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Note

Cycle factorizations of cycle products [1]

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Abstract

Let n and $k_1, k_2, \ldots, k_n$ be integers with $n > 1$ and $k_i \geqslant 2$ for $1 \leqslant i \leqslant n$. We show that there exists a C_s-factorization of $\prod_{i=1}^n C_{2^{k_i}}$ if and only if $s = 2^t$ with $2 \leqslant t \leqslant k_1 + \cdots + k_n$. We also settle the problem of cycle factorizations of the d-cube. © 1998 Elsevier Science B.V. All rights reserved

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1. Graph decompositions and products

A sequence $H_1, H_2, \ldots, H_n$ of graphs with union G is called a *decomposition* of G in case each edge of G is in H_i for exactly one i, and in this case we write $G = H_1 + H_2 + \cdots + H_n$. If in addition the subgraphs H_i are all isomorphic to H, then we write $G = nH$, say that H *divides* G, and write $H|G$.

We call a subgraph F of G a *factor* of G if it contains all the vertices of G. If in addition each component of F is isomorphic to H, we call F an H*-factor* of G. A decomposition of G into H-factors is called an H*-factorization* of G, and in this case we write $H||G$.

Let $H_1, H_2, \ldots, H_n$ be a decomposition of a graph G. We write $G = H_1 +_s H_2 +_s \cdots +_s H_n = \sum s_{i=1}^n H_i$ if each subgraph H_i is a factor of G, and $G = H_1 +_d H_2 +_d \cdots +_d H_n = \sum d_{i=1}^n H_i$ if the subgraphs H_i are pairwise vertex disjoint.

If G_1 and G_2 are graphs with vertex sets V_1 and V_2, respectively, then their *product* is the graph $G_1 \times G_2$ with vertex set $V_1 \times V_2$ and $\{(u_1, u_2), (v_1, v_2)\}$ an edge if and

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only if $\{u_1, v_1\}$ is an edge of G_1 and $u_2 = v_2$ or $u_1 = v_1$ and $\{u_2, v_2\}$ is an edge of G_2. It is easily proved that if G_i has v_i vertices and e_i edges, $1 \leqslant i \leqslant n$, then $\prod G_i$ has $\prod v_i$ vertices and $\sum e_i \prod_{j \neq i} v_j$ edges. We use the terminology of [3].

The decomposition of graphs has been, and remains, the focus of a great deal of research (see [4] for a thorough discussion of the subject). In particular, K_k-decompositions of K_n and C_k-decompositions of K_n have received much attention. For an excellent reference on cycle decompositions, the reader is directed to [8].

In this article we investigate cycle factorizations of $\prod_{i=1}^{n} C_{2^{k_i}}$, where n and $k_1, k_2, \ldots, k_n$ are integers $\geqslant 2$.

We use the following result of Stong [9] on Hamilton decompositions of the product of two Hamilton decomposable graphs. Stong's result subsumes earlier results on the same topic by Kotzig [7], Foregger [5] and Aubert and Schneider [1,2].

Theorem A. *Let G_1 and G_2 be graphs that are decomposable into n and m Hamilton cycles, respectively, with $n \leqslant m$. Then $G_1 \times G_2$ is Hamilton decomposable if one of the following holds:*
(1) $m \leqslant 3n$,
(2) $n \geqslant 3$,
(3) $|V(G_1)|$ *is even, or*
(4) $|V(G_2)| \geqslant 6\lceil m/n \rceil - 3$, *where* $\lceil x \rceil$ *is the least integer greater than or equal to* x.

We state two lemmas which extend in an obvious way to sums with more than two terms. Their straightforward proofs will be omitted.

Lemma 1. *We have* $H \times (G_1 +_d G_2) = (H \times G_1) +_d (H \times G_2)$.

Lemma 2. *We have* $(H_1 +_s H_2) \times (G_1 +_s G_2) = (H_1 \times G_1) +_s (H_2 \times G_2)$.

2. Expanding decompositions

As usual, C_m denotes a cycle with m edges. We allow $m = 2$ in some preliminary results. We identify the vertices of C_m with Z_m, with i adjacent to $i + 1$. If x is a k-tuple, let $x[i]$ denote the ith component of x. Let

$$G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_k}.$$

Let $E = \{\varepsilon \in \{0, 1, -1\}^k : \sum_{i=1}^{k} |\varepsilon[i]| = 1\}$. We can represent any path in G by $P = (v + \sum_{j=1}^{t} \varepsilon_j : t = 0, 1, \ldots, n)$, where $v \in V(G)$ and $\varepsilon_j \in E$ for all j. Call this path *i-closed* if $\sum_{j=1}^{n} \varepsilon_j[i] \equiv 0 \pmod{m_i}$. If P is a cycle, then P is i-closed for all i. Call P *i-local* if $\sum_{j=1}^{n} \varepsilon_j[i] = 0$. Call a decomposition of G into cycles *i-local* if every cycle in the decomposition is i-local.

Theorem 1. *Let G be as above, suppose* $1 \leqslant h \leqslant k$, *and let*

$$G' = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_{h-1}} \times C_{qm_h} \times C_{m_{h+1}} \times \cdots \times C_{m_k},$$

where q is a positive integer. Suppose that G has an h-local C_r*-decomposition* (C_r*-factorization). Then* G' *has a* C_r*-decomposition* (C_r*-factorization). Furthermore, if* $1 \leqslant h' \leqslant k$ *and the decomposition of G is* h'*-local, then so is the decomposition of* G'.

Proof. We will apply the theory of voltage graphs [6]. The base graph will be G and the group Z_q. Orient the edge $e = \{u, u+\varepsilon\}$ of G from u to v if $\varepsilon[i] = 1$ for some i, and assign it the voltage $\alpha(e) \in Z_q$, where $\alpha(e) = 1$ if $i = h$ and $u[i] = -1$, and $\alpha(e) = 0$ otherwise. Then it can easily be seen that the derived graph G^α of the voltage graph $\langle G, \alpha \rangle$ is isomorphic to G' if the orientations of its edges are ignored. Each h-local r-cycle of G has net voltage 0, which has order 1 in Z_q. Then according to Theorem 2.1.3 of [6] each such cycle corresponds to q vertex-disjoint cycles of the same length in G'.

If we have a C_r-factorization of G, we can partition the cycles of the decomposition into edge-disjoint 2-factors, and these lift to edge-disjoint 2-factors of G'.

The last statement of the theorem is clear from how the decomposition of G' is defined. □

3. Factorizations of products of two cycles

In this section we give two theorems and a lemma concerning factorizations of the products of two cycles. We use the abbreviations 1 for $(1,0)$, $\bar{1}$ for $(-1,0)$, 2 for $(0,1)$, and $\bar{2}$ for $(0,-1)$. The path in $C_r \times C_s$ with vertices $v, v+a_1, v+a_1+a_2, \ldots, v+a_1+\cdots+a_n$ will be abbreviated $v + a_1a_2\cdots a_n$, and we use exponents to indicate repeated symbols.

Theorem 2. *Let* $m \geqslant 2$, $n \geqslant 1$ *and* $q \geqslant 1$ *be integers. Then* $C_{4mn} \| C_{2m} \times C_{2nq}$.

Proof. We decompose $C_{2m} \times C_{2n}$ into two Hamilton cycles, namely

$$A = (0,0) + (21\bar{2}1)^{m-1}2\left(2\bar{1}^{2(m-1)}21^{2(m-1)}\right)^{n-1}1\bar{2}^{2n-1}1$$

and

$$B = (0,0) + \left(1\bar{2}^{2n-1}12^{2n-1}\right)^{m-2}1\bar{2}^{2n-1}1(1^2 2\bar{1}^2 2)^{n-1}21\bar{2}12.$$

These are pictured in Fig. 1. (Note that the order of the vertices is changed for B to simplify the picture.) By examining Fig. 1 it can be verified that these are Hamilton cycles. It may also be checked that A and B have no common edges. For example

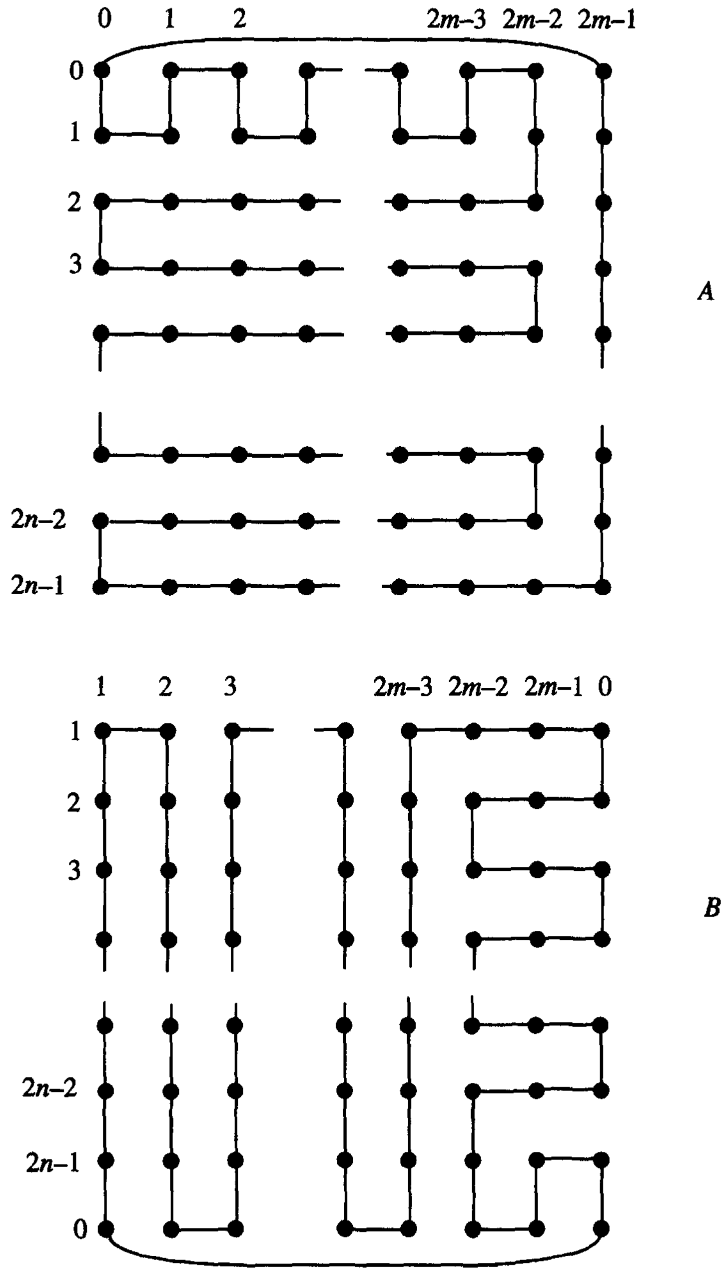

Fig. 1.

the 'horizontal' edges $\{(x,y),(x+1,y)\}$ of A with $y = 0$ are exactly those with $x = 1,3,\ldots,2m-1$, while the corresponding edges of B have $x = 0,2,\ldots,2m-2$. The other rows and columns can be checked similarly.

Since 2 and $\bar{2}$ appear the same number of times in A, and also in B, these cycles are 2-local. Thus, the result follows from Theorem 1. □

Theorem 3. *Let p, q, and m be positive integers. Then $C_{4m}||C_{2mp} \times C_{2mq}$ if either $m > 1$ or else $p > 1$ and $q > 1$.*

Proof. We start with the case $m > 1$, where we will first show that $C_{4m}||C_{2m} \times C_{2m}$. The factors will consist of cycles

$$A_r = (2r, 2r) + 1^{2m-1} 2 \bar{1}^{2m-1} \bar{2}, \quad 0 \leqslant r < m$$

and

$$B_s = (2s, 2s) + \bar{2}^{2m-1} \bar{1} 2^{2m-1} 1, \quad 0 \leqslant s < m.$$

These are easily seen to be cycles of length $4m$. The cycles A_r are vertex disjoint since the second coordinate of each vertex in A_r is either $2r$ or $2r + 1$.

It remains to show that no edge is in both A_r and B_s. Suppose the horizontal edge $\{(x, y), (x + 1, y)\}$ is in both. Note that for this edge to be in B_s we must have $x = 2s - 1$ and $y = 2s$ or $y = 2s + 1$, while for it to be in A_r we must have $x \in \{2r, 2r + 1, \ldots, 2r + 2m - 2\}$ and $y = 2r$ or $y = 2r + 1$. Whether y is even or odd we conclude that $r = s$ and so $x = 2r - 1$, a contradiction. The proof for vertical edges is similar.

Since the cycles in the factorization are both 1-local and 2-local, this case of the theorem follows from an application of Theorem 1.

The remaining case is when $m = 1$ and p and q both exceed 1. Then it may be checked that the factors

$$A_{r,s} = (2r, 2s) + 12\bar{1}\bar{2}, \quad 0 \leqslant r < p, \ 0 \leqslant s < q$$

and

$$B_{u,v} = (2u, 2v) + \bar{1}\bar{2}12, \quad 0 \leqslant u < p, \ 0 \leqslant v < q$$

work. □

Lemma 3. *Let k and l be integers $\geqslant 2$. Then $C_s||C_{2^k} \times C_{2^l}$ if $s = 2^t$ with $2 \leqslant t \leqslant k + l$.*

Proof. Let $s = 2^t$ with $2 \leqslant t \leqslant k + l$. Assume $k \leqslant l$. If $t \leqslant k + 1$, we apply Theorem 3 with $m = 2^{t-2}$, $p = 2^{k+1-t}$ and $q = 2^{l+1-t}$, while if $t \geqslant k + 2$, we apply Theorem 2 with $m = 2^{k-1}$, $n = 2^{t-k-1}$ and $q = 2^{k+l-t}$. □

4. Factorizations of cycle products

Lemma 4. *Let j, k and l be integers $\geqslant 2$. Then $C_s||C_{2^j} \times C_{2^k} \times C_{2^l}$ if $s = 2^t$ with $2 \leqslant t \leqslant j + k + l$.*

Proof. First, we treat the case when $t \leqslant j+l$. Let J represent the graph with 2^k vertices and no edges. We have

$$
\begin{aligned}
C_{2^j} \times C_{2^k} \times C_{2^l} &\cong C_{2^k} \times C_{2^j} \times C_{2^l} \\
&= (C_{2^k} +_s J) \times \left(\textstyle\sum_d C_{2^t} +_s \sum_d C_{2^t} \right) \quad \text{(Lemma 3)} \\
&= (C_{2^k} \times \textstyle\sum_d C_{2^t}) +_s (J \times \sum_d C_{2^t}) \quad \text{(Lemma 2)} \\
&= \textstyle\sum_d (C_{2^k} \times C_{2^t}) +_s \sum_d (J \times C_{2^t}) \quad \text{(Lemma 1).}
\end{aligned}
$$

This case now follows from Lemma 3 and the fact that $C_{2^t} \| J \times C_{2^t}$.

Now, assume $t > j+l$. Let $p = t-(j+l) \geqslant 1$. By Theorem 2 with $m = 2^{j-1}$, $n = 2^{p-1}$, and $q = 1$ we have $C_{2^j} \times C_{2^p} = A +_s B$, where A and B are 2-local Hamilton cycles of length 2^{j+p}, as pictured in Fig. 1. We will extend this to a Hamilton factorization of $C_{2^j} \times C_{2^p} \times C_{2^l} = (A +_s B) \times C$, where $C = C_{2^l}$, using the method of Aubert and Schneider [2]. They factor $G = (A+B) \times C$ into Hamilton cycles by first labeling the vertices of $A+B$ as $v_1, v_2, \ldots$ according to their order in B. In our case we will let $v_1 = (0, 2n-1)$, $v_2 = (0,0)$, etc. (See Fig. 1.) The vertices adjacent to v_1 in A are $a = v_4 = (1, 2n-1)$ and $b = v_{4mn-6} = (0, 2n-2)$.

First, Aubert and Schneider construct a Hamilton cycle $C^{(1)}$ in G in which all edges moving in the second coordinate come from edges of A. Since A has no edges in which the second coordinate changes from $2n-1$ to 0, the cycle $C^{(1)}$ is 2-local.

Then $G \backslash C^{(1)}$ is factored into two more Hamilton cycles, $C^{(2)}$ and $C^{(3)}$. All the edges of $G \backslash C^{(1)}$ that move in the second coordinate come from B except for some corresponding to the edge $\{v_1, b\}$ of A. The latter have the form $\{(0, 2n-1, i), (0, 2n-2, i)\}$ in G. But $G \backslash C^{(1)}$ contains no edges where the second coordinate changes from 0 to 1, and so $C^{(2)}$ and $C^{(3)}$ are also 2-local.

Thus, we have $C_{2^j} \times C_{2^p} \times C_{2^l} = \sum_1^3 C_{2^{j+p+l}} = \sum_1^3 C_{2^t}$, where each factor on the right is 2-local. Note that $p = t-(j+l) \leqslant j+k+l-(j+l) = k$. Then by Theorem 1 with $h=2$ and $q = 2^{k-p}$ we have $C_{2^t} \| C_{2^j} \times C_{2^k} \times C_{2^l}$. $\square$

Theorem 4. *Let n and $k_1, k_2, \ldots, k_n$ be integers with $n > 1$ and $k_i \geqslant 2$ for $1 \leqslant i \leqslant n$. Then $C_s \| \prod_{i=1}^n C_{2^{k_i}}$ if and only if $s = 2^t$ with $2 \leqslant t \leqslant k_1 + \cdots + k_n$.*

Proof. If $C_s \| \prod C_{2^{k_i}}$, then, since the latter graph has $2^{k_1+\cdots+k_n}$ vertices, we must have $s | 2^{k_1+\cdots+k_n}$, from which it follows that $s = 2^t$ with $1 \leqslant t \leqslant k_1 + \cdots + k_n$. Since the product has no parallel edges, $t \geqslant 2$.

The cases $n = 2$ and 3 of the converse are covered by Lemmas 3 and 4. Now assume we know that $C_{2^t} \| \prod_{i=1}^n C_{2^{k_i}}$ whenever $2 \leqslant t \leqslant k_1 + \cdots + k_n$, where n is some integer $\geqslant 3$. We will prove the $n+1$ case.

Set $K = k_1 + \cdots + k_n$. First, assume $2 \leqslant t \leqslant K$. Let J represent the graph with $2^{k_{n+1}}$ vertices and no edges. We have

$$\begin{aligned}
\prod_{i=1}^{n+1} C_{2^{k_i}} &= \prod_{i=1}^{n} C_{2^{k_i}} \times C_{2^{k_{n+1}}} \\
&= \left(\sum_{s,\,i=1}^{n} \sum_d C_{2^t} \right) \times \left(C_{2^{k_{n+1}}} +_s \sum_{s,\,i=2}^{n} J \right) \quad \text{(induction hypothesis)} \\
&= \left(\sum_d C_{2^t} \right) \times C_{2^{k_{n+1}}} +_s \sum_{s,\,i=2}^{n} \left(\sum_d C_{2^t} \right) \times J \quad \text{(Lemma 2)} \\
&= \sum_d (C_{2^t} \times C_{2^{k_{n+1}}}) +_s \sum_{s,\,i=2}^{n} \sum_d (C_{2^t} \times J),
\end{aligned}$$

where the last equality uses Lemma 1. It is easy to see that $C_{2^t} || C_{2^t} \times J$, and so our result now follows from Lemma 3.

Now, assume $K < t \leqslant K + k_{n+1}$, so that $k_n < t - K + k_n \leqslant k_n + k_{n+1}$. Then

$$\begin{aligned}
\prod_{i=1}^{n+1} C_{2^{k_i}} &= \left(\prod_{i=1}^{n-1} C_{2^{k_i}} \right) \times (C_{2^{k_n}} \times C_{2^{k_{n+1}}}) \\
&= \left(\sum_{s,\,i=1}^{n-1} C_{2^{K-k_n}} \right) \times \left(\sum_d C_{2^{t-K+k_n}} +_s \sum_d C_{2^{t-K+k_n}} \right) \quad \text{(Theorem A and Lemma 3)} \\
&= (C_{2^{K-k_n}} \times \sum_d C_{2^{t-K+k_n}}) +_s \left(\left(\sum_{s,\,i=2}^{n-1} C_{2^{K-k_n}} \right) \times \sum_d C_{2^{t-K+k_n}} \right) \quad \text{(Lemma 2)} \\
&= \sum_d (C_{2^{K-k_n}} \times C_{2^{t-K+k_n}}) +_s \sum_d \left(\left(\sum_{s,\,i=2}^{n-1} C_{2^{K-k_n}} \right) \times C_{2^{t-K+k_n}} \right) \quad \text{(Lemma 1)} \\
&= \sum_d \left(\sum_{s,\,i=1}^{2} C_{2^t} \right) +_s \sum_d \left(\sum_{s,\,i=1}^{n-1} C_{2^t} \right) \quad \text{(Lemma 3 and Theorem A)} \\
&= \sum_{s,\,i=1}^{n+1} \sum_d C_{2^t}.
\end{aligned}$$

This completes the proof. □

Theorem 5. *For $j = 1, \ldots, n$, let G_j be a graph such that $G_j \cong \sum_{s,\,i=1}^{r} \sum_d C_{2^{k_j}}$, where $n > 1$ and $k_j \geqslant 2$ for $1 \leqslant j \leqslant n$. Then a sufficient condition that $C_s || \prod_{j=1}^{n} G_j$ is that $s = 2^t$, $2 \leqslant t \leqslant k_1 + \cdots + k_n$. If $\prod_{j=1}^{n} |V(G_j)| = 2^{k_1 + \cdots + k_n}$, then this condition is also necessary.*

Proof. If $2 \leqslant t \leqslant k_1 + \cdots + k_n$, we have

$$\begin{aligned}
\prod_{j=1}^{n} G_j &= \prod_{j=1}^{n} \left(\sum_{s,\,i=1}^{r} \sum_d C_{2^{k_j}} \right) = \sum_{s,\,i=1}^{r} \prod_{j=1}^{n} \sum_d C_{2^{k_j}} \quad \text{(Lemma 2)} \\
&= \sum_{s,\,i=1}^{r} \sum_d \prod_{j=1}^{n} C_{2^{k_j}} \quad \text{(Lemma 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathop{\Sigma_s}_{i=1}^{r} \Sigma_d \mathop{\Sigma_s}_{j=1}^{n} \Sigma_d C_{2^t} \quad \text{(Theorem 4)}\\
&= \mathop{\Sigma_s}_{i=1}^{r} \mathop{\Sigma_s}_{j=1}^{n} \Sigma_d \Sigma_d C_{2^t}\\
&= \mathop{\Sigma_s}_{i=1}^{rn} \Sigma_d C_{2^t}.
\end{aligned}
$$

Conversely, if we have $\prod_{j=1}^{n} |V(G_j)| = 2^{k_1+\cdots+k_n}$, then $\prod_{j=1}^{n} G_j$ has $2^{k_1+\cdots+k_n}$ vertices and no parallel edges, and so $C_s || \prod_{j=1}^{n} G_j$ implies $s = 2^t$ with $2 \leqslant t \leqslant k_1 + \cdots + k_n$. $\square$

5. An application to cubes

The *d-cube*, denoted Q_d, is defined to be $(K_2)^d$. It is easily seen that Q_d has 2^d vertices, each of degree d, and $d2^{d-1}$ edges, and is bipartite. Thus, if $C_s | Q_d$, then s must be even. Since the degree of a vertex of Q_d is twice the number of cycles in the decomposition that contain it, d must also be even.

The following theorem completely settles the question of for which s and d we have $C_s || Q_d$.

Theorem 6. *We have $C_s || Q_d$ if and only if d is even and $s = 2^t$, $2 \leqslant t \leqslant d$.*

Proof. If $C_s || Q_d$, then we have already argued that d is even, say $d = 2n$. Then we have $Q_d = ((K_2)^2)^n = (C_4)^n$, and Theorem 4 shows that $s = 2^t$ with $2 \leqslant t \leqslant d$.

Conversely, if $d = 2n$ and $s = 2^t, 2 \leqslant t \leqslant d$, then Theorem 4 tells us that $C_s || (C_4)^n = Q_d$ except possibly for $n = 1$, which entails only $C_4 || C_4$. $\square$

Although if d is odd we cannot hope to find a cycle factorization of Q_d, we can do so up to a 1-factor.

Theorem 7. *Let $d \geqslant 3$ be odd and suppose $2 \leqslant t \leqslant d$. Then there exists a 1-factor F of Q_d such that $C_{2^t} || Q_d \backslash F$.*

Proof. We can think of $Q_d = Q_{d-1} \times K_2$ as the disjoint union of two graphs G and G', each isomorphic to Q_{d-1}, and to each other via $v \to v'$, along with all edges $\{v, v'\}$. Then if $2 \leqslant t \leqslant d - 1$ we have C_{2^t}-factorizations of G and G' by Theorem 5. These can be combined into a C_{2^t}-factorization of $Q_d \backslash F$, where F is the union of all the edges $\{v, v'\}$.

The case where $t = d$ remains. We know that G has a decomposition into cycles $H_1, H_2, \ldots, H_r$ of length 2^{d-1}, where $r = (d-1)/2$. We claim we can find distinct vertices $v_1, w_1, v_2, w_2, \ldots, v_r, w_r$ such that $\{v_i, w_i\}$ is an edge of H_i for $1 \leqslant i \leqslant r$. For suppose we have found $v_1, w_1, v_2, w_2, \ldots, v_j, w_j$, with $j < r$. Then $2j < 2r =$

$d-1 \leqslant 2^{d-2}$, since $d \geqslant 3$. Thus we have used up fewer than half the vertices of H_{j+1} and can choose v_{j+1} and w_{j+1}. This proves the claim.

Now from each pair H_i, H_i' in G and G' we form a cycle of length 2^d by removing the edges $\{v_i, w_i\}$ and $\{v_i', w_i'\}$ and inserting $\{v_i, v_i'\}$ and $\{w_i, w_i'\}$. Since the v's and w's are distinct this leaves a 1-factor in Q_d. □

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