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Is there a set of reals not in $K(\mathbb{R})$?

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Abstract

We show, using the fine structure of $K(\mathbb{R})$, that the theory ZF + AD + $\exists X \subseteq \mathbb{R}[X \notin K(\mathbb{R})]$ implies the existence of an inner model of ZF + AD + DC containing a measurable cardinal above its Θ, the supremum of the ordinals which are the surjective image of $\mathbb{R}$. As a corollary, we show that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ for some $P \subseteq (\Theta^+)^{K(\mathbb{R})}$, where $K(P)$ is the Dodd–Jensen Core Model relative to P. In conclusion, we show that the theory ZF + AD + $\neg\mathrm{DC}_{\mathbb{R}}$ implies that $\mathbb{R}^\dagger$ (dagger) exists. © 1998 Published by Elsevier Science B.V. All rights reserved.

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0. Introduction

Let ω be the set of all natural numbers. $\mathbb{R} = {}^\omega\omega$ is the set of all functions from ω to ω. We call $\mathbb{R}$ the set of reals and regard $\mathbb{R}$ as a topological space by giving it the product topology using the discrete topology on ω. For a set Y and each $A \subseteq {}^\omega Y$ we associate a two person infinite game on Y, with *payoff* A, denoted by G_A:

$$\begin{array}{lccccc} \mathbf{I} & y(0) & & y(2) & & \\ & & & & & \cdots \\ \mathbf{II} & & y(1) & & y(3) & \end{array}$$

in which player **I** wins if $y \in A$, and player **II** wins if $y \notin A$. We call A *determined* if the corresponding game G_A is determined, that is, either player **I** or **II** has a winning quasi-strategy (see [15, p. 287]). Since we will be working in a context without the Axiom of Choice, we do not require strategies to be single valued.

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The *Axiom of Determinacy* (AD) is a regularity hypothesis about games on ω that states: $\forall A \subseteq \mathbb{R}$ (A is determined). Similarly, we let $\mathrm{AD}_{\mathbb{R}}$ represent the analogous assertion concerning games on $\mathbb{R}$. Given a pointclass $\Gamma \subseteq \mathscr{P}(\mathbb{R})$, we say that Γ – AD or "Γ is determined", to mean that every two-person game on ω with payoff in Γ is determined. Likewise, we let Γ – $\mathrm{AD}_{\mathbb{R}}$ assert that every two-person game on $\mathbb{R}$ with payoff in Γ is determined. (We are assuming a canonical homeomorphism ${}^{\omega}\mathbb{R} \approx \mathbb{R}$.)

Finally, the *Axiom of Dependent Choices* (DC) asserts: For every set X and every relation R on X,

$$(\forall a \in X)(\exists b \in X)R(a,b) \Rightarrow (\exists f : \omega \to X)(\forall n \in \omega)R(f(n), f(n+1)).$$

Let $\mathrm{DC}_{\mathbb{R}}$ be DC restricted to the case of the reals, that is, where $X = \mathbb{R}$.

From the beginning $L(\mathbb{R})$, the smallest inner model of ZF containing the reals $\mathbb{R}$, was seen as the *natural* inner model for determinacy and the structure of this model came to be studied for its own sake. Consequently, a number of theorems have been established resolving many issues *in* $L(\mathbb{R})$. For example, questions about

(a) the extent of scales,

(b) the axiom of dependent choices,

(c) strong partition cardinals,

(d) measurable cardinals below Θ,

(e) the size of Θ,

(f) the first order theory of $\mathrm{HOD}^{L(\mathbb{R})}$ (see [18]), and

(g) various characterizations of determinacy in $L(\mathbb{R})$ (see [8, Sections 30 and 32])

have been answered. A number of the solutions to these problems about $L(\mathbb{R})$ use the fact that $L(\mathbb{R})$ has a fine structure theory analogous (in many ways) to the fine structure theory of L, Gödel's constructible universe. For example, Martin and Steel [13] used the fine structure of $L(\mathbb{R})$ to determine the extent of scales in $L(\mathbb{R})$. In addition, relying on Steel's analysis in [17], Kechris [9] showed that $\mathrm{ZF} + \mathrm{AD} + V = L(\mathbb{R}) \Rightarrow \mathrm{DC}$. This success makes it natural to ask the following:

Question 1. How can one extend the range of what constitutes a "constructible" set of reals beyond $L(\mathbb{R})$ and still be able to resolve the important problems of descriptive set theory?

The real core model $K(\mathbb{R})$, introduced in [1], contains a "constructible" set of reals not in $L(\mathbb{R})$. In addition, a number of descriptive set theoretic problems have also been resolved in $K(\mathbb{R})$. For example, using a mixture of descriptive set theory, fine structure and the theory of iterated ultrapowers, one can produce definable scales in $K(\mathbb{R})$ beyond those in $L(\mathbb{R})$ and prove that $K(\mathbb{R}) \models \mathrm{DC}$ (see [1]). We believe that $K(\mathbb{R})$ is another *natural* inner model (for determinacy) whose structure will also be studied for its own sake, and that $K(\mathbb{R})$ provides a "first step" to answering Question 1. In this paper we pursue the problem of going beyond $K(\mathbb{R})$.

It seems that for an inner model of AD to be "well behaved," it requires a fine structure theory. Coincidentally, fine structure is an important tool in the study of inner models of large cardinal hypotheses and the Axiom of Choice, the smallest of which is the Core Model K. Dodd–Jensen [4, 5] carefully develop the fine structure of the inner model K and use this structure to demonstrate the existence of an inner model $L[\mu]$ with a measurable cardinal κ under certain hypotheses (e.g., "there is a non-trivial $j: K \underset{\Sigma_\omega}{\longrightarrow} K$"). Mitchell and Steel have developed and studied the fine structure of core models larger than K. In fact, using the fine structure of these core models, Steel [18] has shown that $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies the GCH. These techniques developed by Steel very likely can be used to show that $\mathrm{HOD}^{K(\mathbb{R})}$ also satisfies the GCH.

There are a number of large cardinal hypotheses which imply that $L(\mathbb{R})$ is an inner model of determinacy, and stronger cardinal hypotheses exist which yield larger inner models of determinacy (for example, $K(\mathbb{R})$).

Question 2. What other hypotheses allow one to build larger inner models of ZF + AD + DC?

The hypothesis

$$\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\, [X \notin L(\mathbb{R})]$$

is enough to build the real core model $K(\mathbb{R})$ and to show that it is an inner model of ZF + AD + DC (see the proof of Corollary 5.15 in [1]). In this paper we show that the hypothesis

$$\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}[X \notin K(\mathbb{R})] \tag{1}$$

is strong enough to build an even larger inner model $L[\mu](\mathbb{R})$ of ZF + AD + DC containing a measurable cardinal above its Θ. Here, Θ is the supremum of the ordinals which are the surjective image of $\mathbb{R}$. It appears that the hypothesis

$$\mathrm{ZF} + \mathrm{AD} + \text{"no } \textit{fine-structural} \text{ inner model of ZF + AD + DC contains } \mathscr{P}(\mathbb{R})\text{"} \tag{2}$$

allows one to build larger and larger inner models of determinacy and dependent choices. In some sense, (2) is a "large cardinal hypothesis". Here, note that we are considering only fine-structural inner models of ZF + AD + DC *containing* the set of reals $\mathbb{R}$. Another related hypothesis, which implies (2), is

$$\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}. \tag{3}$$

In this paper we are interested in the consequences of hypotheses (1) and (3) above.

The inner models K and $K(\mathbb{R})$ have a similar fine structure but they are quite different in other ways. For example, the Axiom of Choice holds in K, but is "false" in $K(\mathbb{R})$. Also, the methods used to analyze the sets of reals in $K(\mathbb{R})$ are obtained by merging descriptive set theory with large cardinal theory, and these methods do not apply to the inner model K. However, using the fine structure of $K(\mathbb{R})$ under the above hypothesis (1), we can establish that $K(\mathbb{R})$ is "iterable" and that there exists a larger inner model, $L[\mu](\mathbb{R})$, of AD with a measurable cardinal $\kappa > \Theta^{L[\mu](\mathbb{R})}$. Here, μ is an $\mathbb{R}$-complete, normal measure on κ in $L[\mu](\mathbb{R})$ (for the meaning of $\mathbb{R}$-complete in $L[\mu](\mathbb{R})$ see Definition 0.7).

Two consequences of this construction of the inner model $L[\mu](\mathbb{R})$ are:
(a) $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \exists\kappa > \Theta[\kappa \text{ is measurable}])$
(b) $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}) \Rightarrow \mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + (\omega^2 - \boldsymbol{\Pi}^1_1) - \mathrm{AD}_{\mathbb{R}})$.

Solovay [16] shows that the theory ZF + AD does not prove DC, assuming a strong hypothesis. Solovay conjectured that ZF + AD does not prove $\mathrm{DC}_{\mathbb{R}}$. Kechris–Woodin [9] were the first to inquire about the strength of the theory $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$. In particular, they asked:

Question 3. Does $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}})$ imply $\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}})$?

On the other hand, Woodin has asked (see 30.31 of [8]):

Question 4. Does ZF + AD imply $\mathrm{DC}_{\mathbb{R}}$?

The above results (a) and (b) may be a first step in finding a proof of the conjecture (stated in [1]) that $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$ implies the existence of inner models of AD with Woodin cardinals above Θ (see [8] for the definition of a Woodin cardinal). These inner models will (likely) have the form $L[\mathscr{E}](\mathbb{R})$, where $\mathscr{E}$ is a sequence of extenders, each of which is $\mathbb{R}$-complete. The assumption $\mathrm{AD}+\neg\mathrm{DC}_{\mathbb{R}}$ appears to provide a "bootstrapping process" for constructing "larger and larger" inner models of $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$. More specifically, one should be able to construct inner models of ZF + AD+DC containing "more and more" extenders (see Section 26 of [8]) and to construct an extender sequence $\mathscr{E}$, by recursion (via [14]), where $L[\mathscr{E}](\mathbb{R})$ has enough extenders to witness the existence of at least one Woodin cardinal. This "construction process" may have the following form:

Having constructed the inner model M of ZF + AD containing $\mathbb{R}$
(i) prove that $M \models \mathrm{DC}_{\mathbb{R}}$ (this may involve constructing quasi-scales, as in [1]),
(ii) conclude that $\exists X \subseteq \mathbb{R}[X \notin M]$,
(iii) prove that there is a non-trivial elementary embedding $j : M \to M'$ with M' transitive (see Lemma 4.1),
(iv) use the embedding j to construct a larger inner model $N \supset M$ of ZF + AD containing $\mathbb{R}$ (N may contain a set of reals not in M, see Theorems 4.16 and 4.18).

If the above "process" exhausts all sets of reals (that is, every set of reals is in one of the inner models of $\mathrm{DC}_{\mathbb{R}}$ constructed), then one would have a proof of

$$\mathrm{ZF} + \mathrm{AD} \Rightarrow \mathrm{DC}_{\mathbb{R}}$$

and thus a positive answer to Question 4. Otherwise (conceivably), after some stage the models constructed all have the same sets of reals. Are these inner models of

$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$? Another possibility is that one eventually constructs an inner model of $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$ and then the process stops. In any case, the completion of this "process" could lead to some interesting results.

The present paper is organized into four sections. In Section 1 we show that $K(\mathbb{R})$ satisfies a modified "generalized continuum hypothesis" and then use this, together with a theorem of Vopěnka, to give a computation of $\mathrm{HOD}^{K(\mathbb{R})}$. Another computation of $\mathrm{HOD}^{K(\mathbb{R})}$ is given in Section 4 (see discussion below). In Section 2 we define when μ is an M-measure on certain inner models M of AD. The usual construction of iterated ultrapowers of M assumes that M satisfies the Axiom of Choice, as does the standard proof that the associated ultrapower embeddings are elementary. However, the Axiom of Choice is false in an inner model M of AD. So in Section 2 we also show that Łoś' Theorem holds for the ultrapower of these "choiceless" inner models and that the corresponding embeddings are elementary. Furthermore, the typical argument proving that an inner model is "iterable" usually requires DC, the Axiom of Dependent Choices (see Lemma 19.12 of [8]). But in this paper we will not assume DC. So in Section 3 we develop some tools which will be used, in Section 4, to prove that $(K(\mathbb{R}), \mu)$ is weakly iterable without DC, whenever μ is a countably complete $K(\mathbb{R})$-measure on $\kappa > \Theta^{K(\mathbb{R})}$. Finally in Section 4 we show that there exists an inner model of $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ containing a measurable cardinal above its Θ, under the assumption of $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}[X \notin K(\mathbb{R})]$. As a corollary, we show that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ where $K(P)$ is the Dodd–Jensen Core Model relative to P, a set of ordinals. This generalizes a result of Woodin which states that $\mathrm{HOD}^{L(\mathbb{R})} = L(P)$ for some set $P \subseteq \Theta^{L(\mathbb{R})}$ (see Section 1 of [18]).

Remark. Steel and Woodin have established many important theorems using the structure of $\mathrm{HOD}^{L(\mathbb{R})}$. We expect that additional results will arise from the study of the structure of $\mathrm{HOD}^{K(\mathbb{R})}$.

0.1. Preliminaries and notation

We work in ZF and state our additional hypotheses as we need them. We do this to maintain a careful surveillance on the use of determinacy and dependent choice in the proofs of our main theorems. Variables $x, y, z, w \ldots$ generally range over $\mathbb{R}$, while $\alpha, \beta, \gamma, \delta \ldots$ range over OR, the class of ordinals. If $0 \leqslant j \leqslant \omega$ and $1 \leqslant k \leqslant \omega$, then $\omega^j \times ({}^{\omega}\omega)^k$ is recursively homeomorphic to $\mathbb{R}$, and we sometimes tacitly identify the two.

A proper class M is called an *inner model* if and only if M is a transitive $\in$-model of ZF containing all the ordinals. For an inner model M with $X \in M$, we shall write $\mathscr{P}^M(X)$ to denote the power set of X as computed in M. For an ordinal $\kappa \in M$, we shall abuse standard notation slightly and write ${}^{\kappa}M = \{f \in M \mid f : \kappa \to M\}$.

We distinguish between the notations $L[A]$ and $L(A)$. The inner model $L(A)$ is defined to be the class of sets constructible *above* A, whereas the inner model $L[A]$ is defined to be the class of sets constructible *relative* to A (see [8, p. 34]). Thus, one defines $L[A](B)$ to be the class of sets constructible *relative* to A and *above* B.

A pointclass is a class of subsets of $\mathbb{R}$ closed under recursive substitutions. A boldface pointclass is a pointclass closed under continuous substitutions. For a pointclass Γ, we write "$\Gamma-\mathrm{AD}$" or "$\mathrm{Det}(\Gamma)$" to denote the assertion that all games on ω with payoff in Γ are determined. For any notions from Descriptive Set Theory which we have not defined, we refer the reader to Moschovakis [15].

Our general set theoretic notation is standard. Given a function f, we write $\mathrm{dom}(f) = \{x\colon \exists y(f(x)=y)\}$ and $\mathrm{ran}(f)=\{y\colon \exists x(f(x)=y)\}$. We shall write $\langle x_1,\dots,x_n\rangle$ to represent a finite sequence of elements. For any set X, $(X)^{<\omega}$ is the set of all finite sequences of elements of X, $[X]^{<\omega}$ is the set of all finite subsets of X, and $\mathscr{P}(X)$ is the set of all subsets of X. Given two finite sequences s and t, the sequence $s^\frown t$ is the concatenation of s to t. Generally, μ will be a normal measure on $\mathscr{P}(\kappa)$, where κ is an ordinal. For an ordinal α, V_α is the set of all sets of rank less than α. We let $y = T_c(x)$ denote the formula "y is the transitive closure of x". For any model (or inner model) $\mathscr{M}=(M,\in,\dots)$, we write ${}^\kappa M=\{f\in M \mid f:\kappa\to M\}$. In addition, for a model (or inner model) $\mathscr{M}$ having only one "measurable cardinal", we shall write $\kappa^{\mathscr{M}}$ to denote this cardinal in $\mathscr{M}$. Similarly, when $\mathscr{M}$ has only one "measure", we shall write $\mu^{\mathscr{M}}$ to denote this measure.

Given a model $\mathscr{M}=(M,c_1,c_2,\dots,c_m,A_1,A_2,\dots,A_N)$, where the A_i are predicates and the c_i are constants, if $X\subseteq M$ then $\Sigma_n(\mathscr{M},X)$ is the class of relations on M definable over $\mathscr{M}$ by a Σ_n formula from parameters in $X\cup\{c_1,c_2,\dots,c_m\}$. $\Sigma_\omega(\mathscr{M},X)=\bigcup_{n\in\omega}\Sigma_n(\mathscr{M},X)$. We write "$\Sigma_n(\mathscr{M})$" for $\Sigma_n(\mathscr{M},\emptyset)$ and "$\boldsymbol{\Sigma}_n(\mathscr{M})$" for the boldface class $\Sigma_n(\mathscr{M},M)$. Similar conventions hold for Π_n and Δ_n notations. If $\mathscr{M}$ is a substructure of $\mathscr{N}$ and $X\subseteq M\subseteq N$, then "$\mathscr{M}\prec_n^X\mathscr{N}$" means that $\mathscr{M}\models\phi[a]$ if and only if $\mathscr{N}\models\phi[a]$, for all $a\in(X)^{<\omega}$ and for all Σ_n formulae ϕ (the formula ϕ is allowed constants taken from $\{c_1,c_2,\dots,c_m\}$). We write "$\mathscr{M}\prec_n\mathscr{N}$" for "$\mathscr{M}\prec_n^M\mathscr{N}$". In addition, for any two models $\mathscr{M}$ and $\mathscr{N}$, we write $\pi:\mathscr{M}\underset{\Sigma_n}{\longrightarrow}\mathscr{N}$ to indicate that the map π is a Σ_n-elementary embedding, that is, $\mathscr{M}\models\phi[a]$ if and only if $\mathscr{N}\models\phi[\pi(a)]$, for all $a=\langle a_0,a_1,\dots\rangle\in(M)^{<\omega}$ and for all Σ_n formulae ϕ, where $0\leqslant n\leqslant\omega$ and $\pi(a)=\langle\pi(a_0),\pi(a_1),\dots\rangle$.

We now give an overview of the fundamental notions presented in [1, 2], which will be assumed here. The language $\mathscr{L}_N=\{\in,\underline{\mathbb{R}},c_1,\dots,c_m,A_1,\dots,A_N\}$ consists of the constant symbols $\underline{\mathbb{R}}$ and $c_1,\dots,c_m$ together with the membership relation $\in$ and the predicate symbols $A_1,\dots,A_N$. The theory R_N is the deductive closure of the following weak set theory above the reals:

(1)	$\forall x\forall y(x=y\Leftrightarrow\forall z(z\in x\Leftrightarrow z\in y))$	(extensionality)
(2)	$\exists y\forall x(x\notin y)$	($\emptyset$ exists)
(3)	$\forall x(x\neq\emptyset\Rightarrow\exists y(y\in x\wedge x\cap y=\emptyset))$	(foundation)
(4)	$\forall x\forall y\exists z\forall t(t\in z\Leftrightarrow(t=x\vee t=y))$	(pairing)
(5)	$\forall x\exists y\forall t(t\in y\Leftrightarrow\exists z(z\in x\wedge t\in z))$	(union)
(6)	$\exists w(\emptyset\in w\wedge\mathrm{ord}(w)\wedge\lim(w)\wedge\forall\alpha\in w\neg\lim(\alpha))$	(ω exists)
(7)	$\forall u\forall\vec{x}\exists z\forall s(s\in z\Leftrightarrow s\in u\wedge\psi(s,\vec{x}))$	(Σ_0 separation)

(8) $\forall u \forall \vec{x} \forall w \exists y \forall z (z \in y \Leftrightarrow \exists t \in w (z = \{s \in u : \varphi(s, t, \vec{x})\}))$ (Σ_0 closure)

(9) $\forall x (x \in \underline{\mathbb{R}} \Leftrightarrow \forall y \in x \exists n \in \omega \exists z \exists f (\mathrm{Tr}(z) \wedge y \subseteq z \wedge f : n \xrightarrow{\text{onto}} z))$ ($\underline{\mathbb{R}} = V_{\omega+1}$)

where, in (7) and (8), φ and ψ are any Σ_0 formulas. The above predicates ord(w), lim(w), and Tr(w) abbreviate "w is an ordinal", "w is limit", and "w is transitive", respectively.

Of course, $V_{\omega+1}$ can be "constructed" from the set of reals $\mathbb{R}$. It is more convenient, however, to start constructing new sets from the transitive set $V_{\omega+1}$ rather than from $\mathbb{R}$. Since $\mathbb{R}$ is a proper subset of $V_{\omega+1}$ and is easily "separated" from $V_{\omega+1}$, we shall consider $V_{\omega+1}$ as given and we will tacitly identify the two.

The theory $\mathrm{R}^+ (= \mathrm{R}_N^+)$ is R_N together with the Π_2 sentence $V = L[A_1, \ldots, A_N](\underline{\mathbb{R}})$. We are interested in transitive models $\mathscr{M} = (M, \in, \underline{\mathbb{R}}^{\mathscr{M}}, A_1, \ldots, A_N)$ of R_N^+. We shall write $\mathbb{R}^{\mathscr{M}} = \underline{\mathbb{R}}^{\mathscr{M}}$ for $\mathscr{M}$'s version of the reals. For any $\alpha \in \mathrm{OR}^{\mathscr{M}}$ we let $S_\alpha^{\mathscr{M}}(\underline{\mathbb{R}})$ denote the unique set in $\mathscr{M}$ satisfying $\mathscr{M} \models \exists f (\varphi(f) \wedge \alpha \in \mathrm{dom}(f) \wedge S_\alpha^{\mathscr{M}}(\underline{\mathbb{R}}) = f(\alpha))$, where φ is the Σ_0 sentence used to define the sequence $\langle S_\gamma^{\mathscr{M}}(\underline{\mathbb{R}}) : \gamma < \mathrm{OR}^{\mathscr{M}} \rangle$ (see [1, Definition 1.5]). For $\lambda = \mathrm{OR}^{\mathscr{M}}$, let $S_\lambda^{\mathscr{M}}(\underline{\mathbb{R}}) = \bigcup_{\alpha < \lambda} S_\alpha^{\mathscr{M}}(\underline{\mathbb{R}})$. Let $\widehat{\mathrm{OR}}$ denote the class $\{\gamma\colon$ the ordinal $\omega\gamma$ exists$\}$ and let $J_\gamma^{\mathscr{M}}(\underline{\mathbb{R}}) = S_{\omega\gamma}^{\mathscr{M}}(\underline{\mathbb{R}})$, for $\gamma \leqslant \widehat{\mathrm{OR}}^{\mathscr{M}}$. Since $\mathscr{M} \models \mathrm{R}^+$, it follows that $M = J_\alpha^{\mathscr{M}}(\underline{\mathbb{R}})$ where $\alpha = \widehat{\mathrm{OR}}^{\mathscr{M}}$. For $1 < \gamma \leqslant \widehat{\mathrm{OR}}^{\mathscr{M}}$ let $\mathscr{M}^\gamma$ be the substructure of $\mathscr{M}$ defined by $\mathscr{M}^\gamma = (J_\gamma^{\mathscr{M}}(\underline{\mathbb{R}}), \in, \underline{\mathbb{R}}^{\mathscr{M}}, J_\gamma^{\mathscr{M}}(\underline{\mathbb{R}}) \cap A_1, \ldots, J_\gamma^{\mathscr{M}}(\underline{\mathbb{R}}) \cap A_N)$ and let $M^\gamma = J_\gamma^{\mathscr{M}}(\underline{\mathbb{R}})$. We can write $\mathscr{M}^\gamma = (M^\gamma, \in, \underline{\mathbb{R}}^{\mathscr{M}}, A_1, \ldots, A_N)$, as this will cause no confusion. Note that $\mathscr{M}^\gamma$ is *amenable*, that is, $a \cap A_i \in M^\gamma$, for all $a \in M^\gamma$ where $1 \leqslant i \leqslant N$.

Definition 0.1. Let $\mathscr{M}$ be a transitive model of R^+. The *projectum* $\rho_{\mathscr{M}}$ is the least ordinal $\rho \leqslant \widehat{\mathrm{OR}}^{\mathscr{M}}$ such that $\mathscr{P}(\mathbb{R}^{\mathscr{M}} \times \omega\rho) \cap \boldsymbol{\Sigma}_1(\mathscr{M}) \not\subseteq M$, and $p_{\mathscr{M}}$ is the $\leqslant_{BK}$-least $p \in [\mathrm{OR}^{\mathscr{M}}]^{<\omega}$ such that $\mathscr{P}(\mathbb{R}^{\mathscr{M}} \times \omega\rho_{\mathscr{M}}) \cap \Sigma_1(\mathscr{M}, \{p\}) \not\subseteq M$.

The order $<_{BK}$ is the Brouwer–Kleene order on finite sets of ordinals and is a Σ_0 well-order. We now recall the definition of a master code and the notion of acceptability, as stated in [2].

Definition 0.2. The Σ_1-*master code*, $A_{\mathscr{M}}$, of $\mathscr{M}$ is the set

$$A_{\mathscr{M}} = \{(x, s) \in \mathbb{R}^{\mathscr{M}} \times (\omega\rho_{\mathscr{M}})^{<\omega} : \mathscr{M} \models \varphi_{x(0)}(\lambda n . x(n+1), s, p_{\mathscr{M}})\}$$

where $\langle \varphi_i : i \in \omega \rangle$ is a fixed recursive listing of all the Σ_1 formulae of three variables in the language $\mathscr{L}_N$ and $\lambda . n x(n+i)$ is the real y such that $y(n) = x(n+i)$ for all $n \in \omega$.

Definition 0.3. Suppose that $\mathscr{M}$ is a transitive model of R_N^+. We say that $\mathscr{M}$ is *acceptable* (above the reals) provided that whenever $\mathscr{P}(\delta \times \mathbb{R}^{\mathscr{M}}) \cap M^{\nu+1} \not\subseteq M^\nu$ for $\nu < \widehat{\mathrm{OR}}^{\mathscr{M}}$ and $\delta < \mathrm{OR}^{\mathscr{M}^\nu}$, then for each $u \in M^{\nu+1}$ there is an $f \in M^{\nu+1}$ such that

$$f = \langle f_{\xi,x} : \delta \leqslant \xi < \mathrm{OR}^{\mathscr{M}^\nu} \wedge x \in \mathbb{R}^{\mathscr{M}} \rangle$$

and

$$f_{\xi,x}\colon \xi \times \mathbb{R}^{\mathcal{M}} \xrightarrow{\text{onto}} \{\xi\} \cup (\mathscr{P}(\xi \times \mathbb{R}^{\mathcal{M}}) \cap u).$$

Write $\langle J_\xi^{(\mathcal{M},A_{\mathcal{M}})}(\underline{\mathbb{R}})\colon \xi \in \mathrm{OR}\rangle$ for the Jensen hierarchy of sets which are relatively constructible above $\mathbb{R}^{\mathcal{M}}$ from the predicates $A_1 \cap M, \dots, A_N \cap M, A_{\mathcal{M}} \cap M$.

Definition 0.4. Given that $\mathcal{M}$ is acceptable, let $M^* = J_{\rho_{\mathcal{M}}}^{(\mathcal{M},A_{\mathcal{M}})}(\underline{\mathbb{R}})$. The Σ_1-code of $\mathcal{M}$ is the structure $\mathcal{M}^* = (M^*, \in, \underline{\mathbb{R}}, c_1, \dots, c_m, A_1 \cap M^*, \dots, A_N \cap M^*, A_{\mathcal{M}} \cap M^*)$ where the constants have the same interpretation in $\mathcal{M}^*$ as in $\mathcal{M}$.

In order to apply fine structural techniques to the structure $\mathcal{M}$, one must ensure that the two structures $\mathcal{M}$ and $\mathcal{M}^*$ have the same bounded subsets of $\mathbb{R}^{\mathcal{M}} \times \omega\rho_{\mathcal{M}}$. The acceptability of $\mathcal{M}$ implies that $\mathcal{M}^*$ is a substructure of $\mathcal{M}$ containing all the bounded subsets of $\mathbb{R}^{\mathcal{M}} \times \omega\rho_{\mathcal{M}}$ which are elements in $\mathcal{M}$ (see Lemma 1.15 of [2]). Thus, fine structural techniques can be applied to $\mathcal{M}$. In addition, since $\mathcal{M}^*$ is an acceptable model of R_{N+1}^+ (see Lemma 1.17 of [2]), one can "iterate the projectum".

Definition 0.5. Suppose that $\mathcal{M}$ is acceptable. Inductively define on $n \in \omega$ the Σ_n-code of $\mathcal{M}$, denoted by $\mathcal{M}^n$, as follows:
(1) $\mathcal{M}^0 = \mathcal{M}$, $\rho_{\mathcal{M}}^0 = \mathrm{OR}^{\mathcal{M}}$, $p_{\mathcal{M}}^0 = \emptyset$, and $\mathscr{A}_{\mathcal{M}}^0 = \emptyset$.
(2) Assume that $\mathcal{M}^n$ has been defined and that $\rho_{\mathcal{M}^n} > 1$. Define $\mathcal{M}^{n+1} = (\mathcal{M}^n)^*$, $\rho_{\mathcal{M}}^{n+1} = \rho_{\mathcal{M}^n}$, $p_{\mathcal{M}}^{n+1} = p_{\mathcal{M}^n}$, and $\mathscr{A}_{\mathcal{M}}^{n+1} = A_{\mathcal{M}^n}$.

Remark. The above notation is slightly inconsistent with previous notation. Namely, for an ordinal γ, $\mathcal{M}^\gamma$ denotes a structure whose domain consists of the sets constructed in $J_\gamma^{\mathcal{M}}(\underline{\mathbb{R}})$, while $\mathcal{M}^n$ denotes the Σ_n-code of $\mathcal{M}$. However, we shall use integer variables, for example n, exclusively for denoting $\mathcal{M}^n$ the Σ_n-code of $\mathcal{M}$.

There is another way of iterating a "projectum".

Definition 0.6. Let $\mathcal{M}$ be acceptable. Define $\gamma_{\mathcal{M}}^0 = \mathrm{OR}^{\mathcal{M}}$. For $1 \leqslant n < \omega$, define $\gamma_{\mathcal{M}}^n$ to be the least ordinal $\gamma \leqslant \widehat{\mathrm{OR}}^{\mathcal{M}}$ such that $\mathscr{P}(\mathbb{R}^{\mathcal{M}} \times \omega\gamma) \cap \Sigma_n(\mathcal{M}) \not\subseteq M$.

For an arbitrary acceptable $\mathcal{M}$ the connection between $\gamma_{\mathcal{M}}^n$ and $\rho_{\mathcal{M}}^n$ is not clear. However, if $\mathcal{M}$ is a "mouse", then $\gamma_{\mathcal{M}}^n = \rho_{\mathcal{M}}^n$ whenever $\rho_{\mathcal{M}}^n$ is defined (see Definition 0.17 and Theorem 0.19 below).

We shall now review some definitions and results from [1].

Definition 0.7. Let μ be a normal measure on κ. We say that μ is an $\mathbb{R}$-*complete measure* on κ if the following holds:

If $\langle A_x : x \in \mathbb{R}\rangle$ is any sequence such that $A_x \in \mu$ for all $x \in \mathbb{R}$, then $\bigcap_{x\in\mathbb{R}} A_x \in \mu$.

We now focus our attention on transitive models $\mathscr{M}$ of R^+ such that $\mathscr{M}$ believes that one of its predicates is an $\mathbb{R}^{\mathscr{M}}$-complete measure on $\mathscr{P}(\kappa)\cap M$. For this reason we modify our official language by letting $\mathscr{L}_N=\{\in,\underline{\mathbb{R}},\underline{\kappa},\mu,A_1,\dots,A_N\}$, where μ is a new predicate symbol and $\underline{\kappa}$ is a new constant symbol.

Definition 0.8. A model $\mathscr{M}=(M,\in,\underline{\mathbb{R}}^{\mathscr{M}},\underline{\kappa}^{\mathscr{M}},\mu,A_1,\dots,A_N)$ is a *premouse* (above the reals) if
(1) $\mathscr{M}$ is a transitive model of R^+,
(2) $\mathscr{M}\models$ "μ is an $\underline{\mathbb{R}}$-complete measure on $\underline{\kappa}$".

$\mathscr{M}$ is a *pure* premouse if $\mathscr{M}=(M,\underline{\mathbb{R}}^{\mathscr{M}},\underline{\kappa}^{\mathscr{M}},\mu)$. Finally, $\mathscr{M}$ is a *real* premouse if it is pure and $\mathbb{R}^{\mathscr{M}}=\mathbb{R}$.

To distinguish our definition of a premouse from the premice of Dodd–Jensen [5], we may sometimes refer to our version as "premice above the reals".

Definition 0.9. The theory PM is the theory R^+ together with the sentence "μ is an $\underline{\mathbb{R}}$-complete measure on $\underline{\kappa}$".

The statement "μ is an $\underline{\mathbb{R}}$-complete measure on $\underline{\kappa}$" is a Π_1 assertion. The theory PM can be axiomatized by a single Π_2 sentence.

We defined in [1] the ultrapower of a premouse $\mathscr{M}$, denoted by $\mathscr{M}_1$, and showed that a version of Łoś' Theorem holds for this ultrapower. We shall write κ or $\kappa^{\mathscr{M}}$ for $\underline{\kappa}^{\mathscr{M}}$ when the context is clear. We shall slightly abuse standard notation and write ${}^{\kappa}M=\{f\in M\mid f:\kappa\to M\}$. For $f\in{}^{\kappa}M$, we write $[f]$ for the (usual) equivalence class of f (see [1, p. 226]).

Theorem 0.10. *Let $\mathscr{M}$ be a premouse. Then*

$$\mathscr{M}_1\models\varphi([f_1],\dots,[f_n])\Leftrightarrow\mathscr{M}\models\{\xi\in\kappa\colon\varphi(f_1(\xi),\dots,f_n(\xi))\}\in\mu,$$

for every Σ_0 formula φ and for all $f_1,\dots,f_n\in{}^{\kappa}M$.

Proof. See 2.4 of [1]. □

For a premouse $\mathscr{M}$ define $\pi^{\mathscr{M}}:\mathscr{M}\to\mathscr{M}_1$ by $\pi^{\mathscr{M}}(a)=[c_a]$ for $a\in M$. When the context is clear we shall drop the superscript and write π for $\pi^{\mathscr{M}}$.

Corollary 0.11. $\pi:\mathscr{M}\xrightarrow[\Sigma_0]{}\mathscr{M}_1$.

Lemma 0.12. *Let $\mathscr{M}$ be a premouse. Then $\pi:\mathscr{M}\to\mathscr{M}_1$ is cofinal and hence, $\pi:\mathscr{M}\xrightarrow[\Sigma_1]{}\mathscr{M}_1$.*

Proof. See 1.18 and 2.6 of [1]. □

Lemma 0.13. *Let $\mathcal{M}$ be a premouse. Then $\mathcal{M}_1 \models$ PM.*

Proof. See 2.7 of [1]. □

The proofs of Definition 0.9, Theorem 0.10 and Corollary 0.11 use the fact that a premouse $\mathcal{M}$ is transitive. The ultrapower $\mathcal{M}_1$ is not necessarily transitive (or even well-founded). However, $\mathcal{M}_1$ inherits from $\mathcal{M}$ all the properties required to form the ultrapower and to prove that Definition 0.9, Theorem 0.10 and Corollary 0.11 apply to $\mathcal{M}_1$. Thus, we can iterate this ultrapower construction through the ordinals.

Definition 0.14. Let $\mathcal{M}$ be a premouse. The *premouse iteration* of $\mathcal{M}$

$$\langle\langle \mathcal{M}_\alpha\rangle_{\alpha\in\mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathcal{M}_\alpha \underset{\Sigma_1}{\longrightarrow} \mathcal{M}_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

is the commutative system satisfying the inductive definition:
(1) $\mathcal{M}_0 = \mathcal{M}$,
(2) $\pi_{\gamma\gamma} =$ identity map, and $\pi_{\beta\gamma} \circ \pi_{\alpha\beta} = \pi_{\alpha\gamma}$ for all $\alpha\leqslant\beta\leqslant\gamma\leqslant\lambda$,
(3) If $\lambda = \lambda' + 1$, then $\mathcal{M}_\lambda =$ ultrapower of $\mathcal{M}_{\lambda'}$, and $\pi_{\alpha\lambda} = \pi^{\mathcal{M}_{\lambda'}} \circ \pi_{\alpha\lambda'}$ for all $\alpha\leqslant\lambda'$,
(4) If λ is limit, then $\langle \mathcal{M}_\lambda, \langle \pi_{\alpha\lambda} : \mathcal{M}_\alpha \to \mathcal{M}_\lambda\rangle_{\alpha<\lambda}\rangle$ is the direct limit of

$$\langle\langle \mathcal{M}_\alpha\rangle_{\alpha<\lambda}, \langle \pi_{\alpha\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta\rangle_{\alpha\leqslant\beta<\lambda}\rangle.$$

We note that the maps in the above commutative system are cofinal (see Definition 1.18 of [1]) and are Σ_1 embeddings, that is,

$$\pi_{\alpha\beta} : \mathcal{M}_\alpha \overset{\text{cofinal}}{\underset{\Sigma_1}{\longrightarrow}} \mathcal{M}_\beta$$

for all $\alpha\leqslant\beta\in\mathrm{OR}$.

Definition 0.15. A premouse $\mathcal{M}$ is an *iterable premouse* if $\mathcal{M}_\lambda$ is well-founded for all $\lambda\in\mathrm{OR}$.

In the paper [1], with which we shall assume the reader is acquainted, we introduced the Real Core Model, $K(\mathbb{R})$, and showed that $K(\mathbb{R})$ is an inner model containing the reals and definable scales beyond those in $L(\mathbb{R})$. We assumed that $K(\mathbb{R})$ satisfies the Axiom of Determinacy. To establish our results in [1] on the existence of scales, we defined real 1-mice and showed how the basic fine-structural notions of Dodd–Jensen [5] generalize to iterable "premice above the reals". Recall that $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ is a real 1-mouse if $\mathcal{M}$ is an iterable real premouse and $\mathscr{P}(\mathbb{R}\times\kappa)\cap\Sigma_1(M)\not\subseteq M$, where M has the form $J_\alpha[\mu](\mathbb{R})$ and κ is the "measurable cardinal" in $\mathcal{M}$. Real 1-mice suffice to define the real core model and to prove the results in [1] about $K(\mathbb{R})$; however, they are not sufficient to prove all the results in the present paper. That is, some of our results require the full fine-structure of $K(\mathbb{R})$ as developed in [2].

In the paper [2] we generalized Dodd–Jensen's notion of a mouse to that of a *real mouse* $\mathcal{M}$ containing all the reals and having the form $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$. The definition of a real mouse is obtained (roughly) by "advancing" the notion of a real 1-mouse; that is, by

(i) replacing Σ_1 with Σ_n, where n is the smallest integer such that $\mathscr{P}(\mathbb{R} \times \kappa) \cap \Sigma_{n+1}(\mathscr{M}) \not\subseteq M$, and

(ii) defining a stronger iterability condition.

We shall now (more formally) review the definition of a "mouse" and the definition of "mouse iteration".

Definition 0.16. Let $\mathscr{M}$ be an acceptable pure premouse. We say that $\mathscr{M}$ is *critical* if $\rho^{n+1}_{\mathscr{M}} \leqslant \kappa^{\mathscr{M}} < \rho^{n}_{\mathscr{M}}$, for some $n \in \omega$. This integer n is denoted by $n(\mathscr{M})$ and we write $\bar{\mathscr{M}} = \mathscr{M}^{n(\mathscr{M})}$.

Definition 0.17. Let $\mathscr{M}$ be a critical pure premouse. Since $\bar{\mathscr{M}}$ is also a premouse, let

$$\langle \langle \bar{\mathscr{M}}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \bar{\pi}_{\alpha\beta} : \bar{\mathscr{M}}_\alpha \underset{\Sigma_1}{\overset{\text{cofinal}}{\longrightarrow}} \bar{\mathscr{M}}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle \tag{$*$}$$

be the premouse iteration of $\bar{\mathscr{M}}$, as in Definition 0.14. We say that $\mathscr{M}$ is a *mouse* whenever (i) $\bar{\mathscr{M}}$ is an iterable premouse and (ii) the system $(*)$ of transitive models can be extended (by decoding master codes, as in [2]) to a commutative system of transitive structures

$$\langle \langle \mathscr{M}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathscr{M}_\alpha \underset{\Sigma_{n(\mathscr{M})+1}}{\longrightarrow} \mathscr{M}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle. \tag{$\surd$}$$

The system $(\surd)$ is called the *mouse iteration* of $\mathscr{M}$. If, in addition, the mouse $\mathscr{M}$ contains all the reals, that is, if $\mathbb{R}^{\mathscr{M}} = \mathbb{R}$, then $\mathscr{M}$ is said to be a *real mouse*.

Theorem 0.18. *Suppose $\mathscr{M}$ is an acceptable pure premouse and let $n \in \omega$. If $\omega\rho^{n}_{\mathscr{M}} \geqslant \kappa^{\mathscr{M}}$, then $\gamma^{n}_{\mathscr{M}} = \rho^{n}_{\mathscr{M}}$.*

Proof. This follows from Corollaries 2.13 and 1.33 of [2]. □

Theorem 0.19. *Let $n \in \omega$. If $\mathscr{M}$ is a mouse, then $\gamma^{n}_{\mathscr{M}} = \rho^{n}_{\mathscr{M}}$ whenever $\rho^{n}_{\mathscr{M}}$ is defined.*

Proof. This follows from Corollaries 2.13, 1.33 and 2.38 of [2]. □

Remark

(1) An iterable real premouse is acceptable (see Theorem 1.4).

(2) A real 1-mouse $\mathscr{M} = (M, \mathbb{R}, \kappa, \mu)$ is the simplest of real mice, that is, $\mathscr{M}$ is a real 1-mouse if it is an iterable real premouse and $\mathscr{P}(\mathbb{R} \times \kappa) \cap \Sigma_1(M) \not\subseteq M$.

The following non-standard definition of cardinality in a structure "above the reals" is given in [2].

Definition 0.20. Let $\mathscr{M}$ be a premouse.

(1) For $a \in M$ the *$\mathcal{M}$-cardinality* of a, denoted by $|a|_{\mathcal{M}}$, is the least ordinal $\lambda \in \mathrm{OR}^{\mathcal{M}}$ such that $f : \lambda \times \mathbb{R}^{\mathcal{M}} \xrightarrow{\text{onto}} a$ for some $f \in M$.
(2) An ordinal $\lambda \leqslant \mathrm{OR}^{\mathcal{M}}$ is an *$\mathcal{M}$-cardinal* if $\lambda = |\lambda|_{\mathcal{M}}$ or $\lambda = \mathrm{OR}^{\mathcal{M}}$.
(3) For an ordinal $\lambda < \mathrm{OR}^{\mathcal{M}}$, $\lambda^+_{\mathcal{M}}$ is the least $\mathcal{M}$-cardinal greater than λ.

Lemma 1.7 of [1] implies that for a set a in a premouse $\mathcal{M}$, there is a function $f \in \mathcal{M}$ such that $f : [\alpha]^{<\omega} \times \mathbb{R}^{\mathcal{M}} \xrightarrow{\text{onto}} a$, for some $\alpha < \mathrm{OR}^{\mathcal{M}}$. Thus, Lemma 1.4 of [2] shows that the $\mathcal{M}$-cardinality of any set in $\mathcal{M}$ exists.

Remark. In the definition of $\mathcal{M}$-cardinality, we have decided to use the cross product $\lambda \times \mathbb{R}^{\mathcal{M}}$. In the special case where $\lambda = 0$, the cross product $0 \times \mathbb{R}^{\mathcal{M}} = \emptyset$ and hence, does not involve any reals. So, when applying the above definition, we shall abuse cross product notation slightly, and define $0 \times \mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}}$.

In next section, we shall extend the above notion of "cardinality" to certain inner models of ZF containing the reals. This completes our overview of the notions and results presented in [1, 2]. For any terms that are undefined below, we refer the reader to these two papers.

1. $K(\mathbb{R})$ and the GCH

Dodd–Jensen's Core Model K satisfies AC + GCH, that is, the Axiom of Choice and the Generalized Continuum Hypothesis. We recall that the Real Core Model $K(\mathbb{R})$ is the union of real 1-mice, that is,

$$K(\mathbb{R}) = \{x : \exists \mathcal{N}(\mathcal{N} \text{ is a real 1-mouse} \wedge x \in N)\}.$$

$K(\mathbb{R})$ is a natural generalization of the Core Model K. In this section we shall show that $K(\mathbb{R})$ satisfies a "generalized continuum hypothesis" of the form

$$(\forall \lambda \in \mathrm{OR})\,[\text{the } \textit{cardinality} \text{ of } \mathscr{P}(\lambda \times \mathbb{R}) = \text{the first } \textit{cardinal} \text{ larger than } \lambda].$$

However, $K(\mathbb{R})$ does not satisfy the Axiom of Choice (assuming AD). In particular, $\mathscr{P}(\lambda)$ cannot be well-ordered in $K(\mathbb{R})$ for any ordinal $\lambda \geqslant \omega$. So, we shall modify the standard definition of cardinality in $K(\mathbb{R})$.

Definition 1.1. Let M be an inner model containing the reals $\mathbb{R}$.
(1) The *M-cardinality* of a set $a \in M$, denoted by $|a|_M$, is the least ordinal $\lambda \in \mathrm{OR}$ such that $f : \lambda \times \mathbb{R} \xrightarrow{\text{onto}} a$ for some $f \in M$.
(2) An ordinal λ is called an *M-cardinal* if $\lambda = |\lambda|_M$.
(3) For an ordinal λ the least *M-cardinal* greater than λ is denoted by λ^+_M.

Remark. We have chosen (as remarked earlier) to use the cross product $\lambda \times \mathbb{R}$ in our definition of M-cardinality. However, the cross product $0 \times \mathbb{R} = \emptyset$ and does not involve the reals. When applying the above definition, we shall abuse standard notation (in this special case only) and define $0 \times \mathbb{R} = \mathbb{R}$. Consequently, if $f : \mathbb{R} \xrightarrow{\text{onto}} a$ for some $f \in M$, then $|a|_M = 0$.

Since $K(\mathbb{R})$ is the union of real 1-mice, Lemma 1.7 of [1] and Lemma 1.4 of [2] imply that the $K(\mathbb{R})$-cardinality of every set in $K(\mathbb{R})$ exists. However, we must distinguish between the notion of an ordinal λ being a $K(\mathbb{R})$-*cardinal* and the notion of λ being a *standard* cardinal in $K(\mathbb{R})$.

Definition 1.2. Let M be an inner model containing the reals $\mathbb{R}$.
(1) λ is a *cardinal in M* if for no $\xi < \lambda$ does there exist an $f \in M$ such that $f : \xi \xrightarrow{\text{onto}} \lambda$.
(2) λ^+ is the least cardinal in M greater than λ.

Remark. In $K(\mathbb{R})$, $\Theta^+_{K(\mathbb{R})} = \Theta^+$ and for all ordinals $\lambda < \Theta^{K(\mathbb{R})}$
(1) $\lambda^+_{K(\mathbb{R})} = \Theta^{K(\mathbb{R})}$,
(2) $|\lambda|_{K(\mathbb{R})} = 0$,
(3) $|\mathscr{P}(\lambda)|_{K(\mathbb{R})} = 0$, assuming $K(\mathbb{R}) \models \mathrm{AD}$ (see 28.15 of [8]).

Definition 1.3. Let M be an inner model containing the reals $\mathbb{R}$. We shall write $M \models$ "GCH" to denote that

$$M \models (\forall \lambda \in \mathrm{OR})\,[|\mathscr{P}(\lambda \times \mathbb{R})|_M = \lambda^+_M].$$

Note that if $M \models$ "GCH", then $M \models (\forall \lambda \in \mathrm{OR})\,[|\mathscr{P}(\lambda)|_M \leqslant \lambda^+_M]$.

Remark. For any inner model M of ZFC, we shall write $|a|^M$ to denote the *standard* cardinality, in M, of a set $a \in M$. Also, we shall write $M \models \mathrm{GCH}$ to mean that the *standard* Generalized Continuum Hypothesis holds in M, that is, $M \models (\forall \lambda \in \mathrm{OR})[2^\lambda = \lambda^+]$.

We will show that in fact, $K(\mathbb{R}) \models$ "GCH" and that certain generic extensions of $K(\mathbb{R})$ satisfy the standard GCH. We now state a generalization of Lemma 5.21 of Dodd–Jensen [5].

Theorem 1.4. *Suppose that $\mathscr{M}$ is an iterable pure premouse above the reals. Then $\mathscr{M}$ is acceptable.*

Proof (*Sketch*). A proof of this theorem is actually given in [3]. The proof is a generalization of Dodd–Jensen's proof of Lemma 5.21 in [5]. For this reason we only discuss, here, the main ingredients that are used in the proof of Theorem 1.4. Let $\mathscr{M}$ be an iterable pure premouse above the reals and let $\kappa = \kappa^{\mathscr{M}}$. One proves by induction on $\gamma \leqslant \widehat{\mathrm{OR}}^{\mathscr{M}}$ that $\mathscr{M}^\gamma$ is acceptable (see the first two paragraphs of Chapter 11 of [4]). To ensure that $\mathscr{M}^{\gamma+1}$ is acceptable, assuming that $\mathscr{M}^\gamma$ is acceptable, one must prove the following two lemmas (see Definition 0.16).

Lemma 1.5. *Let $\gamma < \widehat{\mathrm{OR}}^{\mathcal{M}}$. Assume that $\mathcal{M}^\gamma$ is acceptable and critical. Let $n = n(\mathcal{M}^\gamma)$. If $\rho^{n+1}_{\mathcal{M}^\gamma} < \kappa$, then $\mathcal{P}(M^\gamma) \cap M^{\gamma+1} \subseteq \Sigma_\omega(\mathcal{M}^\gamma)$.*

For any pure premouse above the reals, say $\mathcal{N}$, define $H^{\mathcal{N}}_\kappa = \{a \in N : |T_c(a)|_{\mathcal{N}} < \kappa^{\mathcal{N}}\}$, where $T_c(a)$ denotes the transitive closure of a. Lemma 1.9 of [1] states that $\mathcal{N} \models \forall a \exists y(y = T_c(a))$ and, in addition, the comment following Definition 0.20 shows that $\mathcal{N} \models \forall a \exists \lambda(\lambda = |T_c(a)|_{\mathcal{N}})$. Thus, the definition of $H^{\mathcal{N}}_\kappa$ is well-defined.

The next lemma is a direct generalization of Lemma 4.9 of Dodd–Jensen [5].

Lemma 1.6. *Let $\gamma < \widehat{\mathrm{OR}}^{\mathcal{M}}$. Assume that $\mathcal{M}^\gamma$ is acceptable and critical. Let $n = n(\mathcal{M}^\gamma)$. If $\rho^{n+1}_{\mathcal{M}^\gamma} = \kappa$, then $H^{\mathcal{M}^\gamma}_\kappa = H^{\mathcal{M}^{\gamma+1}}_\kappa$.*

Using the above two lemmas, the argument proving Lemma 5.21 of Dodd–Jensen [5] can now be adapted to prove that $\mathcal{M}^{\gamma+1}$ is acceptable (in addition, see Chapter 11 of [4]). □

Remark. Lemmas 1.5 and 1.6 are used in [3] to solve a problem for constructing scales in $K(\mathbb{R})$. Let $\mathcal{M}$ be a iterable real premouse and suppose that a set A is constructed in $M^{\gamma+1} \backslash M^\gamma$, where $\kappa^{\mathcal{M}} \leqslant \gamma < \mathrm{OR}^{\mathcal{M}}$. Since we are using the measure $\mu^{\mathcal{M}}$ to construct new sets in $M^{\gamma+1}$, it is possible that $A \in M^{\gamma+1} \backslash \Sigma_\omega(\mathcal{M}^\gamma)$. The above two lemmas are used to show that this cannot happen when A is a set of reals. In particular, when a new set of reals A is constructed in $M^{\gamma+1} \backslash M^\gamma$, then $A \in \Sigma_\omega(\mathcal{M}^\gamma)$. In [3], this fact is important for our determination of whether or not A has a scale of minimal complexity in $K(\mathbb{R})$.

Our next lemma is simply a restatement of Lemma 1.10 of [2].

Lemma 1.7. *Let $\mathcal{M}$ be acceptable. There is a uniformly $\Sigma_1(\mathcal{M})$ sequence*

$$\langle a^\nu_{ijx} \colon \omega \leqslant \nu \wedge i < \nu \wedge \omega j < \nu' \wedge x \in \mathbb{R}^{\mathcal{M}} \rangle,$$

where $\nu < \mathrm{OR}^{\mathcal{M}}$ and $\nu' \leqslant \mathrm{OR}^{\mathcal{M}}$, such that
(1) $\{a^\nu_{ijx} \colon i < \nu \wedge \omega j < \nu' \wedge x \in \mathbb{R}^{\mathcal{M}}\} = \mathcal{P}(\nu \times \mathbb{R}) \cap M$,
(2) $\{a^\nu_{ijx} \colon i < \nu \wedge \omega j < \tau \wedge x \in \mathbb{R}^{\mathcal{M}}\} \in M$ for each $\tau < \nu'$,
(3) $\nu' \leqslant \nu^+_{\mathcal{M}}$.

Proof. See [2]. □

Corollary 1.8. $K(\mathbb{R}) \models$ "GCH".

Proof. Theorem 5.5 of [1] implies that $K(\mathbb{R}) \models$ "V is the union of real 1-mice". We work in $K(\mathbb{R})$. Let λ be an infinite ordinal. Let $\mathcal{M}$ be a real 1-mouse such that $\mathcal{P}(\lambda \times \mathbb{R}) \in \mathcal{M}$ and $|\lambda|^+_{\mathcal{M}} = |\lambda|^+_{K(\mathbb{R})}$. Since $\mathcal{M}$ is acceptable, Lemma 1.7 implies that $|\mathcal{P}(\lambda \times \mathbb{R})|_{\mathcal{M}} \leqslant |\lambda|^+_{\mathcal{M}}$. Because $|\lambda|^+_{K(\mathbb{R})} = |\lambda|^+_{\mathcal{M}}$ and $\mathcal{P}(\lambda \times \mathbb{R}) \in \mathcal{M}$, it follows that $|\mathcal{P}(\lambda \times$

$\mathbb{R})|_{K(\mathbb{R})} \leqslant |\lambda|^+_{K(\mathbb{R})}$ and thus $|\mathscr{P}(\lambda \times \mathbb{R})|_{K(\mathbb{R})} = |\lambda|^+_{K(\mathbb{R})}$ (by Cantor's Theorem). For a finite ordinal λ, $|\mathscr{P}(\lambda \times \mathbb{R})|_{K(\mathbb{R})} = |\mathscr{P}(\omega \times \mathbb{R})|_{K(\mathbb{R})} = |\omega|^+_{K(\mathbb{R})} = |\lambda|^+_{K(\mathbb{R})}$. □

Before we prove our next theorem, we shall first show that generic extensions "preserve" $K(\mathbb{R})$. Let M be an inner model of ZF and let $\mathbb{R}^M$ be the set of reals in M. Definition 2.34 of [1] describes when a structure is a 1-mouse and when it is a real 1-mouse. We shall say that a structure $\mathcal{N}$ is an $\mathbb{R}^M$ *1-mouse* if $\mathcal{N}$ is a 1-mouse and $\mathbb{R}^{\mathcal{N}} = \mathbb{R}^M$. In other words, $\mathcal{N}$ is a 1-mouse having $\mathbb{R}^M$ as its set of reals. Let $K(\mathbb{R}^M)$ be the union of $\mathbb{R}^M$ 1-mice. Our next result shows that forcing over M does not add any new $\mathbb{R}^M$ 1-mice and thus preserves $K(\mathbb{R})^M$.

Lemma 1.9. *Let M be an inner model of* ZF, *let $\mathbb{P} \in M$ be a partial order. Suppose that G is $\mathbb{P}$-generic over M and that $M[G] \models$* ZFC. *Then $K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$.*

Proof. We are assuming that AC holds in the generic extension $M[G]$. However, we do not assume that AC holds in the ground model M. Recall that $K(\mathbb{R})^M$ is the union of real 1-mice in M, and $K(\mathbb{R}^M)^{M[G]}$ is the union of $\mathbb{R}^M$ 1-mice in $M[G]$. So, to prove that $K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$, it is sufficient to show that $M[G]$ does not contain any new $\mathbb{R}^M$ 1-mice. To do this, suppose that $\mathcal{N}$ is an $\mathbb{R}^M$ 1-mouse in $M[G]$. We shall prove that $\mathcal{N} \in M$. Let $\theta > \max(|\mathcal{N}|^{M[G]}, |\mathbb{P}|^{M[G]})$ be a regular cardinal in $M[G]$. Since $\theta > |\mathbb{P}|^{M[G]}$ is a regular cardinal in $M[G]$, it follows that $\mathbb{P}$ is a partial order with the θ-chain condition *in* $M[G]$. One can now prove that for any $C \in M[G]$ if

$$M[G] \models \text{"}C \text{ is closed and unbounded in } \theta\text{"},$$

then there exists a $C' \in M$ such that $C' \subseteq C$ and

$$M \models \text{"}C' \text{ is closed and unbounded in } \theta\text{"}$$

(see the proof of Lemma 10.14 of [8]). Now, in $M[G]$, let

$$\langle \langle \mathcal{N}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathcal{N}_\alpha \underset{\Sigma_1}{\longrightarrow} \mathcal{N}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\mathcal{N}$ and let $\kappa_\alpha = \pi_{0\alpha}(\underline{\kappa}^{\mathcal{N}}) = \underline{\kappa}^{\mathcal{N}_\alpha}$, for each $\alpha \in \mathrm{OR}^{M[G]}$. The θth-premouse iterate of $\mathcal{N}$ is a transitive structure of the form $\mathcal{N}_\theta = (N_\theta, \in, \mathbb{R}^M, \kappa_\theta, \mu^{\mathcal{N}_\theta})$. Lemma 2.37 of [1] asserts that $\mathcal{N}_\theta$ is a 1-mouse and so, let $\mathscr{C} = \mathscr{C}(\mathcal{N}_\theta)$ be the core of $\mathcal{N}_\theta$ (see the bottom of p. 239 of [1]). The core $\mathscr{C}$ is a 1-mouse and Lemma 2.32 of [1] shows that $\mathscr{C}(\mathcal{N}_\theta) = \mathscr{C}(\mathcal{N})$. Theorem 2.39 of [1] implies that there is an ordinal $\xi \in M[G]$ such that $\mathscr{C}_\xi = \mathcal{N}$.

Since $\theta > |\mathcal{N}|^{M[G]}$ is a regular cardinal in $M[G]$, it follows that $\kappa_\theta = \theta$ and that $I = \{\kappa_\beta : \beta < \theta\} \in M[G]$ is closed and unbounded in θ. Corollary 2.14 of [1] shows that, for each $\beta < \theta$, the set $I \backslash \kappa_\beta$ is a set of order $\Sigma_1(\mathcal{N}_\theta, \{\pi_{\beta\theta}(a) : a \in N_\beta\})$ indiscernibles. In addition, for all $X \in \mathscr{P}(\kappa_\theta) \cap N_\theta$

$$X \in \mu^{\mathcal{N}_\theta} \text{ if and only if } \exists \alpha < \theta (X \supseteq I \backslash \alpha).$$

Hence, given I and $\mathbb{R}^M$, one can construct $\mathcal{N}_\theta$ (see the proof of Theorem 2.43 of [1]). In particular, one can construct $\mathcal{N}_\theta$ using the filter μ_I on θ defined by

$$A \in \mu_I \text{ if and only if } \exists\alpha < \theta (A \supseteq I \setminus \alpha).$$

In fact, given any closed unbounded $I' \subseteq I$, one can construct $\mathcal{N}_\theta$ from I' and $\mathbb{R}^M$. Moreover, as noted above, there does exist an $I' \subseteq I$ in M such that

$$M \models \text{“}I' \text{ is closed and unbounded in } \theta\text{”}.$$

It now follows that $\mathcal{N}_\theta \in M$ and hence, $\mathscr{C} = \mathscr{C}(\mathcal{N}_\theta) \in M$. Since ξ is also an ordinal in M, we see that $\mathscr{C}_\xi \in M$ and, because $\mathcal{N} = \mathscr{C}_\xi$, we conclude that $\mathcal{N} \in M$. Therefore, $K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$. □

All the generic extensions discussed in this paper will satisfy the Axiom of Choice and so, Lemma 1.9 shows that these extensions preserve $K(\mathbb{R})$. However, $K(\mathbb{R})$ is preserved even when the generic extension is not a model AC.

Corollary 1.10. *Let M be an inner model of* ZF, *let $\mathbb{P} \in M$ be a partial order. Suppose that G is $\mathbb{P}$-generic over M. Then $K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$.*

Proof (*Sketch*). To prove that $K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$, it is sufficient to show that $M[G]$ does not contain any new $\mathbb{R}^M$ 1-mice. Suppose that $\mathcal{N}$ is an $\mathbb{R}^M$ 1-mouse in $M[G]$. By refining the proof of Lemma 1.9, we shall show that $\mathcal{N} \in M$.

Let $\alpha \in \mathrm{OR}^M$ be such that $\mathbb{P} \in V_\alpha^M$ and let $X = V_{\alpha+2}^M$. In $M[G]$, let $Y = \mathcal{N} \times X$ and let $L(Y)$ be the smallest inner model of $M[G]$ containing Y as a set. Construct a partial order $\mathbb{Q} \in L(Y)$ so that any H, which is $\mathbb{Q}$-generic over $L(Y)$, induces a well-ordering of Y in $L(Y)[H]$. Now, let H be $\mathbb{Q}$-generic over $L(Y)$. So, $L(Y)[H] \models \mathrm{ZFC}$. Let $\theta > \max(|\mathcal{N}|^{L(Y)[H]}, |\mathbb{P}|^{L(Y)[H]})$ be a regular cardinal in $L(Y)[H]$. Since $\theta > |\mathbb{P}|^{L(Y)[H]}$ is a regular cardinal in $L(Y)[H]$, it follows that $\mathbb{P}$ is a partial order with the θ-chain condition *in* $L(Y)[H]$. Since $V_{\alpha+2}^M \in L(Y)[H]$, one can now prove that for any $C \in M[G]$ if

$$M[G] \models \text{“}C \text{ is closed and unbounded in } \theta\text{”},$$

then there exists a $C' \in M$ such that $C' \subseteq C$ and

$$M \models \text{“}C' \text{ is closed and unbounded in } \theta\text{”}$$

(again, see the proof of Lemma 10.14 of [8]). In $L(Y)[H]$, let $\kappa_\theta = \theta$ and $I \subseteq \theta$ be as in the proof of Lemma 1.9. It follows that $I \in M[G]$ and hence, there exists a closed unbounded $I' \subseteq I$ in M. Therefore, $\mathcal{N} \in M$. □

For the remainder of this paper we let $\mathbb{Q} = (Q, \leqslant)$ be the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1; that is,

$Q=\{s\in{}^{\xi}\mathbb{R}\colon \xi\in\omega_1\}$ and for $s,t\in Q$, $s\leqslant t$ if and only if $\mathrm{dom}(s)\geqslant\mathrm{dom}(t)$ and $t=s\restriction\mathrm{dom}(t)$. Let HOD be the class of hereditarily ordinal definable sets.

Theorem 1.11. *Let $V=K(\mathbb{R})$. Suppose that $V\models \mathrm{ZF}+\mathrm{DC}$ and let G be $\mathbb{Q}$-generic over V. Then*
(1) $\mathbb{R}^V=\mathbb{R}^{V[G]}$,
(2) ${\omega_1}^V=\omega_1^{V[G]}$,
(3) $K(\mathbb{R})=K(\mathbb{R})^{V[G]}$,
(4) $\Theta^V=\omega_2^{V[G]}$,
(5) $(\Theta^+)^V=\omega_3^{V[G]}$,
(6) $V[G]\models \mathrm{ZFC}+\mathrm{GCH}$,
(7) $\mathrm{HOD}^V=\mathrm{HOD}^{V[G]}$.

Proof (*Sketch*). Let $V=K(\mathbb{R})$. Since $K(\mathbb{R})\models\mathrm{DC}$, it follows that both $\mathbb{R}$ and ω_1 are preserved under $\mathbb{Q}$-forcing. We now show that $K(\mathbb{R})[G]\models\mathrm{AC}$. An argument is given in [11] (on p. 203) which shows that the Axiom of Choice holds in any forcing extension of a ground model M of ZFC. A simple modification of this argument will prove that the generic extension $K(\mathbb{R})[G]$ satisfies AC. Let X be any set in $K(\mathbb{R})[G]$. We will show that X can be well-ordered in $K(\mathbb{R})[G]$. Let σ be a $\mathbb{Q}$-name for X. In $K(\mathbb{R})$ there is an ordinal α and a function $\pi:\alpha\times\mathbb{R}\xrightarrow{\text{onto}}\mathrm{dom}(\sigma)$. Using π, one can now construct a $\mathbb{Q}$-name τ for a function $f\in K(\mathbb{R})[G]$ such that $f:\alpha\times\mathbb{R}\xrightarrow{\text{onto}}X$. Since there is a bijection in $K(\mathbb{R})[G]$ between $\mathbb{R}$ and ω_1, it follows that there is a function $g\in K(\mathbb{R})[G]$ such that $g:\lambda\xrightarrow{\text{onto}}X$ for some ordinal λ. Thus, X can be well-ordered in $K(\mathbb{R})[G]$.

Lemma 1.9 implies that the generic extension $V[G]$ does not add any new real 1-mice. Thus $K(\mathbb{R})=K(\mathbb{R})^{V[G]}$. Now, since $\mathbb{Q}$ is weakly homogeneous (see p. 129 of [8]) and because V is definable in $V[G]$, it follows that $\mathrm{HOD}^V=\mathrm{HOD}^{V[G]}$.

Claim. *For all ordinals $\xi\geqslant\lambda\geqslant\omega_1$,*

$$(\exists f\in K(\mathbb{R})[G])[f:\lambda\xrightarrow{\text{onto}}\xi]\ \textit{ if and only if }\ (\exists g\in K(\mathbb{R}))[g:\lambda\times\mathbb{R}\xrightarrow{\text{onto}}\xi].$$

Proof. Let $\xi\geqslant\lambda\geqslant\omega_1$.

($\Rightarrow$) Assume that $f\in K(\mathbb{R})[G]$ is such that $f:\lambda\xrightarrow{\text{onto}}\xi$. Let $\hat f$ be a $\mathbb{Q}$-name for $f\in K(\mathbb{R})[G]$, and let $\check a$ be a canonical $\mathbb{Q}$-name for $a\in K(\mathbb{R})$. Let $p\in G$ be such that $p\Vdash$ "$\hat f:\check\lambda\xrightarrow{\text{onto}}\check\xi$". Define the map $h:\lambda\times\mathbb{Q}\xrightarrow{\text{onto}}\xi$ in $K(\mathbb{R})$ by

$$h(\alpha,q)=\begin{cases}\beta, & q\in\mathbb{Q},\ \beta\in\xi \text{ and } p^\frown q\Vdash \text{“}\hat f(\check\alpha)=\check\beta\text{”},\\ 0, & \text{otherwise.}\end{cases}$$

Here $p^\frown q$ is the concatenation of q to p. Since $h\in K(\mathbb{R})$ and there is a map in $K(\mathbb{R})$ from $\mathbb{R}$ onto $\mathbb{Q}$, it follows that there is a map $g\in K(\mathbb{R})$ such that $g:\lambda\times\mathbb{R}\xrightarrow{\text{onto}}\xi$.

($\Leftarrow$) Assume that $g\in K(\mathbb{R})$ is such that $g:\lambda\times\mathbb{R}\xrightarrow{\text{onto}}\xi$. Since there is a bijection in $K(\mathbb{R})[G]$ between $\mathbb{R}$ and ω_1, it follows that there is a map $f\in K(\mathbb{R})[G]$ such that $f:\lambda\xrightarrow{\text{onto}}\xi$. □

Proof of Theorem 1.11 (*Completion*). For any ordinal $\xi \geqslant \omega_1$ the Claim implies that ξ is a cardinal in $K(\mathbb{R})[G]$ if and only if ξ is a $K(\mathbb{R})$-cardinal. Hence, Θ^V and $(\Theta^+)^V$ are collapsed to ω_2 and ω_3, respectively.

Now, since $K(\mathbb{R}) \models$ "GCH", the argument (see [11, pp. 202–203]) showing that the Power Set Axiom holds in the forcing extension $K(\mathbb{R})[G]$ can now be used, together with the above Claim, to establish that $K(\mathbb{R})[G] \models \mathrm{GCH}$. □

Remark.
(1) Note that G in Theorem 1.11 can easily be coded as a subset of $\omega_1^{K(\mathbb{R})}$.
(2) By Theorem 5.14 of [1], the conclusion of Theorem 1.11 also holds under the assumption $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD}$.

For a Boolean Algebra $\mathscr{B}$, let

$$\mathscr{B}_* = \{a \in \mathscr{B} \colon \pi(a) = a \text{ for every automorphism } \pi \text{ of } \mathscr{B}\}.$$

For $\mathscr{B} \in \mathrm{HOD}$, we shall write $\mathscr{B}_*^{\mathrm{HOD}}$ to denote the computation of $\mathscr{B}_*$ in HOD. Given an inner model M of ZF, we say that G is *$\mathscr{B}$-generic over M* when $\mathscr{B}$ is a complete Boolean Algebra in M and G is a generic (over M) ultrafilter.

Our next two lemmas follow directly from a theorem of Vopěnka (see Theorem 59 of [6, p. 269]). For the remainder of this section, $\mathscr{B}$ will denote a Boolean Algebra.

Lemma 1.12 (ZFC). *Suppose $V = \mathrm{HOD}[G]$, where G is $\mathscr{B}$-generic over* HOD. *Then* $G \cap \mathscr{B}_*^{\mathrm{HOD}} \in \mathrm{HOD}$.

Lemma 1.13 (ZFC). *Suppose $V = \mathrm{HOD}[G]$, where G is $\mathscr{B}$-generic over* HOD. *If $M \subseteq$* HOD *is a (transitive) inner model of* ZFC *such that*
(1) $\mathscr{B} \in M$,
(2) $G \cap \mathscr{B}_*^{\mathrm{HOD}} \in M$,
(3) $V = M[G]$,
then $M = \mathrm{HOD}$.

Another theorem of Vopěnka states that if $V = L[A]$ for a set of ordinals A, then V is a generic extension of HOD (see Theorem 65 of [6, p. 293]). Our next lemma generalizes this theorem and gives us a method for "computing" HOD.

Lemma 1.14 (ZFC+GCH). *Assume $V = L[D][A]$ and $L[D] \subseteq \mathrm{HOD}$, where $A \subseteq \kappa \in$* OR *and $D \subseteq V$ is a set or a proper class. Then there is a Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ where $B \leqslant \kappa^{++}$ is an ordinal, and there is a G which is $\mathscr{B}$-generic over* HOD *such that*
(i) $V = \mathrm{HOD}[G]$,
(ii) $G \cap \mathscr{B}_*^{\mathrm{HOD}} \in \mathrm{HOD}$.
In addition, there exists a $b \colon \kappa \to B$ such that
(iii) $\mathrm{HOD} = L[D](\{\mathscr{B}, b, \widehat{B}\}) = L[D](P)$
where $\widehat{B} = G \cap \mathscr{B}_*^{\mathrm{HOD}}$ and $P \subseteq \kappa^{++}$ is a canonical coding of $\{\mathscr{B}, b, \widehat{B}\}$.

Proof. Since the proof of $V = \mathrm{HOD}[G]$ is essentially the same as the proof of Theorem 65 given in [6], we first give a sketch of the proof of (i). Let $C = \mathrm{OD} \cap \mathscr{P}(\mathscr{P}(\kappa))$, where OD is the class of ordinal definable sets. Consider the partial order $\mathbb{C} = (C, \subseteq)$. There is a complete Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ in HOD, where B is an ordinal, such that $\mathbb{C}$ is isomorphic to $\mathscr{B}$ as witnessed by an ordinal definable isomorphism $\pi : \mathbb{C} \to \mathscr{B}$. Define $G = \{\pi(u) \colon u \in C \wedge A \in u\}$. One can show that G is $\mathscr{B}$-generic over HOD, and our next Claim shows that V is the resulting generic extension. This Claim, together with Lemma 1.12, implies that $G \cap \mathscr{B}_*^{\mathrm{HOD}} \in \mathrm{HOD}$.

Claim. $V = \mathrm{HOD}[G]$.

Proof. Define $b : \kappa \to B$ by $b(\alpha) = \pi(\{X \subseteq \kappa \colon \alpha \in X\})$. Clearly, $b \in \mathrm{HOD}$ and for every $\alpha \in \kappa$

$$\alpha \in A \Leftrightarrow b(\alpha) \in G.$$

Therefore, $A \in \mathrm{HOD}[G]$. By assumption, $L[D] \subseteq \mathrm{HOD}$. Thus, $L[D][A] \subseteq \mathrm{HOD}[G]$. Hence, $V = \mathrm{HOD}[G]$. □

Proof of Lemma 1.14 (*Completion*). Since $\mathbb{C}$ satisfies the $(\kappa^{++})^V$-chain condition (see 6.7 of [11, p. 212]), $\mathscr{B}$ has the $(\kappa^{++})^V$-chain condition in HOD. Furthermore, $V \models |B| \leqslant \kappa^{++}$ since $V \models \mathrm{GCH}$. Thus, $\mathrm{HOD} \models |B| \leqslant \left(\kappa^{++}\right)^V$ (see 6.9 of [11, p. 213]). Therefore, the complete Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ in HOD, where B is an ordinal, can be taken so that $B \leqslant \kappa^{++}$.

Finally, to prove (iii), let $M = L[D](\{\mathscr{B}, b, \widehat{B}\})$. Clearly, $M \subseteq \mathrm{HOD}$. Thus, G is $\mathscr{B}$-generic over M and $V = M[G]$. Therefore, $M = \mathrm{HOD}$ by Lemma 1.13. Now let $P \subseteq \kappa^{++}$ be a canonical coding of $\{\mathscr{B}, b, \widehat{B}\}$. Hence, $\mathrm{HOD} = L[D](P)$. □

Suppose that $\mathscr{M}$ is a real 1-mouse. Let $\mathscr{C} = \mathscr{C}(\mathscr{M})$ be the core of $\mathscr{M}$, let $\langle \pi_{\alpha\beta} : \mathscr{C}_\alpha \to \mathscr{C}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}}$ be the premouse iteration of $\mathscr{C}$ and let $\kappa_\alpha = \pi_{0\alpha}(\kappa^{\mathscr{C}})$ for each ordinal α. Define $i(\mathscr{M})$ to be the ordinal λ such that $\mathscr{C}_\lambda = \mathscr{M}$ (see Definition 2.36 and Theorem 2.39 of [1]) and define $I_{\mathscr{M}} = \{\kappa_\alpha \colon \alpha < i(\mathscr{M})\}$.

Definition 1.15. Define the class D to be

$$D = \{(\xi, \kappa) \colon \exists \mathscr{M}\, (\mathscr{M} \text{ is a real 1-mouse} \wedge \kappa = \kappa^{\mathscr{M}} \wedge i(\mathscr{M}) = \omega \wedge \xi \in I_{\mathscr{M}})\}.$$

Suppose that $\mathscr{M}$ and $\mathscr{N}$ are real 1-mice such that $\kappa^{\mathscr{M}} = \kappa^{\mathscr{N}}$ and $i(\mathscr{M}) = i(\mathscr{N}) = \omega$. Then Lemma 5.3 of [1] implies that $\mathscr{M} = \mathscr{N}$.

Lemma 1.16. $K(\mathbb{R}) = L[D](\mathbb{R})$.

Proof. This is a direct analogue of the fact that $K = L[D]$, where K is the Core Model of Dodd–Jensen [5] and D is defined as in Definition 1.15 above, but "without the reals." See Chapter 14 of Dodd [4] for the details. □

We are now in a position to give our first computation of $\mathrm{HOD}^{K(\mathbb{R})}$.

Theorem 1.17. *Suppose that* $K(\mathbb{R}) \models \mathrm{ZF}+\mathrm{DC}$. *Then there exists a* $P \subseteq (\Theta^+)^{K(\mathbb{R})}$ *such that* $\mathrm{HOD}^{K(\mathbb{R})} = L[D](P)$.

Proof. Let $V = K(\mathbb{R})$ and let $\mathbb{Q} \in V$ be the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1. Let A be $\mathbb{Q}$-generic over V. Theorem 1.11 asserts that $V[A] \models \mathrm{ZFC}+\mathrm{GCH}$, $(\Theta^+)^V = \omega_3^{V[A]}$ and $\mathrm{HOD}^V = \mathrm{HOD}^{V[A]}$. So we write $\mathrm{HOD} = \mathrm{HOD}^V = \mathrm{HOD}^{V[A]}$ and shall now compute HOD in $V[A]$. Note that $V[A] = K(\mathbb{R})[A] = L[D](\mathbb{R})[A] = L[D][A]$, where A can be coded as a subset of ω_1. Clearly, $L[D] \subseteq \mathrm{HOD}$. Thus there is a $P \subseteq \omega_3 = (\Theta^+)^V$ such that $\mathrm{HOD} = L[D](P)$, by Lemma 1.14. Therefore, $P \in K(\mathbb{R})$, $P \subseteq (\Theta^+)^{K(\mathbb{R})}$ and $\mathrm{HOD}^{K(\mathbb{R})} = L[D](P)$. $\square$

Remark. Assuming $\mathrm{ZF}+\mathrm{AD}+\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$, we show in Section 4 that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ where $K(P)$ is the Core Model relative to $P \subseteq (\Theta^+)^{K(\mathbb{R})}$.

2. Weak ultrapowers

Let M be an inner model of ZF. We now define what it means for μ to be a "measure" on $\mathscr{P}^M(\kappa) = \mathscr{P}(\kappa) \cap M$, and we shall define the ultrapower of (M,μ). We do not assume that $\mu \in M$.

Definition 2.1. Let M be an inner model of ZF and let $\kappa > \omega$ be an ordinal. We say that μ is an *M-measure on κ* if

(1) μ is a proper subset of $\mathscr{P}^M(\kappa)$ containing no singletons.
(2) $\forall X, Y\,[(X \subseteq Y \in \mathscr{P}^M(\kappa) \wedge X \in \mu) \to Y \in \mu]$.
(3) $\forall X \in \mathscr{P}^M(\kappa)\,[X \in \mu \vee (\kappa \backslash X) \in \mu]$.
(4) If $\lambda < \kappa$, $\langle X_\xi : \xi < \lambda \rangle \in M$ and $(\forall \xi < \lambda) X_\xi \in \mu$, then $\bigcap_{\xi<\lambda} X_\xi \in \mu$.
(5) If $\langle X_\xi : \xi < \kappa \rangle \in M$ and $(\forall \xi < \kappa) X_\xi \in \mu$, then $\{\xi \in \kappa : \xi \in \bigcap_{\eta<\xi} X_\eta\} \in \mu$.

Conditions (1) and (2) ensure that μ is a filter on $\mathscr{P}^M(\kappa)$. Condition (3) asserts that μ is an *ultrafilter* on $\mathscr{P}^M(\kappa)$. Condition (4) is called *κ-completeness* and, in this case, μ is said to be *κ-complete.* Condition (5) is called *normality* and, in this case, μ is said to be *normal.*

Definition 2.2. Let M be an inner model of ZF and let μ be an M-measure on κ. We say that μ is *$\mathbb{R}^M$-complete* if the following holds:

$$\text{If } \langle A_x : x \in \mathbb{R}^M \rangle \text{ is any sequence in } M \text{ such that } A_x \in \mu \text{ for all } x \in \mathbb{R}^M,$$
$$\text{then } \bigcap_{x \in \mathbb{R}^M} A_x \in \mu. \tag{1}$$

Remark. Assuming AD, there are unboundedly many measurable cardinals $\kappa < \Theta$. It is easy to check that their measures are not $\mathbb{R}$-complete in V. However, any (normal) measure on $\kappa > \Theta$ is $\mathbb{R}$-complete in V.

Lemma 2.3. *Suppose that M is an inner model of* ZF+AD+DC *and that μ is an M-measure on $\kappa > \Theta^M$. Then for any $f: \kappa \to \mathscr{P}(\mathbb{R})$ in M, there exists an $A \in \mathscr{P}(\mathbb{R}) \cap M$ such that $\{\xi \in \kappa\colon f(\xi) = A\} \in \mu$.*

Proof. Define in M the relation $\leqslant_{\mathrm{w}}$ on $\mathscr{P}(\mathbb{R})$ by

$$A \leqslant_{\mathrm{w}} B \ \text{ if and only if } A \text{ is Wadge reducible to } B$$

and define $A <_{\mathrm{w}} B$ if and only if $A \leqslant_{\mathrm{w}} B \wedge B \nleqslant_{\mathrm{w}} A$ (see p. 424 of [15]). Because M is an inner model of ZF + AD + DC, it follows that

$$M \models \text{“}<_{\mathrm{w}} \text{ is well-founded and } \operatorname{rank}(<_{\mathrm{w}}) = \Theta\text{”}.$$

For $A \in \mathscr{P}(\mathbb{R}) \cap M$, let $|A|_{\mathrm{w}}^M$ be the $<_{\mathrm{w}}$-rank of A (in M). Define $g: \kappa \to \Theta^M$ by $g(\xi) = |f(\xi)|_{\mathrm{w}}^M$. Since $\Theta^M < \kappa$, there is a $\lambda < \kappa$ such that the set $U = \{\xi \in \kappa\colon g(\xi) = \lambda\}$ is in μ (by κ-completeness). In M, let $W \subseteq \mathbb{R}^M$ be universal for the Wadge degrees of $<_{\mathrm{w}}$-rank less than $\lambda + 1$ and for each $\xi \in U$ define $C_\xi = \{x \in \mathbb{R}^M\colon W_x = f(\xi)\}$.

Claim 1. $(\forall \xi, \xi' \in U)(C_\xi \cap C_{\xi'} \neq \emptyset \Rightarrow C_\xi = C_{\xi'})$.

Proof. Let $\xi, \xi' \in U$ and suppose that $C_\xi \cap C_{\xi'} \neq \emptyset$. Let $x \in C_\xi \cap C_{\xi'}$. Then $f(\xi) = W_x = f(\xi')$. Hence, $C_\xi = C_{\xi'}$. □

Claim 2. $\exists C(\{\xi \in \kappa\colon C_\xi = C\} \in \mu)$.

Proof. Assume for a contradiction that there is no such C. For each $\xi \in \kappa$ define

$$Y_\xi = \begin{cases} \{\eta \in U\colon C_\eta \neq C_\xi\} & \text{if } \xi \in U, \\ \kappa & \text{if } \xi \notin U. \end{cases}$$

Clearly, $\langle Y_\xi : \xi < \kappa \rangle \in M$ and by assumption $(\forall \xi < \kappa)(Y_\xi \in \mu)$. By normality the set $Y = \{\xi \in \kappa\colon \xi \in \bigcap_{\eta < \xi} Y_\eta\}$ is in μ. Claim 1 implies that $C_\xi \cap C_{\xi'} = \emptyset$ for distinct $\xi, \xi' \in Y$. Thus, we can define in M a map from $\mathbb{R}^M$ onto Y. However, $Y \in \mu$ and so, Y has order type κ. Hence, $\kappa < \Theta^M$. Contradiction. □

Proof of Lemma 2.3 (*Conclusion*). Now let C be as in Claim 2, and let A be such that $A = W_x$ for all $x \in C$. So $\{\xi \in \kappa\colon f(\xi) = A\} \in \mu$. □

Corollary 2.4. *Suppose that M is an inner model of* ZF + AD + DC. *If μ is an M-measure on $\kappa > \Theta^M$, then μ is $\mathbb{R}^M$-complete.*

Proof. To prove that μ is $\mathbb{R}^M$-complete, let $\langle A_x : x \in \mathbb{R}^M \rangle$ be any sequence in M such that $A_x \in \mu$ for all $x \in \mathbb{R}^M$. Define the function $f: \kappa \to \mathscr{P}(\mathbb{R})$ in M by

$$f(\xi) = \{x \in \mathbb{R}^M\colon \xi \in A_x\}.$$

By Lemma 2.3 there is an $A \subseteq \mathbb{R}^M$ in M such that $U = \{\xi \in \kappa\colon f(\xi) = A\} \in \mu$. We now show that $A = \mathbb{R}^M$. Let $x \in \mathbb{R}^M$. Since $A_x \in \mu$, $A_x \cap U$ is non-empty. Let $\xi \in A_x \cap U$.

Hence, $x \in f(\xi) = A$. Thus, $A = \mathbb{R}^M$. It follows that $\bigcap_{x \in \mathbb{R}^M} A_x = U \in \mu$. Therefore, μ is $\mathbb{R}^M$-complete. $\square$

Given an inner model M of ZF and μ, an M-measure on κ, we define an ultrapower of (M, μ). We will denote this ultrapower by ${}^\kappa M/\mu$ where ${}^\kappa M = \{ f \in M \colon f : \kappa \to M \}$. For $f, g \in {}^\kappa M$ define

$$f \sim g \text{ if and only if } (M, \mu) \models \{\xi \in \kappa \colon f(\xi) = g(\xi)\} \in \mu.$$

The above set is in M, and $\sim$ is an equivalence relation on ${}^\kappa M$. For $f \in {}^\kappa M$, we denote the equivalence class of f by $[f]$ (implicitly using *Scott's Trick*). Let $M_1 = {}^\kappa M/\mu = \{ [f] \colon f \in {}^\kappa M \}$ and define

$$[f] \in^{M_1} [g] \quad \text{if and only if } (M, \mu) \models \{\xi \in \kappa \colon f(\xi) \in g(\xi) \} \in \mu.$$

The set on the right-hand side is in M and therefore can be measured by μ. For $a \in M$, let $c_a \in {}^\kappa M$ be the constant function defined by $c_a(\xi) = a$ for all $\xi \in \kappa$. Define the *natural embedding* $j : M \to M_1$ by $j(a) = c_a$.

We shall write M for $(M, \in)$ and write M_1 for $(M_1, \in^{M_1})$. Note that $\in^{M_1}$ may not be well-founded. Since μ is only κ-complete for sequences in M, the usual proof that $\in^{M_1}$ is well-founded (which uses DC) may break down. We do not assume that $\mu \in M$.

Lemma 2.5. *Suppose that M is an inner model of* ZFC. *Suppose that μ is an M-measure on κ. Let M_1 be the ultrapower of (M, μ) and let $j : M \to M_1$ be the natural embedding. Then*
(1) $j : M \xrightarrow[\Sigma_\omega]{} M_1$,
(2) $j(\xi) = \xi$ *for all* $\xi < \kappa$,
(3) $(M_1, \in^{M_1}) \models \kappa < j(\kappa)$.

Proof. To prove (1), it is enough to check that Łoś' Theorem holds for this ultrapower. Recall that ${}^\kappa M$ denotes functions in M.

Claim. For every Σ_ω formula φ in the language of set theory,

$$M_1 \models \varphi([f_1], \ldots, [f_n]) \text{ if and only if } (M, \mu) \models \{ \xi \in \kappa \colon \varphi(f_1(\xi), \ldots, f_n(\xi)) \} \in \mu,$$

for all $f_1, \ldots, f_n \in {}^\kappa M$.

Proof. This is done by the usual induction on the complexity of formulae. The existential quantifier step follows because M satisfies the Axiom of Choice. $\square$

Proof of Lemma 2.5 (*Conclusion*). Assertion (2) of the lemma follows from κ-completeness and (3) follows from the normality of μ. $\square$

Remark. We could "iterate" the M-measure μ on κ to an M_1-measure μ_1 on $j(\kappa)$ by the clause

$$[f] \in \mu_1 \quad \text{if and only if } (M, \mu) \models \{\xi \in \kappa \colon f(\xi) \in \mu\} \in \mu$$

if the above subset of κ is a member of M. If $\mu \in M$, then this would be the case. However, we do not assume that $\mu \in M$.

We now show that Łoś' Theorem holds for certain "choiceless" inner models. Let OD(X) be the class of sets which are ordinal definable from an element in X. In particular, OD($\mathbb{R}$) is the class of sets which are ordinal definable from a real.

Lemma 2.6. *Suppose that M is an inner model of* ZF + $V = \mathrm{OD}(\mathbb{R})$. *Suppose that μ is an $\mathbb{R}^M$-complete M-measure on κ. Let M_1 be the ultrapower of (M, μ) and let $j : M \to M_1$ be the natural embedding. Then*
(1) $j : M \underset{\Sigma_\omega}{\longrightarrow} M_1$,
(2) $j(\xi) = \xi$ *for all* $\xi < \kappa$,
(3) $(M_1, \in^{M_1}) \models \kappa < j(\kappa)$,
(4) $\mathbb{R}^M = \mathbb{R}^{M_1}$.

Proof. To prove (1), it is enough to check that Łoś' Theorem holds for this ultrapower.

Claim. *For every Σ_ω formula φ in the language of set theory,*

$$M_1 \models \varphi([f_1], \ldots, [f_n]) \quad \textit{if and only if} \quad (M, \mu) \models \{\xi \in \kappa \colon \varphi(f_1(\xi), \ldots, f_n(\xi))\} \in \mu,$$

for all $f_1, \ldots, f_n \in {}^\kappa M$.

Proof. The proof is by induction on the complexity of φ. The existential quantifier case is the only case which requires checking. The usual proof of this case assumes that M satisfies the Axiom of Choice, but we can get by with a weaker choice principle.

Suppose that φ is the formula $\exists v \psi(v, v_1, \ldots, v_n)$, where ψ is in Σ_ω. Let $f_1, \ldots, f_n \in {}^\kappa M$ and suppose that

$$(M, \mu) \models \{\, \xi \in \kappa \colon \exists v \psi(v, f_1(\xi), \ldots, f_n(\xi)) \,\} \in \mu.$$

We want an $f \in {}^\kappa M$ so that

$$(M, \mu) \models \{\xi \in \kappa \colon \psi(f(\xi), f_1(\xi), \ldots, f_n(\xi)) \,\} \in \mu.$$

Define in M the function $h \colon \kappa \to \mathscr{P}(\mathbb{R})$ by

$$h(\xi) = \{x \in \mathbb{R}^M \colon \exists v[v \in \mathrm{OD}(\{x\}) \wedge \psi(v, f_1(\xi), \ldots, f_n(\xi))]\}.$$

Clearly, $(M, \mu) \models \{\xi \in \kappa \colon h(\xi) \neq \emptyset\} \in \mu$. Because μ is $\mathbb{R}^M$-complete, there is an $x \in \mathbb{R}^M$ such that $(M, \mu) \models \{\xi \in \kappa \colon x \in h(\xi)\} \in \mu$. Fix such a real x and let W be a well-ordering, definable over M, of the class $\mathrm{OD}(\{x\})^M$. Now define the function $f : \kappa \to M$, in M, by

$$f(\xi) = \begin{cases} \text{the } W\text{-least } v \text{ such that } \psi(v, f_1(\xi), \ldots, f_n(\xi)) & \text{if } v \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, f is as desired. $\square$

Proof of Lemma 2.6 (*Conclusion*). Assertion (2) of the lemma, follows from κ-completeness, (3) follows from the normality of μ and (4) is implied by $\mathbb{R}^M$-completeness. □

Corollary 2.7. *Suppose M is an inner model of* ZF + AD + DC + $V = \mathrm{OD}(\mathbb{R})$. *Suppose that μ is an M-measure on $\kappa > \Theta^M$. Let M_1 be the ultrapower of (M,μ) and let $j\colon M \to M_1$ be the natural embedding. Then*
(1) $j : M \underset{\Sigma_\omega}{\longrightarrow} M_1$,
(2) $j(\xi) = \xi$ *for all* $\xi < \kappa$,
(3) $(M_1, \in^{M_1}) \models \kappa < j(\kappa)$,
(4) $\mathbb{R}^M = \mathbb{R}^{M_1}$.

Proof. Since μ is an M-measure on $\kappa > \Theta^M$, Lemma 2.4 implies that μ is $\mathbb{R}^M$-complete. Lemma 2.6 now implies the desired result. □

We will want to be able to iterate the above ultrapower operation on M. To do this we shall need (as remarked above) to ensure that certain subsets of κ are in M.

Definition 2.8. Let μ be an M-measure on κ. If every sequence $\langle A_\xi\colon \xi \in \kappa\rangle$ in $M \cap {}^{\kappa}\mathscr{P}(\kappa)$ satisfies $\{\xi \in \kappa\colon A_\xi \in \mu\} \in M$, then we say that (M,μ) is *good on* κ.

If (M,μ) is good on κ, then (M,μ) is also said to be "weakly amenable" (see p. 244 of [8]).

Definition 2.9. Let μ be an $\mathbb{R}^M$-complete, M-measure on κ. If every sequence $\langle A_{(\xi,x)}\colon (\xi,x) \in \kappa \times \mathbb{R}^M\rangle$ in $M \cap {}^{\kappa\times\mathbb{R}^M}\mathscr{P}(\kappa)$ satisfies $\{(\xi,x) \in \kappa \times \mathbb{R}^M\colon A_{(\xi,x)} \in \mu\} \in M$, then we say that (M,μ) is *really good on* κ.

Clearly, if (M,μ) is really good on κ, then (M,μ) is good on κ.

Let M be an inner model of ZF + $V = \mathrm{OD}(\mathbb{R})$ and suppose that (M,μ) is really good on κ. Let $j^\mu_{01} : M \to M_1$ be natural embedding of M into the ultrapower $M_1 = {}^{\kappa}M/\mu$. Let $\kappa_1 = j^\mu_{01}(\kappa)$ and let

$$\mu_1 = \{[f] \in M_1\colon \{\xi \in \kappa\colon f(\xi) \in \mu\} \in \mu\}.$$

We shall call μ_1 the *first iterate of* μ. One can show that the structure (M_1,μ_1) is "really good on κ_1". We put this in quotes because M_1 may not be well-founded. To illustrate how one can show that (M_1,μ_1) is "really good on κ_1", we prove the following proposition.

Proposition 2.10. *Let (M,μ) and (M_1,μ_1) be as above. Then*
(i) μ_1 *is κ_1-complete, that is, (M_1,μ_1) satisfies* (4) *of Definition* 2.1.
(ii) μ_1 *is $\mathbb{R}^{M_1}$-complete, that is, (M_1,μ_1) satisfies* (1) *of Definition* 2.2.

Proof. First we prove (i). Suppose that $\lambda < j(\kappa)$, $\langle X_\gamma : \gamma < \lambda \rangle \in M_1$ and $(\forall \gamma < \lambda) X_\gamma \in \mu_1$. We need to show that $\bigcap_{\gamma<\lambda} X_\gamma \in \mu_1$. Let $f, g \in M$, where $g : \kappa \to \kappa$ and $f : \kappa \to M$, be such that $[g] = \lambda$ and $[f] = \langle X_\gamma : \gamma < [g] \rangle$. So,

$$\{\xi \in \kappa\colon\ f(\xi) = \langle A_{\xi,\gamma}\colon\ \gamma < g(\xi)\rangle\} \in \mu$$

and we can assume that $A_{\xi,\gamma} \subseteq \kappa$ for all $\xi, \gamma \in \kappa$. The sequence $\langle A_{(\xi,\gamma)} : (\xi,\gamma) \in \kappa \times \kappa \rangle$ is in M and so, it can be coded as a κ-sequence in M. Since (M, μ) is good on κ, it follows that $C = \{(\xi,\gamma)\colon\ A_{\xi,\gamma} \in \mu\} \in M$. Now, define the function $G : \kappa \times \kappa \to \mathscr{P}(\kappa)$ in M by

$$G(\xi,\gamma) = \begin{cases} A_{\xi,\gamma} & \text{if } (\xi,\gamma) \in C, \\ \kappa & \text{otherwise.} \end{cases}$$

Note that $G(\xi,\gamma) \in \mu$ for all $\xi, \gamma \in \kappa$ and, since μ is κ-complete, it follows that $\bigcap_{\gamma<g(\xi)} G(\xi,\gamma) \in \mu$ for all $\xi < \kappa$. We now show that

$$\{\xi \in \kappa\colon\ (\forall \gamma < g(\xi))[G(\xi,\gamma) = A_{\xi,\gamma}]\} \in \mu.$$

Suppose, for a contradiction, that $\{\xi \in \kappa\colon\ (\exists \gamma < g(\xi))[G(\xi,\gamma) \neq A_{\xi,\gamma}]\} \in \mu$. Define the function $\widehat{\gamma} : \kappa \to \kappa + 1$ by

$$\widehat{\gamma}(\xi) = \begin{cases} \gamma & \text{if } \gamma \text{ is the least } \gamma < g(\xi) \text{ such that } G(\xi,\gamma) \neq A_{\xi,\gamma}, \\ \kappa & \text{otherwise.} \end{cases}$$

It follows that $\{\xi\colon \widehat{\gamma}(\xi) < g(\xi)\} \in \mu$ and so, $[\widehat{\gamma}] < \lambda$. Define $\widehat{X}(\xi) = A_{\widehat{\gamma}(\xi),\xi}$ if $\widehat{\gamma}(\xi) < \kappa$, and $\widehat{X}(\xi) = \emptyset$ otherwise. So, $\widehat{X}(\xi) \notin \mu$ for all $\xi \in \kappa$ and hence, $[\widehat{X}] \notin \mu_1$. So $[\widehat{X}] = X_{[\widehat{\gamma}]} \notin \mu_1$ and $[\widehat{\gamma}] < [g] = \lambda$, which contradicts our assumption that $(\forall \gamma < \lambda) X_\gamma \in \mu_1$.

Letting $h : \kappa \to M$ be the function in M defined by

$$h(\xi) = \bigcap_{\gamma < g(\xi)} G(\xi,\gamma),$$

it follows that $[h] \in \mu_1$. In addition, since $h(\xi) = \bigcap_{\gamma<g(\xi)} A_{\xi,\gamma}$ for "almost every" ξ, we see that $[h] = \bigcap_{\gamma<\lambda} X_\gamma$. Thus, (M_1, μ_1) satisfies (4) of Definition 2.1.

The proof of (ii) is very much like the proof of (i), except we use the $\mathbb{R}^M$-completeness of μ and the fact that M is really good on κ. Recall that $\mathbb{R}^M = \mathbb{R}^{M_1}$ and so, we shall assume (for notational simplicity) that $\mathbb{R} = \mathbb{R}^M = \mathbb{R}^{M_1}$. Now suppose that $\langle X_a : a \in \mathbb{R} \rangle \in M_1$ and $(\forall a \in \mathbb{R}) X_a \in \mu_1$. We need to show that $\bigcap_{a \in \mathbb{R}} X_a \in \mu_1$. Let $f \in M$, where $f : \kappa \to M$, be such that $[f] = \langle X_a : a \in \mathbb{R} \rangle$. So,

$$\{\xi \in \kappa\colon\ f(\xi) = \langle A_{\xi,a} : a \in \mathbb{R} \rangle\} \in \mu$$

and we can assume that $A_{\xi,a} \subseteq \kappa$ for all $(\xi,a) \in \kappa \times \mathbb{R}$. Because (M, μ) is really good on κ, it follows that $C = \{(\xi,a)\colon\ A_{\xi,a} \in \mu\} \in M$. Now, define the function $G : \kappa \times \mathbb{R} \to \mathscr{P}(\kappa)$ in M by

$$G(\xi,a) = \begin{cases} A_{\xi,a} & \text{if } (\xi,a) \in C, \\ \kappa & \text{otherwise.} \end{cases}$$

Note that $G(\xi,a)\in\mu$ for all $(\xi,a)\in\kappa\times\mathbb{R}$ and, since μ is $\mathbb{R}$-complete, it follows that $\bigcap_{a\in\mathbb{R}} G(\xi,a)\in\mu$ for all $\xi<\kappa$. We now show that

$$\{\xi\in\kappa\colon (\forall a\in\mathbb{R})[G(\xi,a)=A_{\xi,a}]\}\in\mu.$$

Suppose, for a contradiction, that $\{\xi\in\kappa\colon (\exists a\in\mathbb{R})[G(\xi,a)\neq A_{\xi,a}]\}\in\mu$. By $\mathbb{R}$-completeness, there exists a $b\in\mathbb{R}$ such that $\{\xi\in\kappa\colon G(\xi,b)\neq A_{b,\xi})\}\in\mu$.

Define $\widehat{X}(\xi)=A_{b,\xi}$. So, $\widehat{X}(\xi)\notin\mu$ for all $\xi\in\kappa$ and hence, $[\widehat{X}]\notin\mu_1$. So $[\widehat{X}]=X_b\notin\mu_1$, which contradicts our assumption that $(\forall a\in\mathbb{R})\, X_a\in\mu_1$.

Letting $h:\kappa\to M$ be the function in M defined by

$$h(\xi)=\bigcap_{a\in\mathbb{R}} G(\xi,a),$$

it follows that $[h]\in\mu_1$. In addition, since $h(\xi)=\bigcap_{a\in\mathbb{R}} A_{\xi,a}$ for "almost every" ξ, we see that $[h]=\bigcap_{a\in\mathbb{R}} X_a$. Thus, (M_1,μ_1) satisfies (1) of Definition 2.2. □

So, if (M,μ) is really good on κ, then the structure (M_1,μ_1) inherits all the properties from the structure (M,μ) necessary to construct an ultrapower M_2 of (M_1,μ_1) and to define a natural embedding $j^{\mu}_{12}:M_1\to M_2$. This procedure can be iterated through the ordinals and thus, one obtains the commutative system

$$\langle\langle M_\alpha\rangle_{\alpha\in\mathrm{OR}},\langle j^{\mu}_{\alpha\beta}:M_\alpha \xrightarrow[\Sigma_\omega]{} M_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

such that (M_α,μ_α) is "really good on κ_α" for each $\alpha\in\mathrm{OR}$, where $\kappa_\alpha=j^{\mu}_{0\alpha}(\kappa)$. We write μ_α for the αth *iterate of* μ and write M_α for $(M_\alpha,\in^{M_\alpha})$. We say that (M,μ) is *weakly iterable* if M_α is well-founded for all ordinals α. In this case we always identify M_α with its transitive collapse.

Similarly, let M be an inner model of ZFC and suppose that (M,μ) is good on κ. We get a commutative system

$$\langle\langle M_\alpha\rangle_{\alpha\in\mathrm{OR}},\langle j^{\mu}_{\alpha\beta}:M_\alpha \xrightarrow[\Sigma_\omega]{} M_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle \tag{1}$$

such that for each $\alpha\in\mathrm{OR}$, (M_α,μ_α) is "good on κ_α" where $\kappa_\alpha=j^{\mu}_{0\alpha}(\kappa)$. Again, we write μ_α for the αth iterate of μ and write M_α for $(M_\alpha,\in^{M_\alpha})$. We say that (M,μ) is *weakly iterable* if M_α is well-founded for all ordinals α. In this case we always identify M_α with its transitive collapse.

Remark. Kunen was the first to define the system (1) starting with an inner model M of ZFC and a weakly amenable M-measure on κ (see Section 19 of [8]).

We now present a condition (see Definition 12.15 of [4]) on an inner model M which will ensure that a given M-measure is good (really good) on κ.

Definition 2.11. Let κ be an ordinal and suppose that M is an inner model of ZF. We say that M is *κ-maximal* provided that, whenever $\pi:M\xrightarrow[\Sigma_\omega]{} M'$, M' is transitive and $\pi(\xi)=\xi$ for all $\xi\in\kappa$, then $M=M'$.

Lemma 14.19 of [4] asserts that K, the Core Model, is 0-maximal. Our next result generalizes this result to $K(\mathbb{R})$.

Lemma 2.12. *$K(\mathbb{R})$ is 0-maximal.*

Proof. Suppose $\pi : K(\mathbb{R}) \underset{\Sigma_\omega}{\longrightarrow} M$, where M is a (transitive) inner model. Note that $\mathbb{R} \in M$ and since $K(\mathbb{R}) \models V = K(\mathbb{R})$ (see Theorem 5.5 of [1]), it follows that $M \models V = K(\mathbb{R})$. Let $\mathcal{N} \in M$ be such that $M \models$ "$\mathcal{N}$ is a real 1-mouse". By absoluteness, $\mathcal{N}$ is a real 1-mouse. Thus, $M \subseteq K(\mathbb{R})$. To see that $K(\mathbb{R}) \subseteq M$, let $a \in K(\mathbb{R})$ and let $\mathcal{M} \in K(\mathbb{R})$ be a real 1-mouse. Let $\mathcal{N} = \pi(\mathcal{M})$ and, since π is an elementary embedding of $K(\mathbb{R})$ to M, it follows that $M \models$ "$\mathcal{N}$ is a real 1-mouse". Lemma 2.25 of [1] implies that there is a $\theta \in \mathrm{OR}$ such that the premouse iterates $\mathcal{M}_\theta$ and $\mathcal{N}_\theta$ are comparable. Because π is an elementary embedding, it follows that $\sigma : \mathcal{M} \underset{\Sigma_\omega}{\longrightarrow} \mathcal{N}$, where $\sigma = \pi \restriction \mathcal{M}$. Thus, $\sigma : F^{\mathcal{M}} \to F^{\mathcal{N}}$ is $\leqslant$-extendible (see Definition 2.30 of [1]). By the proof of Theorem 2.31 of [1], there exists an $\in$-order preserving map $\sigma_\theta : \mathrm{OR}^{\mathcal{M}_\theta} \to \mathrm{OR}^{\mathcal{N}_\theta}$ and hence, $\mathcal{M}_\theta$ must be an initial segment of $\mathcal{N}_\theta$. Thus, $\mathcal{M}_\theta \in M$ because $\mathcal{N}_\theta \in M$. Therefore, the core $\mathscr{C}(\mathcal{M}) \in M$ (see Definition 2.36 and Lemma 2.37 of [1]). Because $\mathcal{M}$ is an iterate of its core (by Theorem 2.39 of [1]), it follows that $\mathcal{M} \in M$. Hence, $a \in M$. Therefore, $K(\mathbb{R}) \subseteq M$. $\square$

Given an inner model M with an $\mathbb{R}^M$-complete M-measure μ on κ, the next lemma gives a condition which implies that (M, μ) is really good on κ. This will allow us to generate the iterated ultrapower of (M, μ).

Lemma 2.13. *Suppose that M is an inner model of* ZF + $V = \mathrm{OD}(\mathbb{R})$ *which is κ-maximal. Let μ be an $\mathbb{R}^M$-complete M-measure on κ. If the ultrapower ${}^\kappa M/\mu$ is well-founded, then (M, μ) is really good on κ.*

Proof. Let $M_1 = {}^\kappa M/\mu$, where ${}^\kappa M = \{ f \in M \colon f : \kappa \to M \}$. Since M_1 is well-founded, we identify M_1 with its transitive collapse. Let $j : M \underset{\Sigma_\omega}{\longrightarrow} M_1$ be the natural embedding. Because M is κ-maximal, Lemma 2.6 implies that $M = M_1$. Let $A = \langle A_{(\xi,x)} : (\xi, x) \in \kappa \times \mathbb{R}^M \rangle$ be a sequence in $M \cap {}^{\kappa \times \mathbb{R}^M}\mathscr{P}(\kappa)$. Note that

$$A_{(\xi,x)} \in \mu \quad \text{if and only if} \quad \kappa \in j(A)(\xi, x)$$

for all $(\xi, x) \in \kappa \times \mathbb{R}^M$. Therefore,

$$\{(\xi, x) \in \kappa \times \mathbb{R}^M \colon A_{(\xi,x)} \in \mu\} = \{(\xi, x) \in \kappa \times \mathbb{R}^M \colon \kappa \in j(A)(\xi, x)\}.$$

Since $j(A) \in M$, it follows that $\{(\xi, x) \in \kappa \times \mathbb{R}^M \colon A_{(\xi,x)} \in \mu\} \in M$. Thus, (M, μ) is really good on κ. $\square$

Corollary 2.14. *Suppose that $K(\mathbb{R}) \models$* ZF + AD *and that μ is a $K(\mathbb{R})$-measure on $\kappa > \Theta^{K(\mathbb{R})}$. If the ultrapower ${}^\kappa K(\mathbb{R})/\mu$ is well-founded, then $(K(\mathbb{R}), \mu)$ is really good on κ.*

Proof. Assume that $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD}$ and let μ be a $K(\mathbb{R})$-measure on $\kappa > \Theta^{K(\mathbb{R})}$. Theorem 5.14 of [1] implies that $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. Corollary 2.4 implies that μ is $\mathbb{R}^{K(\mathbb{R})}$-complete. In addition, Lemma 1.16 asserts that $K(\mathbb{R}) \models V = \mathrm{OD}(\mathbb{R})$. Now, because $K(\mathbb{R})$ is 0-maximal, Lemma 2.13 implies that $(K(\mathbb{R}), \mu)$ is really good on κ. □

3. Generic extensions of weak iterated ultrapowers

A typical argument proving that an inner model is "iterable" usually requires DC, the Axiom of Dependent Choices (see pp. 254–257 of [8]). However, we are not assuming DC. So the goal of this section is to develop some tools which will be used in Section 4 to prove that $(K(\mathbb{R}), \mu)$ is weakly iterable without DC, whenever μ is an countably complete, $K(\mathbb{R})$-measure on $\kappa > \Theta^{K(\mathbb{R})}$.

Let M be an inner model of ZFC and suppose that μ is an M-measure on κ. Let $H_\kappa^M = \{x \in M\colon M \models |T_c(x)| < \kappa\}$. Let $\mathbb{P} = (P, \leqslant)$ be a partial order in H_κ^M. Whenever G is $\mathbb{P}$-generic over M, we shall call $M[G]$ a *small* generic extension of M and define $\mu[G] \supseteq \mu$ by

$$\mu[G] = \{X \subseteq \kappa\colon X \in M[G] \wedge \exists Y \in M\,(Y \in \mu \wedge Y \subseteq X)\}. \tag{$*$}$$

We call $\mu[G]$ the *generic expansion* of μ. We will show that $\mu[G]$ is a good $M[G]$-measure on κ, whenever μ is a good M-measure on κ. We will also show that small generic extensions can be used to expand a system of iterated ultrapowers to a larger system of iterated ultrapowers. For the remainder of this section M, $\mathbb{P}$ and G will be as above, and we will not distinguish between $\mathbb{P}$ and its domain P. Let $\hat{X}$ be a $\mathbb{P}$-name for $X \in M[G]$, and let $\check{a}$ be a canonical $\mathbb{P}$-name for $a \in M$.

Lemma 3.1. *If (M, μ) is good on κ, then $(M[G], \mu[G])$ is also good on κ.*

Proof. To prove the lemma it is sufficient to prove that $\mu[G]$ is (1) an ultrafilter, (2) normal and (3) weakly amenable. We do this by means of three claims. The first claim implies that $\mu[G]$ is an ultrafilter.

Claim 1. $(\forall X \in \mathscr{P}^{M[G]}(\kappa))(\exists Y \in \mu \cap M)[Y \subseteq X \vee Y \subseteq \kappa \backslash X]$.

Proof. Let $X \in \mathscr{P}^{M[G]}(\kappa)$ and let $\hat{X}$ be a $\mathbb{P}$-name for X. Let $p \in G$ be such that $p \Vdash \hat{X} \subseteq \check{\kappa}$. For each $q \in \mathbb{P}$, let $A_q = \{\xi \in \kappa\colon q \| \check{\xi} \in \hat{X}\}$ where $q \| \check{\xi} \in \hat{X}$ is an abbreviation for the statement "q *decides* $\check{\xi} \in \hat{X}$", that is, "$q \Vdash \check{\xi} \in \hat{X}$ or $q \Vdash \check{\xi} \notin \hat{X}$". Consider the set $D = \{q \in \mathbb{P}\colon q \leqslant p \wedge A_q \in \mu\}$. Because (M, μ) is good on κ, it follows that $D \in M$. By κ-completeness, D is dense below p. Therefore, let $q \in G \cap D$. Let $B = \{\xi \in A_q\colon q \Vdash \check{\xi} \in \hat{X}\} \in M$ and let $C = \{\xi \in A_q\colon q \Vdash \check{\xi} \notin \hat{X}\} \in M$. Since $A_q = B \cup C$ and $A_q \in \mu$, either $B \in \mu$ or $C \in \mu$. If $B \in \mu$, let $Y = B$. Thus, $Y \in M$ and $Y \subseteq X$. If $C \in \mu$, let $Y = C$. Therefore, $Y \in M$ and $Y \subseteq \kappa \backslash X$. This completes the proof of Claim 1. □

We now show that $\mu[G]$ is normal.

Claim 2. *Let* $\langle X_\xi \colon \xi<\kappa\rangle \in M[G]$ *be such that* $(\forall \xi<\kappa)(X_\xi \in \mu[G])$. *Then* $\{\xi\in\kappa\colon \xi\in \bigcap_{\eta<\xi} X_\eta\}\in\mu[G]$.

Proof. Suppose $\langle X_\xi \colon \xi<\kappa\rangle \in M[G]$ is such that $X_\xi\in\mu[G]$ for all $\xi<\kappa$. Let $X=\{\xi\in\kappa\colon \xi\in\bigcap_{\eta<\xi} X_\eta\}$. We must show that $X\in\mu[G]$. Let $f:\kappa\to\mathscr{P}(\kappa)$ be such that $f\in M[G]$ and $(\forall\xi<\kappa)\, f(\xi)=X_\xi$. Let $\hat{f}$ be a $\mathbb{P}$-name for f and let $p\in G$ be such that $p\Vdash \hat{f}\colon \check{\kappa}\to\mathscr{P}(\check{\kappa})$. Define in M,

$$A_{(q,\xi)}=\{\lambda\in\kappa\colon q\Vdash \check{\lambda}\in\hat{f}(\check{\xi})\}$$

for each $q\leqslant p$ and $\xi\in\kappa$. Since $X_\xi\in\mu[G]$ for each $\xi\in\kappa$, it follows that for all $\xi\in\kappa$ there exists a $q\leqslant p$ with $q\in G$ such that $A_{(q,\xi)}\in\mu$.

Now let

$$B=\{(q,\xi)\in\mathbb{P}\times\kappa\colon q\leqslant p\wedge A_{(q,\xi)}\in\mu\}.$$

Because (M,μ) is good on κ, it follows that $B\in M$. For each $\xi\in\kappa$, define $Y_\xi=\bigcap_{(q,\xi)\in B} A_{(q,\xi)}$. Note that $\langle Y_\xi\colon \xi<\kappa\rangle\in M$ and for each $\xi\in\kappa$, $Y_\xi\subseteq X_\xi$ and $Y_\xi\in\mu$ by κ-completeness. Let $Y=\{\xi\in\kappa\colon \xi\in\bigcap_{\eta<\xi}Y_\eta\}$. Since μ is normal, $Y\in\mu$. So $Y\subseteq X$ and $Y\in\mu$. Therefore, $X\in\mu[G]$. $\square$

Finally, we show that $(M[G],\mu[G])$ is good on κ.

Claim 3. *Let* $\langle X_\xi\colon \xi<\kappa\rangle$ *be a sequence in* $M[G]\cap {}^\kappa\mathscr{P}(\kappa)$. *Then* $\{\xi\in\kappa\colon X_\xi\in\mu[G]\}\in M[G]$.

Proof. Suppose $\langle X_\xi\colon \xi<\kappa\rangle\in M[G]$ is such that $X_\xi\subseteq\kappa$ for all $\xi<\kappa$. Let $X=\{\xi\in\kappa\colon X_\xi\in\mu[G]\}$. We must show that $X\in M[G]$. Let $f:\kappa\to\mathscr{P}(\kappa)$ be such that $f\in M[G]$ and $(\forall\xi<\kappa)\, f(\xi)=X_\xi$. Let $\hat{f}$ be a $\mathbb{P}$-name for f and let $p\in G$ be such that $p\Vdash\hat{f}:\check{\kappa}\to\mathscr{P}(\check{\kappa})$. Define in M,

$$A_{(q,\xi)}=\{\lambda\in\kappa\colon q\Vdash\check{\lambda}\in\hat{f}(\check{\xi})\}$$

for each $q\leqslant p$ and $\xi\in\kappa$. For each $\xi\in\kappa$, note that $X_\xi\in\mu[G]$ if and only if there exists a $q\leqslant p$ such that $q\in G$ and $A_{(q,\xi)}\in\mu$.

Now let

$$B=\{(q,\xi)\in\mathbb{P}\times\kappa\colon q\leqslant p\wedge A_{(q,\xi)}\in\mu\}.$$

Because (M,μ) is good on κ, it follows that $B\in M$. For each $\xi\in\kappa$,

$$X_\xi\in\mu[G] \quad\text{if and only if}\quad (\exists q\in G)[(q,\xi)\in B].$$

Therefore, $X\in M[G]$. $\square$

Proof of Lemma 3.1 (*Completion*). This completes the proof of the lemma. $\square$

Recall that M is an inner model of ZFC, μ is an M-measure on κ, $\mathbb{P}$ is a partial order in H_κ^M and G is $\mathbb{P}$-generic over M. Now, let $j^\mu : M \xrightarrow[\Sigma_\omega]{} M_1$ be the natural embedding of M into the ultrapower $M_1 = {}^\kappa M/\mu$. In addition, let μ_1 be the first iterate of μ.

If (M,μ) is good on κ, then by Lemma 3.1 $(M[G],\mu[G])$ is also good on κ and we can form the ultrapower of $(M[G],\mu[G])$. In this case, let $j^{\mu[G]} : M[G] \xrightarrow[\Sigma_\omega]{} M[G]_1$ be the natural embedding of $M[G]$ into the ultrapower $M[G]_1 = {}^\kappa M[G]/\mu[G]$, where ${}^\kappa M[G] = \{f \in M[G] : f : \kappa \to M[G]\}$.

Lemma 3.2. *Assume (M,μ) is good on κ. If M_1 is transitive, then*
(1) $\mathbb{P} \in M_1$.
(2) *G is also $\mathbb{P}$-generic over M_1.*
(3) *There is an embedding $j^G \supseteq j^\mu$ such that $j^G : M[G] \xrightarrow[\Sigma_\omega]{} M_1[G]$.*
(4) *The ultrapower $M[G]_1$ is well-founded.*
(5) $M[G]_1 = M_1[G]$, $\mu[G]_1 = \mu_1[G]$, *and* $j^{\mu[G]} = j^G$
where $\mu[G]_1$ is the first iterate of $\mu[G]$ and $\mu_1[G]$ is the generic expansion of μ_1.

Proof. Since $\mathbb{P} \in H_\kappa^M$, it follows that $j^\mu(\mathbb{P}) = \mathbb{P}$ and $j^\mu(p) = p$ for all $p \in \mathbb{P}$. Also, because (M,μ) is good on κ, $\mathscr{P}^M(\mathbb{P}) = \mathscr{P}^{M_1}(\mathbb{P})$ (see Lemma 19.1 of [8]). Thus, $\mathbb{P} \in M_1$ and G is $\mathbb{P}$-generic over M_1.

Since G is $\mathbb{P}$-generic over both M and M_1, let i_G^M and $i_G^{M_1}$ be the G-interpretations of the $\mathbb{P}$-names in M and M_1, respectively. Since j^μ is an elementary embedding, $j^\mu(\hat{x})$ is a $\mathbb{P}$-name in M_1 for all $\mathbb{P}$-names $\hat{x}$ in M. We now define an embedding $j^G : M[G] \to M_1[G]$. For $x \in M[G]$, let $\hat{x} \in M$ be any $\mathbb{P}$-name for x (so $i_G^M(\hat{x}) = x$) and define

$$j^G(x) = i_G^{M_1}(j^\mu(\hat{x})).$$

Claim 1. *j^G is well-defined and $j^G : M[G] \xrightarrow[\Sigma_\omega]{} M_1[G]$.*

Proof. To show that j^G is well-defined, let $\hat{x}, \hat{y} \in M$ be $\mathbb{P}$-names such that $i_G^M(\hat{y}) = i_G^M(\hat{x})$. Let $p \in G$ be such that $p \Vdash \hat{x} = \hat{y}$. Since $j^\mu(p) = p$, $j^\mu(\mathbb{P}) = \mathbb{P}$ and j^μ is elementary, we conclude that

$$M_1 \models \text{“}p \Vdash j^\mu(\hat{x}) = j^\mu(\hat{y})\text{”}.$$

Therefore,

$$i_G^{M_1}(j^\mu(\hat{x})) = i_G^{M_1}(j^\mu(\hat{y})).$$

To see that $j^G : M[G] \to M_1[G]$ is elementary, let $x_0, \ldots, x_n \in M[G]$ and let φ be a formula such that

$$M[G] \models \varphi(x_0, \ldots, x_n).$$

Let $\hat{x}_i$ be a $\mathbb{P}$-name such that $i^M_G(\hat{x}_i)=x_i$ for each $i\leqslant n$. Let $p\in G$ be such that $M\models$ "$p\Vdash\varphi(\hat{x}_0,\ldots,\hat{x}_n)$". By applying j^μ, we see that $M_1\models$ "$p\Vdash\varphi(j^\mu(\hat{x}_0),\ldots,j^\mu(\hat{x}_n))$". Therefore,

$$M_1[G]\models\varphi(j^G(x_0),\ldots,j^G(x_n)).\qquad\square$$

Claim 2. $(\forall X\in\cap\mathscr{P}^{M[G]}(\kappa))\,[X\in\mu[G]\Leftrightarrow\kappa\in j^G(X)]$.

Proof. Let $X\in M[G]\cap\mathscr{P}(\kappa)$. We show that $X\in\mu[G]\Leftrightarrow\kappa\in j^G(X)$.

($\Rightarrow$) Suppose that $X\in\mu[G]$. By the definition of $\mu[G]$, there is a $Y\in M$ such that $Y\in\mu$ and $Y\subseteq X$. Because $j^\mu:M\xrightarrow[\Sigma_\omega]{}M_1$ is the natural embedding, $\kappa\in j^\mu(Y)$. Since $j^G:M[G]\xrightarrow[\Sigma_\omega]{}M_1[G]$ and $M[G]\models Y\subseteq X$, we have that $M_1[G]\models j^G(Y)\subseteq j^G(X)$. But $j^G(Y)=j^\mu(Y)$; therefore, $\kappa\in j^G(X)$.

($\Leftarrow$) Suppose that $\kappa\in j^G(X)$. We show that $X\in\mu[G]$. Suppose that $X\notin\mu[G]$. Then $Y=(\kappa\backslash X)\in\mu[G]$. Thus, by the proof of ($\Rightarrow$) above, it follows that $\kappa\in j^G(Y)$. But $j^G(X)\cap j^G(Y)=j^G(X)\cap j^G(\kappa\backslash X)=\emptyset$. Contradiction. $\square$

We now show that $(M[G]_1,\in^{M[G]_1})$ is isomorphic to $(M_1[G],\in^{M_1[G]})$ and hence, the ultrapower of $(M[G],\mu[G])$ is well-founded. Define a map

$$\sigma:(M[G]_1,\in^{M[G]_1})\to(M_1[G],\in^{M_1[G]})$$

as follows: for $a\in M[G]_1$, let $f:\kappa\to M[G]$ be any function in $M[G]$ such that $M[G]_1\models j^{\mu[G]}(f)(\kappa)=a$ (see 5.13 of [8]) and define

$$\sigma(a)=j^G(f)(\kappa).$$

Claim 3. *σ is well-defined and $\sigma:(M[G]_1,\in^{M[G]_1})\to(M_1[G],\in^{M_1[G]})$ is an onto isomorphism.*

Proof. To see that σ is well-defined, let $f:\kappa\to M[G]$ and $g:\kappa\to M[G]$ be any two functions in $M[G]$. Then

$$\begin{aligned}(M[G]_1,\in^{M[G]_1})&\models j^{\mu[G]}(f)(\kappa)=j^{\mu[G]}(g)(\kappa)\\ &\Leftrightarrow\{\xi\in\kappa:\ M[G]\models f(\xi)=g(\xi)\}\in\mu[G]\\ &\Leftrightarrow(M_1[G],\in^{M_1[G]})\models j^G(f)(\kappa)=j^G(g)(\kappa).\end{aligned}\qquad(\blacklozenge)$$

The last equivalence follows from Claim 2. One can also show, arguing as in ($\blacklozenge$), that $M[G]_1\models a\in b\Leftrightarrow M_1[G]\models\sigma(a)\in\sigma(b)$, for all $a,b\in M[G]_1$. Now, to see that σ is onto, let $b\in M_1[G]$ and let $\hat{b}\in M_1$ be a $\mathbb{P}$-name for b. There is a function $g\in M$ such that $g:\kappa\to M$ and $j^\mu(g)(\kappa)=\hat{b}$. Since $\{\xi\in\kappa:\ g(\xi)$ is a $\mathbb{P}$-name$\}\in\mu$, we can assume that $g(\xi)$ is a $\mathbb{P}$-name for all $\xi\in\kappa$. Let $f\in M[G]$ be the function $f:\kappa\to M[G]$ defined by $f(\xi)=i^M_G(g(\xi))$. One can check that $j^G(f)(\kappa)=b$. Hence, σ an onto isomorphism. $\square$

Proof of Lemma 3.2 (*Completion*). The relationships between the maps defined above are outlined in the commutative diagram

$$\begin{tikzcd} M[G] \arrow[r, "j^{\mu[G]}"] \arrow[d, "\mathrm{id}"'] & M[G]_1 \arrow[d, "\sigma"] \\ M[G] \arrow[r, "j^G"'] \arrow[d, "i_G^M"'] & M_1[G] \arrow[d, "i_G^{M_1}"] \\ M^{\mathbb{P}} \arrow[r, "j^\mu"'] & M_1^{\mathbb{P}} \end{tikzcd} \tag{D}$$

where id is the identity map and $M^{\mathbb{P}}$, $M_1^{\mathbb{P}}$ are the classes of $\mathbb{P}$-names in M and M_1, respectively. □

Lemma 3.3. *Assume (M,μ) is good on κ. If (M,μ) is weakly iterable, then $(M[G],$ $\mu[G])$ is also weakly iterable.*

Proof. Assume that (M,μ) is weakly iterable and let

$$\langle\langle M_\alpha\rangle_{\alpha\in\mathrm{OR}}, \langle j^\mu_{\alpha\beta} : M_\alpha \xrightarrow[\Sigma_\omega]{} M_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle \tag{$*$}$$

be the weak iteration of (M,μ). So, M_α is transitive for all $\alpha\in\mathrm{OR}$ and since $\mathbb{P}\in H_\kappa^M$, it follows that $j^\mu_{0\alpha}(\mathbb{P})=\mathbb{P}$, $j^\mu_{0\alpha}(p)=p$ for all $p\in\mathbb{P}$, and $\mathscr{P}^M(\mathbb{P})=\mathscr{P}^{M_\alpha}(\mathbb{P})$. Hence, $\mathbb{P}\in M_\alpha$ and G is $\mathbb{P}$-generic over M_α for all ordinals α.

Lemma 3.1 implies that $(M[G],\mu[G])$ is good on κ. Therefore, let

$$\langle\langle M[G]_\alpha\rangle_{\alpha\in\mathrm{OR}}, \langle j^{\mu[G]}_{\alpha\beta} : M[G]_\alpha \xrightarrow[\Sigma_\omega]{} M[G]_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

be the weak iteration of $(M[G],\mu[G])$. We must show that $M[G]_\alpha$ is well-founded (transitive) for all $\alpha\in\mathrm{OR}$.

Since G is $\mathbb{P}$-generic over M_α for all ordinals α, it follows (see (3) of Lemma 3.2) that the above commutative system $(*)$ can be extended to the commutative system

$$\langle\langle M_\alpha[G]\rangle_{\alpha\in\mathrm{OR}}, \langle j^G_{\alpha\beta} : M_\alpha[G] \xrightarrow[\Sigma_\omega]{} M_\beta[G]\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

where $j^G_{\alpha,\beta}\supseteq j^\mu_{\alpha,\beta}$ and $M_\alpha[G]$ is transitive for all ordinals $\alpha\leqslant\beta$. By generalizing the commutative diagram (D) in the proof of Lemma 3.2, one obtains the commutative diagram

$$\begin{tikzcd}
M[G] \arrow[r, "j_{01}^{\mu[G]}"] \arrow[d, "\mathrm{id}"'] & M[G]_1 \arrow[r, "j_{12}^{\mu[G]}"] \arrow[d, "\sigma_1"] & M[G]_2 \arrow[r] \arrow[d, "\sigma_2"] & \cdots \arrow[r] & M[G]_\lambda \arrow[r, "j_{\lambda,\lambda+1}^{\mu[G]}"] \arrow[d, "\sigma_\lambda"] & M[G]_{\lambda+1} \arrow[d, "\sigma_{\lambda+1}"] & \cdots \\
M[G] \arrow[r, "j_{01}^{G}"'] & M_1[G] \arrow[r, "j_{12}^{G}"'] & M_2[G] \arrow[r] & \cdots \arrow[r] & M_\lambda[G] \arrow[r, "j_{\lambda,\lambda+1}^{G}"'] & M_{\lambda+1}[G] & \cdots \\
M^{\mathbb{P}} \arrow[r, "j_{01}^{\mu}"'] \arrow[u, "i_G^M"] & M_1^{\mathbb{P}} \arrow[r, "j_{12}^{\mu}"'] \arrow[u, "i_G^{M_1}"'] & M_2^{\mathbb{P}} \arrow[r] \arrow[u, "i_G^{M_2}"'] & \cdots \arrow[r] & M_\lambda^{\mathbb{P}} \arrow[r, "j_{\lambda,\lambda+1}^{\mu}"'] \arrow[u, "i_G^{M_\lambda}"'] & M_{\lambda+1}^{\mathbb{P}} \arrow[u, "i_G^{M_{\lambda+1}}"'] & \cdots
\end{tikzcd}$$

One can now prove by induction on β that
(1) the iterate $M[G]_\beta$ is well-founded,
(2) $M[G]_\beta = M_\beta[G]$,
(3) $\mu[G]_\beta = \mu_\beta[G]$,
(4) $j_{\alpha,\beta}^{\mu[G]} = j_{\alpha,\beta}^{G}$
for all ordinals $\alpha \leqslant \beta$, where $\mu[G]_\beta$ is the βth iterate of $\mu[G]$ and $\mu_\beta[G]$ is the generic expansion of μ_β. When $\beta = \lambda$ is a limit ordinal the proof of (1)–(4) is clear, and when $\beta = \lambda + 1$ is a successor ordinal one can establish (1)–(4) as in the proof of Lemma 3.2. Therefore, $(M[G], \mu[G])$ is weakly iterable. □

4. AD and a set of reals not in $K(\mathbb{R})$

We emphasize again that we do not assume DC, the Axiom of Dependent Choices.

Lemma 4.1. *Assume* $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then there exists an ordinal* κ *and a* $\hat{\mu}$ *such that*
(1) $\hat{\mu}$ *is a* $K(\mathbb{R})$*-measure on* κ,
(2) $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$,
(3) $(K(\mathbb{R}), \hat{\mu})$ *is really good on* κ,
(4) $(K(\mathbb{R}), \hat{\mu})$ *is weakly iterable.*

Proof. We are assuming $V \models \mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$ where V is the universe of sets. Let φ be the conclusion of the lemma we want to prove. We will take generic extensions of transitive inner models of V to show that $V \models \varphi$. This is comparable to showing that φ is true in every countable transitive model of a sufficiently large fragment of $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. Thus, $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})] \vdash \varphi$.

By Theorem 5.14 of [1], $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. By assumption, let $X \subseteq \mathbb{R}$ be such that $X \notin K(\mathbb{R})$. Using AD, Wadge's Lemma (see 7D.3 of [15]) implies that every $B \in \mathscr{P}(\mathbb{R}) \cap K(\mathbb{R})$ is a continuous pre-image of X. Thus, $\Theta^{K(\mathbb{R})} < \Theta$. By standard results of AD, it follows that $(\Theta^+)^{K(\mathbb{R})} \leqslant (\Theta^{K(\mathbb{R})})^+ < \Theta$ (see 7D.19 of [15]). In addition, under AD, it follows from work of Kechris [10] that there exists a measurable cardinal κ with measure ν such that $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$. We shall use ν to construct a $K(\mathbb{R})$-measure $\hat{\mu}$ which is really good on κ and then prove that $(K(\mathbb{R}), \hat{\mu})$ is weakly iterable.

Remark. The above consequences of AD, cited in [10, 15], do not require the Axiom of Dependent Choices for their proofs.

Now, let A be $\mathbb{Q}$-generic over $K(\mathbb{R})$, where $\mathbb{Q}$ is the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1. Again, we note that A is essentially a subset of ω_1. Let $\overline{V} = K(\mathbb{R})[A]$. Since $K(\mathbb{R}) \models \mathrm{DC}$, Theorem 1.11 implies that
(1) $\mathbb{R}^{K(\mathbb{R})} = \mathbb{R}^{\overline{V}}$,
(2) ${\omega_1}^{K(\mathbb{R})} = \omega_1^{\overline{V}}$,
(3) $K(\mathbb{R}) = K(\mathbb{R})^{\overline{V}}$,
(4) $\Theta^{K(\mathbb{R})} = \omega_2^{\overline{V}}$,
(5) $(\Theta^+)^{K(\mathbb{R})} = \omega_3^{\overline{V}}$,
(6) $\overline{V} \models \mathrm{ZFC} + \mathrm{GCH}$,
(7) $\mathrm{HOD}^{K(\mathbb{R})} = \mathrm{HOD}^{\overline{V}}$.
By Lemma 1.16, $K(\mathbb{R}) = L[D](\mathbb{R})$ where D is as in Definition 1.15. Let $\mathscr{H} = \mathrm{HOD}^{\overline{V}}$. Note that $\overline{V} = K(\mathbb{R})[A] = L[D](\mathbb{R})[A] = L[D](A)$, $A \subset \omega_1^{\overline{V}}$, and $L[D] \subseteq \mathscr{H}$. Thus, by Lemma 1.14, $\overline{V}$ is a generic extension of $\mathscr{H}$ and one can compute $\mathrm{HOD}^{K(\mathbb{R})}$; that is, there is a Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ where $B \leqslant \omega_3^{\overline{V}}$ is an ordinal, and there is a G which is $\mathscr{B}$-generic over $\mathscr{H}$ such that
(i) $\overline{V} = \mathscr{H}[G]$,
(ii) $G \cap \mathscr{B}_*^{\mathscr{H}} = \widehat{B} \in \mathscr{H}$.
In addition, there exists a $b : \omega_1 \to B$ such that
(iii) $\mathscr{H} = L[D](\{\mathscr{B}, b, \widehat{B}\}) = L[D](P)$
where $P \subseteq {\omega_3}^{\overline{V}}$ is a canonical coding of $\{\mathscr{B}, b, \widehat{B}\}$. Therefore,

$$\mathscr{H} = \mathrm{HOD}^{\overline{V}} = \mathrm{HOD}^{K(\mathbb{R})} = L[D](\{\mathscr{B}, b, \widehat{B}\}) = L[D](P).$$

For the remainder of this section we will not distinguish between the Boolean Algebra $\mathscr{B}$ and its domain B.

Let $N = L[D, \nu](P)$. Note that $N \models \mathrm{ZFC}$ and that $\mathscr{H} = L[D](P)$ is an inner model of N. Since $N \models$ "ν is a measure on κ", one can define in N a normal measure $\mu \in N$ on κ and, by absoluteness, one can show that N is iterable by μ. We first establish that $(\mathscr{H}, \mu)$ is a good on κ, and then we shall show that $(\mathscr{H}, \mu)$ is weakly iterable.

Let $N_1 = {}^{\kappa}N/\mu$ be the ultrapower of (N, μ) and let $\mathscr{H}_1 = {}^{\kappa}\mathscr{H}/\mu$ be the ultrapower of $(\mathscr{H}, \mu)$. Since $\mathscr{H}_1$ can be embedded into N_1, it follows that $\mathscr{H}_1$ is well-founded and, as usual, we identify $\mathscr{H}_1$ with its transitive collapse. Let $j^{\mu} : \mathscr{H} \to \mathscr{H}_1$ be the natural

embedding. Since $\mathscr{H} \models \mathrm{ZFC}$, Lemma 2.5 implies
(1) $j^\mu : \mathscr{H} \xrightarrow[\Sigma_\omega]{} \mathscr{H}_1$,
(2) $j^\mu(\xi) = \xi$ for all $\xi < \kappa$,
(3) $\mathscr{H}_1 \models \kappa < j^\mu(\kappa)$.
Our first goal is to show that $\mathscr{H} = \mathscr{H}_1$, from which we shall conclude that $(\mathscr{H}, \mu)$ is good on κ. The proofs of these two assertions will be accomplished in a series of claims (see Claims 5 and 6 below).

Claim 1. *$j^\mu(P) = P$ and $\mathscr{P}^{\mathscr{H}}(P) = j^\mu(\mathscr{P}^{\mathscr{H}}(P)) = \mathscr{P}^{\mathscr{H}_1}(P)$.*

Proof. Let $\alpha = (\Theta^+)^{K(\mathbb{R})}$ and note that $P \subseteq \alpha < \kappa$. Since

$$N \models \text{“}\mu \text{ is a normal measure on } \kappa\text{”},$$

standard arguments about measurable cardinals show that $\mathscr{P}^{\mathscr{H}}(\alpha) = j^\mu(\mathscr{P}^{\mathscr{H}}(\alpha)) = \mathscr{P}^{\mathscr{H}_1}(\alpha)$. □

Claim 1, together with its proof, implies that
(1) $j^\mu(\mathscr{B}) = \mathscr{B}$,
(2) $\mathscr{P}^{\mathscr{H}}(\mathscr{B}) = j^\mu\left(\mathscr{P}^{\mathscr{H}}(\mathscr{B})\right) = \mathscr{P}^{\mathscr{H}_1}(\mathscr{B})$,
(3) $\mathscr{B}_*^{\mathscr{H}} = \mathscr{B}_*^{\mathscr{H}_1}$,
(4) G is $\mathscr{B}$-generic over $\mathscr{H}_1$.
Lemma 3.2 implies that there is an embedding $j^G \supseteq j^\mu$ such that $j^G : \mathscr{H}[G] \xrightarrow[\Sigma_\omega]{} \mathscr{H}_1[G]$.

Claim 2. *$\mathscr{H}_1 \subseteq \mathrm{HOD}^{\mathscr{H}_1[G]}$.*

Proof. As noted above, $\mathscr{H} = \mathrm{HOD}^{\overline{V}}$ and $\mathscr{H}[G] = \overline{V}$. Let $x \in \mathscr{H}_1$. We shall show that $x \in \mathrm{HOD}^{\mathscr{H}_1[G]}$. Let $f \in {}^\kappa\mathscr{H}$ be such that $j^\mu(f)(\kappa) = x$. Since $f \in \mathscr{H}$, it follows that $\mathscr{H}[G] \models f \in \mathrm{HOD}$. Thus, $\mathscr{H}_1[G] \models j^G(f) \in \mathrm{HOD}$. But $j^\mu(f) = j^G(f)$ and so, $\mathscr{H}_1[G] \models j^\mu(f) \in \mathrm{HOD}$. Therefore, $\mathscr{H}_1[G] \models j^\mu(f)(\kappa) \in \mathrm{HOD}$, that is, $\mathscr{H}_1[G] \models x \in \mathrm{HOD}$. □

Claim 3. *$\mathscr{H}_1 = \mathrm{HOD}^{\mathscr{H}_1[G]}$.*

Proof. Since $j^G : \mathscr{H}[G] \xrightarrow[\Sigma_\omega]{} \mathscr{H}_1[G]$, $j^G(G) = G$ and

$$\mathscr{H}[G] \models \text{“}G \text{ is } \mathscr{B}\text{-generic over HOD and } V = \mathrm{HOD}[G]\text{”},$$

it follows that

$$\mathscr{H}_1[G] \models \text{“}G \text{ is } \mathscr{B}\text{-generic over HOD and } V = \mathrm{HOD}[G]\text{”}.$$

Since $\mathscr{B}, \widehat{B} \in \mathscr{H}_1$, Lemma 1.13 Claim 2 imply that $\mathscr{H}_1 = \mathrm{HOD}^{\mathscr{H}_1[G]}$. □

Claim 4. *$K(\mathbb{R}) = K(\mathbb{R})^{\mathscr{H}_1[G]}$.*

Proof. ($\subseteq$). Let $\mathscr{M} \in K(\mathbb{R})$ be a real 1-mouse. We shall show that $\mathscr{M} \in K(\mathbb{R})^{\mathscr{H}_1[G]}$. Theorem 5.5 (iii) of [1] will then imply that $K(\mathbb{R}) \subseteq K(\mathbb{R})^{\mathscr{H}_1[G]}$. To see that $\mathscr{M} \in$

$K(\mathbb{R})^{\mathscr{H}_1[G]}$ let $\mathscr{C}=\mathscr{C}(\mathscr{M})\in K(\mathbb{R})$ be the core of $\mathscr{M}$ (see Definition 2.36 of [1]), let

$$\langle\langle\mathscr{C}_\alpha\rangle_{\alpha\in\mathrm{OR}},\langle\pi_{\alpha\beta}:\mathscr{C}_\alpha \xrightarrow[\Sigma_1]{} \mathscr{C}_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

be the premouse iteration of $\mathscr{C}$ and let $\kappa_\alpha=\pi_{0\alpha}(\kappa^{\mathscr{C}})$ for each ordinal α. We shall show that $\mathscr{C}_\omega\in K(\mathbb{R})^{\mathscr{H}_1[G]}$ and thus, $\mathscr{C}\in K(\mathbb{R})^{\mathscr{H}_1[G]}$ (see 2.35, 2.37 and 2.38 of [1]). Since every 1-mouse is an iterate of its core (by Theorem 2.39 of [1]), it will then follow that $\mathscr{M}\in K(\mathbb{R})^{\mathscr{H}_1[G]}$. So we shall assume without loss of generality that $\mathscr{M}=\mathscr{C}_\omega$.

Note that $K(\mathbb{R})=K(\mathbb{R})^{\mathscr{H}[G]}$ and thus, $\mathscr{H}[G]\models$"$\mathscr{M}$ is a real 1-mouse". Because $j^G:\mathscr{H}[G]\xrightarrow[\Sigma_\omega]{}\mathscr{H}_1[G]$, it follows that $\mathscr{H}_1[G]\models$"$j^G(\mathscr{M})$ is a real 1-mouse". Let $\mathscr{N}=j^G(\mathscr{M})$ and $\sigma=j^G\restriction\mathscr{M}$. Note that $\sigma:\mathscr{M}\xrightarrow[\Sigma_\omega]{}\mathscr{N}$. Keep in mind that V, $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_1[G]$ all have the same ordinals. In addition, $\mathscr{H}$ and $\mathscr{H}_1$ are inner models of V while $\mathscr{H}_1[G]$ is not an inner model of V. Also note that $\mathscr{M}\in V$ and $\mathscr{N}\in\mathscr{H}_1[G]$. Our plan is to show that $\mathscr{M}$ and $\mathscr{N}$ have comparable premouse iterates. A theorem of ZF (see Lemma 2.25 of [1]) states that any two iterable real premice have comparable iterates. However, before we can apply this theorem we must ensure that both $\mathscr{M}$ and $\mathscr{N}$ are elements in the same transitive model of ZF.

Subclaim. *The structures $\mathscr{M}$, $\mathscr{N}$ and the map $\sigma:\mathscr{M}\xrightarrow[\Sigma_\omega]{}\mathscr{N}$ are all elements in V.*

Proof. That $\mathscr{M}\in V$ is clear. Let $d=\langle(\kappa_n,\kappa_\omega):n\in\omega\rangle$. Note that $\mathscr{M}$ can be completely constructed from d and $\mathbb{R}$; that is, define μ_d by

$$A\in\mu_d \text{ if and only if } A\subseteq\kappa_\omega\wedge\exists n\in\omega\ \forall k>n\,(\kappa_k\in A)$$

and observe that $\mathscr{M}$ can be constructed (literally) relative to μ_d and above $\mathbb{R}$. Lemma 5.3 of [1] implies that $d\in\mathrm{HOD}=\mathscr{H}$. Since $j^\mu\subseteq j^G$, it follows that $j^G(d)=j^\mu(d)$. Hence, $j^G(d)\in\mathscr{H}_1$. Also note that $j^G(d)$ and $\mathbb{R}$ completely determine the construction of $\mathscr{N}$ in $\mathscr{H}_1[G]$. Because $j^G(d)\in V$, we see that $\mathscr{N}\in V$. In addition, d and $j^G(d)$ together with $\mathbb{R}$ completely determine the construction of the map $\sigma:\mathscr{M}\xrightarrow[\Sigma_\omega]{}\mathscr{N}$ and thus, $\sigma\in V$. □

Proof of Claim 4 (*Completion*). Because $\mathscr{M}$ and $\mathscr{N}$ are both in V, Lemma 2.25 of [1] implies that there is a $\theta\in\mathrm{OR}$ such that the premouse iterates $\mathscr{M}_\theta$ and $\mathscr{N}_\theta$ are comparable in V. Also, since $\sigma:\mathscr{M}\xrightarrow[\Sigma_\omega]{}\mathscr{N}$ is in V, it follows (see the proof of Lemma 2.12) that $\mathscr{M}_\theta$ must be an initial segment of $\mathscr{N}_\theta$. Thus, $\mathscr{M}_\theta\in\mathscr{H}_1[G]$ because $\mathscr{N}_\theta\in\mathscr{H}_1[G]$. Therefore, the core $\mathscr{C}(\mathscr{M})\in\mathscr{H}_1[G]$ (see Lemma 2.37 of [1]). Since $\mathscr{M}$ is an iterate of its core, $\mathscr{M}\in\mathscr{H}_1[G]$. Hence, $\mathscr{M}\in K(\mathbb{R})^{\mathscr{H}_1[G]}$. Therefore, $K(\mathbb{R})\subseteq K(\mathbb{R})^{\mathscr{H}_1[G]}$.

($\supseteq$) Let $\mathscr{M}\in K(\mathbb{R})^{\mathscr{H}_1[G]}$, that is, $\mathscr{H}_1[G]\models$"$\mathscr{M}$ is a real 1-mouse". We shall show that $\mathscr{M}\in K(\mathbb{R})$. Let $\mathscr{C}=\mathscr{C}(\mathscr{M})\in\mathscr{H}_1[G]$ be the core of $\mathscr{M}$ and let

$$\langle\langle\mathscr{C}_\alpha\rangle_{\alpha\in\mathrm{OR}},\langle\pi_{\alpha\beta}:\mathscr{C}_\alpha \xrightarrow[\Sigma_1]{} \mathscr{C}_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

be the premouse iteration of $\mathscr{C}$ and let $\kappa_\alpha=\pi_{0\alpha}(\kappa^{\mathscr{C}})$ for each ordinal α. As above, we shall assume without loss of generality that $\mathscr{M}=\mathscr{C}_\omega$. Let $d\in\mathscr{H}_1[G]$ be defined as in the proof of the above Subclaim. We note that $d \in \mathrm{HOD}^{\mathscr{H}_1[G]}$ which follows from Lemma 5.3 of [1]. By Claim 3, $\mathrm{HOD}^{\mathscr{H}_1[G]} = \mathscr{H}_1$. Therefore, $d \in V$. Since $\mathscr{M}$ can be constructed completely from d and $\mathbb{R}$, we conclude that $\mathscr{M} \in V$. Hence, $\mathscr{M} \in K(\mathbb{R})$. Thus, $K(\mathbb{R})^{\mathscr{H}_1[G]} \subseteq K(\mathbb{R})$. □

Claim 5. $\mathscr{H} = \mathscr{H}_1$.

Proof. Since $\mathscr{H}[G] \models$ "$\mathrm{HOD} = \mathrm{HOD}^{K(\mathbb{R})}$" and $j^G : \mathscr{H}[G] \underset{\Sigma_\omega}{\longrightarrow} \mathscr{H}_1[G]$, it follows that $\mathscr{H}_1[G] \models$ "$\mathrm{HOD} = \mathrm{HOD}^{K(\mathbb{R})}$". Now, because $\mathscr{H} = \mathrm{HOD}^{\mathscr{H}[G]}$ and $\mathscr{H}_1 = \mathrm{HOD}^{\mathscr{H}_1[G]}$, Claim 4 implies that $\mathscr{H} = \mathscr{H}_1$. □

Claim 6. *$(\mathscr{H}, \mu)$ is good on κ.*

Proof. Recall that $\mathscr{H}_1 = {}^\kappa\mathscr{H}/\mu$, where ${}^\kappa\mathscr{H} = \{ f \in \mathscr{H} \colon f \colon \kappa \to \mathscr{H} \}$. Since $\mathscr{H} = \mathscr{H}_1$, an argument analogous to the proof of Lemma 2.13 (using Lemma 2.5) shows that $(\mathscr{H}, \mu)$ is *good on* κ. □

Claim 6 implies that we can form the commutative system

$$\langle \langle \mathscr{H}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle j^\mu_{\alpha\beta} \colon \mathscr{H}_\alpha \underset{\Sigma_\omega}{\longrightarrow} \mathscr{H}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

(see Section 2). Recall that (N, μ) is iterable and, since each weak iterate of $(\mathscr{H}, \mu)$ can be embedded into an iterate of (N, μ), we have that $(\mathscr{H}, \mu)$ is weakly iterable. Therefore, $(\mathscr{H}[G], \mu[G])$ is good on κ and is weakly iterable by Lemmas 3.1 and 3.3, respectively. Let

$$\langle \langle \mathscr{H}[G]_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle j^{\mu[G]}_{\alpha\beta} \colon \mathscr{H}[G]_\alpha \underset{\Sigma_\omega}{\longrightarrow} \mathscr{H}[G]_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the weak iteration of $(\mathscr{H}[G], \mu[G])$. Recall [see (3.0) in Section 3] that

$$\mu[G] = \{ X \subseteq \kappa \colon X \in \mathscr{H}[G] \wedge \exists Y \in \mathscr{H} \, (Y \in \mu \wedge Y \subseteq X) \}.$$

Let

$$\hat{\mu} = \{ X \subseteq \kappa \colon X \in K(\mathbb{R}) \wedge \exists Y \in \mathscr{H} \, (Y \in \mu \wedge Y \subseteq X) \}$$

that is, $\hat{\mu} = \mu[G] \cap K(\mathbb{R})$. Note that $\mu \cap \mathscr{H} \subseteq \hat{\mu} \subseteq \mu[G]$ and that $\hat{\mu} \in V$.

Claim 7. *$(K(\mathbb{R}), \hat{\mu})$ is really good on κ.*

Proof. By Claim 4, $K(\mathbb{R}) = K(\mathbb{R})^{\mathscr{H}_1[G]}$. Since $\mu[G]$ is an $\mathscr{H}[G]$-measure on κ, we see that $\hat{\mu}$ is a $K(\mathbb{R})$-measure on κ. Let $K(\mathbb{R})_1 = {}^\kappa K(\mathbb{R})/\hat{\mu}$ be the ultrapower of $(K(\mathbb{R}), \hat{\mu})$. Because $K(\mathbb{R})_1$ can be embedded into the well-founded ultrapower ${}^\kappa\mathscr{H}[G]/\mu = \mathscr{H}[G]_1$, it follows that $K(\mathbb{R})_1$ is well-founded. Corollary 2.14 implies that $(K(\mathbb{R}), \hat{\mu})$ is really good on κ. □

Claim 8. *$(K(\mathbb{R}), \hat{\mu})$ is weakly iterable.*

Proof. By Claim 7, $(K(\mathbb{R}), \hat{\mu})$ is really good on κ. Thus, let

$$\langle\langle K(\mathbb{R})_\alpha\rangle_{\alpha\in\mathrm{OR}}, \langle j^{\hat{\mu}}_{\alpha\beta} : K(\mathbb{R})_\alpha \xrightarrow[\Sigma_\omega]{} K(\mathbb{R})_\beta\rangle_{\alpha\leqslant\beta\in\mathrm{OR}}\rangle$$

be the weak iteration of $(K(\mathbb{R}), \hat{\mu})$. Since $(\mathscr{H}[G], \mu[G])$ is weakly iterable, we have that $\mathscr{H}[G]_\alpha$ is transitive for all ordinals α. Because $K(\mathbb{R}) \subseteq \mathscr{H}[G]$ and $\hat{\mu} \subseteq \mu[G]$, one can define (by induction on α) embeddings

$$\tau_\alpha : K(\mathbb{R})_\alpha \xrightarrow[\Sigma_0]{} \mathscr{H}[G]_\alpha$$

satisfying

(1) τ_0 is the identity map,
(2) $\tau_\alpha : (K(\mathbb{R})_\alpha, \hat{\mu}_\alpha) \xrightarrow[\Sigma_0]{} (\mathscr{H}[G]_\alpha, \mu[G]_\alpha)$,
(3) $\tau_\beta \circ j^{\hat{\mu}}_{\alpha\beta} = j^{\mu[G]}_{\alpha\beta} \circ \tau_\alpha$ for all ordinals $\alpha \leqslant \beta$.

Therefore, $(K(\mathbb{R}), \hat{\mu})$ is weakly iterable. □

Proof of Lemma 4.1 (*Conclusion*). This completes the proof of the lemma. □

Definition 13.8 of [4] states that if μ is a normal measure in $L[\mu]$, then $L[\mu]$ is called a ρ-model. We now extend this terminology to the analogous inner model above the reals.

Definition 4.2. If μ is an $\mathbb{R}$-complete $L[\mu](\mathbb{R})$-measure on κ, then $L[\mu](\mathbb{R})$ is said to be a ρ-model with critical point κ.

Let $\nu = \mu \cap L[\mu](\mathbb{R})$. Note that $\nu \in L[\mu](\mathbb{R})$ and that

$$L[\mu](\mathbb{R}) \models \text{“}\nu \text{ is an } \mathbb{R}\text{-complete measure on } \kappa\text{”}.$$

Lemma 4.3. *Suppose that $(K(\mathbb{R}), \mu)$ is really good on κ. If $(K(\mathbb{R}), \mu)$ is weakly iterable, then there is a ρ-model $L[\nu](\mathbb{R})$ with critical point $\lambda > \kappa$ such that*
(1) $L[\nu](\mathbb{R})$ is iterable,
(2) $\mathscr{P}(\lambda \times \mathbb{R}) \cap L[\nu](\mathbb{R}) \subseteq K(\mathbb{R})$.

Proof. Let φ be the sentence we want to prove, that is,

$$\forall\mu\forall\kappa(\text{“}(K(\mathbb{R}), \mu) \text{ is really good on } \kappa \text{ and is weakly iterable”} \Rightarrow \psi(\kappa))$$

where $\psi(\kappa)$ is the assertion

$$\begin{aligned}&\exists\nu\exists\lambda[L[\nu](\mathbb{R}) \text{ is a } \rho\text{-model with critical point } \lambda > \kappa\\ &\qquad \wedge L[\nu](\mathbb{R}) \text{ is iterable}\\ &\qquad \wedge \mathscr{P}(\lambda \times \mathbb{R}) \cap L[\nu](\mathbb{R}) \subseteq K(\mathbb{R})].\end{aligned}$$

We now show that $V \models \varphi$ for any countable transitive model V of (a sufficiently large fragment of) ZF. Thus, ZF $\vdash \varphi$.

Let V be a countable transitive model of ZF. Suppose $\kappa, \mu \in V$ are such that

$$V \models \text{“}(K(\mathbb{R}), \mu) \text{ is really good on } \kappa \text{ and weakly iterable”}.$$

In V, let $M = L[D, \mu](\mathbb{R})$ where D is as in Definition 1.15. By absoluteness,

$$M \models \text{“}(K(\mathbb{R}), \mu) \text{ is really good on } \kappa \text{ and weakly iterable”}.$$

If we can show that $M \models \psi(\kappa)$, then $V \models \psi(\kappa)$ by absoluteness and thus $V \models \varphi$, completing the proof. Hence, our task is to prove that $M \models \psi(\kappa)$.

In M, let $\mathbb{P}$ be the partial order $\mathbb{P} = (\mathbb{R}^{<\omega}, \leqslant)$ where $p \leqslant q$ if and only if q is an initial segment of p, for all $p, q \in \mathbb{R}^{<\omega}$. Let G be $\mathbb{P}$-generic over M. Clearly, $M[G] = L[D, \mu](G)$ and $M[G] \models \mathrm{ZFC}$.

Definition 2.34 of [1] describes when a structure is a 1-mouse and when it is a real 1-mouse. Definition 2.18 of [2] generalizes these concepts and defines when a structure is a mouse and when it is a real mouse. We shall say that a structure $\mathcal{N}$ is an $\mathbb{R}^M$ *mouse* if $\mathcal{N}$ is a mouse and $\mathbb{R}^{\mathcal{N}} = \mathbb{R}^M$. In other words, $\mathcal{N}$ is a mouse having $\mathbb{R}^M$ as its set of reals. Let $K(\mathbb{R}^M)$ be the union of $\mathbb{R}^M$ 1-mice. We note that the forcing extension $M[G]$ does not contain any new $\mathbb{R}^M$ 1-mice (see Lemma 1.9). Therefore, $\mathcal{N}$ is an $\mathbb{R}^M$ 1-mouse in $M[G]$ if and only if $\mathcal{N}$ is a real 1-mouse in V. Thus, $K(\mathbb{R})^V = K(\mathbb{R})^M = K(\mathbb{R}^M)^{M[G]}$. By absoluteness, we conclude that

$$M[G] \models \text{“}(K(\mathbb{R}^M), \mu) \text{ is really good on } \kappa \text{ and weakly iterable”}.$$

Until further notice we shall work in $M[G]$, a transitive model of ZFC. Let $\lambda > \kappa$ be a regular cardinal. Let $(K(\mathbb{R}^M)_\lambda, \mu_\lambda)$ and κ_λ be the λth iterates of $(K(\mathbb{R}^M), \mu)$ and κ, respectively. Note that $K(\mathbb{R}^M)_\lambda = K(\mathbb{R}^M)$ (see Lemma 2.12), $\kappa_\lambda = \lambda$, and $\mu_\lambda \subseteq F_\lambda$ where F_λ is the closed unbounded filter on λ. It follows that μ_λ is countably complete. Let $\nu = \mu_\lambda$ and note that $\nu \in M$.

Let $\lambda^+ = \lambda^+_{L[\nu](\mathbb{R}^M)}$ be as computed in $L[\nu](\mathbb{R}^M)$ (see Definition 1.1).

Claim 1.
(1) $\mathscr{P}(\lambda \times \mathbb{R}^M) \cap L[\nu](\mathbb{R}^M) \subseteq J_{\lambda^+}[\nu](\mathbb{R}^M)$,
(2) $(\forall \alpha < \lambda^+)(\exists \beta \geqslant \alpha)(\beta < \lambda^+ \wedge \mathscr{P}(\lambda \times \mathbb{R}^M) \cap J_{\beta+1}[\nu](\mathbb{R}^M) \setminus J_\beta[\nu](\mathbb{R}^M) \neq \emptyset)$.

Proof. Gödel's argument for proving that the GCH holds in L can be used to show that $\mathscr{P}(\lambda \times \mathbb{R}^M) \cap L[\nu](\mathbb{R}^M) \subseteq J_{\lambda^+}[\nu](\mathbb{R}^M)$. We work in $L[\nu](\mathbb{R}^M)$ and so, we shall assume that $L[\nu](\mathbb{R})$ is the universe.

To prove (1), let $Y \in \mathscr{P}(\lambda \times \mathbb{R})$. Let $\eta \in \mathrm{OR}$ be such that $Y, \nu \in J_\eta[\nu](\mathbb{R})$. Corollary 1.8 of [1] and Lemma 1.4 of [2] implies that there exists a function $f : \omega\eta \times \mathbb{R} \xrightarrow{\text{onto}} J_\eta[\nu](\mathbb{R})$. Therefore, we can form a Σ_1 Skolem Hull of $\{Y\} \cup \mathbb{R} \cup \lambda$ in the structure $(J_\eta[\nu](\mathbb{R}), \nu)$. Thus there exists a Σ_1-elementary substructure $\mathscr{H} = (H, \nu^H)$ of $(J_\eta[\nu](\mathbb{R}), \nu)$ containing $\{Y\} \cup \mathbb{R} \cup \lambda$ such that $|H|_{L[\nu](\mathbb{R})} = |\lambda|_{L[\nu](\mathbb{R})}$ (because $Y \subseteq \lambda \times \mathbb{R}$). Since $\mathscr{H} \models R^+$ (see Lemma 1.17 of [1]), the transitive collapse of (H, ν^H) has the form $J_\delta[\nu](\mathbb{R})$. It follows that $Y \in J_\delta[\nu](\mathbb{R})$ and $\delta < \lambda^+$.

To prove (2), note that $\mathscr{P}(\lambda \times \mathbb{R}) \subseteq J_{\lambda^+}[v](\mathbb{R})$. Let $\alpha < \lambda^+$ be an ordinal. Since there exists a function $f : \alpha \times \mathbb{R} \xrightarrow{\text{onto}} J_\alpha[v](\mathbb{R})$, it follows that there is a function $g : \lambda \times \mathbb{R} \xrightarrow{\text{onto}} J_\alpha[v](\mathbb{R})$. Cantor's Theorem applies and shows that $\mathscr{P}(\lambda \times \mathbb{R}) \not\subseteq J_\alpha[v](\mathbb{R})$. Thus, an ordinal $\beta \geqslant \alpha$ witnessing (2) must exist. □

Claim 2. $(\forall \beta \geqslant \lambda)[\mathscr{P}(\lambda \times \mathbb{R}^M) \cap J_{\beta+1}[v](\mathbb{R}^M) \backslash J_\beta[v](\mathbb{R}^M) \neq \emptyset \Rightarrow \mathscr{P}(\lambda \times \mathbb{R}^M) \cap \Sigma_\omega(J_\beta[v](\mathbb{R}^M), v) \not\subseteq J_\beta[v](\mathbb{R}^M)]$.

Proof (*Sketch*). Let $\beta \geqslant \lambda$ be an ordinal. Let $\mathscr{N} = (J_\beta[v](\mathbb{R}^M), v) = (N, v)$ and $\mathscr{N}^+ = (J_{\beta+1}[v](\mathbb{R}^M), v) = (N^+, v)$. Assume that there exists a set $A \subseteq \lambda \times \mathbb{R}^M$ in $N^+ \backslash N$. Suppose, for a contradiction, that $\gamma^n_{\mathscr{N}} > \lambda$ for all $n \in \omega$ (see Definition 0.6). Let $\bar{v} = v \cap N$. One can now show that the structure $(J_{\beta+1}[\bar{v}](\mathbb{R}^M), v)$ is amenable in v (see the proof of Lemma 11.22 of [4]). Therefore, $J_{\beta+1}[\bar{v}](\mathbb{R}^M) = J_{\beta+1}[v](\mathbb{R}^M)$. Thus, $A \in \Sigma_\omega(\mathscr{N})$. Since $\gamma^n_{\mathscr{N}} > \lambda$ for all $n \in \omega$, it follows that $A \in N$. Contradiction. □

Remark. Let $\mathscr{N} = (J_\beta[v](\mathbb{R}^M), v) = (N, v)$ and $\mathscr{N}^+ = (J_{\beta+1}[v](\mathbb{R}^M), v) = (N^+, v)$ be as in the proof of Claim 2. If any set $A \subseteq \lambda \times \mathbb{R}^M$ is constructed in $N^+ \backslash N$, then Claim 2 states there must exist a set $B \subseteq \lambda \times \mathbb{R}^M$ in $\Sigma_\omega(\mathscr{N}) \backslash N$. Claim 2 does not assert that $A \in \Sigma_\omega(\mathscr{N})$.

Claim 3. $\forall \beta < \lambda^+ [J_\beta[v](\mathbb{R}^M) \subseteq K(\mathbb{R}^M)]$.

Proof. Suppose for a contradiction that $\exists \beta < \lambda^+ (J_\beta[v](\mathbb{R}^M) \not\subseteq K(\mathbb{R}^M))$. Hence, $\mathscr{P}(\lambda \times \mathbb{R}^M) \cap L[v](R^M) \not\subseteq K(\mathbb{R}^M)$. Let $\alpha < \lambda^+$ be the largest ordinal such that $J_\alpha[v](\mathbb{R}^M) \subseteq K(\mathbb{R}^M)$. Now let β be the smallest ordinal such that $\alpha \leqslant \beta < \lambda^+$ and $\mathscr{P}(\lambda \times \mathbb{R}^M) \cap J_{\beta+1}[v](\mathbb{R}^M) \backslash J_\beta[v](\mathbb{R}^M) \neq \emptyset$. Such a β exists, as noted in (2) of Claim 1. Let $\mathscr{N} = (J_\beta[v](\mathbb{R}^M), v) = (N, v)$ and let $\mathscr{N}^+ = (J_{\beta+1}[v](\mathbb{R}^M), v) = (N^+, v)$. Since $\mathscr{P}(\lambda \times \mathbb{R}^M) \cap J_\beta[v](\mathbb{R}^M) \subseteq K(\mathbb{R}^M)$, it follows that $\mathscr{N} \models$ "v is $\mathbb{R}$-complete" and that $\mathbb{R}^{\mathscr{N}} = \mathbb{R}^M$. DC and the countable completeness of v, implies that $\mathscr{N}$ is iterable. Therefore, $\mathscr{N}$ is acceptable above the reals (see Theorem 1.4). Since there exists an $A \subseteq \lambda \times \mathbb{R}^M$ in $N^+ \backslash N$, it follows $\mathscr{N}$ is critical, that is, $\rho^n_{\mathscr{N}} \leqslant \lambda$ for some $n \in \omega$. [To see this, suppose for a contradiction, that $\rho^n_{\mathscr{N}} > \lambda$ for all $n \in \omega$. Theorem 0.18 implies that $\gamma^n_{\mathscr{N}} = \rho^n_{\mathscr{N}} > \lambda$ for all $n \in \omega$. Claim 2 implies that $A \in N$. Contradiction. Therefore, $\mathscr{N}$ is critical.] Again, by DC and the countable completeness of v, the mouse iterates of $\mathscr{N}$ are well-founded. Thus, $\mathscr{N}$ is an $\mathbb{R}^M$ mouse. Theorem 2.49 of [2] implies that $\mathscr{N} \in K(\mathbb{R}^M)$. Since $(K(\mathbb{R}^M), v)$ is really good on λ (see Definition 2.9), it follows that $J_{\beta+1}[v](\mathbb{R}^M) \subseteq K(\mathbb{R}^M)$. But $\alpha < \beta + 1$, and this contradicts our choice of α. □

Remark. Dodd–Jensen, in their analysis of the Core Model K, originated the argument used in the proof of the above Claim 3 (see Lemma 16.11 of [4]).

Proof of Lemma 4.3 (*Conclusion*). Claim 3 implies that

$$\mathscr{P}(\lambda \times \mathbb{R}^M) \cap L[v](\mathbb{R}^M) \subseteq K(\mathbb{R}^M).$$

Hence,

$$L[\nu](\mathbb{R}^M) \models \text{“}\nu \text{ is an } \mathbb{R}\text{-complete measure on } \lambda\text{”}.$$

By DC and the countable completeness of ν, it follows that $L[\nu](\mathbb{R}^M)$ is iterable.

We note, stepping out of $M[G]$, that the construction of $L[\nu](\mathbb{R}^M)$ is absolute between M and $M[G]$. Therefore,

$$\begin{aligned} M \models (&L[\nu](\mathbb{R}) \text{ is a } \rho\text{-model with critical point } \lambda > \kappa \\ &\wedge L[\nu](\mathbb{R}) \text{ is iterable} \\ &\wedge \mathscr{P}(\lambda \times \mathbb{R}) \cap L[\nu](\mathbb{R}) \subseteq K(\mathbb{R})), \end{aligned}$$

that is, $M \models \psi(\kappa)$. This completes the proof of the lemma. □

Theorem 4.4. *Assume* ZF+AD+$\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then there is a ρ-model $L[\nu](\mathbb{R})$ of* ZF + DC + AD *with critical point* $\lambda > \Theta^{L[\nu](\mathbb{R})}$.

Proof. Assume ZF + AD + $\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. Lemma 4.1 implies that there exists an ordinal κ and a $K(\mathbb{R})$-measure $\hat{\mu}$ on κ such that $(K(\mathbb{R}), \hat{\mu})$ is weakly iterable and $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$. Lemma 4.3 now implies that there is ρ-model $L[\nu](\mathbb{R})$ with critical point $\lambda > \kappa$ such that $\mathscr{P}(\mathbb{R}) \cap L[\nu](\mathbb{R}) \subseteq K(\mathbb{R})$. Hence, $\Theta^{L[\nu](\mathbb{R})} = \Theta^{K(\mathbb{R})} < \lambda$. Since $K(\mathbb{R}) \models \mathrm{DC}_{\mathbb{R}}$ (by Theorem 5.14 of [1]), it follows that $L[\nu](\mathbb{R}) \models \mathrm{DC}_{\mathbb{R}}$. Hence, $L[\nu](\mathbb{R}) \models \mathrm{DC}$ (see the proof of Theorem 5.5(iv) and the subsequent remark in [1]). Consequently, $L[\mu](\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. □

Remark. In the proof of Theorem 4.4, one obtains a measurable cardinal κ, where $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$, and first shows that a weakly iterable $K(\mathbb{R})$-measure exists on κ. Then one constructs a ρ-model with critical point $\lambda > \kappa$. The method used to obtain this ρ-model does not produce an immediate bound on λ; for example, it may be the case that $\lambda \geqslant \Theta$. A little refining of the proof, yields a ρ-model with critical point λ where $\kappa < \lambda < \Theta$. However, since κ is a measurable cardinal greater than $(\Theta^+)^{K(\mathbb{R})}$, one suspects that there should be a ρ-model with critical point κ. Corollary 4.17 (below) asserts that this is, in fact, the case.

Corollary 4.5. *Assume* ZF + AD + $\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then there is an inner model of* ZF + AD + DC + $\exists \kappa > \Theta$ [κ *is measurable*].

Corollary 4.6. Con(ZF + AD + $\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$) $\Rightarrow$ Con(ZF + AD + DC + $(\boldsymbol{\Pi}^1_1 - \omega^2) - \mathrm{AD}_{\mathbb{R}}$).

Proof. Assume ZF+AD+$\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. By Theorem 4.4, there exists a measure ν such that

$$L[\nu](\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC} + \text{“}\nu \text{ is an } \mathbb{R}\text{-complete measure on } \kappa\text{”}.$$

The techniques of Martin [12] generalize and prove that

$$L[\nu](\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC} + (\boldsymbol{\Pi}^1_1 - \omega^2) - \mathrm{AD}_{\mathbb{R}}. \qquad \square$$

Assuming AD, Theorem 4.4 implies that if there is a set of reals not in $K(\mathbb{R})$, then there exists an iterable ρ-model $L[\nu](\mathbb{R})$. Our next theorem shows that the converse does not hold.

Theorem 4.7. *Suppose that $L[\mu](\mathbb{R})$ is an iterable ρ-model with critical point κ. Then*
(1) $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$,
(2) $\mathscr{P}(\kappa \times \mathbb{R}) \cap L[\mu](\mathbb{R}) = \mathscr{P}(\kappa \times \mathbb{R}) \cap K(\mathbb{R})$.

Proof (*Sketch*). We first prove (1), that is, we prove that $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$. Let $\mathscr{M} \in K(\mathbb{R})$ be a real 1-mouse. We shall show that $\mathscr{M} \in L[\mu](\mathbb{R})$. Theorem 5.5(iii) of [1] will then imply that $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$. Let

$$\langle \langle \mathscr{M}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathscr{M}_\alpha \underset{\Sigma_1}{\longrightarrow} \mathscr{M}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\mathscr{M}$. Since $(L[\mu](\mathbb{R}), \mu)$ is iterable, let

$$\langle \langle L[\mu](\mathbb{R})_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle j^\mu_{\alpha\beta} : L[\mu](\mathbb{R})_\alpha \underset{\Sigma_\omega}{\longrightarrow} L[\mu](\mathbb{R})_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle \qquad (\maltese)$$

be the iterated ultrapower of $(L[\mu](\mathbb{R}), \mu)$. Note that $L[\mu](\mathbb{R})_\alpha = L[\mu_\alpha](\mathbb{R})$ where μ_α is the αth iterate of μ. Let θ be an ordinal such that $\mathscr{M}_\theta$ and $L[\mu_\theta](\mathbb{R})$ are comparable (the proof of Lemma 2.25 in [1] can be modified to prove, in ZF, that such a ordinal θ exists). It follows that $\mathscr{M}_\theta$ must be an initial segment of $L[\mu_\theta](\mathbb{R})$. Hence, $\mathscr{M}_\theta \in L[\mu_\theta](\mathbb{R})$. Since $\mathscr{M}$ is a real 1-mouse, it now follows that $\mathscr{M} \in L[\mu_\theta](\mathbb{R})$. But $L[\mu_\theta](\mathbb{R})$ is an inner model of $L[\mu](\mathbb{R})$. Thus, $\mathscr{M} \in L[\mu](\mathbb{R})$.

We now prove (2), that is, we prove that $\mathscr{P}(\kappa \times \mathbb{R}) \cap L[\mu](\mathbb{R}) = \mathscr{P}(\kappa \times \mathbb{R}) \cap K(\mathbb{R})$. Let $\kappa^+ = \kappa^+_{L[\mu](\mathbb{R}^M)}$ be as computed in $L[\mu](\mathbb{R}^M)$ (see Definition 1.1). The arguments used to prove Claims 1–3 of Lemma 4.3 can be used to prove (2). In particular, the argument proving Claim 1 shows that $\mathscr{P}(\kappa \times \mathbb{R}) \cap L[\mu](\mathbb{R}) \subseteq J_{\kappa^+}[\mu](\mathbb{R})$ and that $(\forall \alpha < \kappa^+)(\exists \beta \geqslant \alpha)(\beta < \kappa^+ \wedge \mathscr{P}(\kappa \times \mathbb{R}) \cap J_{\beta+1}[\mu](\mathbb{R}) \backslash J_\beta[\mu](\mathbb{R}) \neq \emptyset)$. Now, the proof of Claim 3 (of Lemma 4.3) can be adapted to prove our next result.

Claim. $\forall \beta < \kappa^+ \, [J_\beta[\nu](\mathbb{R}) \subseteq K(\mathbb{R})]$.

Proof. Suppose for a contradiction that $\exists \beta < \kappa^+ (J_\beta[\mu](\mathbb{R}) \not\subseteq K(\mathbb{R}))$. Hence, $\mathscr{P}(\kappa \times \mathbb{R}) \cap L[\mu](R) \not\subseteq K(\mathbb{R})$. Let $\alpha < \kappa^+$ be the largest ordinal such that $J_\alpha[\mu](\mathbb{R}) \subseteq K(\mathbb{R})$. Now let β be the smallest ordinal such that $\alpha \leqslant \beta < \kappa^+$ and $\mathscr{P}(\kappa \times \mathbb{R}) \cap J_{\beta+1}[\mu](\mathbb{R}) \backslash J_\beta[\mu](\mathbb{R}) \neq \emptyset$. Let $\mathscr{M} = (J_\beta[\mu](\mathbb{R}), \mu) = (M, \mu)$ and let $\mathscr{M}^+ = (J_{\beta+1}[\mu](\mathbb{R}), \mu) = (M^+, \mu)$. Since $L[\mu](\mathbb{R})$ is a ρ-model, it follows that $\mathscr{M} \models$ "μ is $\mathbb{R}$-complete" and that $\mathbb{R}^{\mathscr{M}} = \mathbb{R}$. Since $L[\mu](\mathbb{R})$ is iterable, it follows that $\mathscr{M}$ is premouse iterable; because, the premouse iterates of $\mathscr{M}$ can be embedded into the iterates of $L[\mu](\mathbb{R})$. Therefore, $\mathscr{M}$ is acceptable above the reals (see Theorem 1.4). Since there exists an $A \subseteq \kappa \times \mathbb{R}$ in $M^+ \backslash M$, it follows that $\mathcal{M}$ is critical, that is, $\rho^n_{\mathcal{M}} \leqslant \kappa$ for some $n \in \omega$ (see the proof of Claim 3 of Lemma 4.3). We now show that the mouse iterates of $\mathcal{M}$ can also be embedded into the iterates of $L[\mu](\mathbb{R})$ and thus, $\mathcal{M}$ is a real mouse. Let $\bar{\mathcal{M}}$ be as defined in Definition 0.16. Let

$$\langle \langle \bar{\mathcal{M}}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \bar{\pi}_{\alpha\beta} : \bar{\mathcal{M}}_\alpha \xrightarrow[\Sigma_1]{\text{cofinal}} \bar{\mathcal{M}}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\bar{\mathcal{M}}$. Note that $\bar{\mathcal{M}} \in L[\mu](\mathbb{R})$ and that

$$L[\mu](\mathbb{R}) \models \text{“}\bar{\mathcal{M}} \text{ is a well-founded structure”}.$$

Thus,

$$L[\mu](\mathbb{R})_\alpha \models \text{“}j^\mu_{0\alpha}(\bar{\mathcal{M}}) \text{ is a well-founded structure”}$$

(see (✠) above). Now, since each $\bar{\mathcal{M}}_\alpha$ can be embedded into $j^\mu_{0\alpha}(\bar{\mathcal{M}})$, it follows that $\bar{\mathcal{M}}_\alpha$ is well-founded for each ordinal α. In addition, let $E^{\bar{\mathcal{M}}}$ be as in Definition 2.15 of [2] (recall that the relation $E^{\bar{\mathcal{M}}}$ codes the “$\in$-order” on the ordinals in $\mathcal{M}$). Since $E^{\bar{\mathcal{M}}} \in L[\mu](\mathbb{R})$ and

$$L[\mu](\mathbb{R}) \models \text{“}E^{\bar{\mathcal{M}}} \text{ is a well-founded”},$$

it follows that

$$L[\mu](\mathbb{R})_\alpha \models \text{“}j^\mu_{0\alpha}(E^{\bar{\mathcal{M}}}) \text{ is a well-founded”}.$$

Since each $E^{\bar{\mathcal{M}}_\alpha}$ can be embedded into $j^\mu_{0\alpha}(E^{\bar{\mathcal{M}}})$, it follows that $E^{\bar{\mathcal{M}}_\alpha}$ is well-founded for each ordinal α. Therefore, $\mathcal{M}$ is a real mouse. Theorem 2.49 of [2] implies that $\mathcal{M} \in K(\mathbb{R})$. Because $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$, it also follows that the ultrapower ${}^\kappa K(\mathbb{R})/\mu$ is well-founded. Thus, $(K(\mathbb{R}), \mu)$ is really good on κ by Corollary 2.14. Therefore, $J_{\beta+1}[\mu](\mathbb{R}) \subseteq K(\mathbb{R})$. But $\alpha < \beta + 1$, and this contradicts our choice of α. □

Proof of Theorem 4.7 (*Conclusion*). Since $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$, the above Claim implies (2). □

Theorem 1.17 gives one computation of $\mathrm{HOD}^{K(\mathbb{R})}$. Our next theorem presents another computation of $\mathrm{HOD}^{K(\mathbb{R})}$ under the assumption of $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. In this case we show that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ where $K(P)$ is the Core Model relative to P, a set of ordinals. Before we prove this, we shall give an overview of how $K(P)$ is defined together with some of its properties.

Until further notice we shall let $P \subseteq \eta$ for some fixed ordinal η. We shall write $J_\alpha[\nu][P]$ to denote the αth level of constructibility relative to both ν and P. We shall write $\mathcal{N}$ to denote a structure of the form $\mathcal{N} = (J_\alpha[\nu][P], \kappa, P, \nu) = (N, \kappa, P, \nu)$, where $\eta < \kappa < \alpha$ are ordinals.

Definition 4.8. The *projectum* $\rho_{\mathcal{N}}$ is the least ordinal $\rho \leqslant \alpha$ such that $\mathscr{P}(\omega\rho) \cap \Sigma_1(\mathcal{N}) \not\subseteq N$, and $p_{\mathcal{N}}$ is the $\leqslant_{BK}$-least $p \in [\omega\alpha]^{<\omega}$ such that $\mathscr{P}(\omega\rho_{\mathcal{N}}) \cap \Sigma_1(\mathcal{N}, \{p\}) \not\subseteq N$.

Definition 4.9. We say that $\mathcal{N} = (J_\alpha[\nu][P], \kappa, P, \nu)$ is a P 1-*mouse* if
(1) $\mathcal{N}$ is a premouse, that is, $\mathcal{N} \models$ "ν is a normal measure on κ",
(2) $\mathcal{N}$ is premouse iterable,
(3) $\rho_{\mathcal{N}} \leqslant \kappa$.

Definition 4.10. The P Core Model is the class

$$K(P) = \{x\colon \exists \mathcal{N}(\mathcal{N} \text{ is a } P \text{ 1-mouse} \wedge x \in N)\}.$$

The arguments given by Dodd–Jensen about the Core Model K can be used to give proofs of the next three lemmas.

Lemma 4.11. *There exists a P 1-mouse if and only if $P^\#$ exists.*

Therefore, in the case where there are no P 1-mice, one assumes the convention that $K(P) = L(P)$.

Lemma 4.12. $K(P) \models \mathrm{ZFC}$.

Recall that $P \subseteq \eta$ where η is a fixed ordinal.

Definition 4.13. An inner model $L[\mu](P)$ of ZFC is called a P-model with critical point κ, if $L[\mu](P) \models$ "μ is a normal measure on $\kappa > \eta$".

Lemma 4.14. *If $L[\mu](P)$ is a P-model with critical point κ, then $(L[\mu](P), \mu)$ is iterable, $K(P) \subseteq L[\mu](P)$ and $\mathscr{P}(\kappa) \cap L[\nu](P) \subseteq K(P)$.*

We now present another computation of $\mathrm{HOD}^{K(\mathbb{R})}$.

Theorem 4.15. *Assume* $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then there exists a $P \subseteq (\Theta^+)^{K(\mathbb{R})}$ such that* $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$.

Proof. Lemmas 4.1 and 4.3 imply that there is an iterable ρ-model $L[\mu](\mathbb{R})$ with critical point $\kappa > (\Theta^+)^{K(\mathbb{R})}$. Theorem 4.7 implies that $K(\mathbb{R}) \subseteq L[\mu](\mathbb{R})$. By Theorem 5.14 of [1], it follows that $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. Let A be $\mathbb{Q}$-generic over $K(\mathbb{R})$, where $\mathbb{Q}$ is the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1. Again, we note that A is essentially a subset of ω_1. Let $\bar{V} = K(\mathbb{R})[A]$. Since $K(\mathbb{R}) \models \mathrm{DC}$, Theorem 1.11 implies that
(1) $\mathbb{R}^{K(\mathbb{R})} = \mathbb{R}^{\bar{V}}$,
(2) ${\omega_1}^{K(\mathbb{R})} = \omega_1^{\bar{V}}$,
(3) $K(\mathbb{R}) = K(\mathbb{R})^{\bar{V}}$,
(4) $\Theta^{K(\mathbb{R})} = \omega_2^{\bar{V}}$,

(5) $(\Theta^+)^{K(\mathbb{R})} = \omega_3^{\bar{V}}$,
(6) $\bar{V} \models \mathrm{ZFC} + \mathrm{GCH}$,
(7) $\mathrm{HOD}^{K(\mathbb{R})} = \mathrm{HOD}^{\bar{V}}$.

By Lemma 1.16, $K(\mathbb{R}) = L[D](\mathbb{R})$ where D is as in Definition 1.15. Let $\mathscr{H} = \mathrm{HOD}^{\bar{V}}$. Note that $\bar{V} = K(\mathbb{R})[A] = L[D](\mathbb{R})[A] = L[D](A)$, $A \subseteq \omega_1^{\bar{V}}$, and $L[D] \subseteq \mathscr{H}$. Thus, by Lemma 1.14, $\bar{V}$ is a generic extension of $\mathscr{H}$; that is, there is a Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ where $B \leqslant \omega_3{}^{\bar{V}}$ is an ordinal, and there is a G which is $\mathscr{B}$-generic over $\mathscr{H}$ such that

(i) $\bar{V} = \mathscr{H}[G]$,
(ii) $G \cap \mathscr{B}_*^{\mathscr{H}} = \widehat{B} \in \mathscr{H}$.

In addition, letting $b : \omega_1 \to B$ be as in the proof of Lemma 1.14,
(iii) $\mathscr{H} = L[D](\{\mathscr{B}, b, \widehat{B}\}) = L[D](P)$
where $P \subseteq (\Theta^+)^{K(\mathbb{R})} = \omega_3{}^{\bar{V}}$ is a canonical coding of $\{\mathscr{B}, b, \widehat{B}\}$.

Claim 1. $K(P) \subseteq K(\mathbb{R})$.

Proof. Let $\mathscr{N}$ be a P 1-mouse in $K(P)$. We show that $\mathscr{N} \in K(\mathbb{R})$. Let

$$\langle \langle \mathscr{N}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathscr{N}_\alpha \xrightarrow[\Sigma_1]{} \mathscr{N}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\mathscr{N}$. Also, since $(L[\mu](\mathbb{R}), \mu)$ is iterable, let

$$\langle \langle L[\mu](\mathbb{R})_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle j^{\mu}_{\alpha\beta} : L[\mu](\mathbb{R})_\alpha \xrightarrow[\Sigma_\omega]{} L[\mu](\mathbb{R})_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the iterated ultrapower of $(L[\mu](\mathbb{R}), \mu)$. Note that $L[\mu](\mathbb{R})_\alpha = L[\mu_\alpha](\mathbb{R})$ where μ_α is the αth iterate of μ. Let λ be an ordinal such that $\kappa^{\mathscr{N}_\lambda} = \kappa^{L[\mu_\lambda](\mathbb{R})} = \lambda$. For each $\alpha < \lambda$, let $\kappa_\alpha = \pi_{0\alpha}(\kappa^{\mathscr{N}})$ and let $C = \{\kappa_\alpha : \alpha < \lambda\}$. We note that $\mathscr{N}_\lambda$ is completely determined from P and C. Since $K(P) \subseteq L[\mu_\lambda](\mathbb{R})$, it follows that $\mathscr{N} \in L[\mu_\lambda](\mathbb{R})$ and therefore, $C \in L[\mu_\lambda](\mathbb{R})$. Because $\mathscr{P}(\lambda \times \mathbb{R}) \cap L[\mu_\lambda](\mathbb{R}) \subseteq K(\mathbb{R})$, we have that $P, C \in K(\mathbb{R})$. Thus, $\mathscr{N}_\lambda \in K(\mathbb{R})$ and so, $\mathscr{N} \in K(\mathbb{R})$. $\square$

Claim 2. $K(P) \subseteq \mathrm{HOD}^{K(\mathbb{R})}$.

Proof. Let $\mathscr{N}$ be a P 1-mouse in $K(P)$. By Claim 1, $\mathscr{N} \in K(\mathbb{R})$. We show that $\mathscr{N} \in \mathrm{HOD}^{K(\mathbb{R})}$. Let $\mathscr{C}$ be the "P-core" of $\mathscr{N}$ and let

$$\langle \langle \mathscr{C}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} : \mathscr{C}_\alpha \xrightarrow[\Sigma_1]{} \mathscr{C}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\mathscr{C}$. Let $\kappa_\alpha = \pi_{0\alpha}(\kappa^{\mathscr{C}})$ for each ordinal α, and define $d = \langle (\kappa_n, \kappa_\omega) : n \in \omega \rangle$. It follows that $\mathscr{C}_\omega$ can be completely constructed (defined) from d and P. In addition, d is definable from P and κ_ω (see Lemma 5.3 of [1]). Since $P \in \mathrm{HOD}^{K(\mathbb{R})}$, it follows that $d \in \mathrm{HOD}^{K(\mathbb{R})}$. Therefore, $\mathscr{C}_\omega \in \mathrm{HOD}^{K(\mathbb{R})}$ and so, $\mathscr{C} \in \mathrm{HOD}^{K(\mathbb{R})}$. Because $\mathscr{N}$ is an iterate of its P-core $\mathscr{C}$, we conclude that $\mathscr{N} \in \mathrm{HOD}^{K(\mathbb{R})}$. $\square$

Recall that G is $\mathscr{B}$-generic over $\mathscr{H} = \mathrm{HOD}^{K(\mathbb{R})}$ and $\bar{V} = K(\mathbb{R})[A] = \mathscr{H}[G]$. Claim 2 implies that G is also $\mathscr{B}$-generic over $K(P)$. Our final claim will be used to conclude that $K(P) = \mathrm{HOD}^{K(\mathbb{R})}$.

Claim 3. $\bar{V} = K(P)[G]$.

Proof. We know that $\bar{V} = K(\mathbb{R})[A]$ and we want to prove that $\bar{V} = K(P)[G]$. Because $b : \omega_1 \to B$ is encoded by P, it follows that A is definable from P and G (see the proof of the Claim presented as part of the proof of Lemma 1.14). Hence, $A \in K(P)[G]$. To show that $\bar{V} = K(P)[G]$, it is sufficient to prove that $K(\mathbb{R}) \subseteq K(P)[G]$. Let $\mathscr{M} \in K(\mathbb{R})$ be a real 1-mouse. We show that $\mathscr{M} \in K(P)[G]$. Let

$$\langle \langle \mathscr{M}_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle \pi_{\alpha\beta} \colon \mathscr{M}_\alpha \xrightarrow[\Sigma_1]{} \mathscr{M}_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the premouse iteration of $\mathscr{M}$. Also let $\kappa_\alpha = \pi_{0\alpha}(\kappa^{\mathscr{M}})$ for each ordinal α. We prove that $\mathscr{M}_\lambda \in K(P)[G]$ for some ordinal λ and thus, $\mathscr{M} \in K(P)[G]$.

$L[\mu](P)$ is a P-model with critical point $\kappa > (\Theta^+)^{K(\mathbb{R})}$. Let

$$\langle \langle L[\mu](P)_\alpha \rangle_{\alpha \in \mathrm{OR}}, \langle j^\mu_{\alpha\beta} \colon L[\mu](P)_\alpha \xrightarrow[\Sigma_\omega]{} L[\mu](P)_\beta \rangle_{\alpha \leqslant \beta \in \mathrm{OR}} \rangle$$

be the iterated ultrapower of $(L[\mu](P), \mu)$. Note that $L[\mu](P)_\alpha = L[\mu_\alpha](P)$ where μ_α is the αth iterate of μ. Since $\mathscr{B} = (B, \leqslant)$, $B \subseteq \kappa$ and $\mathscr{P}(B)^{L[\mu_\alpha](P)} = \mathscr{P}(B)^{K(P)}$ for all ordinals α, it follows that G is $\mathscr{B}$-generic over $L[\mu_\alpha](P)$ for all ordinals α. Thus, $K(\mathbb{R}) \subseteq L[\mu_\alpha](\mathbb{R}) \subseteq L[\mu_\alpha](P)[G]$ for each ordinal α. In particular, $\mathscr{M} \in L[\mu](P)[G]$. In $L[\mu](P)[G]$ let $\lambda > \max(\kappa, |\mathscr{M}|)$ be a regular cardinal, where $|\mathscr{M}|$ is the standard cardinality of $\mathscr{M}$. Because $\mathscr{M}_\lambda \in K(\mathbb{R})$, we have that $\mathscr{M}_\lambda \in L[\mu_\lambda](P)[G]$. Let $C = \{\kappa_\alpha \colon \alpha < \lambda\}$. Thus, $C \in L[\mu_\lambda](P)[G]$ and

$$L[\mu_\lambda](P)[G] \models \text{“}C \text{ is closed and unbounded in } \lambda\text{”}.$$

Recall that, for all $X \in \mathscr{P}(\kappa_\lambda) \cap \mathscr{M}_\lambda$

$$X \in \mu^{\mathscr{M}_\lambda} \quad \text{if and only if } \exists \alpha < \lambda\, (X \supseteq C \setminus \alpha).$$

Hence, given C and $\mathbb{R}$, one can construct $\mathscr{M}_\lambda$. In fact, given a closed unbounded C', where $C' \subseteq C$, one can build $\mathscr{M}_\lambda$ from C' and $\mathbb{R}$ (see the proof of Lemma 1.9).

By a standard forcing argument (see Lemma 10.14 of [8]), there is a $C' \subseteq C$ such that $C' \in L[\mu_\lambda](P)$ and

$$L[\mu_\lambda](P) \models \text{“}C' \text{ is closed and unbounded in } \lambda\text{”}.$$

Since $\mathscr{P}(\lambda)^{L[\mu_\lambda](P)} = \mathscr{P}(\lambda)^{K(P)}$ and $\lambda = \kappa_\lambda$, we have that $C' \in K(P)$. Because $C', \mathbb{R} \in K(P)[G]$, it follows that $\mathscr{M}_\lambda \in K(P)[G]$. Thus, $\mathscr{M} \in K(P)[G]$. $\square$

Proof of Theorem 4.5 (*Conclusion*). Since $\bar{V} = \mathscr{H}[G]$ and $\mathscr{B}, \hat{B} \in K(P) \subseteq \mathscr{H} = \mathrm{HOD}^{\bar{V}}$, Lemma 1.13 and Claim 3 imply that $K(P) = \mathscr{H} = \mathrm{HOD}^{K(\mathbb{R})}$. $\square$

Remark. Given that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ for some $P \subseteq (\Theta^+)^{K(\mathbb{R})}$, one cannot find such a $P \subseteq \gamma < \Theta^{K(\mathbb{R})}$. Otherwise, $P^\dagger$ (dagger) is in $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ (assuming $\mathrm{AD}^{K(\mathbb{R})}$). However, as noted in the introduction, Woodin has proven that $\mathrm{HOD}^{L(\mathbb{R})} = L(W)$ for some $W \subseteq \Theta^{L(\mathbb{R})}$. The ideas used in Woodin's proof are (probably) flexible enough to allow one to show that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$ for some $P \subseteq \Theta^{K(\mathbb{R})}$. Note that $\Theta^{L(\mathbb{R})} < \Theta^{K(\mathbb{R})}$ (when $\mathbb{R}^\#$ exists) and so, one cannot take $P = W$.

Theorem 4.16. *Assume* ZF + AD + $\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then there is a ρ-model $L[\mu](\mathbb{R})$ with critical point κ such that $(\Theta^+)^{L[\mu](\mathbb{R})} < \kappa < \Theta$ and $L[\mu](\mathbb{R}) \models$* ZF + AD + DC.

Proof. By Theorem 4.15, there exists a $P \subseteq (\Theta^+)^{K(\mathbb{R})}$ such that $\mathrm{HOD}^{K(\mathbb{R})} = K(P)$. We now review how P was obtained in the proof of Theorem 4.15. We have that $K(\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ and A is $\mathbb{Q}$-generic over $K(\mathbb{R})$, where $\mathbb{Q} \in K(\mathbb{R})$ is the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1. Recall that $P \in \mathrm{HOD}^{K(\mathbb{R})}$ is a subset of $(\Theta^+)^{K(\mathbb{R})}$ and gives a canonical coding of a Boolean Algebra $\mathscr{B} = (B, \leqslant) \in \mathrm{HOD}^{K(\mathbb{R})}$. In addition, there exists a G which is $\mathscr{B}$-generic over $\mathrm{HOD}^{K(\mathbb{R})}$ and $K(\mathbb{R})[A] = \mathrm{HOD}^{K(\mathbb{R})}[G]$. Also, recall that $B \subseteq (\Theta^+)^{K(\mathbb{R})}$.

Let κ be a measurable cardinal, where $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$, and let ν be a measure on κ (see the proof of Lemma 4.1). Consider the inner model $L[\nu](P)$. We now assume that $\nu = L[\nu](P) \cap \nu$. Since $L[\nu](P) \models \mathrm{ZFC}$, we can assume without loss of generality that

$$L[\nu](P) \models \text{“}\nu \text{ is a normal measure on } \kappa\text{”}.$$

Hence, $\mathscr{P}(B) \cap L[\nu](P) \subseteq K(P)$ and, since G is $\mathscr{B}$-generic over $K(P) = \mathrm{HOD}^{K(\mathbb{R})}$, it follows that G is also $\mathscr{B}$-generic over $L[\nu](P)$. Let

$$\nu[G] = \{X \subseteq \kappa \colon X \in L[\nu](P)[G] \wedge \exists Y \in L[\nu](P)\,(Y \in \nu \wedge Y \subseteq X)\}$$

be the *generic expansion* of ν. The proof of Lemma 3.1, implies that

$$L[\nu](P)[G] \models \text{“}\nu[G] \text{ is a normal measure on } \kappa\text{”}.$$

Let $\mu = \nu[G] \cap L[\,\nu[G]\,](\mathbb{R})$. One can check that $\mu \in V$. Note that

$$L[\nu](P)[G] \models \text{“}L[\mu](\mathbb{R}) \text{ is an iterable } \rho\text{-model with critical point } \kappa\text{”}.$$

By absoluteness then, $L[\mu](\mathbb{R})$ is an iterable ρ-model with critical point κ. Theorem 4.7 implies that

$$\mathscr{P}(\kappa \times \mathbb{R}) \cap L[\mu](\mathbb{R}) = \mathscr{P}(\kappa \times \mathbb{R}) \cap K(\mathbb{R}).$$

Therefore, $(\Theta^+)^{L[\mu](\mathbb{R})} = (\Theta^+)^{K(\mathbb{R})}$ and so, $(\Theta^+)^{L[\mu](\mathbb{R})} < \kappa < \Theta$. Since $K(\mathbb{R}) \models \mathrm{DC}_{\mathbb{R}}$, it follows that $L[\mu](\mathbb{R}) \models \mathrm{DC}_{\mathbb{R}}$. Hence, $L[\mu](\mathbb{R}) \models \mathrm{DC}$. Consequently, $L[\mu](\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. $\square$

Assuming AD, there are unboundedly many measurable cardinals $\kappa < \Theta$. It is easy to check that their measures are not $\mathbb{R}$-complete in V. However, our next result shows that most of these measures are $\mathbb{R}$-complete in certain inner models of V.

Corollary 4.17 (ZF + AD). *Assume* $(\Theta^+)^{K(\mathbb{R})} < \kappa < \Theta$ *where* κ *is a measurable cardinal. Then there is a* ρ*-model* $L[\mu](\mathbb{R})$ *of* ZF + AD + DC *with critical point* κ.

Proof. This follows as in the proof of Theorem 4.16. □

Theorem 4.18. *Assume* $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. *Then* $\mathbb{R}^\dagger$ *(dagger) exists.*

Proof. Assume $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$. By Theorem 4.16, there exists a ρ-model $L[\mu](\mathbb{R})$ with critical point κ such that $(\Theta^+)^{L[\mu](\mathbb{R})} < \kappa < \Theta$ and $L[\mu](\mathbb{R}) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. As in the proof of Lemma 4.1, it follows from work of Kechris [10] that there exists a measurable cardinal λ with measure ν such that $(\Theta^+)^{L[\mu](\mathbb{R})} < \kappa < \lambda < \Theta$.

Let A be $\mathbb{Q}$-generic over $L[\mu](\mathbb{R})$, where $\mathbb{Q}$ is the standard partial order that produces (under DC) a generic enumeration of all the reals in length ω_1. Again, we note that A is essentially a subset of ω_1. Let $\bar{V} = L[\mu](\mathbb{R})[A]$. Since $L[\mu](\mathbb{R}) \models \mathrm{DC}$, the proof of Theorem 1.11 applies and shows that

(1) $\mathbb{R}^{L[\mu](\mathbb{R})} = \mathbb{R}^{\bar{V}}$,
(2) ${\omega_1}^{L[\mu](\mathbb{R})} = \omega_1^{\bar{V}}$,
(3) $L[\mu](\mathbb{R}) = L[\mu](\mathbb{R})^{\bar{V}}$,
(4) $\Theta^{L[\mu](\mathbb{R})} = \omega_2^{\bar{V}}$,
(5) $(\Theta^+)^{L[\mu](\mathbb{R})} = \omega_3^{\bar{V}}$,
(6) $\bar{V} \models \mathrm{ZFC} + \mathrm{GCH}$,
(7) $\mathrm{HOD}^{L[\mu](\mathbb{R})} = \mathrm{HOD}^{\bar{V}}$.

Let $\mathscr{H} = \mathrm{HOD}^{\bar{V}}$. Note that $\bar{V} = L[\mu](\mathbb{R})[A] = L[\mu][A]$, $A \subset \omega_1^{\bar{V}}$ and that $L[\mu] \subseteq \mathrm{HOD}^{\bar{V}}$. Thus, by Lemma 1.14, $\bar{V}$ is a generic extension of $\mathscr{H}$ and one can compute $\mathrm{HOD}^{L[\mu](\mathbb{R})}$; that is, there is a Boolean Algebra $\mathscr{B} = (B, \leqslant_{\mathscr{B}})$ where $B \leqslant {\omega_3}^{\bar{V}}$ is an ordinal, and there is a G which is $\mathscr{B}$-generic over $\mathscr{H}$ such that

(i) $\bar{V} = \mathscr{H}[G]$,
(ii) $\mathscr{H} = L[\mu](P)$ for some $P \subseteq (\Theta^+)^{L[\mu](\mathbb{R})} = \omega_3^{\bar{V}}$.

Let $N = L[\mu, \nu](P)$. Note that $N \models \mathrm{ZFC}$ and that $\mathscr{H} = L[\mu](P)$ is an inner model of N. Since $\nu \in N$ and $N \models$ "ν is a measure on λ", one can assume that ν is a normal measure on λ in N and, by absoluteness, one can show that N is iterable by ν. Since $L[\mu](P)$ is λ-maximal (see Definition 2.11), it follows that $(\mathscr{H}, \nu)$ is good on λ by an argument analogous to the proof of Lemma 2.13 (using Lemma 2.5). Therefore, one can form the weak iteration of $(\mathscr{H}, \nu)$. Each weak iterate of $(\mathscr{H}, \nu)$ can be embedded into an iterate of (N, ν). Thus, $(\mathscr{H}, \nu)$ is weakly iterable. Let

$$\nu[G] = \{X \subseteq \lambda \colon X \in L[\mu](P)[G] \wedge \exists Y \in L[\mu](P)\,(Y \in \nu \wedge Y \subseteq X)\}$$

be the *generic expansion* of ν. Lemmas 3.1 and 3.3 imply that

$$(L[\mu](P)[G], \nu[G]) \text{ is weakly iterable.} \qquad (\star)$$

Let $\hat{\nu} = \nu[G] \cap L[\mu](\mathbb{R})$. Note that $\nu \cap \mathscr{H} \subseteq \hat{\nu} \subseteq \nu[G]$ and that $\hat{\nu} \in V$. Since ($\bigstar$) implies that $(L[\mu](\mathbb{R}), \hat{\nu})$ is weakly iterable, it follows that $\mathbb{R}^\dagger$ exists (see Lemma 13.19 of [4]). □

Corollary 4.19 (ZF + AD). *Assume $\Theta^{K(\mathbb{R})} < \Theta$. Then $\mathbb{R}^\dagger$ (dagger) exists.*

Proof. If $\Theta^{K(\mathbb{R})} < \Theta$, then $\exists X \subseteq \mathbb{R}[X \notin K(\mathbb{R})]$. Theorem 4.18 implies that $\mathbb{R}^\dagger$ exists. □

Corollary 4.20. *Suppose that $L[\mu](\mathbb{R})$ is a ρ-model of* ZF + DC. *Then there exists a $P \subseteq (\Theta^+)^{L[\mu](\mathbb{R})}$ such that $\mathrm{HOD}^{L[\mu](\mathbb{R})} = L[\mu](P)$.*

Proof. This follows as in the proof of Theorem 4.18. □

The theory $\mathrm{ZF} + \mathrm{AD} + \exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$ implies many of the results in this paper. Thus, our next theorem shows that all of these results are also implied by the theory $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$.

Theorem 4.21. *Assume* $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$. *Then $\exists X \subseteq \mathbb{R}\,[X \notin K(\mathbb{R})]$.*

Proof. Theorem 5.14 of [1] shows that $\mathrm{ZF} + \mathrm{AD} \Rightarrow K(\mathbb{R}) \models \mathrm{DC}$. So the assumption $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$ implies that there is a set of reals X not in $K(\mathbb{R})$. □

In particular, Theorems 4.4, 4.18 and 4.21 imply our final two results.

Theorem 4.22. *Assume* $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$. *Then there is an inner model of* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \exists\kappa > \Theta$ [*κ is measurable*].

Theorem 4.23. *Assume* $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$. *Then $\mathbb{R}^\dagger$ exists.*

Just as $\mathbb{R}^\#$ gives rise to iterable real premice (see Lemma 5.2 of [1]), $\mathbb{R}^\dagger$ gives rise to iterable "double" real premice, that is, a real premouse with two measures. The theory of double real mice will not be developed here, but the theory is similar to that of single real mice and can be used to show that $\mathbb{R}^\dagger$ has a quasi-scale, assuming that $L[\mu](\mathbb{R}) \models \mathrm{AD}$ (compare this with the remark following Corollary 4.9 of [1]). Assuming $\mathrm{ZF} + \mathrm{AD} + \neg\mathrm{DC}_{\mathbb{R}}$, it follows that $L(\mathbb{R}^\dagger) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. Thus, $L(\mathbb{R}^\dagger)$ is another *fine-structural* inner model of determinacy. Hence, the "bootstrapping process", identified in the introduction, continues.

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