ON FIBRE SPACES WITH CROSS-SECTIONS

BY

SZE–TSEN HU

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1. *Introduction*

Throughout the present paper, we are concerned with a fibre space E over a base space B with projection

$$p : E \to B$$

in the sense of J.–P. SERRE, [3, p. 443]. Assume that there is a cross-section

$$q : B \to E,$$

that is to say, q is a continuous map of B into E such that the composition pq is the identity map on B. As an immediate consequence, q is a homeomorphism of B onto a subspace $q(B)$ of E. Hence we may identify B with $q(B)$ by means of q; then B becomes a subspace of E and the cross-section q becomes the inclusion map. Furthermore, the projection p becomes a retraction of E onto B, and hence B is a closed subspace of E provided that E is a Hausdorff space.

Let A be a subspace of B and let $D = p^{-1}(A) \subset E$. Then $B \cap D = A$. Thus we have

$$A \subset B \subset E, \qquad A \subset D \subset E.$$

Let A_0 be a subspace of A and pick a point $a_0 \in A_0$ as the base point for all homotopy groups involved in this paper. Denote by F the fibre over the point a_0, that is to say,

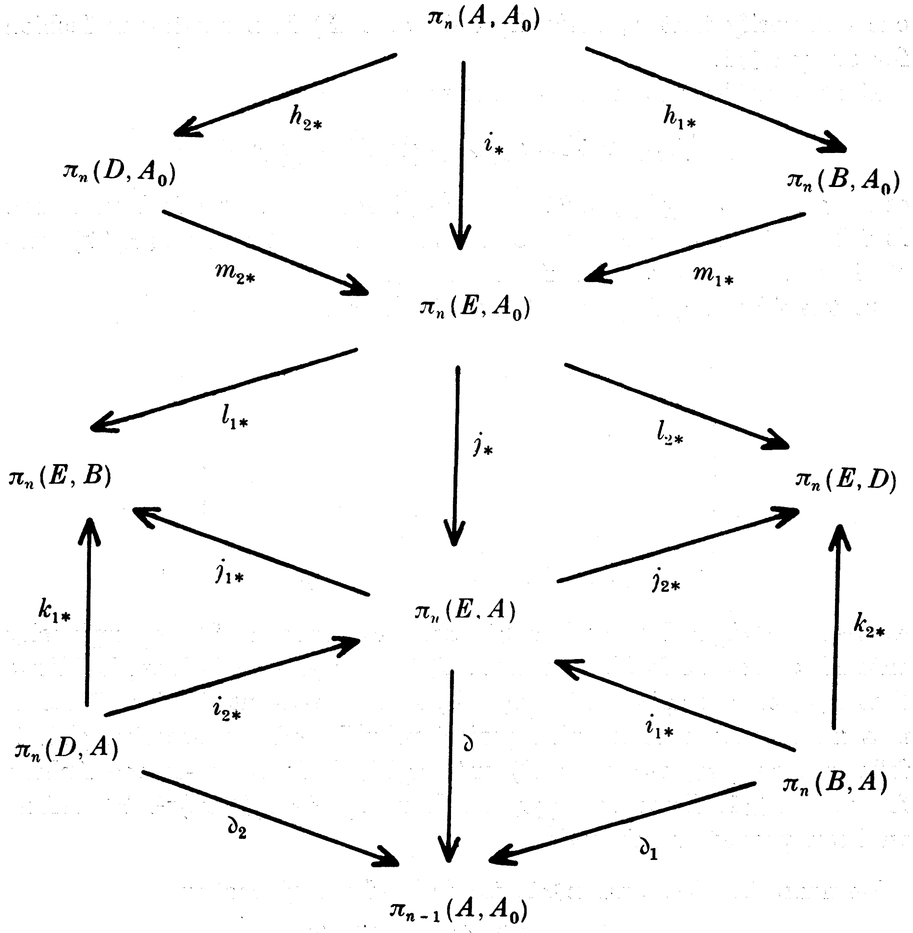
$$F = p^{-1}(a_0) \subset D.$$

The purpose of this paper is to establish an exact sequence

$$\dots \leftarrow \pi_{n-1}(A, A_0) \overset{\Delta}{\leftarrow} \pi_n(E, A_0) \overset{\varphi}{\leftarrow} \pi_n(B, A_0) \times \pi_n(D, A_0) \overset{\psi}{\leftarrow} \pi_n(A, A_0) \leftarrow \dots$$

where Δ, φ, ψ are homomorphisms defined as follows. Consider the following diagram (see next page) where all homomorphisms other than the boundary operators $\partial, \partial_1, \partial_2$ are induced by inclusion maps. In § 2, we shall prove that k_{1*} and k_{2*} are isomorphisms. Then, the homomorphisms Δ, φ, ψ are given by the following formulae:

$$\begin{aligned} \psi u &= (h_{1*}u, h_{2*}u^{-1}), && u \in \pi_n(A, A_0), \\ \varphi(v_1, v_2) &= (m_{1*}v_1)(m_{2*}v_2), && v_1 \in \pi_n(B, A_0), \; v_2 \in \pi_n(D, A_0), \\ \Delta w &= \partial_2 k_{1*}^{-1} l_{1*} w, && w \in \pi_n(E, A_0). \end{aligned}$$



Observe the apparent similarity between this exact sequence and the Mayer–Vietoris sequence of a proper triad in homology theory, [1, p. 39].

2. *The isomorphisms k_{1*} and k_{2*}*

Consider the transformations

$$p_* : \pi_n(E) \to \pi_n(B), \quad q_* : \pi_n(B) \to \pi_n(E)$$

induced by the projection $p: E \to B$ and the cross-section $q: B \to E$ for each $n \geqq 0$. Since pq is the identity map on B, it follows that p_* carries $\pi_n(E)$ onto $\pi_n(B)$ and q_* sends $\pi_n(B)$ into $\pi_n(E)$ in a one-to-one fashion. Therefore, in case $n \geqq 1$, p_* is an epimorphism and q_* is a monomorphism. Then, from the exactness of the homotopy sequence

$$\ldots \leftarrow \pi_{n-1}(E) \overset{q_*}{\leftarrow} \pi_{n-1}(B) \overset{\partial_*}{\leftarrow} \pi_n(E, B) \overset{r_*}{\leftarrow} \pi_n(E) \overset{q_*}{\leftarrow} \pi_n B) \leftarrow \ldots$$

of the triplet (E, B, a_0), [2, p. 493], we deduce that, for every $n \geqq 1$, ∂_* sends $\pi_n(E, B)$ into the neutral element of $\pi_{n-1}(B)$ and r_* carries $\pi_n(E)$ onto $\pi_n(E, B)$. Let K_n denote the kernel of p_* in $\pi_n(E)$. Then, one can verify that r_* sends K_n onto $\pi_n(E, B)$ in a one-to-one fashion for every $n \geqq 1$.

Next, consider the exact homotopy sequence

$$\dots \xleftarrow{d_*} \pi_n(B) \xleftarrow{p_*} \pi_n(E) \xleftarrow{s_*} \pi_n(F) \xleftarrow{d_*} \pi_{n+1}(B) \xleftarrow{p_*} \dots$$

of the given fibering, [4, p. 91], where s_* is induced by the inclusion map. It follows from the exactness of the sequence that s_* sends $\pi_n(F)$ onto K_n in a one-to-one fashion for each $n \geqq 0$.

In the following diagram of inclusion maps

$$\begin{array}{ccccc} D & \xleftarrow{\ \sigma\ } & F & \xrightarrow{\ s\ } & E \\ \downarrow{\scriptstyle \varrho} & & & & \downarrow{\scriptstyle r} \\ (D, A) & & \xrightarrow{\qquad k_1 \qquad} & & (E, B) \end{array}$$

we have $rs = k_1 \varrho \sigma$. For each $n \geqq 1$, since s_* sends $\pi_n(F)$ isomorphically onto K_n and r_* sends K_n onto $\pi_n(E, B)$ in a one-to-one fashion, it follows that $r_* s_*$ carries $\pi_n(F)$ onto $\pi_n(E, B)$ in a one-to-one fashion. Since D is a fibre space over A with projection $p|D$ and cross-section $q|A$, it follows that $\varrho_* \sigma_*$ carries $\pi_n(F)$ onto $\pi_n(D, A)$ in a one-to-one fashion for each $n \geqq 1$. Since $rs = k_1 \varrho \sigma$, we have $k_{1*} = r_* s_* (\varrho_* \sigma_*)^{-1}$. Therefore, we have proved the following

Lemma 1. *For each $n \geqq 1$, the induced transformation*

$$k_{1*} : \pi_n(D, A) \to \pi_n(E, B)$$

sends $\pi_n(D, A)$ onto $\pi_n(E, B)$ in a one-to-one fashion.

Now, let us turn to k_{2*}. By the fibering theorem in homotopy theory, [2, p. 495], the induced transformation

$$p_* : \pi_n(E, D) \to \pi_n(B, A), \qquad n \geqq 1,$$

of the projection $p : (E, D) \to (B, A)$ sends $\pi_n(E, D)$ onto $\pi_n(B, A)$ in a one-to-one fashion. Since pk_2 is the identity map on (B, A), we have $k_{2*} = p_*^{-1}$. Therefore, we have proved the following

Lemma 2. *For each $n \geqq 1$, the induced transformation*

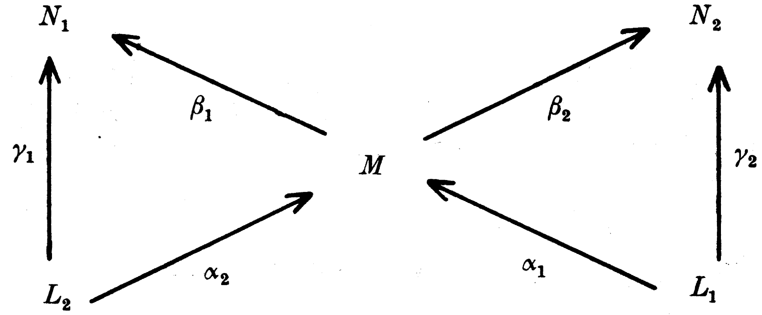
$$k_{2*} : \pi_n(B, A) \to \pi_n(E, D)$$

sends $\pi_n(B, A)$ onto $\pi_n(E, D)$ in a one-to-one fashion.

Thus, in case $n > 1$, both k_{1*} and k_{2*} are isomorphisms. Then, the construction of the sequence described in § 1 and the proof of its exactness can be carried out just like those of the Mayer–Vietoris sequence except one must take care of the fact that some groups in the present sequence might fail to be abelian.

3. *Algebraic lemmas*

Lemma 3. *In the following diagram of groups and homomorphisms*



if the two diagonals are exact, the two triangles are commutative, and γ_1, γ_2 *are isomorphisms, then we have the following conclusions:*

(3.1) α_1, α_2 *are monomorphisms and* β_1, β_2 *are epimorphisms.*

(3.2) M *is the direct product of the images of* α_1 *and* α_2.

(3.3) β_1 *carries* $Im(\alpha_2)$ *isomorphically onto* N_1 *and* β_2 *carries* $Im(\alpha_1)$ *isomorphically onto* N_2.

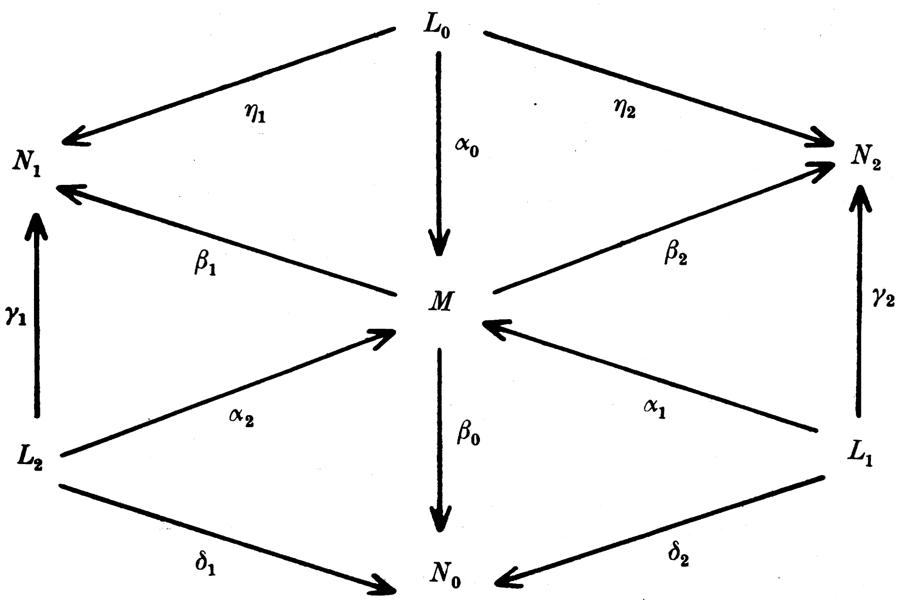
(3.4) *If* $x_1 \in Im(\alpha_1)$, *then* $\alpha_1\gamma_2^{-1}\beta_2 x_1 = x_1$. *If* $x_2 \in Im(\alpha_2)$, *then* $\alpha_2\gamma_1^{-1}\beta_1 x_2 = x_2$.

(3.5) *If* $x \in M$, *then* $x = (\alpha_1\gamma_1^{-1}\beta_2 x)(\alpha_2\gamma_1^{-1}\beta_1 x)$.

If all the groups are abelian, then this lemma reduces to a standard one in algebraic topology, [1, p. 32]. The proof of this lemma is left to the reader.

The following hexagonal lemma is an immediate consequence of Lemma 3 as in the abelian case, [1, p. 38].

Lemma 4. *In the following diagram of groups and homomorphisms*



if all the diagonals are exact, all the triangles are commutative, and γ_1, γ_2 *are isomorphisms, then for each* $w \in L_0$ *we have*

$$\delta_2\gamma_1^{-1}\,\eta_1 w = (\delta_1\gamma_2^{-1}\,\eta_2 w)^{-1}.$$

Remark. This lemma remains valid if the hypothesis $Im(\alpha_0) = Ker(\beta_0)$ is weakened to $Im(\alpha_0) \subset Ker(\beta_0)$. In fact, it is this weakened hypothesis that is needed to prove the conclusion.

4. *Construction of the sequences*

Firstly, we define for each $n \geqq 1$ a transformation

$$\psi : \pi_n(A, A_0) \to \pi_n(B, A_0) \times \pi_n(D, A_0)$$

by setting $\psi u = (h_{1*}u, h_{2*}u^{-1})$ for each $u \in \pi_n(A, A_0)$.

If $n \geqq 2$ or $A_0 = a_0$, then $\pi_n(A, A_0)$ and the direct product $\pi_n(B, A_0) \times \times \pi_n(D, A_0)$ are groups and ψ is a homomorphism.

Secondly, if $n \geqq 2$ or if $n = 1$ with $A_0 = a_0$, then $\pi_n(E, A_0)$ is a group and, therefore, we may define a transformation

$$\varphi : \pi_n(B, A_0) \times \pi_n(D, A_0) \to \pi_n(E, A_0)$$

by setting $\varphi(v_1, v_2) = (m_{1*}v_1)(m_{2*}v_2)$ for each $v_1 \in \pi_n(B, A_0)$ and $v_2 \in \pi_n(D, A_0)$. If $n > 2$ or if $n = 2$ with $A_0 = a_0$, then $\pi_n(E, A_0)$ is abelian and hence φ is a homomorphism.

Finally, if $n \geqq 2$ or if $n = 1$ with $A_0 = a_0$, we may define a transformation.

$$\Delta : \pi_n(E, A_0) \to \pi_{n-1}(A, A_0)$$

by taking $\Delta = \partial_2 k_{1*}^{-1}\, l_{1*}$. If $n > 2$ or if $n = 2$ with $A_0 = a_0$, then Δ is a homomorphism and

$$\Delta w = \partial_2 k_{1*}^{-1}\, l_{1*} w = (\partial_1 k_{2*}^{-1}\, l_{2*} w)^{-1}$$

for each $w \in \pi_n(E, A_0)$ since the lower hexagon of the diagram in § 1 satisfies the hypothesis of Lemma 4.

Therefore, we have constructed the following sequence

$$\pi_2(E, A_0) \overset{\varphi}{\leftarrow} \pi_2(B, A_0) \times \pi_2(D, A_0) \overset{\psi}{\leftarrow} \pi_2(A, A_0) \overset{\Delta}{\leftarrow} \ldots$$

$$\ldots \leftarrow \pi_{n-1}(A, A_0) \overset{\Delta}{\leftarrow} \pi_n(E, A_0) \overset{\varphi}{\leftarrow} \pi_n(B, A_0) \times \pi_n(D, A_0) \overset{\psi}{\leftarrow} \pi_n(A, A_0) \leftarrow \ldots$$

which will be referred to as the *relative sequence.*

If $A_0 = a_0$, then we have a longer sequence

$$\pi_1(E) \overset{\varphi}{\leftarrow} \pi_1(B) \times \pi_1(D) \overset{\psi}{\leftarrow} \pi_1(A) \overset{\Delta}{\leftarrow} \pi_2(E) \leftarrow \ldots$$

$$\ldots \leftarrow \pi_{n-1}(A) \overset{\Delta}{\leftarrow} \pi_n(E) \overset{\varphi}{\leftarrow} \pi_n(B) \times \pi_n(D) \overset{\psi}{\leftarrow} \pi_n(A) \leftarrow \ldots$$

which will be referred to as the *absolute sequence.*

5. *Exactness of the sequences*

Theorem. *The absolute sequence and the relative sequence are both exact.*

The proof of this theorem is analogous to that of the exactness of the Mayer–Vietoris sequence, [1, p. 40], except that the first three groups might not be abelian and the first transformation φ is usually not a homomorphism. Hence we leave the proof to the reader.

If we take $A = a_0$ in the absolute sequence, we obtain $D = F$ and the well-known isomorphism

$$\pi_n(E) \approx \pi_n(B) \times \pi_n(F)$$

for every $\pi \geqq 2$, [4, p. 92]. On the other hand, if E is contractible to a point, then we have

$$\pi_n(A) \approx \pi_n(B) \times \pi_n(D)$$

for every $n \geqq 1$.

Similarly, if we take $A = A_0$ in the relative sequence, we get

$$\varphi : \pi_n(B, A) \times \pi_n(D, A) \approx \pi_n(E, A)$$

for every $n \geqq 3$.

6. *The first transformation φ*

In this final section, we are concerned with the first transformation φ in the relative sequence and that in the absolute sequence.

The following proposition is an obvious sequence of the definition of φ.

Proposition 1. *$\varphi : \pi_2(B, A_0) \times \pi_2(D, A_0) \to \pi_2(E, A_0)$ is a homomorphism if $\pi_2(E, A_0)$ is abelian.*

Proposition 2. *φ carries $\pi_2(B, A_0) \times \pi_2(D, A_0)$ onto $\pi_2(E, A_0)$ if $\pi_1(A, A_0)$ consists of a single element.*

Proof. Let w be an arbitrary element of $\pi_2(E, A_0)$. Since $\pi_1(A, A_0) = 0$, we have

$$\partial_1 k_{2*}^{-1}\, l_{2*} w = 0 = \partial_2 k_{1*}^{-1}\, l_{1*} w$$

in the diagram of § 1. By exactness of the homotopy sequences, there exist elements $z_1 \in \pi_2(B, A_0)$ and $z_2 \in \pi_2(D, A_0)$ such that

$$n_{1*} z_1 = k_{2*}^{-1}\, l_{2*} w, \qquad n_{2*} z_2 = k_{1*}^{-1}\, l_{1*} w,$$

where $n_1 : (B, A_0) \subset (B, A)$ and $n_2 : (D, A_0) \subset (D, A)$ denote the inclusion maps. By (3.5), we have

$$\begin{aligned} j_* w &= (i_{1*} k_{2*}^{-1}\, j_{2*} j_* w)(i_{2*} k_{1*}^{-1}\, j_{1*} j_* w) \\ &= (i_{1*} k_{2*}^{-1}\, l_{2*} w)(i_{2*} k_{1*}^{-1}\, l_{1*} w) \\ &= (i_{1*} n_{1*} z_1)(i_{2*} n_{2*} z_2) \\ &= (j_* m_{1*} z_1)(j_* m_{2*} z_2). \end{aligned}$$

By exactness of the homotopy sequence of (E, A, A_0), there is an element $u \in \pi_2(A, A_0)$ such that

$$i_* u = (m_{1*} z_1)^{-1} w (m_{2*} z_2)^{-1}.$$

If we let $v_1 = z_1 \cdot h_{1*} u$ and $v_2 = z_2$, it follows that

$$\varphi(v_1, v_2) = (m_{1*} v_1)(m_{2*} v_2) = (m_{1*} z_1)(i_* u)(m_{2*} z_2) = w.$$

This completes the proof.

Similarly, we have the following propositions for the first φ in the absolute sequence.

Proposition 3. *$\varphi : \pi_1(B) \times \pi_1(D) \to \pi_1(E)$ is a homomorphism if $\pi_1(E)$ is abelian.*

Proposition 4. *φ carries $\pi_1(B) \times \pi_1(D)$ onto $\pi_1(E)$ if $A = a_0$.*

Wayne State University,
Detroit 2, Michigan

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