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# A duality between pairs of split decompositions for a $Q$-polynomial distance-regular graph

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ABSTRACT

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $D \geq 3$ and standard module $V$. Recently, Ito and Terwilliger introduced four direct sum decompositions of $V$; we call these the $(\mu, \nu)$-*split decompositions* of $V$, where $\mu, \nu \in \{\downarrow, \uparrow\}$. In this paper we show that the $(\downarrow, \downarrow)$-split decomposition and the $(\uparrow, \uparrow)$-split decomposition are dual with respect to the standard Hermitian form on $V$. We also show that the $(\downarrow, \uparrow)$-split decomposition and the $(\uparrow, \downarrow)$-split decomposition are dual with respect to the standard Hermitian form on $V$.



## 1. Introduction

We consider a distance-regular graph $\Gamma$ with vertex set $X$ and diameter $D \geq 3$ (see Section 3 for formal definitions). We assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Let $V$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We call $V$ the *standard module*. We endow $V$ with the Hermitian form $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$. We call this form the *standard Hermitian form* on $V$. Recently, Ito and Terwilliger introduced four direct sum decompositions of $V$ [12]; we call these the $(\mu, \nu)$-*split decompositions* of $V$, where $\mu, \nu \in \{\downarrow, \uparrow\}$. These are defined as follows. Fix a vertex $x \in X$. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ that represents the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$. For $-1 \leq i, j \leq D$, we define

$$V_{i,j}^{\downarrow\downarrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV),$$
$$V_{i,j}^{\uparrow\downarrow} = (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV),$$
$$V_{i,j}^{\downarrow\uparrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V),$$
$$V_{i,j}^{\uparrow\uparrow} = (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).$$

For $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$, we have $V_{i-1,j}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$ and $V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$; therefore $V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$. Let $\tilde{V}_{i,j}^{\mu\nu}$ denote the orthogonal complement of $V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu}$ in $V_{i,j}^{\mu\nu}$ with respect to the standard Hermitian form. By [12, Lemma 10.3],

$$V = \sum_{i=0}^{D}\sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \quad \text{(direct sum)}.$$

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We call the above sum the $(\mu, \nu)$-*split decomposition* of $V$ with respect to $x$. We show that with respect to the standard Hermitian form the $(\downarrow, \downarrow)$-split decomposition (resp. $(\downarrow, \uparrow)$-split decomposition) and the $(\uparrow, \uparrow)$-split decomposition (resp. $(\uparrow, \downarrow)$-split decomposition) are dual in the following sense.

**Theorem 1.1.** *With the above notation, the following* (i), (ii) *hold for* $0 \leq i, j, r, s \leq D$.

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\uparrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.
(ii) $\tilde{V}_{i,j}^{\downarrow\uparrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\downarrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.

To prove Theorem 1.1 we use a result about tridiagonal pairs (Theorem 2.9) which may be of independent interest. We also use some results about the subconstituent algebra of $\Gamma$.

## 2. Tridiagonal pairs

In this section we consider a tridiagonal pair for which the underlying vector space supports a certain Hermitian form. Throughout this section $V$ denotes a vector space over $\mathbb{C}$ with finite positive dimension. We start with the definition of a tridiagonal pair.

**Definition 2.1** (*[10, Definition 1.1]*). Let $V$ denote a vector space over $\mathbb{C}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on $V$ we mean an ordered pair $A, A^*$ of linear transformations on $V$ that satisfy the following four conditions.

(i) $A$ and $A^*$ are both diagonalizable on $V$.
(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of $A$ such that
$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$
where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of $A^*$ such that
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$
where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace $W$ of $V$ such that both $AW \subseteq W$ and $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

**Note 2.2.** According to a common notational convention, $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$, the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

With reference to Definition 2.1, we have $d = \delta$ [10, Lemma 4.5]; we call this common value the *diameter* of $A, A^*$. See [10,11] for more information on tridiagonal pairs.

With reference to Definition 2.1, by the construction we have the direct sum decompositions $V = \sum_{i=0}^{d} V_i$ and $V = \sum_{i=0}^{d} V_i^*$. We now recall four more direct sum decompositions of $V$ called the split decompositions.

**Lemma 2.3** (*[11, Lemma 4.2]*). *With reference to Definition* 2.1, *for* $\mu, \nu \in \{\downarrow, \uparrow\}$, *we have*
$$V = \sum_{i=0}^{d} U_i^{\mu\nu} \quad (\textit{direct sum}),$$
*where*
$$U_i^{\downarrow\downarrow} = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}),$$
$$U_i^{\uparrow\downarrow} = (V_{d-i}^* + \cdots + V_d^*) \cap (V_0 + \cdots + V_{d-i}),$$
$$U_i^{\downarrow\uparrow} = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d),$$
$$U_i^{\uparrow\uparrow} = (V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d).$$

**Definition 2.4.** By a *Hermitian form* on $V$ we mean a function $(\ ,\ ) : V \times V \to \mathbb{C}$ such that, for all $u, v, w$ in $V$ and all $\alpha \in \mathbb{C}$,

(i) $(u + v, w) = (u, w) + (v, w)$,
(ii) $(\alpha u, v) = \alpha(u, v)$,
(iii) $(v, u) = \overline{(u, v)}$.

**Definition 2.5.** Let $(\ ,\ )$ denote a Hermitian form on $V$. By Definition 2.4(iii), we have $(v, v) \in \mathbb{R}$ for $v \in V$. We say that $(\ ,\ )$ is *positive definite* whenever $(v, v) > 0$ for all nonzero $v \in V$.

**Lemma 2.6.** *Let* $(\ ,\ )$ *denote a positive definite Hermitian form on* $V$. *Suppose that we are given a linear transformation* $A : V \to V$ *satisfying*
$$(Au, v) = (u, Av) \quad u, v \in V. \tag{1}$$
*Then all the eigenvalues of* $A$ *are in* $\mathbb{R}$.

**Proof.** Let $\lambda$ denote an eigenvalue of $A$. We show that $\lambda \in \mathbb{R}$. Since $\mathbb{C}$ is algebraically closed there exists a nonzero $v \in V$ such that $Av = \lambda v$. By (1), $(Av, v) = (v, Av)$. Evaluating this using Definition 2.4(ii),(iii), we have $(\lambda - \overline{\lambda})(v, v) = 0$. But $(v, v) \neq 0$, since $( \, , \, )$ is positive definite so $\lambda = \overline{\lambda}$. Therefore $\lambda \in \mathbb{R}$. □

**Assumption 2.7.** Let $A, A^*$ denote a tridiagonal pair on $V$ as in Definition 2.1. For $0 \leq i \leq d$, let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $V_i$ (resp. $V_i^*$). We remark that $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct. We assume that there exists a positive definite Hermitian form $( \, , \, )$ on $V$ satisfying

$$(Au, v) = (u, Av) \quad u, v \in V, \tag{2}$$

$$(A^*u, v) = (u, A^*v) \quad u, v \in V. \tag{3}$$

**Lemma 2.8.** *With reference to Assumption 2.7, the following* (i), (ii) *hold.*

(i) *The eigenspaces $V_0, V_1, \ldots, V_d$ are mutually orthogonal with respect to $( \, , \, )$.*
(ii) *The eigenspaces $V_0^*, V_1^*, \ldots, V_d^*$ are mutually orthogonal with respect to $( \, , \, )$.*

**Proof.** (i) For distinct $i, j$ $(0 \leq i, j \leq d)$, and for $u \in V_i$, $v \in V_j$, we show that $(u, v) = 0$. By (2), $(Au, v) = (u, Av)$. Evaluating this using Definition 2.4(ii), (iii), we find $(\theta_i - \overline{\theta_j})(u, v) = 0$. But $\overline{\theta_j} = \theta_j$ by Lemma 2.6, and $\theta_i \neq \theta_j$, so $(u, v) = 0$.
(ii) Similar to the proof of (i). □

**Theorem 2.9.** *With reference to Lemma 2.3 and Assumption 2.7, the following* (i), (ii) *hold for $0 \leq i, j \leq d$ such that $i + j \neq d$.*

(i) *The subspaces $U_i^{\downarrow\downarrow}$ and $U_j^{\uparrow\uparrow}$ are orthogonal with respect to $( \, , \, )$.*
(ii) *The subspaces $U_i^{\downarrow\uparrow}$ and $U_j^{\uparrow\downarrow}$ are orthogonal with respect to $( \, , \, )$.*

**Proof.** (i) We consider two cases: $i + j < d$ and $i + j > d$. First suppose that $i + j < d$. By Lemma 2.3, $U_i^{\downarrow\downarrow} \subseteq V_0^* + \cdots + V_i^*$ and $U_j^{\uparrow\uparrow} \subseteq V_{d-j}^* + \cdots + V_d^*$. Observe that $V_0^* + \cdots + V_i^*$ is orthogonal to $V_{d-j}^* + \cdots + V_d^*$ by Lemma 2.8(ii), and since $i < d - j$. Therefore $U_i^{\downarrow\downarrow}$ is orthogonal to $U_j^{\uparrow\uparrow}$. Next, suppose that $i + j > d$. By Lemma 2.3, $U_i^{\downarrow\downarrow} \subseteq V_0 + \cdots + V_{d-i}$ and $U_j^{\uparrow\uparrow} \subseteq V_j + \cdots + V_d$. Observe that $V_0 + \cdots + V_{d-i}$ is orthogonal to $V_j + \cdots + V_d$ by Lemma 2.8(i), and since $d - i < j$. Therefore $U_i^{\downarrow\downarrow}$ is orthogonal to $U_j^{\uparrow\uparrow}$.
(ii) Similar to the proof of (i). □

## 3. Subconstituent algebra of distance-regular graphs

In this section we review some definitions and basic concepts concerning subconstituent algebra of distance-regular graphs.

Let $X$ denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the *standard module*. We endow $V$ with the Hermitian form $\langle \, , \, \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where $t$ denotes transpose and $\overline{\phantom{x}}$ denotes complex conjugation. Observe that $\langle \, , \, \rangle$ is positive definite. We call this form the *standard Hermitian form* on $V$. Observe that, for $B \in \mathrm{Mat}_X(\mathbb{C})$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle \quad u, v \in V. \tag{4}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call $D$ the *diameter* of $\Gamma$. We say that $\Gamma$ is *distance-regular* whenever, for all integers $h, i, j$ $(0 \leq h, i, j \leq D)$, and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of $x$ and $y$. The $p_{ij}^h$ are called the *intersection numbers* of $\Gamma$.

For the rest of this paper we assume that $\Gamma$ is distance-regular with diameter $D \geq 3$.

We recall the Bose–Mesner algebra of $\Gamma$. For $0 \leq i \leq D$, let $A_i$ denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $xy$ entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call $A_i$ the $i$th *distance matrix* of $\Gamma$. We abbreviate $A := A_1$ and call this the *adjacency matrix* of $\Gamma$. We observe that (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (iv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (v) $A_iA_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \leq i, j \leq D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra $M$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M$ the *Bose–Mesner algebra* of $\Gamma$. It turns out that $A$ generates $M$

[1, p. 190]. By (4), and since $A$ is real and symmetric,

$$\langle Au, v\rangle = \langle u, Av\rangle \quad u, v \in V. \tag{5}$$

By [3, p. 45], $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (iv) $E_i^t = E_i$ $(0 \le i \le D)$; (v) $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of $\Gamma$.

We recall the eigenvalues of $\Gamma$. Since $E_0, E_1, \ldots, E_D$ form a basis for $M$, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^{D} \theta_iE_i$. Observe that $AE_i = E_iA = \theta_iE_i$ for $0 \le i \le D$. We call $\theta_i$ the *eigenvalue* of $\Gamma$ associated with $E_i$ $(0 \le i \le D)$. By Lemma 2.6 and (5), the eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe that $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct, since $A$ generates $M$. Observe that

$$V = E_0V + E_1V + \cdots + E_DV \quad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$, the space $E_iV$ is the eigenspace of $A$ associated with $\theta_i$.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij}A_i$ for $0 \le i, j \le D$, so $M$ is closed under $\circ$. Thus there exist complex scalars $q_{ij}^h$ $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \le i, j \le D).$$

By [2, p. 170], $q_{ij}^h$ is real and nonnegative for $0 \le h, i, j \le D$. The $q_{ij}^h$ are called the *Krein parameters*. The graph $\Gamma$ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever, for $0 \le h, i, j \le D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \ne 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two [3, p. 59]. See [1,4,5,9,13,14] for more information on the $Q$-polynomial property. From now on we assume that $\Gamma$ is $Q$-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We recall the dual Bose–Mesner algebra of $\Gamma$. Fix a vertex $x \in X$. We view $x$ as a "base vertex". For $0 \le i \le D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $yy$ entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \ne i \end{cases} \quad (y \in X). \tag{6}$$

We call $E_i^*$ the $i$th *dual idempotent* of $\Gamma$ with respect to $x$ [15, p. 378]. We observe that (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^* (0 \le i \le D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$; (iv) $E_i^*E_j^* = \delta_{ij}E_i^*$ $(0 \le i, j \le D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M^*$ the *dual Bose–Mesner algebra* of $\Gamma$ with respect to $x$ [15, p. 378]. For $0 \le i \le D$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $yy$ entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A_0^*, A_1^*, \ldots, A_D^*$ form a basis for $M^*$ [15, p. 379]. Moreover, (i) $A_0^* = I$; (ii) $\overline{A_i^*} = A_i^*$ $(0 \le i \le D)$; (iii) $A_i^{*t} = A_i^*$ $(0 \le i \le D)$; (iv) $A_i^*A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \le i, j \le D)$ [15, p. 379]. We call $A_0^*, A_1^*, \ldots, A_D^*$ the *dual distance matrices* of $\Gamma$ with respect to $x$. We abbreviate $A^* := A_1^*$ and call this the *dual adjacency matrix* of $\Gamma$ with respect to $x$. The matrix $A^*$ generates $M^*$ [15, Lemma 3.11]. By (4), and since $A^*$ is real and symmetric,

$$\langle A^*u, v\rangle = \langle u, A^*v\rangle \quad u, v \in V. \tag{7}$$

We recall the dual eigenvalues of $\Gamma$. Since $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for $M^*$, and since $A^*$ is real, there exist real scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ such that $A^* = \sum_{i=0}^{D} \theta_i^*E_i^*$. Observe that $A^*E_i^* = E_i^*A^* = \theta_i^*E_i^*$ for $0 \le i \le D$. We call $\theta_i^*$ the *dual eigenvalue* of $\Gamma$ associated with $E_i^*$ $(0 \le i \le D)$. Observe that $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are mutually distinct, since $A^*$ generates $M^*$.

We recall the subconstituents of $\Gamma$. For all $y \in X$, let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. From (6), we find

$$E_i^*V = \mathrm{span}\{\hat{y} \mid y \in X,\ \partial(x, y) = i\} \quad (0 \le i \le D). \tag{8}$$

By (8), and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for $V$, we find

$$V = E_0^*V + E_1^*V + \cdots + E_D^*V \quad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$, the space $E_i^*V$ is the eigenspace of $A^*$ associated with $\theta_i^*$. We call $E_i^*V$ the $i$th *subconstituent* of $\Gamma$ with respect to $x$.

We recall the subconstituent algebra of $\Gamma$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the *subconstituent algebra* (or *Terwilliger algebra*) of $\Gamma$ with respect to $x$ [15, Definition 3.3]. We observe that $T$ is generated by $A, A^*$. We observe that $T$ has finite dimension. Moreover, $T$ is semi-simple since it is closed under the conjugate transpose map [7, p. 157]. See [6,8,15–17] for more information on the subconstituent algebra.

For the rest of this paper we adopt the following notational convention.

**Notation 3.1.** We assume that $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \ge 3$. We assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. We fix $x \in X$ and write $A^* = A^*(x)$, $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. We abbreviate $V = \mathbb{C}X$. For notational convenience we define $E_{-1} = 0$, $E_{D+1} = 0$ and $E_{-1}^* = 0$, $E_{D+1}^* = 0$.

We recall some useful results on $T$-modules. With reference to Notation 3.1, by a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let $W$ denote a $T$-module. Then $W$ is said to be *irreducible* whenever $W$ is nonzero and $W$ contains no $T$-modules other than 0 and $W$. Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Then the orthogonal complement of $W'$ in $W$ is a $T$-module [8, p. 802]. It follows that each $T$-module is an orthogonal direct sum of irreducible $T$-modules. In particular, $V$ is an orthogonal direct sum of irreducible $T$-modules. Let $W$ denote an irreducible $T$-module. By the *endpoint* of $W$ we mean $\min\{i | 0 \le i \le D,\ E_i^*W \ne 0\}$. By the *diameter* of $W$ we mean $|\{i | 0 \le i \le D,\ E_i^*W \ne 0\}| - 1$. By the *dual endpoint* of $W$ we mean $\min\{i | 0 \le i \le D, E_iW \ne 0\}$. By the *dual diameter* of $W$ we mean $|\{i | 0 \le i \le D, E_iW \ne 0\}| - 1$. The diameter of $W$ is equal to the dual diameter of $W$ [13, Corollary 3.3].

**Remark 3.2.** With reference to Notation 3.1, let $W$ denote an irreducible $T$-module. Then $A$ and $A^*$ act on $W$ as a tridiagonal pair in the sense of Definition 2.1. This follows from [15, Lemma 3.4, Lemma 3.9, Lemma 3.12], [18, Lemma 3.2], and since $A, A^*$ generate $T$.

**Lemma 3.3.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then, for $\mu, \nu \in \{\downarrow, \uparrow\}$, we have*

$$W = \sum_{h=0}^{d} W_h^{\mu\nu} \quad (\textit{direct sum}), \tag{9}$$

*where for $0 \le h \le d$,*

$$\begin{aligned}
W_h^{\downarrow\downarrow} &= (E_\rho^*W + \cdots + E_{\rho+h}^*W) \cap (E_\tau W + \cdots + E_{\tau+d-h}W),\\
W_h^{\uparrow\downarrow} &= (E_{\rho+d-h}^*W + \cdots + E_{\rho+d}^*W) \cap (E_\tau W + \cdots + E_{\tau+d-h}W),\\
W_h^{\downarrow\uparrow} &= (E_\rho^*W + \cdots + E_{\rho+h}^*W) \cap (E_{\tau+h}W + \cdots + E_{\tau+d}W),\\
W_h^{\uparrow\uparrow} &= (E_{\rho+d-h}^*W + \cdots + E_{\rho+d}^*W) \cap (E_{\tau+h}W + \cdots + E_{\tau+d}W).
\end{aligned}$$

**Proof.** Immediate from Lemma 2.3 and Remark 3.2. □

We remark that the sum (9) is not orthogonal in general. However, we do have the following result.

**Lemma 3.4.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with diameter $d$. Then the following* (i), (ii) *hold for $0 \le h, \ell \le d$ such that $h + \ell \ne d$.*

(i) *The subspaces $W_h^{\downarrow\downarrow}$ and $W_\ell^{\uparrow\uparrow}$ are orthogonal with respect to the standard Hermitian form.*
(ii) *The subspaces $W_h^{\downarrow\uparrow}$ and $W_\ell^{\uparrow\downarrow}$ are orthogonal with respect to the standard Hermitian form.*

**Proof.** Combine Theorem 2.9, (5), (7), Remark 3.2, and Lemma 3.3. □

## 4. The split decompositions of the standard module

In this section we recall the four split decompositions for the standard module and discuss their basic properties.

**Definition 4.1** (*[12, Definition 10.1]*). With reference to Notation 3.1, for $-1 \le i, j \le D$, we define

$$\begin{aligned}
V_{i,j}^{\downarrow\downarrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV),\\
V_{i,j}^{\uparrow\downarrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV),\\
V_{i,j}^{\downarrow\uparrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V),\\
V_{i,j}^{\uparrow\uparrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).
\end{aligned}$$

In each of the above four equations, we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

**Definition 4.2** (*[12, Definition 10.2]*). With reference to Notation 3.1 and Definition 4.1, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \le i, j \le D$, we have $V_{i-1,j}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$ and $V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$. Therefore,

$$V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}.$$

Referring to the above inclusion, we define $\tilde{V}_{i,j}^{\mu\nu}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is,

$$\tilde{V}_{i,j}^{\mu\nu} = (V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu})^\perp \cap V_{i,j}^{\mu\nu}.$$

**Lemma 4.3** (*[12, Lemma 10.3]*). *With reference to Notation 3.1 and Definition 4.2, the following holds for $\mu, \nu \in \{\downarrow, \uparrow\}$:*

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \quad (\textit{direct sum}). \tag{10}$$

**Definition 4.4.** We call the sum (10) the $(\mu, \nu)$-*split decomposition* of $V$ with respect to $x$.

**Remark 4.5.** The decomposition (10) is not orthogonal in general.

**Lemma 4.6.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then, for $0 \le h \le d$ and $0 \le i, j \le D$, the following (i)–(iv) hold.*

(i) $W_h^{\downarrow\downarrow} \subseteq \tilde{V}_{i,j}^{\downarrow\downarrow}$ *if and only if* $i = \rho + h$ *and* $j = \tau + d - h$.
(ii) $W_h^{\uparrow\downarrow} \subseteq \tilde{V}_{i,j}^{\uparrow\downarrow}$ *if and only if* $i = D - \rho - d + h$ *and* $j = \tau + d - h$.
(iii) $W_h^{\downarrow\uparrow} \subseteq \tilde{V}_{i,j}^{\downarrow\uparrow}$ *if and only if* $i = \rho + h$ *and* $j = D - \tau - h$.
(iv) $W_h^{\uparrow\uparrow} \subseteq \tilde{V}_{i,j}^{\uparrow\uparrow}$ *if and only if* $i = D - \rho - d + h$ *and* $j = D - \tau - h$.

**Proof.** Immediate from [12, Lemma 11.4] and (10). □

**Lemma 4.7.** *With reference to Notation 3.1, fix an orthogonal direct sum decomposition of the standard module $V$ of $\Gamma$ into irreducible $T$-modules:*

$$V = \sum_W W. \tag{11}$$

*Then the following (i)–(iv) hold for $0 \le i, j \le D$.*

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow} = \sum W_h^{\downarrow\downarrow}$, *where the sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le i$, dual endpoint $\tau = i + j - \rho - d$, diameter $d \ge i - \rho$, and $h = i - \rho$.*
(ii) $\tilde{V}_{i,j}^{\uparrow\downarrow} = \sum W_h^{\uparrow\downarrow}$, *where the sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le D - i$, dual endpoint $\tau = i + j + \rho - D$, diameter $d \ge D - \rho - i$, and $h = \rho + d - D + i$.*
(iii) $\tilde{V}_{i,j}^{\downarrow\uparrow} = \sum W_h^{\downarrow\uparrow}$, *where the sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le i$, dual endpoint $\tau = \rho + D - i - j$, diameter $d \ge i - \rho$, and $h = i - \rho$.*
(iv) $\tilde{V}_{i,j}^{\uparrow\uparrow} = \sum W_h^{\uparrow\uparrow}$, *where the sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le D - i$, dual endpoint $\tau = 2D - \rho - d - i - j$, diameter $d \ge D - \rho - i$, and $h = \rho + d - D + i$.*

**Proof.** (i) For $0 \le i, j \le D$ define

$$v_{i,j} = \sum W_h^{\downarrow\downarrow}, \tag{12}$$

where the sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le i$, dual endpoint $\tau = i + j - \rho - d$, diameter $d \ge i - \rho$, and $h = i - \rho$. We show that $\tilde{V}_{i,j}^{\downarrow\downarrow} = v_{i,j}$. We first show that $\tilde{V}_{i,j}^{\downarrow\downarrow} \supseteq v_{i,j}$. Let $W_h^{\downarrow\downarrow}$ denote one of the terms in the sum on the right in (12). We show that $W_h^{\downarrow\downarrow}$ is contained in $\tilde{V}_{i,j}^{\downarrow\downarrow}$. Let $\rho, \tau, d$ denote the endpoint, dual endpoint, and diameter of $W$, respectively. By construction, $\tau = i + j - \rho - d$ and $h = i - \rho$. Subtracting the second equation from the first equation we find $j = \tau + d - h$. Now $W_h^{\downarrow\downarrow}$ is contained in $\tilde{V}_{i,j}^{\downarrow\downarrow}$ by Lemma 4.6(i). We have now shown that $\tilde{V}_{i,j}^{\downarrow\downarrow} \supseteq v_{i,j}$. We can now easily show that $\tilde{V}_{i,j}^{\downarrow\downarrow} = v_{i,j}$. Expanding the sum (11) using Lemma 3.3, we get

$$\begin{aligned} V &= \sum_W W \quad \text{(direct sum)} \\ &= \sum_W \sum_h W_h^{\downarrow\downarrow} \quad \text{(direct sum)}, \end{aligned}$$

where the second sum is over the integer $h$ from 0 to the diameter of $W$. In the above sum we change the order of summation to get

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \sum W_h^{\downarrow\downarrow} \quad \text{(direct sum)},$$

where the third sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \le i$, dual endpoint $\tau = i + j - \rho - d$, diameter $d \ge i - \rho$, and $h = i - \rho$. In other words,

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} v_{i,j} \quad \text{(direct sum)}.$$

By this, (10), and since $\tilde{V}_{i,j}^{\downarrow\downarrow} \supseteq v_{i,j}$ for $0 \le i, j \le D$, we find $\tilde{V}_{i,j}^{\downarrow\downarrow} = v_{i,j}$ for $0 \le i, j \le D$.
(ii), (iii), (iv) Similar to the proof of (i). □

Now we have the main result.

**Theorem 4.8.** *With reference to Notation 3.1 and Definition 4.2, the following* (i), (ii) *hold for* $0 \leq i, j, r, s \leq D$.

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\uparrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.
(ii) $\tilde{V}_{i,j}^{\downarrow\uparrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\downarrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.

**Proof.** (i) Assume that $i+r \neq D$ or $j+s \neq D$. We show that $\tilde{V}_{i,j}^{\downarrow\downarrow}$ and $\tilde{V}_{r,s}^{\uparrow\uparrow}$ are orthogonal. To do this we will use Lemma 4.7(i), (iv). Let $W_h^{\downarrow\downarrow}$ (resp. $W'^{\uparrow\uparrow}_{h'}$) denote one of the terms in the sum in Lemma 4.7(i) (resp. Lemma 4.7(iv)). We show that $W_h^{\downarrow\downarrow}$ and $W'^{\uparrow\uparrow}_{h'}$ are orthogonal. There are two cases to consider. First, assume that $W \neq W'$. Then $W$ and $W'$ are orthogonal so $W_h^{\downarrow\downarrow}$ and $W'^{\uparrow\uparrow}_{h'}$ are orthogonal. Next, assume that $W = W'$. Let $\rho, \tau, d$ denote the corresponding endpoint, dual endpoint, and diameter. By Lemma 4.7(i),

$$\tau = i + j - \rho - d, \qquad h = i - \rho. \tag{13}$$

By Lemma 4.7(iv),

$$\tau = 2D - \rho - d - r - s, \qquad h' = \rho + d - D + r. \tag{14}$$

Adding the equations on the right in (13) and (14), we get

$$i + r - D = h + h' - d. \tag{15}$$

Subtracting the equation on the left in (13) from the equation on the left in (14), and evaluating the result using (15), we get

$$j + s - D = d - h - h'. \tag{16}$$

By (15), (16), and since $i + r \neq D$ or $j + s \neq D$, we find $h + h' \neq d$. Now $W_h^{\downarrow\downarrow}$ and $W_{h'}^{\uparrow\uparrow}$ are orthogonal by Lemma 3.4(i).
(ii) Similar to the proof of (i). □

**Corollary 4.9.** *With reference to Notation 3.1 and Definition 4.2, the following* (i), (ii) *hold for* $0 \leq i, j \leq D$.

(i) $\dim \tilde{V}_{i,j}^{\downarrow\downarrow} = \dim \tilde{V}_{D-i,D-j}^{\uparrow\uparrow}$.
(ii) $\dim \tilde{V}_{i,j}^{\downarrow\uparrow} = \dim \tilde{V}_{D-i,D-j}^{\uparrow\downarrow}$.

**Proof.** Immediate from Theorem 4.8 and elementary linear algebra. □


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