_n) - AP_CSu_j\| + \beta_n\|x_n - P_CSu_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|P_CSx_n - P_CSu_j\| \\
&\leq \alpha_n\|\gamma f(x_n) - AP_CSu_j\| + \beta_n\|x_n - P_CSu_j\| + (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - u_j\| \\
&= \alpha_n(\|\gamma f(x_n) - AP_CSu_j\| - \overline{\gamma}\|x_n - u_j\|) + \beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\| \\
&= \delta_n + \beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\|
\end{aligned}$$

where $\delta_n = \alpha_n(\|\gamma f(x_n) - AP_CSu_j\| - \overline{\gamma}\|x_n - u_j\|)$. From $\lim_{n\to\infty}\alpha_n = 0$, we have $\delta_n \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned}
\|x_{n+1} - P_CSu_j\|^2 &= (\delta_n + \beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\|)^2 \\
&= (\beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\|)^2 + 2(\beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \\
&= \beta_n^2\|x_n - P_CSu_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 + 2\beta_n(1-\beta_n)\|x_n - P_CSu_j\|\|x_n - u_j\| + \sigma_n
\end{aligned}$$

where $\sigma_n = 2(\beta_n\|x_n - P_CSu_j\| + (1-\beta_n)\|x_n - u_j\|)\delta_n + \delta_n^2 \to 0$ as $n \to \infty$, and hence

$$\begin{aligned}
\|x_{n+1} - P_CSu_j\|^2 &\leq \beta_n^2\|x_n - P_CSu_j\|^2 + (1-\beta_n)^2\|x_n - u_j\|^2 \\
&\quad + \beta_n(1-\beta_n)(\|x_n - P_CSu_j\|^2 + \|x_n - u_j\|^2) + \sigma_n \\
&= \beta_n\|x_n - P_CSu_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n. \qquad (3.15)
\end{aligned}$$

From (3.25), we have

$$\begin{aligned}
\|x_n - P_CSu_j\|^2 &= \|(x_n - x_{n+1}) + (x_{n+1} - P_CSu_j)\|^2 \\
&= \|x_{n+1} - P_CSu_j\|^2 + 2\langle x_{n+1} - P_CSu_j, x_n - x_{n+1}\rangle + \|x_n - x_{n+1}\|^2 \\
&= \|x_{n+1} - P_CSu_j\|^2 + 2\|x_{n+1} - P_CSu_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2, \\
&\leq \beta_n\|x_n - P_CSu_j\|^2 + (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_CSu_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2
\end{aligned}$$

and hence

$$(1-\beta_n)\|x_n - P_CSu_j\|^2 \leq (1-\beta_n)\|x_n - u_j\|^2 + \sigma_n + 2\|x_{n+1} - P_CSu_j\|\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|^2.$$

For any Banach limit μ and $\sigma_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, we have

$$\mu_n\|x_n - P_CSu_j\|^2 \leq \mu_n\|x_n - u_j\|^2. \qquad (3.16)$$

Since $u_j - x_n = \frac{1}{j}(\gamma f(u_j) + (I-A)P_CSu_j - x_n) + (1-\frac{1}{j})(P_CSu_j - x_n)$, we have

$$\left(1 - \frac{1}{j}\right)(x_n - P_CSu_j) = (x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_CSu_j - x_n).$$

It follows from Lemma 2.1(ii) that

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - P_CSu_j\|^2 &= \|(x_n - u_j) + \frac{1}{j}(\gamma f(u_j) + (I-A)P_CSu_j - x_n)\|^2 \\
&\geq \|x_n - u_j\|^2 + \frac{2}{j}\langle(\gamma f(u_j) + (I-A)P_CSu_j - x_n), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_CSu_j - u_j - (x_n - u_j), x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_CSu_j - u_j, x_n - u_j\rangle - \frac{2}{j}\langle x_n - u_j, x_n - u_j\rangle \\
&= \|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_CSu_j - u_j, x_n - u_j\rangle - \frac{2}{j}\|x_n - u_j\|^2 \\
&= \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_CSu_j - u_j, x_n - u_j\rangle. \qquad (3.17)
\end{aligned}$$

So, by (3.16) and (3.17), we have

$$\begin{aligned}
\left(1 - \frac{1}{j}\right)^2\|x_n - u_j\|^2 &\geq \left(1 - \frac{1}{j}\right)^2\|P_CSu_j - x_n\|^2 \\
&\geq \left(1 - \frac{2}{j}\right)\|x_n - u_j\|^2 + \frac{2}{j}\langle\gamma f(u_j) + (I-A)P_CSu_j - u_j, x_n - u_j\rangle
\end{aligned}$$

and hence

$$\frac{1}{j^2}\|x_n - u_j\|^2 \geq \frac{2}{j}\langle \gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.$$

This implies that

$$\frac{2}{j}\mu_n\|x_n - u_j\|^2 \geq \mu_n\langle \gamma f(u_j) + (I-A)P_C S u_j - u_j, x_n - u_j\rangle.$$

From Lemmas 2.8 and 2.10, $u_j \to p \in F(T) = F(P_C S)$ *as* $j \to \infty$, we get

$$\mu_n\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.18}$$

where p is the solution of variational inequality (1.9). Next, we show that

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0,$$

where $p \in F(T)$, where p is the solution of variational inequality (1.9). From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have

$$\limsup_{n\to\infty}|\langle \gamma f(p) - Ap, x_{n+1} - p\rangle - \langle \gamma f(p) - Ap, x_n - p\rangle| = 0. \tag{3.19}$$

Hence it follows from (3.18) and (3.19) and Lemma 2.11 that

$$\limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0, \tag{3.20}$$

and from (3.14), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(p) - Ap, P_C S x_n - p\rangle &= \limsup_{n\to\infty}\langle \gamma f(p) - Ap, (P_C S x_n - x_n) + (x_n - p)\rangle \\ &= \limsup_{n\to\infty}\langle \gamma f(p) - Ap, x_n - p\rangle \leq 0.\end{aligned} \tag{3.21}$$

By the same argument as used in Theorem 3.1, we have that the sequence $\{x_n\}$ converges strongly to a fixed point p of T, which is the solution of variational inequality (1.9). This completes the proof. □

Theorem 3.4. *Let H be a Hilbert space, C be a nonempty closed convex subset of H such that $C \pm C \subset C$, and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)P_C S x_n, \end{cases} \tag{3.22}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\Sigma_{i=1}^N \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, \dots, N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^N$, which solves the following solution of the variational inequalities:

$$\langle (A - \gamma f)p, p - x\rangle \leq 0, \quad \forall x \in \cap_{i=1}^N F(T_i). \tag{3.23}$$

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{i=1}^N \eta_i T_i x$. By Lemmas 2.12 and 2.13, we conclude that : $C \to H$ is a $k-$strictly pseudo-contractive mapping with $k = \max\{k_i : i = 1, 2, \dots, N\}$ and $F(T) = F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$. From Theorem 3.1, we can obtain desired conclusion easily. This completes the proof. □

If $\beta_n \equiv 0$, Theorem 3.4 reduces to the following corollary.

Corollary 3.5 (*[6]*). *Let H be a Hilbert space, K be a nonempty closed convex subset of H such that $K \pm K \subset K$, and $T_i : K \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \leq k_i < 1$ and $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let A be strongly positive* *bounded linear operator on K with coefficient $\overline{\gamma} > 0$ and $f : K \to K$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.24}$$

where $S : K \to H$ is a mapping defined by $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, \ldots, N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^{N}$, which solves the following solution of the variational inequalities:

$$\langle (A - \gamma f)p, p - x\rangle \le 0, \forall x \in \bigcap_{i=1}^{N} F(T_i).$$

From the proof of Theorem 3.3, we can obtain the following theorem.

Theorem 3.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H such that $C \pm C \subset C$, and $T_i : C \to H$ be a k_i-strictly pseudo-contractive mapping with a fixed point for some $0 \le k_i < 1$ and $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let A be strongly positive bounded linear operator on C with coefficient $\overline{\gamma} > 0$ and $f : C \to C$ be a contraction with the contractive constant $(0 < \alpha < 1)$ such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) P_C S x_n, \end{cases} \tag{3.25}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)\sum_{i=1}^{N} \eta_i T_i x$ and $k = \max\{k_i : i = 1, 2, \ldots, N\}$. If the control sequence $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a common fixed point p of $\{T_i\}_{i=1}^{N}$, which solves the following solution of variational inequalities (3.23).

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