On Green’s Function for the Biharmonic Equation in a Right Angle Wedge*

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1. Introduction

Several classical questions on the calculus of variations involve nodal properties of the first eigenfunction for the problem

\[ \Delta \Delta u = 0 \quad \text{in } D, \]
\[ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D, \]  

(1.1)

where \( D \) is an arbitrary region in the plane bounded by a simple curve \( C \). Under the assumption that the first eigenfunction has no nodal lines, Szegö [7] showed that for all \( D \) of a given area, the circle has the lowest eigenvalue.

Recent numerical work by Bauer and Reiss [1] indicates that this assumption is false for a square, i.e., nodal lines appear near the corner. Duffin and Shaffer [3] announced that for the annular region with inner radius \( r_0 \) and outer radius 1, the principal eigenfunction has a diametral nodal line if \( r_0 \) is small enough (\( r_0 < 1/715 \)). This seems to be the only example of a domain for which the existence of nodal lines has been proved.

A related question involves the Green’s function for this problem. It has been conjectured by Hadamard [5] that the Green’s function for this problem is positive. This conjecture has the interpretation that when we place a downward point load at \( \vec{x}_0 \in D \) upon an infinitesimally thin elastic plate clamped at \( \partial D \), the resulting deflection \( G(\vec{x}, \vec{x}_0) \) is always directed downward. Duffin [2] has shown that this function can become negative when \( D \) is an infinite strip, while Loewner and Szegö have exhibited bounded regions for which the conjecture is untrue, although these regions are not convex. The statement is true for a circle. Garabedian [4] has shown that the Green’s

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function of an ellipse whose major axis is not even twice as long as its minor axis takes negative values when the arguments $x$ and $x_0$ are sufficiently near the vertices.

In this paper we consider an inhomogeneous problem (1.1) in the quarter plane $0 \leq x, y$. We shall write out the asymptotic expansions of the solutions to the inhomogeneous problem,

$$\Delta \Delta u = f \quad \text{in } D,$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D,$$

both near and far from the corner in terms of powers of $r$, respectively positive and negative. The coefficients depend on $\theta$ ($r$ and $\theta$ are, of course, the polar coordinates). We shall use this expansion to show that the Green's function changes sign and in fact oscillates an infinite number of times as $r \to 0$ and $r \to \infty$. In Theorem 2.6, we write out the expansions in detail. The powers of $r$ in the expansions involve the solutions of a certain transcendental equation.

In a future paper we expect to use the Schwarz alternating procedure in order to investigate these questions in a rectangle. We conjecture that the asymptotic representation near the corners does not change much, and, hence, the Green's function again oscillates near the corners. We also shall attempt to verify analytically the numerical results of Bauer and Reiss.

2. Calculation of the Solution

The problem is now

$$\Delta \Delta u = f(r, \theta), \quad 0 < r, \quad 0 < \theta < \pi/2,$$

$$u(r, 0) = u(r, \pi/2) = u_0(r, 0) + u_\theta(r, \pi/2) = 0. \quad (2.1)$$

We make the change of variables in the complex plane

$$- \ln z = \eta + i \zeta \quad (2.2)$$

or

$$z = e^{-(\eta + i \zeta)}, \quad x = e^{-n} \cos \zeta, \quad y = -e^{-n} \sin \zeta.$$

The equation becomes

$$\left( \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \right) \left( \left( \frac{\partial}{\partial \eta} + 2 \right)^2 + \frac{\partial^2}{\partial \zeta^2} \right) = e^{-4ng}(e^{-n} \cos \zeta, -e^{-n} \sin \zeta), \quad (2.3)$$
with \( f(r, \theta) = g(x, y) \), in the region

\[-\infty < \eta < \infty, \quad -\pi/2 < \zeta < 0,\]

with the boundary conditions

\[ u(\eta, 0) = u(\eta, -\pi/2) = u(\eta, 0) = u(\eta, -\pi/2) = 0. \]

We next apply a Fourier transform with respect to \( \eta \), and call the Fourier variable \( \gamma \). The equation is now

\[
\left( \frac{\partial^2}{\partial \zeta^2} - \gamma^2 \right) \left( \frac{\partial^2}{\partial \gamma^2} + (i\gamma + 2)^2 \right) u = h(\gamma, \zeta), \quad -\pi/2 < \zeta < 0,
\]

\[ h(\gamma, \zeta) = \mathcal{F} e^{4\pi g(e^{-\gamma} \cos \zeta, -e^{-\gamma} \sin \zeta)}, \]

with the same boundary conditions.

This simple approach was used by Kondrat'ev [6] in his work on more general elliptic equations in conical regions.

We can solve this ordinary differential equation with boundary conditions uniquely for all real \( \gamma \), obtaining (we suppress the \( \gamma \) dependence in \( u, u_0 \), and \( h \))

\[
u = u_0(\zeta) + u_0(-\pi/2) \frac{\sinh \gamma \zeta}{\sinh \gamma(\pi/2)}
\]

\[ + D [\sinh \gamma(\zeta + \pi/2) + \sinh(\gamma - 2i)(\zeta + \pi/2)] \]

\[ + C [\sinh \gamma \zeta + \sinh(\gamma - 2i) \zeta], \]

with

\[
u_0(\zeta) = \frac{i}{4} \int_0^\zeta h(t) \left[ \frac{\sinh \gamma(t - \zeta)}{\gamma(t - i)} - \frac{\sinh(\gamma - 2i)(t - \zeta)}{(\gamma - i)(\gamma - 2i)} \right].
\]

Let

\[ \rho(\gamma) = -2[\cosh^2 \gamma \pi/2 + (\gamma - i)^2]. \]

Then

\[
D = \frac{1}{\rho(\gamma)} \left[ u_0'(-\pi/2)(\gamma - i) + u_0(-\pi/2) \coth(\gamma \pi/2) \gamma(\gamma - 2i) \right]
\]

\[ + \begin{cases} 
-\frac{i}{2} u_0'(-\pi/2) \cosh \gamma \pi/2 \\
+ u_0(-\pi/2) \left[ \gamma^2 - i\gamma[(1 + \cosh^2 \gamma \pi/2) \cdot (\sinh \gamma \pi/2)^{-1}] \right]
\end{cases}
\]

\[ \frac{1}{\rho(\gamma)} \]

and, for completeness,

\[
u_0(\zeta) = \frac{i}{4} \int_0^\zeta \frac{h(t)}{(\gamma - 1)} \left[ -\cosh \gamma(t - \zeta) + \cosh(\gamma - 2i)(t - \zeta) \right] dt.
\]
We shall see later that for reasonable \( h \), the asymptotic expansions depend on the zeros of \( \rho(y) \). We notice immediately two facts: \( \rho(y) = 0 \) if and only if both

\[
\begin{align*}
(\text{a}) \quad & \rho(2i - y) = 0, \\
(\text{b}) \quad & \rho(-y) = 0.
\end{align*}
\]

Lemma 2.1. Equation (2.10) has roots exactly at 0, \( i \), \( 2i \) and

\[
\gamma_{\pm}^{(j)} = \pm \gamma_1^{(j)} + i\gamma_2^{(j)}.
\]

Each \( \gamma_1^{(j)} > 0 \) and \( \gamma_2^{(j)} \) lies in some interval \( 4K - 1 < y_2 < 4K \), or \( 4K + 1 < y_2 < 4K + 2 \), or \( 4K + 3 > y_2 > -4K + 2 \), or finally \( -4K + 1 > y_2 > -4K \), for \( K = 1, 2, 3, \ldots \). All the roots except \( y = i \) have multiplicity 1, and \( y = i \) has multiplicity two.

Proof. We wish to solve

\[
\cosh \frac{\gamma_2 \pi}{2} = \pm i(y - i)
\]

or

\[
\begin{align*}
(e^{(\gamma_1 \pi/2)} + e^{(-\gamma_1 \pi/2)}) \cos(\gamma_2 \pi/2) = & \pm 2(1 - \gamma_2), \\
(e^{(\gamma_1 \pi/2)} - e^{(-\gamma_1 \pi/2)}) \sin(\gamma_2 \pi/2) = & \pm 2\gamma_1.
\end{align*}
\]

We see immediately that if \( \cos(\gamma_2 \pi/2) = 0 \), then \( y_2 = 1 \), hence, \( \gamma_1 = 0 \). Otherwise, we have

\[
e^{(\gamma_1 \pi/2)} = \pm \frac{(1 - \gamma_2)}{\cos(\gamma_2 \pi/2)} \pm \left( \frac{(1 - \gamma_2)^2}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2}.
\]

This is a real number; hence, we must have

\[
|1 - \gamma_2| \geq \left| \frac{\gamma_2 \pi}{2} \right|.
\]

Thus, it follows that \( \gamma_2 \) does not lie in the open interval \( 0 < \gamma_2 < 2 \)

\[
e^{(-\gamma_1 \pi/2)} = \pm \frac{(1 - \gamma_2)}{\cos(\gamma_2 \pi/2)} \pm \left( \frac{(1 - \gamma_2)^2}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2},
\]

hence,

\[
\pm \left( \frac{(1 - \gamma_2)^2}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \sin(\gamma_2 \pi/2)
\]

\[
= \pm \left( \frac{(1 - \gamma_2)^2}{\cos(\gamma_2 \pi/2)} \pm \left( \frac{(1 - \gamma_2)^2}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \right.
\]

\[
\left. \right) \quad \left( \left( \frac{(1 - \gamma_2)^2}{\cos(\gamma_2 \pi/2)} \pm \left( \frac{(1 - \gamma_2)^2}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \right)\right).
\]
or
\[
\left( \frac{(1 - \gamma_2)}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \sin(\gamma_2 \pi/2)
= \pm (2/\pi) \ln \left( \pm \frac{(1 - \gamma_2)}{\cos(\gamma_2 \pi/2)} + \left( \frac{(1 - \gamma_2)}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \right). \tag{2.17}
\]

Because of (2.10) and (2.14), we need only seek solutions for which \( \gamma_2 \geq 2 \) and \( \cos(\gamma_2 \pi/2) \neq 0 \). From (2.12), if \( \sin(\gamma_2 \pi/2) = 0 \), then \( \gamma_1 = 0 \) and \( \cos(\gamma_2 \pi/2) = \pm 1 \), so \( \gamma_2 = 0, 2, \gamma_1 = 0 \) are solutions. The quantity on the right side in (2.17) must be real. Then \( \pm (1 - \gamma_2) \cdot (\cos(\gamma_2 \pi/2))^{-1} \) must be positive. Thus, if \( 4K - 1 < \gamma_2 < 4K + 1 \), we take the \( - \) sign; if \( 4K - 3 < \gamma_2 < 4K - 1 \), we take the \( + \) sign. The quantity inside the \( \ln \) brackets is greater than 1. Thus, if the minus sign is taken, we need \( 4K - 2 < \gamma_2 < 4K \); if the plus sign is taken, \( 4K < \gamma_2 < 4K + 2 \). Hence, the only possible roots for \( \gamma_2 > 2 \) occur when \( 4K - 1 < \gamma_2 < 4K \) (minus sign), and \( 4K + 1 < \gamma_2 < 4K + 2 \) (plus sign).

Now we shall show that there exists exactly one solution to (2.17) for each \( K = 1, 2, \ldots \), and, thus, there exists two \( \gamma \)'s with this imaginary part defined by (2.13). If we take the minus sign, we see that the quantity
\[
\sin(\gamma_2 \pi/2) \left( \frac{(1 - \gamma_2)}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2}
+ (2/\pi) \ln \left( - \frac{(1 - \gamma_2)}{\cos(\gamma_2 \pi/2)} + \left( \frac{(1 - \gamma_2)}{\cos^2(\gamma_2 \pi/2)} - 1 \right)^{1/2} \right)
\tag{2.18}
\]
approaches \( -\infty \) as \( \gamma_2 \downarrow 4K - 1 \) and becomes positive as \( \gamma_2 \uparrow 4K \). We, thus, have at least one root in this interval. Differentiate this expression with respect to \( \gamma_2 \). Let
\[
\frac{\gamma_2 - 1}{\cos(\gamma_2 \pi/2)} = f(\gamma_2), \tag{2.19}
\]
then the derivative becomes
\[
(\pi/2) \cos(\gamma_2 \pi/2) \left( f^2 - 1 \right)^{1/2} + \sin(\gamma_2 \pi/2) \frac{ff'}{(f^2 - 1)^{1/2}}
+ (2/\pi) \left( f - (f^2 - 1)^{1/2} \right) \left( f' + \frac{ff'}{(f^2 - 1)^{1/2}} \right)
= \frac{f'}{(f^2 - 1)^{1/2}} (2/\pi + f \sin(\gamma_2 \pi/2)) + (\pi/2) \cos(\gamma_2 \pi/2) (f^2 - 1)^{1/2}
= (2/\pi) \frac{(f')^2}{(f^2 - 1)^{1/2}} \cos(\gamma_2 \pi/2) + (\pi/2) \cos(\gamma_2 \pi/2) (f^2 - 1)^{1/2} > 0
\tag{2.20}
\]
in this interval. Thus, we have exactly one root here.
A very similar argument works for the other case. Next, we show that the multiplicity of all the roots of (2.7) is one, except for \( \gamma = i \) which has multiplicity two. Differentiate (2.7) obtaining

\[
- 2 \left( \pi \cosh \frac{\gamma \pi}{2} \sinh \frac{\gamma \pi}{2} + 2(\gamma - 1) \right). 
\]

(2.21)

This expression vanishes if \( \gamma = i \), but its derivative does not. We know that

\[
(\gamma - i) = \pm i \cosh(\gamma \pi/2). 
\]

(2.22)

Thus, if \( \gamma \neq i \), (2.21) vanishes if and only if

\[
(\pi/2) \sinh(\gamma \pi/2) \pm i = 0. 
\]

(2.23)

In this case

\[
-(\gamma - i)^2 + (4/\pi^2) = 1, 
\]

(2.24)

or

\[
\gamma = i(1 \pm (1 - (4/\pi^2))^{1/2}), 
\]

(2.25)

but this is impossible. Q.E.D.

We next wish to compute the residues of the inverse Fourier transform of \( u(\gamma, \xi) \). We begin by assuming \( e^{i\gamma \xi} h(\gamma, \xi) \) is analytic at the poles of \( u \). These occur only at the zeros of \( p(\gamma) \) and \( \sinh(\gamma \pi/2) \). This assumption will be justified later.

**Lemma 2.2.** The residues of \( e^{i\gamma \xi} u(\gamma, \xi) \) at \( \gamma = 0, i, 2i \) are all zero.

**Proof.** The statement is immediately true for \( u_0(\gamma, \xi) \). The term \( u_0(\gamma, - (\pi/2)) (\sinh \gamma \xi) \cdot (\sinh(\gamma \pi/2))^{-1} \) might give a contribution only at \( \gamma = 2i \), in fact it is

\[
-(2i/\pi) u_0(2i, - \pi/2) \sin 2\xi = - (2i/\pi) u_0(2i, - \pi/2) \sin 2\xi. 
\]

(2.26)

The contributions for the next two terms at \( \gamma = 0 \) are

\[
(1/p'(0)) [u_0'(0, - \pi/2) (- i) + u_0(0, - \pi/2) (2/\pi) (- 2i)] i \sin 2\xi \\
+ (1/p'(0)) [- i u_0'(0, - \pi/2) + u_0(0, - \pi/2) i(4/\pi)] (- i \sin 2\xi) \\
- 0. 
\]

(2.27)

At \( \gamma = i \), \( p(\gamma) \) has a double zero. If we can show that \( D \) and \( C \) have single zeros there, we are finished. This follows immediately from (2.8).
Last, we consider the contributions at \( \gamma = 2i \). We have

\[- \frac{2i}{\pi} u_0(2i, - \pi/2) \sin 2\xi \]

\[+ \frac{1}{\rho'(2i)} \left[ u_0'(2i, - \pi/2) + u_0(2i, - \pi/2) \left(-1\right) 2i(-2/\pi) \right] \left(-i \sin 2\xi\right) \]

\[+ \frac{1}{\rho'(2i)} \left[ iu_0'(2i, - \pi/2) + u_0(2i, - \pi/2) 2i(-2/\pi) i \sin 2\xi \right]. \tag{2.28}\]

The \( u_0' \) terms drop out. Also \( \rho'(2i) = -4i \); thus, the \( u_0 \) terms drop out. Q.E.D.

Next, we realize that for \( \gamma \) real and \( |\gamma| \to \infty \), the hyperbolic cosine and sine terms become unbounded in (2.5), and, thus, the Fourier inversion becomes difficult. This difficulty is overcome by means of the following lemma.

**Lemma 2.3.** We may rewrite (2.5) as

\[ u = \frac{i}{4} \int_{-\pi/2}^{\pi/2} \frac{h(\gamma, t)}{(\gamma - i) \sinh(\gamma \pi/2)} \]

\[\times \sinh \gamma \xi \left[ \frac{\sinh \gamma(t + (\pi/2)) - \sinh(\gamma - 2i)(t + (\pi/2))}{\gamma} \right] \]

\[+ \frac{i}{8} \int_0^\xi \frac{h(\gamma, t)}{\gamma(\gamma - 1)} \left[ \frac{\cosh \gamma(t + \xi + (\pi/2)) - \coth \gamma(t - \xi + (\pi/2))}{\sinh(\gamma \pi/2)} \right] dt \]

\[+ D[\sinh(\gamma \xi + (\pi/2)) + \sinh(\gamma - 2i)(\xi + (\pi/2))] \]

\[+ \frac{\sinh \gamma \xi + \sinh(\gamma - 2i) \xi}{\rho(\gamma) \sinh(\gamma \pi/2)} u_0(-\pi/2) \gamma(\gamma - i) \]

\[- \frac{i}{4} \left[ \frac{\sinh \gamma \xi + \sinh(\gamma - 2i) \xi}{(\gamma - i)(\gamma - 2i)} \right] \left[ \frac{\coth(\gamma \pi/2)}{\cosh^2(\gamma \pi/2) + (\gamma - i)^2} \right] \]

\[\cdot \int_{-\pi/2}^{\pi/2} dt \ h(\gamma, t) \left[ \frac{\cosh(\gamma \pi/2) \sinh(\gamma - 2i)(t + (\pi/2))}{\left[\cosh^2(\gamma \pi/2) + (\gamma - i)^2\right] \sinh(\gamma \pi/2)} \right] \]

\[\cdot \int_0^\xi h(\gamma, t) \left[ \frac{\sinh(\gamma - 2i)(t - \xi) \sinh(\gamma \pi/2)}{\left[\cosh^2(\gamma \pi/2) + (\gamma - i)^2\right] \sinh(\gamma \pi/2)} \right] \]

\[- \frac{i}{8} \cosh^2(\gamma \pi/2) \left[ \left[\cosh^2(\gamma \pi/2) + (\gamma - i)^2\right](\gamma - i)(\gamma - 2i)^{-1} \right] \]

\[\cdot \int_0^\xi \left[ \cosh((\gamma - 2i)(t - \xi) - (\gamma \pi/2)) + \cosh(\gamma - 2i)(t + (\pi/2) + \xi) \right] dt. \tag{2.29}\]
Each of the terms multiplying \( h(\gamma, t) \) is, thus, bounded for \( |\gamma_1| \) large and \( \gamma_2 \) fixed if \( \gamma_2 \) is not a solution of (2.17).

Proof. The proof of this Lemma is very technical, but straightforward and will be left to the reader.

We shall show that the sum of the residues of \( e^{i\gamma_2 u(\gamma, \xi)} \) at \( \gamma_+ = \gamma_1 + i\gamma_2 \), and \( \gamma_- = -\gamma_1 + i\gamma_2 \), for \( \gamma \pm \), the roots of \( p(\gamma) \) with \( \gamma_2 \) a solution of (2.17), is pure imaginary for real \( f \). Moreover, for all but a finite number of \( \xi \) for which it vanishes, each term oscillates sinusoidally in \( \ln r \) with decaying amplitude as \( \gamma_2 \eta \rightarrow \infty \) if we fix \( \xi \).

**Lemma 2.4.** If \( \gamma_2 \) is such that \( e^{(\gamma_2 - 4)\xi} \) is smooth, real, and belongs to \( L_2(\mathbb{R}) \) for each \( t, 0 \leq t \leq (\pi/2) \), then (unless \( g \) is such that it vanishes identically) the sum of the residues of \( e^{\gamma_2 u(\gamma, \xi)} \) at \( \gamma_+ \) and \( \gamma_- \) is pure imaginary and has the oscillating property mentioned previously. The exact expression is

\[
\pm \frac{i}{2} \Re \left\{ \frac{1}{p'(\gamma_+)} \frac{\sinh(\gamma_+ \pi/2)}{\sinh(\gamma_+ \pi/2)} \right\} \int_{-\pi/2}^{\pi/2} dt \int_{-\infty}^{\infty} \frac{d\psi}{\psi} e^{i\gamma_+ (\psi - 4) \xi} \frac{g(e^{-\psi} \cos t, e^{-\psi} \sin t)}{\sinh(\gamma_+ \pi/2)}.
\]

where \( p' \) and \( q \) are defined later.

Proof. We are concerned with a \( \gamma \) which solves (2.10) and, hence, (2.11). For such \( \gamma \),

\[
p' = -2[\pi \cosh(\gamma \pi/2) \sinh(\gamma \pi/2) + 2(i - i)]
\]

We have

\[
\frac{1}{p'(\gamma)} [\sinh(\gamma \xi) + (\pi/2)] + \sinh(\gamma - 2i) (\xi + (\pi/2))
\]

\[
\times \left[ u_0(\gamma, -(\pi/2)) (\gamma - i) \pm \frac{u_0(\gamma, -(\pi/2)) (\gamma - i) (\gamma - 2i)}{\sinh(\gamma \pi/2)} \right]
\]

\[
+ [\sinh(\gamma \xi) + \sinh(\gamma - 2i) \xi]
\]

\[
\times \left[ \frac{1}{p'}(\gamma, -(\pi/2)) (\gamma - i) \pm \frac{u_0(\gamma, -(\pi/2)) (\gamma - 2i) (\gamma - i)}{\sinh(\gamma \pi/2)} \right]
\]

\[
= \left( (\gamma - i)/p'(\gamma) \right) [\sinh(\gamma \xi) + \sinh(\gamma - 2i) \xi]
\]

\[
\pm \sinh(\gamma (\xi + (\pi/2))) \pm \sinh(\gamma - 2i) (\xi + (\pi/2))]
\]

\[
\times \left[ \frac{u_0(\gamma, -(\pi/2)) (\gamma - 2i)}{\sinh(\gamma \pi/2)} \pm u_0(\gamma, -(\pi/2)) \right]
\]
\[ \frac{1}{4p(\gamma)} [\sinh \gamma \zeta + \sinh(\gamma - 2i) \zeta \pm \sinh \gamma (\zeta + (\pi/2)) \]
\[ \pm \sinh(\gamma - 2i) (\zeta + (\pi/2))] \cdot \int_{0}^{\pi/2} h(\gamma, t) \]
\[ \times \left( \frac{-(\gamma - 2i) \sinh \gamma(t + (\pi/2)) + \gamma \sinh(\gamma - 2i)(t + (\pi/2))}{\sinh(\gamma \pi/2)} \right) \]
\[ \mp i \cosh \gamma(t + (\pi/2)) \pm i \cosh(\gamma - 2i)(t + (\pi/2)) \, dt. \tag{2.32} \]

The quantity under the integral sign may be rewritten
\[ \int_{0}^{\pi/2} h(\gamma, t) \left[ i \left[ \frac{\sinh \gamma(t + (\pi/2)) + \sinh(\gamma - 2i)(t + (\pi/2))}{\sinh(\gamma \pi/2)} \right] \]
\[ \pm \frac{i}{\sinh(\gamma \pi/2)} [\cosh(\gamma \pi/2) \sinh \gamma(t + (\pi/2)) \]
\[ - \cosh(\gamma \pi/2) \sinh(\gamma - 2i)(t + (\pi/2)) - \sinh(\gamma \pi/2) \cosh \gamma(t + (\pi/2)) \]
\[ + \cosh(\gamma - 2i)(t + (\pi/2)) \sinh(\gamma \pi/2)] \, dt \]
\[ \pm \sinh \gamma t \pm \sinh(\gamma - 2i)t. \tag{2.33} \]

Let
\[ q(\gamma, \zeta) = \sinh \gamma \zeta + \sinh(\gamma - 2i) \zeta. \tag{2.34} \]

Then we have
\[ \frac{\pm i}{4p'(\gamma) \sinh(\gamma \pi/2)} [q(\gamma, \zeta) \pm q(\gamma, \zeta + (\pi/2))] \]
\[ \cdot \int_{0}^{\pi/2} (q(\gamma, t) \pm q(\gamma, t + (\pi/2))) h(\gamma, t) \, dt. \tag{2.35} \]

Next we may obtain the desired sum
\[ \frac{\pm i}{4p'(\gamma_+) \sinh(\gamma \pi/2)} [e^{i\gamma_+ \eta} q(\gamma_+, \zeta) \pm q(\gamma_+, \zeta + (\pi/2))] \]
\[ \cdot \int_{0}^{\pi/2} dt (q(\gamma_+, t) \pm q(\gamma_+, t + (\pi/2))) \tag{2.36} \]
\[ \cdot \int_{-\infty}^{\infty} e^{-i\gamma_+ \nu - 4\pi} g(e^{-\nu} \cos t, -e^{-\nu} \sin t) \, dv \]
\[ + \text{the same terms with } \gamma_- \text{ replacing } \gamma_. \]
We now notice that

\[
\begin{align*}
\text{i}p'(y_+) &= + \text{i}p'(y_-) \\
\sinh(y_+\pi/2) &= - \sinh(y_-\pi/2) \\
\text{e}^{iy_+(n-v)} &= \text{e}^{iy_-(n-v)} \\
q(y_+, \zeta) &= - q(y_-, \zeta).
\end{align*}
\tag{2.37}
\]

Thus, if we perform the previous addition, we obtain

\[
\pm \frac{i}{2} \Re \frac{1}{p'(y_+) \sinh(y_+\pi/2)} \int_0^{-\pi/2} dt \int_{-\infty}^{\infty} dve^{iy_+(n-v)} g(e^{-v} \cos t, e^{-v} \sin t)
\times [q(y_+, \zeta) \pm q(y_+, \zeta + (\pi/2))] [q(y_+, t) \pm q(y_+, t + (\pi/2)).
\tag{2.38}
\]

If \(q(y_+, \zeta) \pm q(y_+, \zeta + (\pi/2)) = 0\), for an infinite number of \(\zeta\) in the interval \(-\pi/2 < \zeta < 0\), the equality would be true for all \(\zeta\), which is clearly false. Thus, for all but at most a finite number of \(\zeta\), the argument of the foregoing expression exists; hence, the result follows.

We must also consider the singularities in the expression for \(u(y, \zeta)\) due to the zeros of \(\sinh(y\pi/2)\).

**Lemma 2.5.** The residue of \(u(y, \zeta)\) at \(y = 2Ki\), for \(K\) any integer, is zero.

**Proof.** We have already proved this for \(K\) equals 0 and 1 in Lemma 2.2. Otherwise, we have

\[
\begin{align*}
u_0(2Ki, -(\pi/2)) \sinh \frac{2Ki\zeta}{(\pi/2) \cosh K\pi i} + \frac{1}{p(2Ki)} u_0(2Ki, -(\pi/2)) \left( \frac{2Ki(2K - 2) i}{\pi/2} \right) \\
\times (\sinh(2Ki\zeta) \cosh K\pi i - \sinh(2K - 2) \zeta) \cosh K\pi i \\
+ \frac{1}{p(2Ki)} u_0(2Ki, -(\pi/2)) \left( \frac{2Ki(2K - 2) i}{\pi/2} \cosh K\pi i \right) \\
\times (\sinh 2Ki\zeta + \sinh(2K - 2) i\zeta) \\
= \left( \frac{\sinh 2Ki\zeta}{\cosh K\pi i} \right) u_0(2Ki, -(\pi/2)) \left[ 1 - \frac{2K(2K - 2) 2}{2[1 - (2K - 1)^2]} \right] = 0.
\end{align*}
\]

We may now state our first theorem.
THEOREM 2.6. (a) If $r^{3-\gamma_2}(r, \theta)$ is in $L_2$ of the quarter plane for all $\gamma_2$ with $0 \leq \gamma_2 \leq \gamma_2'$, then the asymptotic expression for the solution

$$
\lim_{r \to \infty} r^{-\gamma_2(K)} \left[ u(r, \theta) + \sum_{j=1}^{K} \frac{(-1)^{j+1}}{2} \right] \text{Re} \left[ \left( q\left(\gamma_2^+, -\theta\right) + (-1)^j q\left(\gamma_2^+(\pi/2), -\theta\right) \right) p'\left(\gamma_2^+(\sigma/2)\right) \sinh \gamma_2^+(\sigma/2) \right] 
\times \int_0^\infty \int_{-\pi/2}^{\pi/2} d\theta_1 \left[ q\left(\gamma_2^+, -\theta_1\right) + (-1)^j q\left(\gamma_2^+(\pi/2), -\theta_1\right) \right] 
\times \left[ r^{-\gamma_2(-j)} r_1^{-\gamma_2^+(j)} \right] f(r_1, \theta_1) = 0 \text{ a.e.} \quad (2.40)
$$

is valid, where $\gamma_2^{(j)}$ is the $j$th root of (2.17) for $\gamma_2^{(j)} > 2$, and $K$ is any positive integer such that $\gamma_2^{(K)} < \gamma_2'$.

(b) If $r^{3-\gamma_2}(r, \theta)$ is in $L_2$ of the quarter plane for all $\gamma_2$ with $0 \leq \gamma_2 \leq \gamma_2'$, then the asymptotic expression for the solution

$$
\lim_{r \to \infty} r^{-\gamma_2(K)} \left[ u(r, \theta) + \sum_{j=1}^{K} (-1)^j \right] 
\times \text{Re} \left[ \left( q\left(\gamma_2^{(-j)}, -\theta\right) + (-1)^j q\left(\gamma_2^{(-j)}(\pi/2), -\theta\right) \right) p'\left(\gamma_2^{(-j)}(\sigma/2)\right) \sinh \gamma_2^{(-j)}(\sigma/2) \right] 
\times \int_0^\infty \int_{-\pi/2}^{\pi/2} d\theta_1 \left[ q\left(\gamma_2^{(-j)}, -\theta_1\right) + (-1)^j q\left(\gamma_2^{(-j)}(\pi/2), -\theta_1\right) \right] 
\times \left[ r^{-\gamma_2(-j)} r_1^{-\gamma_2^+(j)} \right] f(r_1, \theta_1) = 0 \text{ a.e.} \quad (2.41)
$$

is valid, where $\gamma_2^{(-j)}$ is the $j$th root of (2.17) for $\gamma_2^{(-j)} < -1$, and $K$ is any positive integer such that $-\gamma_2^{(-K)} < \gamma_2'$.

Proof. If we inverse Fourier transform expression (2.29) for $u(\gamma, \zeta)$ we are dealing with terms like

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyh} dy \int_{-\pi/2}^{\pi/2} h(\gamma, t) dt \frac{\sinh \gamma \zeta}{\sinh \gamma (\pi/2)} \times \left[ \sinh \frac{\gamma (t + (\pi/2))}{\gamma} - \frac{\sinh (\gamma - 2i)(t + (\pi/2))}{(\gamma - 2i)} \right]. \quad (2.42)
$$

The hypothesis in part (a) implies that, for almost all $\zeta$,

$$
e^{(\gamma \zeta - u)} g(e^{-u} \cos \zeta, - e^{-u} \sin \zeta)
$$

is square integrable if $0 \leq \gamma_2 \leq \gamma_2'$. Thus, $h(\gamma, \zeta)$ has an analytic extension in $\gamma$ for $0 < \gamma_2 < \gamma_2'$ and is square integrable in $\gamma_2$ for $0 \leq \gamma_2 \leq \gamma_2'$. We may,
thus, deform the path of integration to $\gamma_1 + i(\gamma_2^{(K)} + \epsilon)$ for $\epsilon > 0$ and small enough so that $\gamma_2^{(K)} + \epsilon < \gamma'_2$. We then pick up $i$ times the residues in the asymptotic expansion plus, by Schwarz inequality, a remainder term of order $O(r^*)$ as $r \to 0$. Part (b) follows in an analogous manner.

**Theorem 2.7.** The Green's function for this problem $G(r, \theta, r_0, \theta_0)$, changes sign an infinite number of times both as $r/r_0 \to 0$ and $r/r_0 \to \infty$, except perhaps, for a finite number of $\theta_0$ and $\theta$.

**Proof.** We merely choose

$$f(r, \theta) = \text{Re}(q(\gamma_+^{(1)}, \theta) - q(\gamma'_+, (\pi/2) - \theta))$$

in an interval $\theta_0 - \epsilon < \theta < \theta + \epsilon$, $r_0 - \epsilon < r < r_0 + \epsilon$, zero elsewhere, where

$$0 < \epsilon < \min(r_0, \theta_0, (\pi/2) - r_0),$$

and

$$\text{Re}[q(\gamma_+^{(1)}, \theta) - q(\gamma'_+, (\pi/2) - \theta)] \neq 0$$

in this interval. The result then follows as $r/r_0 \to 0$ from (2.40).

Similar arguments work as $r/r_0 \to \infty$. Q.E.D.

**References**