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Biordered Sets Come from Semigroups

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1. INTRODUCTION

Information about a semigroup can often be gleaned from its partial algebra of idempotents.

For example, the idempotents of an inverse semigroup form a semilattice. All isomorphisms between principal ideals of a semilattice E form the Munn inverse semigroup T_E, which contains the fundamental image of every inverse semigroup whose idempotents form the semilattice E [11, 12]. Thus semilattices give rise to all fundamental inverse semigroups.

Successful efforts have been made to generalize the Munn construction to the wider class of regular semigroups [1, 6–9, 13, 14]. Nambooripad achieved this using the concept of a *regular biordered set*. The *biordered set of a semigroup* S means simply the partial algebra consisting of the set $E = E(S)$ of idempotents of S with multiplication restricted to

$$D_E = \{(e, f) \in E \times E \mid ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

Thus the product of two idempotents is defined in this partial algebra if and only if one is a right or left zero of the other. The *biorder* on a semigroup S refers to the quasi-orders $\rightarrow$ and $\succ\!\!-$ defined on $E(S)$ by, for e and f in $E(S)$,

$$e \rightarrow f \qquad \text{if and only if} \quad fe = e,$$

and

$$e \succ\!\!- f \qquad \text{if and only if} \quad ef = e.$$

If the semigroup is inverse then the idempotents commute, so that these quasi-orders coincide, forming a semilattice. Nambooripad defines an abstract *biordered set* to be a partial algebra satisfying certain axioms (see below). A biordered set in which every *sandwich set* (for the definition see

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[14] or [4]) is non-empty is called *regular*. Nambooripad shows that regular biordered sets abstractly characterize all biordered sets of regular semigroups ([14], and see also [4]), and moreover uses them as a basis for his own generalization of the Munn inverse semigroup [13, 14].

The purpose of this paper is to show that the abstract definition of a biordered set characterizes biordered sets of arbitrary semigroups. Generalizations of the Munn semigroup beyond the class of regular semigroups are explored in a further paper of the author [5].

Given a biordered set E, we construct a semigroup S by taking the free semigroup on E factored out by all the relations holding in E. We show E is isomorphic to the biordered set of S, so that all biordered sets arise as biordered sets of semigroups. In so doing we produce the freest semigroup with biordered set E, which Pastijn [15] has studied, when it is already known that E comes from some semigroup.

2. Preliminaries

The letters $\mathscr{D}$, $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ will always denote Green's relations on a semigroup, and may be subscripted to distinguish semigroups. Standard terminology and basic results about semigroups and Green's relations, as given in [2] or [10], will be used without comment. All terminology involving biordered sets is included in this section.

Let E be a set with a partial multiplication with domain $D_E \subset E \times E$. If $(e, f) \in D_E$ then the product of e and f will be denoted by $e * f$. Define relations $\rightarrow$ and $\succ\!\!-$ each contained in $E \times E$ by

$$e \rightarrow f \qquad \text{if and only if } (f, e) \in D_E \text{ and } f * e = e$$

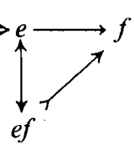
and

$$e \succ\!\!- f \qquad \text{if and only if } (e, f) \in D_E \text{ and } e * f = e.$$

Call E a *biordered set* if E satisfies the following, for $e, f, g \in E$:

(B1) The relations $\rightarrow$ and $\succ\!\!-$ are reflexive and transitive and $D_E = \rightarrow \cup \succ\!\!- \cup (\rightarrow \cup \succ\!\!-)^{-1}$.

(B21) $e \rightarrow f \Rightarrow$



(B21)*
$$e \rightarrowtail f \Rightarrow \begin{array}{ccc} e & \rightarrowtail & f \\ \updownarrow & \nearrow & \\ fe & & \end{array}$$

(B22)
$$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & \rightarrowtail & g \end{array} \Rightarrow f * e \rightarrowtail g * e$$

(B22)*
$$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ f & \longrightarrow & g \end{array} \Rightarrow e * f \to e * g$$

(B31) $$e \to f \to g \Rightarrow (e * g) * f = e * f$$

(B31)* $$e \rightarrowtail f \rightarrowtail g \Rightarrow f * (g * e) = f * e$$

(B32)
$$\begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f & \rightarrowtail & g \end{array} \Rightarrow (g * f) * e = (g * e) * (f * e)$$

(B32)*
$$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ f & \longrightarrow & g \end{array} \Rightarrow e * (f * g) = (e * f) * (e * g)$$

(B4) $$f \to e \leftarrow g \quad \text{and} \quad f * e \rightarrowtail g * e$$
$$\Rightarrow (\exists f' \in E) \quad \begin{array}{ccc} & e & \\ \nearrow & & \nwarrow \\ f' & \rightarrowtail & g \end{array} \quad \text{and} \quad f' * e = f * e.$$

(B4)* $$f \rightarrowtail e \leftarrowtail g \quad \text{and} \quad e * f \to e * g$$
$$\Rightarrow (\exists f' \in E) \quad \begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ f' & \longrightarrow & g \end{array} \quad \text{and} \quad e * f' = e * f.$$

Note (B4) and (B4)* appear differently here from the corresponding axioms given originally in [14]. However, by [14, Proposition 2.4], the complete set of axioms here is equivalent to the axioms for a biordered set given in [14].

If E and F are biordered sets and $\theta\colon E \to F$ is a mapping then θ is called a *morphism* if for $e, f \in E$

$$(e, f) \in D_E \Rightarrow (e\theta, f\theta) \in D_F \qquad \text{and} \qquad (e * f)\,\theta = e\theta * f\theta.$$

Call θ an *isomorphism* if θ is a bijection and both θ and θ^{-1} are morphisms. We call F a *biordered subset* of a biordered set E if $F \subset E$, F is a partial subalgebra of E, in the sense that $D_F = D_E \cap (F \times F)$, and F satisfies the biordered set axioms with respect to the restrictions of $\to$ and $>\!\!-$ to F.

THEOREM 2.1 [14, 1.1]. *Let S be a semigroup and $E(S)$ the set of idempotents of S. Then $E(S)$ forms a biordered set by restricting the semigroup multiplication to*

$$D_{E(S)} = \{(e, f) \in E(S) \times E(S) \mid ef = e \text{ or } ef = f \text{ or } fe = e \text{ or } fe = f\}.$$

Henceforth we call $E(S)$, equipped with the above partial product, the *biordered set of S*. The aim of this paper is to show that every biordered set arises as the biordered set of some semigroup.

Let E be a biordered set, so $>\!\!-\!\!<$ and $\leftrightarrow$ are equivalence relations on E. Let $L[R]$ denote an arbitrary member of $E/>\!\!-\!\!<\,[E/\leftrightarrow]$, and for $e \in E$ let $L_e[R_e]$ denote the $>\!\!-\!\!<[\leftrightarrow]$-class containing e. Denote the full transformation semigroup on a set X by $\mathscr{T}(X)$, and the dual transformation semigroup by $\mathscr{T}^*(X)$. If $\alpha \in \mathscr{T}(X)$ then α^* denotes the corresponding element of $\mathscr{T}^*(X)$. Put $X = E/>\!\!-\!\!< \cup \{\infty\}$ and $Y = E/\leftrightarrow \cup \{\infty\}$, where ∞ is a new symbol. Define

$$\begin{array}{llll}
\text{(i)} & \rho\colon E \to \mathscr{T}(X) & \text{by, for } e \in E, \\
& \rho_e\colon L \mapsto L_{xe} & \text{if } x \to e \text{ for some } x \in L \\
& \qquad\mapsto \infty & \text{otherwise} \\
& \quad \infty \mapsto \infty & \\
\text{(ii)} & \lambda\colon E \to \mathscr{T}(Y) & \text{by, for } e \in E, \\
& \lambda_e\colon R \mapsto R_{ex} & \text{if } x >\!\!- e \text{ for some } x \in R \\
& \qquad\mapsto \infty & \text{otherwise} \\
& \quad \infty \mapsto \infty & \\
\text{(iii)} & \phi\colon E \to \mathscr{T}(X) \times \mathscr{T}^*(Y) & \text{by, for } e \in E \\
& \quad e \mapsto \phi_e = (\rho_e, \lambda_e^*). &
\end{array}$$

THEOREM 2.2 [3, Theorem 2]. *If E is a biordered set, and ϕ is defined as above, then $E\phi$ is a biordered subset of $E(\mathscr{T}(X) \times \mathscr{T}^*(Y))$ and $\phi\colon E \to E\phi$ is an isomorphism.*

LEMMA 2.3 ([4, Lemma 4], Due to T. E. Hall). *Let E be a biordered set. If $\alpha \in E$ ($\langle E\phi \rangle$) and $\phi_x \mathscr{L}\, \alpha\, \mathscr{R} \phi_y$ for some $x, y \in E$, then $\alpha \in E\phi$.*

3. BIORDERED SETS COME FROM SEMIGROUPS

Suppose throughout this section that E is a biordered set. Let $F[F^1]$ denote the free semigroup [monoid] on the set E. Elements of $E[F^1]$ will be called *letters* [*words*]. Throughout e, f, g, x, y, and z [u, v, and w], with or without subscripts and/or superscripts, will denote letters [words]. The *length* of a word u, denoted by $l(u)$, is the number of letters in u.

Multiplication within F^1 will be denoted by juxtaposition, so if $(f, g) \in D_E$ then the expression fg is a word of length two whilst the expression $f * g$ is a single letter.

Call v a *subword* of w if $w = uvu'$ for some (possibly empty) words u and u'. We say the words $w_1, ..., w_n$ *cover* w if there exists subwords $w'_1, ..., w'_n$ of $w_1, ..., w_n$, respectively, such that $w = w'_1 \cdots w'_n$.

Define a relation σ on F by

$$\sigma = \{(fg, f * g) \mid (f, g) \in D_E\},$$

and let $\sigma^{\#}$ denote the smallest congruence containing σ. Elementary σ-transitions will be denoted by T with or without subscripts, and if T transforms w into w' then we write $T\colon w \mapsto w'$. Hence, for some u, v, f, and g, T is always of the form

$$ufgv \mapsto uf * gv,$$

or

$$uf * gv \mapsto ufgv.$$

Call T *of type* (1) if $f \to g$ or $f \succ\!\!- g$, whence $f \leftrightarrow f * g$, and *of type* (2) if $f \leftarrow g$ or $f -\!\!\prec g$, whence $g \succ\!\!\prec f * g$.

Our aim in what follows is to show E is isomorphic to $E(F/\sigma^{\#})$.

LEMMA 3.1. *If $f_1, ..., f_n, g_1, ..., g_m$ are letters such that $\sigma^{\#}(f_1 \cdots f_n) = \sigma^{\#}(g_1 \cdots g_m)$ then $\phi_{f_1} \cdots \phi_{f_n} = \phi_{g_1} \cdots \phi_{g_m}$.*

Proof. This follows since ϕ is a morphism, by Theorem 2.2, and using a simple induction on the number of elementary σ-transitions used to transform the word $f_1 \cdots f_n$ into the word $g_1 \cdots g_m$.

The following was communicated privately to the author by T. E. Hall:

LEMMA 3.2. *Suppose $\sigma^{\#}(w)$ is an idempotent of $F/\sigma^{\#}$ such that $\sigma^{\#}(w) \mathscr{D} \sigma^{\#}(e)$ for some letter e. Then $\sigma^{\#}(w)$ contains some letter.*

Proof. Let $w=e_1\cdots e_N$. We can find $f_1,\ldots,f_n$, $g_1,\ldots,g_m$ such that $\sigma^{\#}(f_1\cdots f_n)$ is an inverse of $\sigma^{\#}(g_1\cdots g_m)$, $\sigma^{\#}(w)=\sigma^{\#}(f_1\cdots f_n g_1\cdots g_m)$, and $\sigma^{\#}(e)=\sigma^{\#}(g_1\cdots g_m f_1\cdots f_n)$. By Lemma 3.1 then $\phi_{e_1}\cdots\phi_{e_N}$ is idempotent and $\phi_e=\phi_{g_1}\cdots\phi_{g_m}\phi_{f_1}\cdots\phi_{f_n}$. In particular $L_e\rho_e\neq\infty$, so $L_e\rho_{g_1}\cdots\rho_{g_m}\neq\infty$. Hence

$$\begin{array}{ll} x_1\to g_1 & \text{for some } x_1\in L_e \\ x_2\to g_2 & \text{for some } x\in L_{x_1*g_1} \\ \vdots & \quad\vdots \\ x_m\to g_m & \text{for some } x_m\in L_{x_{m-1}*g_{m-1}}. \end{array}$$

Put $S=F/\sigma^{\#}$. By repeated use of Green's lemma we have

$$\sigma^{\#}(x_m * g_m)\,\mathscr{L}_S\sigma^{\#}(eg_1\cdots g_m)=\sigma^{\#}(g_1\cdots g_m)\,\mathscr{L}_S\sigma^{\#}(w).$$

By Lemma 3.1, putting $x=x_m * g_m$,

$$\phi_x\mathscr{L}_{\langle E\phi\rangle}\phi_{e_1}\cdots\phi_{e_N}.$$

Dually there exists y such that $\sigma^{\#}(y)\,\mathscr{R}_S\sigma^{\#}(w)$ and $\phi_y\mathscr{R}_{\langle E\phi\rangle}\phi_{e_1}\cdots\phi_{e_N}$. Hence by Lemma 2.3 there exists z such that $\phi_z=\phi_{e_1}\cdots\phi_{e_N}$, so

$$\phi_x \succ\!\!\!-\!\!\!\prec \phi_z\leftrightarrow\phi_y.$$

Thus, since ϕ^{-1} is a morphism by Theorem 2.2,

$$x \succ\!\!\!-\!\!\!\prec z\leftrightarrow y$$

so that $\sigma^{\#}(x)\,\mathscr{L}_S\sigma^{\#}(z)\,\mathscr{R}_S\sigma^{\#}(y)$. Thus $\sigma^{\#}(z)\,\mathscr{H}_S\sigma^{\#}(w)$, so $\sigma^{\#}(z)=\sigma^{\#}(w)$, since both are idempotents, which proves the lemma.

THEOREM 3.3. *The biordered set E is isomorphic to $E(F/\sigma^{\#})$, the biordered set of $F/\sigma^{\#}$, and any idempotent-generated semigroup T such that $E=E(T)$ is a homomorphic image of $F/\sigma^{\#}$.*

Proof. Let $\eta: E\to E(F/\sigma^{\#})$ be the mapping which sends each e to $\sigma^{\#}(e)$. We show η is a biordered set isomorphism onto $E(F/\sigma^{\#})$. That $E\eta$ is a biordered subset of $E(F/\sigma^{\#})$ and η is a biordered set isomorphism onto $E\eta$ follow by Theorem 2.2 and Lemma 3.1. It remains therefore to show $E\eta=E(F/\sigma^{\#})$.

Suppose in what follows that $\sigma^{\#}(e_1\cdots e_n)$ is an idempotent. We need to find some e such that $\sigma^{\#}(e_1\cdots e_n)=\sigma^{\#}(e)$. However, by Lemma 3.2 it is sufficient to find e such that

$$\sigma^{\#}(e_1\cdots e_n)\,\mathscr{D}\sigma^{\#}(e). \tag{1}$$

Since $\sigma^{\#}(e_1\cdots e_n)$ is idempotent we have $e_1\cdots e_n\sigma^{\#}(e_1\cdots e_n)^n$, so there is a sequence of words $w_1,\dots, w_N$ and transitions $T_k\colon w_k\mapsto w_{k+1}$ for $k=1$ to $N-1$, where $w_1=e_1\cdots e_n$ and $w_N=(e_1\cdots e_n)^n$.

The main idea in what follows is to cover w_k, for $k=1$ to N, by subwords $w_k^1,\dots, w_k^n$ (defined below), such that each $\sigma^{\#}(w_k^i)$ lies in some $\mathscr{D}$-class of an element of $E\eta$.

By an inductive definition we can locate particular subwords by noting positions of letters, from 1 up to $l(w_k)$. Define, for $i=1$ to n, $\alpha_1^i=\beta_1^i=\gamma_1^i=i$. For each i, make the following definition, inductive in the subscripts:

$$\beta_{k+1}^i=\begin{cases}\beta_k^i & \text{if } T_k\colon ufgv\mapsto uf*gv \\ & \quad\text{where } l(u)\geqslant\beta_k^i-1 \\ & \text{or } T_k\colon uf*gv\mapsto ufgv \\ & \quad\text{where } l(u)\geqslant\beta_k^i \\ & \quad\text{or } l(u)=\beta_k^i-1 \text{ and } T_k \text{ is of type (1)} \\ \beta_k^i-1 & \text{if } T_k\colon ufgv\mapsto uf*gv \\ & \quad\text{where } l(u)\leqslant\beta_k^i-2 \\ \beta_k^i+1 & \text{if } T_k\colon uf*gv\mapsto ufgv \\ & \quad\text{where } l(u)\leqslant\beta_k^i-2 \\ & \quad\text{or } l(u)=\beta_k^i-1 \text{ and } T_k \text{ is not of type (1)}\end{cases}$$

$$\alpha_{k+1}^i=\begin{cases}\alpha_k^i & \text{if } T_k\colon ufgv\mapsto uf*gv \\ & \quad\text{where } l(u)\geqslant\alpha_k^i-1 \\ & \quad\text{or } l(u)=\alpha_k^i-2,\ \alpha_k^i<\beta_k^i, \text{ and} \\ & \qquad T_k \text{ is not of type (2)} \\ & \text{or } T_k\colon uf*gv\mapsto ufgv \\ & \quad\text{where } l(u)\geqslant\alpha_k^i-1 \\ \alpha_k^i-1 & \text{if } T_k\colon ufgv\mapsto uf*gv \\ & \quad\text{where } l(u)\leqslant\alpha_k^i-3 \\ & \quad\text{or } l(u)=\alpha_k^i-2 \text{ and either} \\ & \qquad \alpha_k^i=\beta_k^i \text{ or } T_k \text{ is of type (2)} \\ \alpha_k^i+1 & \text{if } T_k\colon uf*gv\mapsto ufgv \\ & \quad\text{where } l(u)\leqslant\alpha_k^i-2\end{cases}$$

$$\gamma^i_{k+1} = \begin{cases} \gamma^i_k & \text{if } T_k\colon ufgv \mapsto uf * gv \\ & \quad \text{where } l(u) \geqslant \gamma^i_k \\ & \quad \text{or } l(u) = \gamma^i_k - 1 \text{ and either} \\ & \qquad \gamma^i_k = \beta^i_k \text{ or } T_k \text{ is of type (1)} \\ & \text{or } T_k\colon uf * gv \mapsto ufgv \\ & \quad \text{where } l(u) \geqslant \gamma^i_k \\ \gamma^i_k - 1 & \text{if } T_k\colon ufgv \mapsto uf * gv \\ & \quad \text{where } l(u) \leqslant \gamma^i_k - 2 \\ & \quad \text{or } l(u) = \gamma^i_k - 1,\ \beta^i_k < \gamma^i_k, \text{ and} \\ & \qquad T_k \text{ is not of type (1)} \\ \gamma^i_k + 1 & \text{if } T_k\colon uf * gv \mapsto ufgv \\ & \quad \text{where } l(u) \leqslant \gamma_k - 1. \end{cases}$$

For integers i and j where $i \leqslant j$, let $[i, j]$ denote all integers from i up to j. From the above definitions it is immediate that, for $k = 1$ to N,

$$\beta^1_k \leqslant \beta^2_k \leqslant \cdots \leqslant \beta^n_k$$

and for each $i = 1$ to n,

$$\alpha^i_k \leqslant \beta^i_k \leqslant \gamma^i_k.$$

Further, for each k,

$$[1, l(w_k)] = \bigcup_{i=1}^{n} [\alpha^i_k, \gamma^i_k].$$

Let e^i_k denote the β^i_kth letter of w_k and let w^i_k denote the subword of w_k obtained by deleting all letters to the left of the α^i_kth letter and to the right of the γ^i_kth letter. The previous observations show that each w_k is covered by the subwords $w^1_k, \ldots, w^n_k$.

In particular w_N is covered by $w^1_N, \ldots, w^n_N$. We now claim that

$$\textit{for some } i,\ e_1 \cdots e_n \textit{ is a subword of } w^i_N. \tag{2}$$

Suppose (2) is false. Then w^1_N does not cover $e_1 \cdots e_n$. Make the inductive hypothesis that $w^1_N, \ldots, w^i_N$ do not cover $(e_1 \cdots e_n)^i$. Then since w^{i+1}_N does not cover $e_1 \cdots e_n$ we have $w^1_N, \ldots, w^{i+1}_N$ do not cover $(e_1 \cdots e_n)^{i+1}$. By induction $w^1_N, \ldots, w^n_N$ do not cover w_N, contradicting the previous paragraph. Hence

(2) must be true. (The reader may note that (2) is even more immediate if our original choice for w_N had been $(e_1 \cdots e_n)^{2n}$ or a higher power.)

Each w^i_k can be written in the form

$$w^i_k = u^i_k e^i_k v^i_k$$

for some (possibly empty) words u^i_k and v^i_k. We now prove, for each i and k,

$$\sigma^{\#}(e^i_k) \mathscr{R} \sigma^{\#}(e^i_k v^i_k) \tag{3}$$

and

$$\sigma^{\#}(e^i_k) \mathscr{L} \sigma^{\#}(u^i_k e^i_k). \tag{4}$$

We prove (3) by induction on k. For $k=1$ we have $w^i_1 = e_i$ for each i, so (3) holds. Suppose (3) holds for k; we show (3) holds for $k+1$. The only cases for T_k we need consider are the following:

(a) $u'\underbrace{ufe^i_k}_{w^i_k}vv' \to u'\underbrace{uf * e^i_k}_{w^i_{k+1}}vv'$

(b) $u'\underbrace{fe^i_k}_{w^i_k}vv' \to u'\underbrace{f * e^i_k}_{w^i_{k+1}}vv'$

(c) $u'\underbrace{ue^i_k f}_{w^i_k}vv' \to u'\underbrace{ue^i_k * f}_{w^i_{k+1}}vv'$

(d) $u'\underbrace{ue^i_k f}_{w^i_k}v \to u'\underbrace{ue^i_k * f}_{w^i_{k+1}}v$

(e) $u'\underbrace{ue^i_k v}_{w^i_k}fgv' \to \begin{cases} u'\underbrace{ue^i_k vf * g}_{w^i_{k+1}}v & \text{of type (1)} \\ u'\underbrace{ue^i_k v}_{w^i_{k+1}}f * gv & \text{not of type (1)} \end{cases}$

(f) $u'\underbrace{ue^i_k}_{w^i_k}vv' \to u'\underbrace{ufg}_{w^i_{k+1}}vv'$ where $e^i_k = f * g$.

Cases (a) and (b): $e^i_{k+1} = f * e^i_k$ and $v^i_{k+1} = v$, so (3) follows for $k+1$ since $\mathscr{R}$ is a left congruence.

Case (c): $e^i_{k+1} = e^i_k * f$ and $v^i_{k+1} = v$. Hence $\sigma^{\#}(e^i_{k+1} v^i_{k+1}) = \sigma^{\#}(e^i_k fv) \mathscr{R} \sigma^{\#}(e^i_k)$, so also $\sigma^{\#}(e^i_k) \mathscr{R} \sigma^{\#}(e^i_k f) = \sigma^{\#}(e^i_{k+1})$, yielding the required result.

Case (d): (3) follows immediately for $k+1$ since v^i_{k+1} is the empty word.

Case(e): If T_k is not of type (1) then $\sigma^{\#}(e^i_{k+1}v^i_{k+1}) = \sigma^{\#}(e^i_k v)\,\mathscr{R}\,\sigma^{\#}(e^i_k) = \sigma^{\#}(e^i_{k+1})$. If T_k is of type (1) then $f \leftrightarrow f * g$, so $\sigma^{\#}(e^i_{k+1}v^i_{k+1}) = \sigma^{\#}(e^i_k vf * g)\,\mathscr{R}\,\sigma^{\#}(e^i_k vf)\,\mathscr{R}\,\sigma^{\#}(e^i_k) = \sigma^{\#}(e^i_{k+1})$.

Case (f): If T_k is of type (1) then $e^i_{k+1} = f$ and $e^i_{k+1} \leftrightarrow e^i_k$ so $\sigma^{\#}(e^i_{k+1}v^i_{k+1}) = \sigma^{\#}(f * gv)\,\mathscr{R}\,\sigma^{\#}(e^i_k)\,\mathscr{R}\,\sigma^{\#}(e^i_{k+1})$. If T_k is not of type (1) then $e^i_{k+1} = g$ and $e^i_{k+1} \succ\!\!\prec e^i_k$, so $\sigma^{\#}(e^i_{k+1}v^i_{k+1}) = \sigma^{\#}(gv) = \sigma^{\#}(ge^i_k v)\,\mathscr{R}\,\sigma^{\#}(ge^i_k) = \sigma^{\#}(g) = \sigma^{\#}(e^i_{k+1})$.

Thus (3) follows by induction. The proof of (4) is similar.

By (2), for some i, j, k, α, and β we have

$$w^i_N = e_j \cdots e_n (e_1 \cdots e_n)^{\alpha} e_1 \cdots e_n (e_1 \cdots e_n)^{\beta} e_1 \cdots e_k$$

(where w^0 denotes the empty word).
Hence

$$\begin{aligned}\sigma^{\#}(w^i_N) &= \sigma^{\#}(e_j \cdots e_n e_1 \cdots e_n e_1 \cdots e_k)\\ &\mathscr{R}\,\sigma^{\#}(e_j \cdots e_n e_1 \cdots e_n)\,\mathscr{L}\,\sigma^{\#}(e_1 \cdots e_n),\end{aligned}$$

that is,

$$\sigma^{\#}(w^i_N)\,\mathscr{D}\,\sigma^{\#}(e_1 \cdots e_n).$$

From (3) and (4) we have

$$\sigma^{\#}(u^i_N e^i_N)\,\mathscr{L}\,\sigma^{\#}(e^i_N)\,\mathscr{R}\,\sigma^{\#}(e^i_N v^i_N),$$

so that

$$\sigma^{\#}(u^i_N e^i_N)\,\mathscr{R}\,\sigma^{\#}(u^i_N e^i_N v^i_N) = \sigma^{\#}(w^i_N),$$

that is,

$$\sigma^{\#}(w^i_N)\,\mathscr{D}\,\sigma^{\#}(e^i_N).$$

Hence

$$\sigma^{\#}(e_1 \cdots e_n)\,\mathscr{D}\,\sigma^{\#}(e^i_N),$$

which gives (1) by putting $e = e^i_N$.

This proves that E and $E(F/\sigma^{\#})$ are isomorphic. If also $E = E(T)$ for some idempotent-generated semigroup T, then $T = F/\tau$ for some congruence $\tau \supset \sigma$, so $\tau \supset \sigma^{\#}$, which proves the last statement of the theorem.

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