Discrete Applied Mathematics 158 (2010) 1006–1011

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

k-tuple total domination in graphs

Michael A. Henning [a,*,1], Adel P. Kazemi [b]

[a] *School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg campus, South Africa*

[b] *Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box 5619911367, Ardabil, Iran*

ARTICLE INFO

Article history:
Received 5 May 2009
Received in revised form 19 January 2010
Accepted 21 January 2010
Available online 18 February 2010

Keywords:
Total domination
k-tuple total domination

ABSTRACT

A set S of vertices in a graph G is a k-tuple total dominating set, abbreviated kTDS, of G if every vertex of G is adjacent to least k vertices in S. The minimum cardinality of a kTDS of G is the k-tuple total domination number of G. For a graph to have a kTDS, its minimum degree is at least k. When $k = 1$, a k-tuple total domination number is the well-studied total domination number. When $k = 2$, a kTDS is called a double total dominating set and the k-tuple total domination number is called the double total domination number. We present properties of minimal kTDS and show that the problem of finding kTDSs in graphs can be translated to the problem of finding k-transversals in hypergraphs. We investigate the k-tuple total domination number for complete multipartite graphs. Upper bounds on the k-tuple total domination number of general graphs are presented.

1. Introduction

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7,8]. For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S. Equivalently, S is a dominating set of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 1$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A set $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S, while the minimum cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G.

In [6] Harary and Haynes defined a generalization of domination as follows: a subset S of V is a *k-tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq k$, that is, v is in S and has at least $k - 1$ neighbors in S or v is in $V - S$ and has at least k neighbors in S. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G. Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$. For a graph to have a k-tuple dominating set, its minimum degree is at least $k - 1$. Hence for trees, $k \leq 2$. A k-tuple dominating set where $k = 2$ is called a *double dominating set* (DDS). The concept of k-tuple domination has been studied by several authors (see, for example, [4,5,9,10], and elsewhere).

In this paper, we study k-tuple total domination in graphs. A subset S of V is a *k-tuple total dominating set* of G, abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \geq k$, that is, S is a kTDS if every vertex has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ is the minimum cardinality of a kTDS of G. We remark that a 1-tuple total domination is the well-studied total domination number. Thus, $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a k-tuple total dominating set, its minimum degree is at least k. Since every $(k + 1)$-tuple total dominating set is also a k-tuple total dominating set, we note that $\gamma_{\times(k+1),t}(G) \leq \gamma_{\times k,t}(G)$ for all graphs with minimum degree at least k. A kTDS of cardinality $\gamma_{\times k,t}(G)$ we call a $\gamma_{\times k,t}(G)$-set. When $k = 2$, a k-tuple total dominating set is called a *double total dominating set*, abbreviated DTDS, and the k-tuple

* Corresponding author.
E-mail addresses: mahenning@uj.ac.za (M.A. Henning), adelpkazemi@yahoo.com (A.P. Kazemi).

1 Present address: Department of Mathematics, University of Johannesburg, P.O. Box 524, Auckland Park, 2006, South Africa.

0166-218X/$ – see front matter
doi:10.1016/j.dam.2010.01.009

total domination number is called the *double total domination number*. The redundancy involved in k-tuple total domination makes it useful in many applications.

Let $G = (V, E)$ be a graph and let $S \subseteq V$. For each k-element subset $S' \subseteq S$, we define the *(S, k)-private neighborhood of S'*, denoted by $\mathrm{pn}_k(S', S)$, to be the set of all vertices v in G such that $N(v) \cap S = S'$ (possibly, $v \in S$). Further, we define the *open k-boundary of S*, denoted by $\mathrm{OB}_k(S)$, to be the set of all vertices v in G such that $v \in \mathrm{pn}_k(S', S)$ for some k-element subset $S' \subseteq S$.

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of arbitrary subsets of V, called *edges*. A *transversal* in H is a subset $S \subseteq V$ such that $|S \cap e| \geq 1$ for every edge $e \in E$, that is, the set S meets every edge in H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A k-uniform hypergraph is a hypergraph in which every edge has size k. Every (simple) graph is a 2-uniform hypergraph. Thus, graphs are special hypergraphs. For a graph $G = (V, E)$, we denote by H_G the open neighborhood hypergraph of G, that is, $H_G = (V, C)$ is the hypergraph with the vertex set V and with the edge set C consisting of the open neighborhoods of vertices of V in G.

For notation and graph theory terminology we in general follow [7]. Specifically, let $G = (V, E)$ be a graph with the vertex set V of order n and the edge set E. The minimum degree among the vertices of G is denoted by $\delta(G)$. A cycle on n vertices is denoted by C_n, and a path on n vertices by P_n. A vertex of degree k is called a *degree-k vertex*.

2. Observations and preliminary results

We begin with the following trivial observation about the k-tuple total domination number of a graph. The proof follows readily from the definitions and is omitted.

Observation 1. *Let G be a graph of order n with $\delta(G) \geq k$, and let S be a kTDS in G.*

(a) $\max\{\gamma_{\times k}(G), k+1\} \leq \gamma_{\times k,t}(G) \leq |V(G)|$.
(b) *If G is a spanning subgraph of a graph H, then $\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G)$.*
(c) *If v is a degree-k vertex in G, then $N_G(x) \subseteq S$.*

A kTDS S in a graph G is a *minimal kTDS* if no proper subset of S is a kTDS in G. We next present necessary and sufficient conditions for a kTDS to be minimal.

Theorem 2. *Let G be a graph of order n with $\delta(G) \geq k$, and let S be a kTDS in G.*

(a) *The set S is a minimal kTDS if and only if $S \setminus \{x\}$ is not a kTDS for every $x \in S$.*
(b) *If S is a minimal kTDS, then for each vertex $x \in S$, there exists a k-element subset $S_x \subseteq S$ such that $x \in S_x$ and $|\mathrm{pn}_k(S_x, S)| \geq 1$.*
(c) *S is a minimal kTDS in G if and only if $\mathrm{OB}_k(S)$ dominates S.*

Proof. (a) If S is a minimal kTDS, then no proper subset of S is a kTDS of G. In particular, $S \setminus \{x\}$ is not a kTDS for every $x \in S$. Conversely, suppose $S \setminus \{x\}$ is not a kTDS for every $x \in S$. For sake of contradiction, suppose that S is not a minimal kTDS. Then, there exists a proper subset S' of S that is a kTDS of G. Let $x \in S \setminus S'$ and let $S_x = S \setminus \{x\}$. Then, $S' \subseteq S_x$. If $S' = S_x$, then S_x is a kTDS of G, a contradiction. Hence, $S' \subset S_x$. Since every superset of a kTDS in G is also a kTDS of G, the set S_x is a kTDS of G, once again producing a contradiction. Hence, S is a minimal kTDS. This establishes part (a).

(b) Suppose that S is a minimal kTDS. Let $x \in S$. By part (a), $S \setminus \{x\}$ is not a kTDS in G. Hence there exists a vertex v such that $|N(v) \cap (S \setminus \{x\})| < k$. However, $|N(v) \cap S| \geq k$. Consequently, $|N(v) \cap S| = k$ and $x \in N(v)$. Let $S_x = N(v) \cap S$. Then, S_x is a k-element subset of S such that $x \in S_x$ and $v \in \mathrm{pn}_k(S_x, S)$. This establishes part (b).

(c) Suppose that S is a minimal kTDS. Let $x \in S$. By part (b), there exists a k-element subset $S_x \subseteq S$ such that $x \in S_x$ and $|\mathrm{pn}_k(S_x, S)| \geq 1$. Let $v \in \mathrm{pn}_k(S_x, S) \subseteq \mathrm{OB}_k(S)$. Then, $N(v) \cap S = S_x$. In particular, xv is an edge of G, and so x is dominated by the open k-boundary $\mathrm{OB}_k(S)$ of S. Hence, $\mathrm{OB}_k(S)$ dominates S. Conversely, suppose that $\mathrm{OB}_k(S)$ dominates S. Let $x \in S$ and let v be a vertex in $\mathrm{OB}_k(S)$ that dominates x. Thus, $N(v) \cap S = S_x$ for some k-element subset $S_x \subseteq S$ and $x \in S_x$. But then v contains fewer that k neighbors in the set $S \setminus \{x\}$, and so $S \setminus \{x\}$ is not a kTDS of G. This is true for an arbitrary vertex x in S. Thus, by part (a), S is a minimal kTDS. This establishes part (c). □

As observed in Observation 1(a), if G is a graph with $\delta(G) \geq k$, then $\gamma_{\times k,t}(G) \geq k+1$. We next characterize graphs with $\gamma_{\times k,t}(G) = k+1$. For this purpose, we define the k-join of a graph G to a graph H of order at least k to be the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H. We denote the k-join of G to H by $G \circ_k H$.

Theorem 3. *Let G be a graph with $\delta(G) \geq k$. Then, $\gamma_{\times k,t}(G) = k+1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F.*

Proof. If $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F, then $V(K_{k+1})$ is a kTDS of size $k+1$, and so $\gamma_{\times k,t}(G) \leq k+1$. Consequently, by Observation 1(a), $\gamma_{\times k,t}(G) = k+1$. Conversely, suppose that $\gamma_{\times k,t}(G) = k+1$. Let S be a $\gamma_{\times k,t}(G)$-set and let $G = (V, E)$. Then, $|S| = k+1$. Every vertex has at least k neighbors in S. In particular, every vertex in S is adjacent to all other k vertices in S, and so $G[S] = K_{k+1}$. If $|V| = k+1$, then $G = K_{k+1}$. If $|V| > k+1$, then let $F = G[V \setminus S]$. Since every vertex in $V \setminus S$ has at least k neighbors in S in the graph G, we have that $G = F \circ_k K_{k+1}$. □

As an immediate consequence of Theorem 3, we have the following result.

Corollary 4. *If G is a complete p-partite graph where $p \geq k+1$, then $\gamma_{\times k,t}(G) = k+1$.*

For an integer $k \geq 1$, we define a *k-transversal* in a hypergraph $H = (V, E)$ as a subset $S \subseteq V$ such that $|S \cap e| \geq k$ for every edge $e \in E$, that is, every edge in H contains at least k vertices from the set S. We define the *k-transversal number* $\tau_k(H)$ of H to be the minimum cardinality of a k-transversal in H. Perhaps much of the recent interest in total domination in graphs arises from the fact that total domination in graphs can be translated to the problem of finding transversals in hypergraphs. We show next that the problem of finding k-tuple total dominating sets in graphs can be translated to the problem of finding k-transversals in hypergraphs.

Theorem 5. *If G is a graph with minimum degree at least k and H_G is the open neighborhood hypergraph of G, then $\gamma_{\times k,t}(G) = \tau_k(H_G)$.*

Proof. On the one hand, every kTDS in G contains at least k vertices from the open neighborhood of each vertex in G and is therefore a k-transversal in H_G. In particular, if S is a $\gamma_{\times k,t}(G)$-set, then S is a k-transversal in H_G, and so $\tau_k(H_G) \leq |S| = \gamma_{\times k,t}(G)$. On the other hand, every k-transversal in H_G contains at least k vertices from the open neighborhood of each vertex of G, and is therefore a kTDS in G. In particular, if T is a $\tau_k(H_G)$-transversal, then T is a kTDS in G, and so $\gamma_{\times k,t}(G) \leq |T| = \tau_k(H_G)$. Consequently, the k-transversal number of the open neighborhood hypergraph of a graph is precisely the k-tuple total domination number of the graph. □

3. Complete multipartite graphs

In this section, we determine the k-tuple total domination number of a complete multipartite graph for $k \geq 2$. By Corollary 4, if G is a complete p-partite graph where $p \geq k+1$, then $\gamma_{\times k,t}(G) = k+1$. Hence in this section we restrict our attention to complete p-partite graphs where $p \leq k$. We first determine the k-tuple total domination number of a complete k-partite graph with minimum degree at least k.

Theorem 6. *Let $k \geq 2$ be an integer and let $G = K(n_1, n_2, \ldots, n_k)$ be a complete k-partite graph where $n_1 \leq n_2 \leq \cdots \leq n_k$. If $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k,t}(G) = k+2$.*

Proof. We note that $\delta(G) \geq k$ if and only if $\sum_{i=1}^{k-1} n_i \geq k$. Hence the condition $\sum_{i=1}^{k-1} n_i \geq k$ guarantees that G has a kTDS. We show that $\gamma_{\times k,t}(G) = k+2$. Let G have vertex partition $V = (V_1, V_2, \ldots, V_k)$, where $|V_i| = n_i$ for $i = 1, 2, \ldots, k$. Our degree condition $\sum_{i=1}^{k-1} n_i \geq k$ implies that $n_k \geq n_{k-1} \geq 2$. The set S consisting of two vertices from both sets V_{k-1} and V_k, and exactly one vertex from each of the remaining $k-2$ partite sets of G, forms a kDTS of G of size $k+2$, and so $\gamma_{\times k,t}(G) \leq k+2$. Hence it suffices for us to show that $\gamma_{\times k,t}(G) \geq k+2$. Let D be a $\gamma_{\times k,t}(G)$-set. By Observation 1(a), $|D| = \gamma_{\times k,t}(G) \geq k+1$. By the Pigeonhole principle, $|D \cap V_r| \geq 2$ for some r, $1 \leq r \leq k$. Since every vertex of V_r must be dominated by at least k vertices of $D \setminus V_r$, we have that $\gamma_{\times k,t}(G) = |D| \geq |D \setminus V_r| + |D \cap V_r| \geq k+2$, as desired. □

In view of Theorem 6, we assume in what follows that G is a complete p-partite graph where $p \leq k-1$.

Theorem 7. *Let $2 \leq p < k$ and let $G = K(n_1, n_2, \ldots, n_p)$ be a complete p-partite graph where $\lceil k/(p-1) \rceil \leq n_1 \leq n_2 \leq \cdots \leq n_p$. Then, $\gamma_{\times k,t}(G) = \lceil kp/(p-1) \rceil$.*

Proof. Let G have vertex partition $V = (V_1, V_2, \ldots, V_p)$, where $|V_i| = n_i$ for $i = 1, 2, \ldots, p$. Let S be a $\gamma_{\times k,t}(G)$-set. For $i = 1, 2, \ldots, p$, let $S_i = S \cap V_i$ and let $|S_i| = s_i$. Then, $|S| = \sum_{i=1}^{p} s_i$. Since each vertex in the partite set V_i, $1 \leq i \leq p$, is dominated by at least k vertices in $S \setminus S_i$, for each $i = 1, 2, \ldots, p$ we have that $|S| - s_i \geq k$. Hence, summing over all i, we have $(p-1)|S| \geq kp$, and so $\gamma_{\times k,t}(G) = |S| \geq \lceil kp/(p-1) \rceil$. It therefore suffices for us to show that there exists a kTDS of G of size $\lceil kp/(p-1) \rceil$.

If $(p-1) \mid k$, then we choose $k/(p-1)$ vertices from each of the p partite sets to form a kTDS of G of size $kp/(p-1)$, as desired. Hence we may assume that $(p-1) \nmid k$. Thus, $k = (p-1) \cdot \lfloor k/(p-1) \rfloor + r$ for some integer r where $1 \leq r \leq p-2$. For $i = 1, \ldots, r+1$, let D_i be a subset of V_i of size $\lceil k/(p-1) \rceil = \lfloor k/(p-1) \rfloor + 1$. For $i = r+2, \ldots, p$, let D_i be a subset of V_i of size $\lfloor k/(p-1) \rfloor$. Let $D = \bigcup_{i=1}^{p} D_i$. Then

$$|D| = (r+1)\left(\left\lfloor \frac{k}{p-1} \right\rfloor + 1\right) + (p-r-1)\left\lfloor \frac{k}{p-1} \right\rfloor = \left\lceil \frac{kp}{p-1} \right\rceil.$$

Further, every vertex in G is totally dominated by at least $(p-r-1)\lfloor k/(p-1) \rfloor + r(\lfloor k/(p-1) \rfloor + 1) = (p-1) \cdot \lfloor k/(p-1) \rfloor + r = k$ vertices in D. Hence, D is a kTDS of G of size $\lceil kp/(p-1) \rceil$, as desired. □

We remark that if the lower bound $n_1 \geq \lceil k/(p-1) \rceil$ in the statement of Theorem 7 is relaxed to $n_1 \geq \lfloor k/(p-1) \rfloor$, then it is no longer necessarily true that $\gamma_{\times k,t}(G) = \lceil kp/(p-1) \rceil$. For example, let $p \geq 4$ and let $k = (p-1)\ell + p - 2$ for some integer $\ell \geq 1$. Let G be a complete p-partite graph with $p-2$ partite sets of size $\lfloor k/(p-1) \rfloor = \ell$ and the remaining two partite sets of size $\ell + p - 2$. (Note that $4 \leq p < k$ and $G = K(n_1, n_2, \ldots, n_p)$ is a complete p-partite graph where $\lfloor k/(p-1) \rfloor \leq n_1 \leq n_2 \leq \cdots \leq n_p$.) Then, $\gamma_{\times k,t}(G) = |V(G)| = p(\ell+1) + p - 4 = \lceil kp/(p-1) \rceil + p - 3 > \lceil kp/(p-1) \rceil$.

4. Double total domination

In this section, we consider the 2-tuple total domination number, or, equivalently, the double total domination of a graph. As remarked earlier, for a graph to have a double total dominating set, which we abbreviate by DTDS, its minimum degree is at least 2. Both neighbors of every degree-2 vertex belong to every DTDS of the graph. In particular, we observe that $\gamma_{\times 2,t}(C_n) = n$. However even if we allow vertices of large degree, the presence of degree-2 vertices may force the double total domination to be the order of the graph.

Theorem 8. *If G is a graph of order n with a minimum degree 2 and a maximum degree at most $n/2$, then $\gamma_{\times 2,t}(G) = n$.*

Proof. Clearly, $\gamma_{\times 2,t}(G) \le n$. That this bound is sharp may be seen as follows. Let G be a graph obtained from $k \ge 1$ disjoint paths P_4 on four vertices by forming a clique on the $2k$ vertices of degree 1. Then, G is a graph of order $n = 4k$ with $\delta(G) = 2$, $\Delta(G) = n/2$ and with $n/2$ vertices of maximum degree. The neighbors of the degree-2 vertices in G partition the vertex set $V(G)$. Hence since both neighbors of every degree-2 vertex belong to every DTDS of the graph, we have that $V(G)$ is the only possible DTDS in G, and so $\gamma_{\times 2,t}(G) = n$. □

We show next that if we restrict our attention to bipartite graphs in which the degree-2 vertices form one of the partite sets, then the bound of Theorem 8 can be improved. For this purpose, we need the following lemma about domination in bipartite graphs which is a slight strengthening of a recent result due to Archdeacon et al. [1] (which itself is actually a special case of a result in [2]).

Lemma 9 ([1]). *Let $\delta \in \{1, 2, 3\}$ and let G be a bipartite graph with partite sets (X, Y) whose vertices in Y are of degree at least δ. Then there exists a set $A \subseteq X$ of size at most $|X \cup Y|/(\delta + 1)$ that dominates Y.*

We are now in a position to present the following results.

Theorem 10. *Let G be a bipartite graph of order n with a minimum degree 2. If the vertices of the minimum degree 2 form a partite set of G, then $\gamma_{\times 2,t}(G) \le 9n/10$.*

Proof. Let G have partite sets (X, Y) such that every vertex of degree 2 belongs to X. Thus the vertices in Y are of degree at least 3. Then, $n = |X| + |Y|$ and counting edges between X and Y, we have that $3|Y| \le 2|X| = 2(n - |Y|)$, whence $|Y| \le 2n/5$. Every DTDS of G must contain every vertex in Y in order to double total dominate the set X. The vertices in Y are double total dominated by a subset of vertices in X. By Lemma 9, there exists a set $A \subseteq X$ of size at most $|X \cup Y|/4 = n/4$ that dominates Y. Let $X_1 = X \setminus A$. Let Y_2 be the set of vertices of Y that are adjacent to at least two vertices in A, and let $Y_1 = Y \setminus Y_2$. If $Y = Y_2$, then $A \cup Y$ is a DTDS of G, whence $\gamma_{\times 2,t}(G) \le |A| + |Y| \le n/4 + 2n/5 = 3n/4 < 9n/10$. Hence we may assume that $Y_1 \neq \emptyset$. Let G_1 be the bipartite graph induced by the set $X_1 \cup Y_1$. Since A dominates Y, every vertex in Y_1 is adjacent to exactly one vertex of A and therefore to at least two vertices of X_1. By Lemma 9, there exists a set $B \subseteq X_1$ of size at most $|X_1 \cup Y_1|/3 \le (|X| - |A| + |Y|)/3 = (n - |A|)/3$ that dominates Y_1. Hence, every vertex in Y is double total dominated by the set $A \cup B$, and so the set $A \cup B \cup Y$ is a DTDS of G. We note that $|A \cup B| = |A| + |B| \le |A| + (n - |A|)/3 = (n + 2|A|)/3$. Hence since $|A| \le n/4$, we have that $|A \cup B| \le n/2$. Therefore, $\gamma_{\times 2,t}(G) \le |A| + |B| + |Y| \le n/2 + 2n/5 = 9n/10$. □

Theorem 11. *If G is a cubic bipartite graph of order n, then $\gamma_{\times 2,t}(G) \le 8n/9$.*

Proof. Let G have partite sets (X, Y). Let $|X| = x$ and $|Y| = y$. Since G is a cubic graph, we have that $x = y$. Every regular bipartite graph has a perfect matching. In particular, G has a perfect matching M. Let F be the 2-regular graph obtained from G by removing the edges of the perfect matching M. Then, F consists of disjoint cycles.

Let C be a cycle in F and let $X_C = X \cap V(C)$ and $Y_C = Y \cap V(C)$. Let C have length k. Since C has even length, either $k = 4\ell$ or $k = 4\ell + 2$ for some integer $\ell \ge 1$. If $k = 4\ell$, then there exists a set A_C of ℓ vertices in X_C that dominates the 2ℓ vertices in Y_C. Note that in this case, the subgraph of F induced by the edges incident with these ℓ vertices in D is a disjoint union of ℓ copies of P_3, where the central vertices of the P_3s form the set A_C. If $k = 4\ell + 2$, then there exists a set A_C of $\ell + 1$ vertices in X_C that dominates the $2\ell + 1$ vertices in Y_C. Note that in this case, the subgraph of F induced by the edges incident with the vertices in D is a disjoint union of a path P_5 and ℓ copies of P_3, where the second and fourth vertex on the P_5 and the central vertices of the P_3s form the set A_C. Let A be the union of the sets A_C over all cycles C in the graph F and let H be the subgraph of F induced by the edges incident with the vertices in D. Then, $H = k_1P_3 \cup k_2P_5$ for some integers k_1 and k_2.

Note that $A \subset X$ and $|A| = k_1 + 2k_2$. Further, the set A consists of the central vertices of the P_3-components in H and the second and fourth vertex in every P_5-component in H. The set A dominates the set Y in H (and therefore in G), and so $y = 2k_1 + 3k_2$. Let Y' be the set of central vertices from all the P_5-components in H. Then, every vertex in Y' is dominated by exactly two vertices of A in H, while every vertex in $Y \setminus Y'$ is dominated by exactly one vertex of A in H. Thus, k_2 vertices of Y are dominated by exactly two vertices of A in H, while $2k_1 + 2k_2$ vertices of Y are dominated by exactly one vertex of A in H. If we now add to H the $|A| = k_1 + 2k_2$ edges of the perfect matching M that are incident with vertices in A, we note that resulting graph H' is a subgraph of G and that at most $k_1 + k_2$ vertices of Y are dominated by exactly one vertex of A in H'. Let Y_1 denote the set of vertices of Y that are dominated by exactly one vertex of A in H'. Then, $|Y_1| \le k_1 + k_2$.

Let $X_1 = X \setminus A$ and let G_1 be the (bipartite) subgraph of G induced by the sets $X_1 \cup Y_1$. Since $|X| = x = y = 2k_1 + 3k_2$ while $|A| = k_1 + 2k_2$, we have that $|X_1| = |X| - |A| = k_1 + k_2$. Every vertex in Y_1 is adjacent to exactly two vertices of X_1. By Lemma 9, there exists a set $B \subseteq X_1$ of size at most $|X_1 \cup Y_1|/3 \le 2(k_1 + k_2)/3$ that dominates Y_1. Hence, every vertex in Y is double total dominated by the set $A \cup B$. We note that $|A \cup B| = |A| + |B| \le (k_1 + 2k_2) + 2(k_1 + k_2)/3 = (5k_1 + 8k_2)/6 = 5(2k_1 + 3k_2)/6 + k_2/6 = 5x/6 + k_2/6$. Since $x = 2k_1 + 3k_2 \ge 3k_2$, we have that $k_2 \le x/3$, whence $|A \cup B| \le 5x/6 + x/18 = 8x/9$. Hence we have shown that there exists a set in X of size at most $8x/9$ that double dominates the set Y in G. Similarly, there exists a set in Y of size at most $8y/9$ that double dominates the set X in G. Combining these sets produces a DTDS of G of size at most $8x/9 + 8y/9 = 8n/9$, as desired. □

5. An upper bound

In this section, we present an upper bound for the k-tuple total domination number. Our proofs are along similar lines to that presented by Cockayne and Thomason [3] for the k-tuple domination number.

Lemma 12. *Let G be a graph of order n with a minimum degree $\delta \ge k$ and let $0 \le p \le 1$. Then,*

$$\gamma_{\times k,t}(G) \le \left(p + \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta - i} \right) n.$$

Proof. Let $G = (V, E)$. For each vertex $v \in V$, pick a set N_v consisting of δ neighbors of v. Hence if D is a subset of vertices of G with $|D \cap N_v| \ge k$ for every $v \in V$, then D is a kTDS of G. Form a random set X of the vertices of G by independently placing each vertex into X with probability p. For $i = 0, 1, \ldots, k-1$, we then define the set $V_i = \{v \in V : |N_v \cap X| = i\}$ for $0 \le i \le k-1$. For each $i = 0, 1, \ldots, k-1$, form the set X_i by placing in it $k - i$ vertices from the set $N_v \setminus X$ for each $v \in V_i$. Then, $|X_i| \le (k-i)|V_i|$. Now the set

$$D = X \cup \left(\bigcup_{i=0}^{k-1} X_i \right)$$

is a kTDS of G. By the linearity of expectation,

$$E(|D|) \le E(|X|) + \sum_{i=0}^{k} E(|X_i|) \le E(|X|) + \sum_{i=0}^{k-1} (k-i) E(|V_i|). \tag{1}$$

For each vertex $v \in V$, we have that $P(v \in X) = p$ and $P(v \in V_i) = \binom{\delta}{i} p^i (1-p)^{\delta - i}$. Hence using the well-known fact that for a random subset M of a given finite set N,

$$E(|M|) = \sum_{n \in N} P(n \in M),$$

we have that

$$E(|X|) \le np \quad \text{and} \quad E(|V_i|) \le \binom{\delta}{i} p^i (1-p)^{\delta - i} n.$$

Thus, by Eq. (1), we have that

$$E(|D|) \le np + \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta - i} n.$$

The expectation being an average value, there is consequently a k-tuple total dominating set of G of cardinality at most $E(|D|)$. Hence, $\gamma_{\times k,t} \le E(|D|)$, and the desired upper bound follows. □

We are now in a position to establish the following upper bound on the double total domination number.

Theorem 13. *If G is a graph of order n with a minimum degree $\delta \ge 2$, then*

$$\gamma_{\times 2,t} \le \left(\frac{\ln(\delta + 2) + \ln \delta + 1}{\delta} \right) n.$$

Proof. Since $1 - x \le e^{-x}$ for $x \in \mathbb{R}$, Lemma 12 implies that

$$\begin{aligned} \gamma_{\times 2,t} &\le \left(p + (2 + p\delta)(1-p)^{\delta} \right) n \\ &\le \left(p + (2 + p\delta) e^{-p\delta} \right) n \quad (\text{since for } x \in \mathbb{R},\ 1 - x \le e^{-x}) \\ &\le \left(p + (2 + \delta) e^{-p\delta} \right) n \quad (\text{since } p \le 1). \end{aligned}$$

Setting $p = (\ln(\delta + 2) + \ln \delta)/\delta$, we obtain the desired upper bound. □

For k fixed and sufficiently large minimum degree δ, we have the following upper bound on the k-tuple total domination number of a graph.

Theorem 14. *Let G is a graph of order n with a minimum degree δ. If k is fixed and δ is sufficiently large, then*

$$\gamma_{\times k,t} \leq \left(\frac{\ln\delta + (k-1+\mathrm{o}(1))\ln\ln\delta}{\delta}\right)n.$$

Proof. Since $1 - x \leq \mathrm{e}^{-x}$ for $x \in \mathbb{R}$, Lemma 12 implies that

$$\begin{aligned}\gamma_{\times k,t} &\leq \left(p + \sum_{i=0}^{k-1}(k-i)\binom{\delta}{i}p^i(1-p)^{\delta-i}\right)n \\ &\leq \left(p + k\sum_{i=0}^{k-1}(\delta p)^i \mathrm{e}^{-p(\delta-i)}\right)n \\ &\leq \left(p + k^2\,(\delta p)^{k-1}\,\mathrm{e}^{-p(\delta-k+1)}\right)n.\end{aligned}$$

Now let $\epsilon > 0$ be given and set

$$p = \left(\frac{\ln\delta + (k-1+\epsilon)\ln\ln\delta}{\delta-k+1}\right).$$

Then since

$$(\delta p)^{k-1}\,\mathrm{e}^{-p(\delta-k+1)} = (1+\mathrm{o}(1))(\ln\delta)^{k-1}\delta^{-1}(\ln\delta)^{-(k-1+\epsilon)} < \frac{\epsilon}{\delta}$$

for sufficiently large δ, we have

$$\gamma_{\times k,t} \leq \left(p + \frac{k^2\epsilon}{\delta}\right)n.$$

Since this inequality holds for any given ϵ provided that δ is sufficiently large, we obtain the desired upper bound. □

Acknowledgements

First author's research was supported in part by the South African National Research Foundation.

References

[1] D. Archdeacon, J. Ellis-Monaghan, D. Fischer, D. Froncek, P.C.B. Lam, S. Seager, B. Wei, R. Yuster, Some remarks on domination, J. Graph Theory 46 (2004) 207–210.
[2] V. Chvátal, C. McDiarmid, Small transversals in hypergraphs, Combinatorica 12 (1992) 19–26.
[3] E.J. Cockayne, A.G. Thomason, An upper bound for the *k*-tuple domination number, J. Combin. Math. Combin. Comput. 64 (2008) 251–254.
[4] J. Harant, M.A. Henning, On double domination in graphs, Discuss. Math. Graph Theory 25 (2005) 29–34.
[5] J. Harant, M.A. Henning, A realization algorithm for double domination in graphs, Util. Math. 76 (2008) 11–24.
[6] F. Harary, T.W. Haynes, Double domination in graphs, Ars Combin. 55 (2000) 201–213.
[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.
[9] C.S. Liao, G.J. Chang, Algorithmic aspects of *k*-tuple domination in graphs, Taiwanese J. Math. 6 (2002) 415–420.
[10] C.S. Liao, G.J. Chang, *k*-Tuple domination in graphs, Inform. Process. Lett. 87 (2003) 45–50.