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On blocks with trivial source simple modules

Lluis Puig [a,*], Yuanyang Zhou [b]

[a] *CNRS, Institut de Mathématiques de Jussieu, 6 Av Bizet, 94340 Joinville-le-Pont, France*
[b] *Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China*

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Abstract

Motivated by an observation in Danz and Külshammer (2009) [3], we determine the *source algebra*, and therefore all the structure, of the blocks without *essential Brauer pairs* where the simple modules of all the *Brauer corespondents* have trivial sources.

Keywords: Finite group; Block; Source

1. Introduction

1.1. In [3] Danz and Külshammer, investigating the simple modules for the large Mathieu groups, have found two blocks with noncyclic defect groups of order 9 where all the simple modules have trivial sources and whose source algebras are isomorphic to the source algebras of the corresponding blocks of their *inertial subgroups* [3, Theorems 4.3 and 4.4].[1]

1.2. In their Introduction they note that, in general, any simple module with a trivial source determines an Alperin *weight* [1] — for instance, this follows from [8, Proposition 1.6] — and therefore, in a block with Abelian defect groups and all the simple modules with trivial sources, Alperin's conjecture in [1] forces a canonical bijection between the sets of isomorphism classes of simple modules of the block and of the corresponding block of its *inertial subgroup*. From

* Corresponding author.
E-mail addresses: puig@math.jussieu.fr (L. Puig), zhouyy74@163.com (Y. Zhou).
[1] As a matter of fact, from [12, Corollary 3.6] one easily may find infinitely many examples of such blocks.

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this remark, they raise the question whether, behind this bijection, it should be a true Morita equivalence between both blocks.

1.3. Recently, Zhou proved that, in a suitable inductive context, the answer is in the affirmative [18, Theorem B]; our purpose here is to prove the same fact without any hypothesis on the defect group. In order to explicit our result we need some notation; let p be a prime number, k an algebraically closed field of characteristic p, G a finite group, b a primitive idempotent of the center $Z(kG)$ of the group algebra of G — for short, a *block* of G — and P_γ a *defect pointed group* of b; that is to say, P is a *defect group* of this block in Brauer's terms and γ is a conjugacy class of primitive idempotents i in $(kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$; here, Br_P denotes the usual *Brauer homomorphism*

$$\mathrm{Br}_P : (kG)^P \to (kG)(P) = (kG)^P / \sum_Q (kG)_Q^P \cong kC_G(P) \tag{1.3.1}$$

where Q runs over the set of proper subgroups of P. Recall that the P*-interior algebra* $(kG)_\gamma = i(kG)i$ is called a *source algebra* of b and that its underlying k-algebra is *Morita equivalent* to kGb [8, Definition 3.2 and Corollary 3.5].

1.4. If G' is a second finite group and b' a block of G' admitting the *same* defect group P, it follows from [13, Corollary 7.4 and Remark 7.5] that the source algebras of b and b' are isomorphic — as P*-interior algebras* — if and only if the categories of finitely generated kGb- and $kG'b'$-modules are equivalent to each other *via* a $kGb \otimes_k kG'b'$-module admitting a $P \times P$-stable basis, a fact firstly proved by Leonard Scott [17, Lemma][2]; in this case, we simply say that the blocks b and b' are *identical*. More generally, we say that the blocks b and b' are *stably identical* if the categories of finitely generated kGb- and $kG'b'$-modules are *stably equivalent* to each other — namely, equivalent to each other up to projective modules — *throughout* a $kGb \otimes_k kG'b'$-module admitting a $P \times P$-stable basis.

1.5. Set $N = N_G(P_\gamma)$ — often called the *inertial subgroup* of b — and denote by e the block of $C_G(P)$ determined by *the local point* γ (cf. (1.3.1)). Recall that e is also a block of N and that $k\bar{C}_G(P)\bar{e}$ is a simple k-algebra, where we set $\bar{C}_G(P) = C_G(P)/Z(P)$ and denote by $\bar{e}$ the image of e in $k\bar{C}_G(P)$; then, the action of N on the simple k-algebra $k\bar{C}_G(P)\bar{e}$ determines a central k^*-extension $\hat{E}$ of $E = N/P \cdot C_G(P)$ — often called the *inertial quotient* of b. Setting $\hat{L} = P \rtimes \hat{E}^\circ$ for a lifting of the canonical homomorphism $\hat{E} \to \mathrm{Out}(P)$ to $\mathrm{Aut}(P)$, it follows from [11, Proposition 14.6] that the corresponding *twisted* group algebra $k_*\hat{L}$ is isomorphic to a source algebra of the block e of N.

1.6. Recall that a *Brauer* (b, G)*-pair* (Q, f) is formed by a p-subgroup Q of G such that $\mathrm{Br}_Q(b) \neq 0$ and by a block f of $C_G(Q)$ fulfilling $\mathrm{Br}_Q(b)f = f$ [2, Definition 1.6]; note that f is also a block for any subgroup H of $N_G(Q, f)$ containing $C_G(Q)$. Thus, (P, e) is a Brauer (b, G)-pair and, as a matter of fact, there is $x \in G$ such that [2, Theorem 1.14]

$$(Q, f) \subset (P, e)^x. \tag{1.6.1}$$

[2] Strictly speaking, in [17, Lemma] Scott only considers the case where the *block algebras* kGb and $kG'b'$ are isomorphic.

Then, the *Frobenius category* $\mathcal{F}_{(b,G)}$ of b [16, 3.1] is the category where the objects are the Brauer (b, G)-pairs (Q, f) and the morphisms are the homomorphisms between the corresponding p-groups induced by the *inclusion* between Brauer (b, G)-pairs and the G-conjugation.

1.7. For short, let us say that the block b is *inertially controlled* whenever the Frobenius categories $\mathcal{F}_{(b,G)}$ and $\mathcal{F}_{\hat{L}}$ are equivalent to each other — note that the unity element is the unique block of $\hat{L}$ and we omit it, writing $\mathcal{F}_{\hat{L}}$ instead of $\mathcal{F}_{(1,\hat{L})}$; moreover, since $k_*\hat{L}$ is isomorphic to a source algebra of the block e of N, the Frobenius categories $\mathcal{F}_{(e,N)}$ and $\mathcal{F}_{\hat{L}}$ are always equivalent to each other, so that e is always *inertially controlled*. Similarly, let us say that b is a *block of G with trivial source simple modules* if all the simple kGb-modules have trivial sources.

Theorem 1.8. *With the notation above, the source algebra $(kG)_\gamma$ of the block b of G is isomorphic to $k_*\hat{L}$ if and only if the block b of G is inertially controlled and, for any Brauer (b, G)-pair (Q, f) contained in (P, e), f is a block of $C_G(Q) \cdot N_P(Q)$ with trivial source simple modules.*

1.9. The main tools in proving this result are Linckelmann's Equivalence Criterion on *stable equivalences* [7, Proposition 2.5], the *strict semicovering* homomorphisms that we recall in Section 3 below, and the general criterion on *stable equivalences* in [13, Theorem 6.9], which in our context is summarized by the following result.

Theorem 1.10. *With the notation above, the blocks b of G and e of N are stably identical if and only if, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), the block f of $C_G(Q)$ admits $C_P(Q)$ as a defect p-subgroup and a source algebra isomorphic to $k_*(C_{\hat{L}}(Q))$.*

1.11. Note that $C_{\hat{E}}(Q)$ acts faithfully on $C_P(Q)$ since any $(p'$-$)$subgroup of $C_{\hat{E}}(Q)$ acting trivially on $C_P(Q)$ still acts trivially on P [5, Ch. 5, Theorem 3.4], and that we actually have

$$C_{\hat{L}}(Q) \cong C_P(Q) \rtimes C_{\hat{E}^\circ}(Q). \tag{1.11.1}$$

Moreover, if the defect group P is Abelian then, for any Brauer (b, G)-pair (Q, f) contained in (P, e), P is clearly a defect group of the block f of $C_G(Q)$. Finally, although we only work over k, Lemma 7.8 in [10] allows us to lift all the isomorphisms between *block source algebras* over k above to the corresponding block source algebras over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero having the *residue field* k.

2. Notation and quoted results

2.1. Let A be a finitely dimensional k-algebra; we denote by 1_A the unity element of A and by A^* the multiplicative group of A. An algebra homomorphism f from A to another finitely dimensional k-algebra A' is not necessarily unitary and we say that f is an *embedding* whenever

$$\mathrm{Ker}(f) = \{0\} \quad \text{and} \quad \mathrm{Im}(f) = f(1_A)A'f(1_A). \tag{2.1.1}$$

Following Green, a *G-algebra* is a finitely dimensional k-algebra A endowed with a G-action; recall that, for any subgroup H of G, a *point* α of H on A is an $(A^H)^*$-conjugacy class of primitive idempotents of A^H and the pair H_α is called a *pointed group* on A [8, 1.1]; we denote by $A(H_\alpha)$ the *simple quotient* of A^H determined by α. A second pointed group K_β on A is *contained* in H_α if $K \subset H$ and, for any $i \in \alpha$, there is $j \in \beta$ such that [8, 1.1]

$$ij = j = ji. \tag{2.1.2}$$

2.2. Following Broué, for any p-subgroup P of G we consider the *Brauer quotient* and the *Brauer homomorphism*

$$\mathrm{Br}_P^A : A^P \to A(P) = A^P \Big/ \sum_Q A_Q^P, \tag{2.2.1}$$

where Q runs over the set of proper subgroups of P and A_Q^P is the ideal formed by the sums $\sum_u a^u$ where a runs over A^Q and $u \in P$ over a set of representatives for P/Q; we call *local* any point γ of P on A not contained in $\mathrm{Ker}(\mathrm{Br}_P^A)$ [8, 1.1]. Let us say that A is a *p-permutation G-algebra* if a Sylow p-subgroup of G stabilizes a basis of A; in this case, recall that if P is a p-subgroup of G and Q a normal subgroup of P then the corresponding Brauer homomorphisms induce a k-algebra isomorphism [2, Proposition 1.5]

$$\big(A(Q)\big)(P/Q) \cong A(Q). \tag{2.2.2}$$

Obviously, the group algebra $A = kG$ is a p-permutation G-algebra and the composition of the inclusion $kC_G(Q) \subset A^Q$ with Br_Q^A is an isomorphism which allows us to identify $kC_G(Q)$ with $A(Q)$; then any local point δ of Q on kG determines a block b_δ of $kC_G(Q)$ such that $b_\delta \,\mathrm{Br}_Q^{kG}(\delta) = \mathrm{Br}_Q^{kG}(\delta)$.

2.3. We are specially interested in the G-algebras A endowed with a group homomorphism $\rho : G \to A^*$ inducing the action of G on A — called *G-interior algebras*. In this case, for any pointed group H_α on A and any $i \in \alpha$, the subalgebra $A_\alpha = iAi$ has a structure of *H-interior algebra* mapping $y \in H$ on $\rho(y)i = i\rho(y)$; moreover, setting $x \cdot a \cdot y = \rho(x)a\rho(y)$ for any $a \in A$ and any $x, y \in G$, a G-interior algebra homomorphism from A to another G-interior algebra A' is a G-algebra homomorphism $f : A \to A'$ fulfilling

$$f(x \cdot a \cdot y) = x \cdot f(a) \cdot y. \tag{2.3.1}$$

We also consider the *mixed* situation of an *H-interior G-algebra* B where H is a subgroup of G and B is a G-algebra endowed with a *compatible* H-interior algebra structure, in such a way that the kG-module $B \otimes_{kH} kG$ endowed with the product

$$(a \otimes x).(b \otimes y) = ab^{x^{-1}} \otimes xy, \tag{2.3.2}$$

for any $a, b \in B$ and any $x, y \in G$, and with the group homomorphism mapping $x \in G$ on $1_B \otimes x$ becomes a G-interior algebra — simply noted $B \otimes_H G$. For instance, for any p-subgroup P of G, $A(P)$ is a $C_G(P)$-interior $N_G(P)$-algebra.

2.4. In particular, if H_α and K_β are two pointed groups on A, we say that an injective group homomorphism $\varphi : K \to H$ is an *A-fusion from K_β to H_α* whenever there is a K-interior algebra *embedding*

$$f_\varphi : A_\beta \to \mathrm{Res}^H_K(A_\alpha) \tag{2.4.1}$$

such that the inclusion $A_\beta \subset A$ and the composition of f_φ with the inclusion $A_\alpha \subset A$ are A^*-conjugate; we denote by $F_A(K_\beta, H_\alpha)$ the set of H-conjugacy classes of A-fusions from K_β to H_α and we write $F_A(H_\alpha)$ instead of $F_A(H_\alpha, H_\alpha)$. If $A_\alpha = iAi$ for $i \in \alpha$, it follows from [9, Corollary 2.13] that we have a group homomorphism

$$F_A(H_\alpha) \to N_{A^*_\alpha}(H \cdot i)/H \cdot \big(A^H_\alpha\big)^*. \tag{2.4.2}$$

2.5. Let b be a block of G; then $\alpha = \{b\}$ is a *point* of G on kG and we let P_γ be a local pointed group contained in G_α which is maximal with respect to the inclusion of pointed groups; namely P_γ is a *defect pointed group* of b. Note that, for any p-subgroup Q of G and any subgroup H of $N_G(Q)$ containing Q, we have

$$\mathrm{Br}_Q\big((kG)^H\big) = \big(kC_G(Q)\big)^H; \tag{2.5.1}$$

thus, we have an injection from the set of points of H on $kC_G(Q)$ to the set of points of H on kG such that the corresponding points β° and β fulfill $\mathrm{Br}^{kG}_Q(\beta) = \mathrm{Br}^{kC_G(Q)}_Q(\beta^\circ)$; moreover, this injection preserves the localness and the inclusion of pointed groups [16, 1.19]. In particular, if P is Abelian and Q_δ is a local pointed group on kG contained in P_γ, a point μ of $C_G(Q)$ on kG fulfilling

$$Q_\delta \subset P_\gamma \subset C_G(Q)_\mu \tag{2.5.2}$$

is the *unique* point determined by the block b_δ of $C_G(Q)$ and therefore P is a defect group of this block (cf. 1.8).

2.6. Set $e = b_\gamma$ and $N = N_G(P_\gamma)$; thus, e is a block of N, it determines a point ν of N on kG (cf. 2.5) and P is a defect group of this block; moreover, we have (cf. (1.3.1))

$$(kN)(P) \cong kC_N(P) = kC_G(P) \cong (kG)(P), \tag{2.6.1}$$

there is a local point $\hat\gamma$ of P on $kN \subset kG$ such that $\mathrm{Br}_P(\hat\gamma) = \mathrm{Br}_P(\gamma)$ and it follows from [4, Proposition 4.10] that, for any $\hat\imath \in \hat\gamma$ and any $\ell \in \nu$, the idempotent $\hat\imath\ell$ belongs to γ and that the multiplication by ℓ defines a unitary P-interior algebra homomorphism (cf. 1.5)

$$k_*\hat L \cong (kN)_{\hat\gamma} \to (kG)_\gamma \tag{2.6.2}$$

which is actually a *direct injection* of $k(P \times P)$-modules.

2.7. For any pair of *local pointed groups* Q_δ and R_ε on kG, we denote by $E_G(R_\varepsilon, Q_\delta)$ the set of Q-conjugacy classes of group homomorphisms $\varphi : R \to Q$ induced by conjugation with some $x \in G$ fulfilling $R_\varepsilon \subset (Q_\delta)^x$, and write $E_G(Q_\delta)$ instead of $E_G(Q_\delta, Q_\delta)$; it follows from [9, Theorem 3.1] that

$$E_G(R_\varepsilon, Q_\delta) = F_{kG}(R_\varepsilon, Q_\delta) \tag{2.7.1}$$

and if P_γ contains Q_δ and R_ε then they can be considered as local pointed groups on $(kG)_\gamma$ and it follows from [9, Proposition 2.14] that

$$E_G(R_\varepsilon, Q_\delta) = F_{kG}(R_\varepsilon, Q_\delta) = F_{(kG)_\gamma}(R_\varepsilon, Q_\delta). \tag{2.7.2}$$

In particular, it is clear that $N_G(Q_\gamma)/Q \cdot C_G(Q) \cong E_G(Q_\delta)$ and the action of $N_G(Q_\delta)$ on the simple k-algebra $(kG)(Q_\delta)$ (cf. 2.1) determines a central k^*-extension $\hat{E}_G(Q_\delta)$ of $E_G(Q_\delta)$.

2.8. Recall that a Brauer (b, G)-pair (Q, f) is called *selfcentralizing* if, setting $\bar{C}_G(Q) = C_G(Q)/Z(Q)$ and denoting by $\bar{f}$ the image of f in $k\bar{C}_G(Q_\delta)$, the k-algebra $k\bar{C}_G(Q)\bar{f}$ is simple [14, 1.6], so that $k\bar{C}_G(Q)\bar{f} \cong (kG)(Q_\delta)$ for a local point δ of Q on kG clearly determined by f; we also say that Q_δ is a *selfcentralizing pointed group* on kG; thus we have a bijection, which preserves *inclusion* and *G-conjugacy*, between the sets of selfcentralizing pointed groups on kGb and of selfcentralizing Brauer (b, G)-pairs. Moreover, according to [14, Theorem A.9], an *essential pointed group* on kG is a selfcentralizing pointed group Q_δ on kG fulfilling the following condition.

2.8.1. *$E_G(Q_\delta)$ admits a proper subgroup M such that p divides $|M|$ and does not divide $|M \cap M^\sigma|$ for any $\sigma \in E_G(Q_\delta) - M$.*

Then, from [14, Corollary A.12] and [16, Corollary 5.14], it is not difficult to prove that the block b of G is *inertially controlled* (cf. 1.7) if and only if there are *no essential pointed groups* on kGb; thus, if the defect group P is Abelian the block b of G is inertially controlled.

Lemma 2.9. *With the notation above, the block b of G is inertially controlled if and only if, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), the block f of $C_G(Q)$ admits $C_P(Q)$ as a defect group and it is inertially controlled.*

Proof. Firstly assume that b is inertially controlled; let (Q, f) be a Brauer (b, G)-pair contained in (P, e) and choose a maximal Brauer $(f, Q \cdot C_G(Q))$-pair (R, g); since (Q, f) is also a Brauer $(f, Q \cdot C_G(Q))$-pair, (R, g) necessarily contains (Q, f) (cf. (1.6.1)) and therefore it is also a Brauer (b, G)-pair; hence, there is $x \in G$ such that (cf. (1.6.1))

$$(Q, f)^x \subset (R, g)^x \subset (P, e) \tag{2.9.1}$$

and therefore we get $x = zn$ for suitable $z \in C_G(Q)$ and $n \in N$; so that the maximal Brauer $(f, Q \cdot C_G(Q))$-pair $(R, g)^z$ is contained in (P, e).

Moreover, if (T, h) is a Brauer $(f, C_G(Q))$-pair, it is clear that $(Q \cdot T, h)$ is a Brauer (b, G)-pair; conversely, by the argument above, $(C_P(Q), g^x)$ is a maximal Brauer $(f, C_G(Q))$-pair; then, if $(C_P(Q), g^x)$ contains (T, h) and $(T, h)^z$ with $z \in C_G(Q)$, it is easily checked that (P, e) contains $(Q \cdot T, h)$ and $(Q \cdot T, h)^z$ and therefore we still get $z = wm$ for suitable $w \in C_G(Q \cdot T)$ and $m \in N$, so that m actually belongs to $C_N(Q)$; consequently, since we have $N/C_G(P) \cong L/C_L(P)$, the block f of $C_G(Q)$ is inertially controlled.

Conversely, arguing by contradiction, assume that Q_δ is an essential pointed group contained in P_γ. According to [10, Lemma 3.10], we may assume that the image of $N_P(Q)$ is a Sylow p-subgroup of $E_G(Q_\delta)$ and, since a proper subgroup M of $E_G(Q_\delta)$ fulfilling condition 2.8.1 above contains a Sylow p-subgroup of $E_G(Q_\delta)$, we still may assume that M contains the image of $N_P(Q)$. Moreover, it follows again from [10, Lemma 3.10] that there is a local pointed group R_ε containing and normalizing Q_δ such that its image in $E_G(Q_\delta)$ is not contained in M; then, R centralizes some nontrivial subgroup Z of $Z(Q)$ and, denoting by f the unique block of $H = C_G(Z)$ such that (P, e) contains (Z, f), it follows from our hypothesis that $H \cap P$ is a defect group of this block.

Consequently, denoting by h the block of $C_G(H \cap P)$ such that (P, e) contains $(H \cap P, h)$, this pair is a maximal Brauer (f, H)-pair; moreover, H contains R and $C_G(Q)$, and in particular we have

$$(kH)(Q) \cong (kG)(Q), \tag{2.9.2}$$

so that $\mathrm{Br}_Q(\delta)$ determines a local point $\hat{\delta}$ of Q on kH fulfilling

$$E_H(Q_{\hat{\delta}}) \subset E_G(Q_\delta); \tag{2.9.3}$$

then, applying again [10, Lemma 3.10], we may assume that the image of $N_{H\cap P}(Q)$ in the intersection $E_H(Q_{\hat{\delta}}) \cap M$ is a Sylow p-subgroup of $E_H(Q_{\hat{\delta}})$, whereas this intersection does not contain the image of R; hence, $Q_{\hat{\delta}}$ is an essential pointed group on kHf, which contradicts our hypothesis. We are done. $\square$

3. Strict semicovering homomorphism

3.1. Let P be a finite p-group, B and $\hat{B}$ two P-algebras and $g : B \to \hat{B}$ a unitary P-algebra homomorphism; we say that g is a *strict semicovering* if, for any subgroup Q of P, we have $\mathrm{Ker}(g)^Q \subset J(B^Q)$ and the image $g(j)$ of a primitive idempotent j of B^Q is still primitive in $\hat{B}^Q$ [6, 3.10]; namely if g induces a homomorphism from the maximal semisimple quotient of B^Q to the maximal semisimple quotient of $\hat{B}^Q$, mapping primitive idempotents on primitive idempotents.

3.2. In other words, g is a strict semicovering if and only if, for any subgroup Q of P, it induces a surjective map from the set of points of Q on B to the set of points of Q on $\hat{B}$ and, for any pair of mutually corresponding such points δ and $\hat{\delta}$, it induces a k-algebra embedding [6, 3.10]

$$g(Q_\delta) : B(Q_\delta) \to \hat{B}(Q_{\hat{\delta}}). \tag{3.2.1}$$

3.3. Explicitly, if g is a strict semicovering then, for any pointed group Q_δ on B, there is a unique point $\hat{\delta}$ of Q on $\hat{B}$ fulfilling $g(\delta) \subset \hat{\delta}$; moreover, this correspondence preserves *inclusion* and *localness* [6, Proposition 3.15]. The composition of strict semicoverings is clearly a strict semicovering but, more precisely, the *strictness* provides a converse [6, Proposition 3.6].

Proposition 3.4. *With the notation above, let $\hat{g} : \hat{B} \to \hat{\hat{B}}$ a second unitary P-algebra homomorphism. Then, $\hat{g} \circ g$ is a strict semicovering if and only if $\hat{g}$ and g are so.*

3.5. The fact for a P-algebra homomorphism of being a strict semicovering is essentially of "local" nature as it shows the following result [6, Theorem 3.16].

Theorem 3.6. *With the notation above, the unitary P-algebra homomorphism g is a strict semicovering if and only if, for any p-subgroup Q of P, the $\{1\}$-algebra homomorphism*

$$g(Q) : B(Q) \to \hat{B}(Q) \tag{3.6.1}$$

induced by g is a strict semicovering.

3.7. Here, we may restrict ourselves to consider the following situation. Let G be a finite group, H a normal subgroup of G such that G/H is a p-group, P a p-subgroup of G and Z a subgroup of $H \cap P$ normal in G and central in H; set $\bar{G} = G/Z$ and $\bar{P} = P/Z$.

Proposition 3.8. *With the notation above, the canonical $\bar{P}$-algebra homomorphism $kH \to k\bar{G}$ is a semicovering.*

Proof. For any subgroup $\bar{Q} = Q/Z$ of $\bar{P}$, we have (cf. (1.3.1))

$$(kH)(\bar{Q}) \cong kC_H(Q) \quad \text{and} \quad (k\bar{G})(\bar{Q}) \cong kC_{\bar{G}}(\bar{Q}); \tag{3.8.1}$$

thus, a p'-subgroup K of the converse image of $C_{\bar{G}}(\bar{Q})$ centralizes Q [5, Ch. 5, Theorem 3.2] and therefore it is contained in $C_H(Q)$; that is to say, setting $\overline{C_H(Q)} = C_H(Q)/Z$, the quotient $C_{\bar{G}}(\bar{Q})/\overline{C_H(Q)}$ is a p-group.

Then, it follows from Lemma 3.9 below that any simple $kC_{\bar{G}}(\bar{Q})$-module M has the form

$$M \cong \mathrm{Ind}_{kC_{\bar{G}}(\bar{Q})_N}^{kC_{\bar{G}}(\bar{Q})}(\hat{N}) \tag{3.8.2}$$

where N is a simple $k\overline{C_H(Q)}$-module, $kC_{\bar{G}}(\bar{Q})_N$ the stabilizer in $kC_{\bar{G}}(\bar{Q})$ of the isomorphism class of N and $\hat{N}$ the extended $kC_{\bar{G}}(\bar{Q})_N$-module. Moreover, any simple $kC_H(Q)$-module is also a simple $k\overline{C_H(Q)}$-module and it appears in some simple $kC_{\bar{G}}(\bar{Q})$-module. All this amounts to saying that the canonical $\{1\}$-algebra homomorphism

$$kC_H(Q) \to kC_{\bar{G}}(\bar{Q}) \tag{3.8.3}$$

induces a homomorphism between the corresponding semisimple quotients preserving primitivity and then it suffices to apply Theorem 3.6. □

Lemma 3.9. *Let X be a finite group and Y a normal subgroup of X such that X/Y is a p-group. Then, any simple kY-module N can be extended to the stabilizer X_N in X of the isomorphism class of N and, denoting by $\hat{N}$ the extended kX_N-module, $\mathrm{Ind}_{X_N}^X(\hat{N})$ is a simple kX-module. Moreover, all the simple kX-modules have this form.*

Proof. Straightforward. □

Corollary 3.10. *With the same notation, let $\alpha = \{b\}$ be a point of G on kH and assume that P_γ is a defect pointed group of G_α; denote by $\bar{b}$ and $\bar{\gamma}$ the respective images in $k\bar{G}$ of b and γ. Then, b and $\bar{b}$ are respective blocks of G and $\bar{G}$, γ and $\bar{\gamma}$ are respectively contained in local points $\tilde{\gamma}$ and $\tilde{\bar{\gamma}}$ of P and $\bar{P}$ on kG and $k\bar{G}$, and moreover $P_{\tilde{\gamma}}$ and $\bar{P}_{\tilde{\bar{\gamma}}}$ are respective defect pointed groups* *of these blocks. In particular, setting $Q = H \cap P$, $\bar{H} = H/Z$ and $\bar{Q} = Q/Z$, the respective P- and $\bar{P}$-interior algebras*

$$(kH)_\gamma \otimes_Q P = \bigoplus_u (kH)_\gamma \cdot u \quad \textit{and} \quad (k\bar{H})_{\bar{\gamma}} \otimes_{\bar{Q}} \bar{P} = \bigoplus_{\bar{u}} (k\bar{H})_{\bar{\gamma}} \cdot \bar{u}, \tag{3.10.1}$$

where $u \in P$ runs over a set of representatives for P/Q and $\bar{u}$ is the image in $\bar{P}$ of u, are respective source algebras of these blocks.

Proof. Since any block of G is a k-linear combination of p'-elements of G, kH contains all the blocks of G and therefore b is primitive in $Z(kG)$; moreover, it is easily checked that $(kH)^G$ maps surjectively onto $(k\bar{H})^{\bar{G}}$ and therefore $\bar{\alpha} = \{\bar{b}\}$ is also a point of $\bar{G}$ on $k\bar{H}$, so that $\bar{b}$ is a block of $\bar{G}$.

Moreover, it follows from Propositions 3.4 and 3.8 that the canonical $\bar{P}$-algebra homomorphisms

$$kH \to kG \quad \text{and} \quad kH \to k\bar{G} \tag{3.10.2}$$

are strict semicovering; hence, γ is contained in a local point $\tilde{\gamma}$ of P on kG and $\bar{\gamma}$ in a local point $\tilde{\bar{\gamma}}$ of $\bar{P}$ on $k\bar{G}$; we claim that $P_{\tilde{\gamma}}$ and $\bar{P}_{\tilde{\bar{\gamma}}}$ are maximal local pointed groups on kG and $k\bar{G}$ respectively.

Indeed, since the canonical homomorphism $kH \to kG$ is a semicovering, a local pointed group $P'_{\tilde{\gamma}'}$ on kG containing $P_{\tilde{\gamma}}$ comes from a local pointed group $P'_{\gamma'}$ on kH and it is easily checked that $P'_{\gamma'} \subset G_\alpha$, so that we have $P'_{\gamma'} \subset (P_\gamma)^x$ for a suitable $x \in G$, which forces $P'_{\gamma'} = P_\gamma$; since $\bar{\alpha}$ is a point of $\bar{G}$ on $k\bar{H}$, the same argument proves that $\bar{P}_{\tilde{\bar{\gamma}}}$ is a maximal local pointed group on $k\bar{G}$.

The proof of the last statement is straightforward. We are done. □

4. Stable embeddings: the proof of Theorem 1.10

4.1. Let G be a finite group and A a G-interior algebra; we say that a point β of H on A is *projective* if it is contained in A_1^H or, equivalently, if it has a trivial defect group. Let $\hat{A}$ be a second G-interior algebra and $f : \hat{A} \to A$ a G-interior algebra homomorphism; following [13, 6.4], we say that f is a *stable embedding* if $\mathrm{Ker}(f)$ and $f(1_{\hat{A}})Af(1_{\hat{A}})/f(\hat{A})$ are projective $k(G \times G)$-modules or, equivalently, if the class of the $k(G \times G)$-module homomorphism

$$f : \hat{A} \to f(1_{\hat{A}})Af(1_{\hat{A}}) \tag{4.1.1}$$

in the *stable category* of $k(G \times G)$-modules is an isomorphism.

4.2. In this case, if f is unitary, the exact sequence of $k(G \times G)$-modules

$$0 \to \mathrm{Ker}(f) \to \hat{A} \xrightarrow{f} A \to A/f(\hat{A}) \to 0 \tag{4.2.1}$$

is split [13, 6.4.1] and therefore, for any subgroup H of G, f induces a *$C_G(H)$-interior $N_G(H)$-algebra isomorphism*

$$\hat{A}^H/\hat{A}_1^H \cong A^H/A_1^H; \tag{4.2.2}$$

in particular, f induces a bijection between the sets of *nonprojective points* of H on $\hat{A}$ and on A and, for any pair of corresponding nonprojective points $\hat{\beta}$ and β, we have $N_G(H_{\hat{\beta}}) = N_G(H_\beta)$, f induces a $C_G(H)$-interior $N_G(H_\beta)$-algebra isomorphism [13, 4.6.2]

$$f(H_\beta) : \hat{A}(H_{\hat{\beta}}) \cong A(H_\beta) \tag{4.2.3}$$

and this isomorphism determines a central k^*-extension isomorphism

$$\hat{f}(H_\beta) : \hat{\hat{N}}_G(H_{\hat{\beta}}) \cong \hat{\hat{N}}_G(H_\beta). \tag{4.2.4}$$

Moreover, this correspondence preserves *inclusion*, *localness* and *fusions*.

4.3. We are ready to prove Theorem 1.10; thus, b is a block of G, P_γ is a defect pointed group of b, we set $N = N_G(P_\gamma)$, e is the corresponding block of N, ν is the point of N on kG determined by e, $\hat{\gamma}$ is the local point of P on kN fulfilling $\mathrm{Br}_P(\hat{\gamma}) = \mathrm{Br}_P(\gamma)$ and we denote by (cf. (2.6.2))

$$g : (kN)_{\hat{\gamma}} \to (kG)_\gamma \tag{4.3.1}$$

the unitary P-interior algebra homomorphism determined as above by the multiplication by $\ell \in \nu$; note that the restriction throughout g induces a functor from the category of kGb-modules to the category of kNe-modules which actually coincides with the functor determined by the $k(N \times G)$-module $\ell(kG)$. Firstly, we prove a stronger form of the converse part.

Proposition 4.4. *With the notation above, assume that the blocks b of G and e of N are stably identical. Then, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), $N_P(Q)$ is a defect group of the block f of $C_G(Q) \cdot N_P(Q)$ and a source algebra of this block is isomorphic to $k_*(C_{\hat{L}}(Q) \cdot N_P(Q))$ via an isomorphism inducing a $C_P(Q)$-interior algebra isomorphism from a source algebra of the block f of $C_G(Q)$ onto $k_*C_{\hat{L}}(Q)$.*

Proof. We can apply Theorem 6.9 and Corollary 7.4 in [13] to the Morita stable equivalences between b and e, and between e and b; in our present situation, b and e have the same defect group P and, with the notation in [13], we may assume that $\ddot{P} = P$ and then $\ddot{S} = k$ is the trivial P-interior algebra and $\sigma = \sigma' = \mathrm{id}_P$. Consequently, it follows from [13, 7.6.6] that the block b of G is *inertially controlled* and, for any nontrivial subgroup Q of P, from [13, 6.9.1] we get *$C_P(Q)$-interior $N_P(Q)$-algebra embeddings*

$$(kG)_\gamma(Q) \to (kN)_{\hat{\gamma}}(Q) \quad \text{and} \quad (kN)_{\hat{\gamma}}(Q) \to (kG)_\gamma(Q), \tag{4.4.1}$$

so that both are isomorphisms.

Now, we have *$C_P(Q)$-interior $N_P(Q)$-algebra isomorphisms*

$$(kG)_\gamma(Q) \cong (kN)_{\hat{\gamma}}(Q) \cong k_*C_{\hat{L}}(Q) \tag{4.4.2}$$

and therefore the unity element is primitive in the k-algebra

$$(kG)_\gamma(Q)^{C_P(Q)} \cong (kN)_{\hat{\gamma}}(Q)^{C_P(Q)}; \tag{4.4.3}$$

thus, denoting by f the block of $C_G(Q)$ such that $(Q, f) \subset (P, e)$ [2, Theorem 1.8], it is quite clear that the $C_P(Q)$-interior algebra $(kG)_\gamma(Q)$ is a source algebra of this block and it is indeed isomorphic to $k_* C_{\hat{L}}(Q)$.

Moreover, it follows from Corollary 3.10 above, applied to the groups $C_G(Q) \cdot N_P(Q)$ and $C_G(Q)$, that $N_P(Q)$ is a defect group of the block f of $C_G(Q) \cdot N_P(Q)$ and that the $N_P(Q)$-interior algebra

$$(kG)_\gamma(Q) \otimes_{C_P(Q)} N_P(Q) = \bigoplus_u (kG)_\gamma(Q) \cdot u, \tag{4.4.4}$$

where $u \in N_P(Q)$ runs over a set of representatives for $N_P(Q)/C_P(Q)$, is a source algebra of this block; thus, according to isomorphisms (4.4.2), this $N_P(Q)$-interior algebra is isomorphic to $k_*(C_{\hat{L}}(Q) \cdot N_P(Q))$. We are done. □

Theorem 4.5. *With the notation above, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), assume that $C_P(Q)$ is a defect group of the block f of $C_G(Q)$ and that a source algebra of this block is isomorphic to $k_*(C_{\hat{L}}(Q))$. Then, g is a stable embedding.*

Proof. Since g is a *direct injection* of $k(P \times P)$-modules (cf. 2.6), we have $\mathrm{Ker}(g) = \{0\}$ and the quotient

$$M = (kG)_\gamma / g\big((kN)_{\hat{\gamma}}\big) \tag{4.5.1}$$

is a direct summand of $(kG)_\gamma$ as $k(P \times P)$-modules; hence, since $(kG)_\gamma$ is a permutation $k(P \times P)$-module, it suffices to prove that $M(W) = \{0\}$ for any nontrivial subgroup W of $P \times P$. Actually, we have $(kG)(W) = \{0\}$ unless

$$W = \Delta_\varphi(Q) = \big\{\big(u, \varphi(u)\big)\big\}_{u \in Q} \tag{4.5.2}$$

for some subgroup Q of P and some group homomorphism $\varphi : Q \to P$ induced by the conjugation by some $x \in G$.

More precisely, choosing $i \in \gamma$, the multiplication by x on the right determines a k-linear isomorphism

$$(kG)_\gamma\big(\Delta_\varphi(Q)\big) \cong \big(i(kG)ix\big)(Q); \tag{4.5.3}$$

thus, denoting by f the block of $C_G(Q)$ such that (P, e) contains (Q, f) or, equivalently, such that $f\mathrm{Br}_Q(i) \neq 0$, if we have $(kG)_\gamma(\Delta_\varphi(Q)) \neq \{0\}$, we still have $f\mathrm{Br}_Q(i^x) \neq 0$ or, equivalently, $(P, e)^x$ contains (Q, f) which amounts to saying that $\varphi : Q \to P$ is an $\mathcal{F}_{(b,G)}$-morphism (cf. 2.9). Hence, it suffices to prove that, for any nontrivial subgroup Q of P and any $\mathcal{F}_{(b,G)}$-morphism $\varphi : Q \to P$, we have $M(\Delta_\varphi(Q)) = \{0\}$; but, always since g is a *direct injection* of $k(P \times P)$-modules, g induces an injective homomorphism

$$g\big(\Delta_\varphi(Q)\big) : (kN)_{\hat{\gamma}}\big(\Delta_\varphi(Q)\big) \to (kG)_\gamma\big(\Delta_\varphi(Q)\big); \tag{4.5.4}$$

consequently, it suffices to prove that

$$\dim\bigl((kN)_{\hat\gamma}\bigl(\Delta_\varphi(Q)\bigr)\bigr)=\dim\bigl((kG)_\gamma\bigl(\Delta_\varphi(Q)\bigr)\bigr) \tag{4.5.5}$$

and we argue by induction on $|P:Q|$.

Since we have a P-interior algebra isomorphism $k_*\hat L\cong(kN)_{\hat\gamma}$, we still have

$$(k_*\hat L)\bigl(\Delta_\varphi(Q)\bigr)\cong(kN)_{\hat\gamma}\bigl(\Delta_\varphi(Q)\bigr); \tag{4.5.6}$$

moreover, it is clear that $N_P(Q)$ centralizes a nontrivial subgroup Z of $Z(Q)$ and then, according to our hypothesis, the $C_P(Z)$-interior algebra $k_*C_{\hat L}(Z)$ is isomorphic to a source algebra of the block h of $C_G(Z)$ such that (P,e) contains (Z,h); in particular, setting $H=C_G(Z)$, (Q,f) is also a Brauer (h,H)-pair, we have $C_H(Q)=C_G(Q)$ and $N_P(Q)$ remains a defect group of the block f of $C_H(Q)\cdot N_P(Q)$. Consequently, it easily follows from Proposition 4.4 above, applied to the block h of H, that a source algebra of the block f of $C_G(Q)\cdot N_P(Q)$ is isomorphic to $k_*(C_{\hat L}(Q)\cdot N_P(Q))$.

At this point, we claim that in $(kG)_\gamma(Q)^{N_P(Q)}$ the unity element is primitive; since the point γ is local, it follows from isomorphism (2.2.2) that there is a primitive idempotent $\bar\ell$ of $(kG)_\gamma(Q)^{N_P(Q)}$ determining a local point of $N_P(Q)$ on $(kG)_\gamma(Q)$; but, according to our induction hypothesis, for any subgroup R of $N_P(Q)$ strictly containing Q we may assume that $(kN)_{\hat\gamma}(R)\cong(kG)_\gamma(R)$ (cf. (4.5.4)) and, since $\mathrm{Br}_{\bar R}^{(kG)_\gamma(Q)}(\bar\ell)\neq 0$ where we set $\bar R=R/Q$ (cf. isomorphism (2.2.2)), we necessarily have

$$\mathrm{Br}_{\bar R}^{(kG)_\gamma(Q)}(1_{(kG)_\gamma(Q)}-\bar\ell)=0; \tag{4.5.7}$$

thus, the idempotent $1_{(kG)_\gamma(Q)}-\bar\ell$ belongs to [2, Lemmas 1.11 and 1.12]

$$\bigcap_R \mathrm{Ker}\bigl(\mathrm{Br}_{\bar R}^{(kG)_\gamma(Q)}\bigr)=\bigl((kG)_\gamma(Q)\bigr)_Q^{N_P(Q)}=\mathrm{Br}_Q\bigl(\bigl((kG)_\gamma\bigr)_Q^P\bigr) \tag{4.5.8}$$

where R runs over the set of subgroups of $N_P(Q)$ strictly containing Q; but 0 is the unique idempotent in $((kG)_\gamma)_Q^P$; hence, we get $1_{(kG)_\gamma(Q)}=\bar\ell$, proving the claim.

Consequently, it follows from Corollary 3.10 above, applied to the groups $C_G(Q)\cdot N_P(Q)$ and $C_G(Q)$, that the $N_P(Q)$-interior algebra (cf. 2.3)

$$(kG)_\gamma(Q)\otimes_{C_P(Q)}N_P(Q)=\bigoplus_u (kG)_\gamma(Q)\cdot u, \tag{4.5.9}$$

where $u\in N_P(Q)$ runs over a set of representatives for $N_P(Q)/C_P(Q)$, is a source algebra of the block f of $C_G(Q)\cdot N_P(Q)$; hence, according to our hypothesis, we have an $N_P(Q)$-interior algebra isomorphism

$$k_*\bigl(C_{\hat L}(Q)\cdot N_P(Q)\bigr)\cong(kG)_\gamma(Q)\otimes_{C_P(Q)}N_P(Q); \tag{4.5.10}$$

now, according to isomorphism (4.5.6) and equality (4.5.9), we actually get

$$\dim\big((kN)_{\hat{\gamma}}(Q)\big) = \dim\big(k_*C_{\hat{L}}(Q)\big) = \dim\big((kG)_\gamma(Q)\big) \tag{4.5.11}$$

and therefore $g(Q)$ is an isomorphism.

In particular, the interior $C_P(Q)$-algebra $(kG)_\gamma(Q) \cong k_*C_{\hat{L}}(Q)$ is actually a source algebra of the block f of $C_G(Q)$ and therefore, since we have (cf. (1.11.1))

$$C_{\hat{L}}(Q) \cong C_P(Q) \rtimes C_{\hat{E}}(Q), \tag{4.5.12}$$

it follows from equalities (2.7.2) that there is *no* essential pointed groups on $kC_G(Q)f$, so that the block f of $C_G(Q)$ is *inertially controlled* (cf. 2.9); hence, it follows from Lemma 2.9 and from our hypothesis that the block b of G is also *inertially controlled*.

Consequently, the $\mathcal{F}_{(b,G)}$-morphism $\varphi: Q \to P$ above is induced by some element $n \in N$ and therefore there is an invertible element $a \in (kG)^P$ fulfilling $i^n = i^a$, so that the multiplication by na^{-1} on the right still determines a k-linear isomorphism

$$(kG)_\gamma\big(\Delta_\varphi(Q)\big) \cong (kG)_\gamma(Q); \tag{4.5.13}$$

similarly, we also get

$$(kN)_{\hat{\gamma}}\big(\Delta_\varphi(Q)\big) \cong (kN)_{\hat{\gamma}}(Q); \tag{4.5.14}$$

finally, equality (4.5.5) follows from these isomorphisms and equality (4.5.11).

Corollary 4.6. *With the notation above, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), assume that $C_P(Q)$ is a defect group of the block f of $C_G(Q)$ and that a source algebra of this block is isomorphic to $k_*(C_{\hat{L}}(Q))$. Then, the restriction throughout g induces a stable equivalence between the categories of $(kG)_\gamma$- and $(kN)_{\hat{\gamma}}$-modules. In particular, the blocks b of G and e of N are stably identical.*

Proof. With the notation in 4.3 above, the indecomposable $k(N \times G)$-module $\ell(kG)$ defined by the left-hand and the right-hand multiplication has the p-group $\Delta(P) = \{(u,u) | u \in P\}$ as a vertex and the trivial $k\Delta(P)$-module k as a source. Then this corollary follows from Theorem 4.5 above and [13, Theorem 6.9] applied to the case where $\ddot{M} = \ell(kG)$, $b = e$, $b' = b$, $P_\gamma = P_{\hat{\gamma}}$, $P'_{\gamma'} = P_\gamma$ and $\ddot{S} = k$. □

5. An inductive context: the proof of Theorem 1.8

5.1. Let G be a finite group, b a block of G and P_γ a defect pointed group of b; with the notation in 1.5 above, consider the following condition

5.1.1. *The block b of G is inertially controlled and, for any Brauer (b, G)-pair (Q, f) contained in (P, e), f is a block of $C_G(Q) \cdot N_P(Q)$ with trivial source simple modules.*

First of all, we claim that if the block b of G fulfills this condition then, for any Brauer (b, G)-pair (R, h) contained in (P, e), the block h of the group $H = C_G(R)$ fulfills the corresponding condition.

5.2. Indeed, it follows from Lemma 2.9 that the block h of H is inertially controlled and that $T = C_P(R)$ is a defect group of the block h of H; thus, denoting by ℓ the block of $C_G(R \cdot T)$ such that $(R \cdot T, \ell) \subset (P, e)$ [2, Theorem 1.8], (T, ℓ) is a maximal Brauer (h, H)-pair and, if (Q, f) is a Brauer (h, H)-pair contained in (T, ℓ), $(R \cdot Q, f)$ is a Brauer (b, G)-pair still contained in $(R \cdot T, \ell) \subset (P, e)$ and therefore f is a block of $C_G(R \cdot Q) \cdot N_P(R \cdot Q)$ with trivial source simple modules. Then, since $C_H(Q) \cdot N_T(Q)$ is clearly subnormal in $C_G(R \cdot Q) \cdot N_P(R \cdot Q)$, it follows from Lemma 3.9, possibly applied more than once, that f is still a block of $C_H(Q) \cdot N_{C_P(R)}(Q)$ with trivial source simple modules.

5.3. At this point, assuming that the block b of G fulfills condition 5.1.1 and that, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), we have $C_G(Q) \neq G$, it suffices to argue by induction on $|G|$ to get the hypothesis of Theorem 4.5, namely to get that, for any nontrivial Brauer (b, G)-pair (Q, f) contained in (P, e), $C_P(Q)$ is a defect group of the block f of $C_G(Q)$ (cf. Lemma 2.9) and that a source algebra of this block is isomorphic to $k_*(C_{\hat{L}}(Q))$.

5.4. In this situation, it follows from this theorem and from [13, Theorem 6.9] that the blocks b of G and e of N are *stably identical* (cf. 1.4); more precisely, if M is a simple kGb-module of vertex $Q \subset P$ and f is the block of $C_G(Q)$ such that (P, e) contains (Q, f), on the one hand it follows from [8, Proposition 1.6] that the Brauer (b, G)-pair (Q, f) is selfcentralizing, so that $C_P(Q) = Z(Q)$ [16, 4.8 and Corollary 7.3] and, on the other hand, it easily follows from Theorem 4.5 that the kNe-module $\ell \cdot M$, which is actually indecomposable [7, Theorem 2.1], has also vertex Q; moreover, since we are assuming that the trivial kQ-module k is a source of M, it is clear that the trivial kQ-module k is also a source of $\ell \cdot M$.

5.5. Then, it follows again from [8, Proposition 1.6] applied to the N-interior algebra $\mathrm{End}_k(\ell \cdot M)$, that the quotient $N_N(Q)/Q$, and therefore the quotient $N_N(Q)/Q \cdot C_N(Q)$ [15, Theorem 3.6], admit blocks of *defect zero* — namely, with trivial defect groups — which forces [16, 1.19]

$$\mathbb{O}_p\big(N_N(Q)/Q \cdot C_N(Q)\big) = \{1\}; \tag{5.5.1}$$

but we have [11, Proposition 14.6]

$$C_P(Q) = Z(Q) \quad \text{and} \quad (kN)_{\hat{\gamma}} \cong k_*\hat{L} = k_*\big(P \rtimes \hat{E}^\circ\big); \tag{5.5.2}$$

hence, denoting by $\hat{\delta}$ the unique local point of Q on kNe such that $P_{\hat{\gamma}}$ contains $Q_{\hat{\delta}}$ (cf. 2.8), it follows from (2.7.2) and from the isomorphism in (5.5.2) that, as in (1.11.1), we get [5, Ch. 5, Theorem 3.4]

$$N_N(Q)/Q \cdot C_N(Q) \cong E_N(Q_{\hat{\delta}}) = F_{(kN)_{\hat{\gamma}}}(Q_{\hat{\delta}}) \cong \big(N_P(Q)/Q\big) \rtimes N_{\hat{E}^\circ}(Q) \tag{5.5.3}$$

and, since $\mathbb{O}_p(N_N(Q)/Q \cdot C_N(Q)) = \{1\}$, we still get $N_P(Q) = Q$ which forces $P = Q$. In conclusion, $\ell \cdot M$ admits P as a vertex and it has a trivial source, so that it is a simple kNe-module according again to isomorphism (5.5.2).

5.6. Finally, since the *stable equivalence* induced by the restriction throughout g (cf. (4.3.1)) sends any simple kGb-module to a simple kNe-module, it follows from [7, Proposition 2.5] that the restriction throughout g actually induces an equivalence of categories; moreover, since this equivalence is defined by a $k(G \times N)$-module admitting a $P \times P$-stable basis (cf. 4.3), it follows from [13, Corollary 7.4 and Remark 7.5] that the source algebras of the blocks b of G and e of N are isomorphic.

5.7. Assume now that the block b of G fulfills condition 5.1.1 and that there is an Abelian subgroup Z of P such that $G = C_G(Z)$; we are in the situation considered in 3.7 above with $H = G$; hence, it follows from Corollary 3.10 that $\bar{b}$ is a block of $\bar{G}$ and that $\bar{\gamma}$ is contained in a local point $\bar{\bar{\gamma}}$ of $\bar{P}$ on $k\bar{G}$ such that $\bar{P}_{\bar{\bar{\gamma}}}$ is a defect pointed group of $\bar{b}$; denote by $\bar{e}$ the block of $C_{\bar{G}}(\bar{P})$ determined by the point $\bar{\bar{\gamma}}$.

5.8. We claim that the block $\bar{b}$ of $\bar{G}$ fulfills the corresponding condition 5.1.1. Indeed, if $(\bar{Q}, \bar{f})$ is a Brauer $(\bar{b}, \bar{G})$-pair contained in $(\bar{P}, \bar{e})$ and Q is the inverse image of $\bar{Q}$ in G, the image of $C_G(Q)$ in $C_{\bar{G}}(\bar{Q})$ is a normal subgroup and, once again, the corresponding quotient is a p-group [5, Ch. 5, Theorem 3.4]; hence, it follows again from Corollary 3.10 that $\bar{f}$ is the image in $kC_{\bar{G}}(\bar{Q})$ of a block f of the inverse image C of $C_{\bar{G}}(\bar{Q})$ in G and then, since $C_G(Q)$ is normal in C, it is quite clear that $f = \mathrm{Tr}^C_{C_{\tilde{f}}}(\tilde{f})$ for a suitable block $\tilde{f}$ of $C_G(Q)$ where $C_{\tilde{f}}$ denotes the stabilizer of $\tilde{f}$ in C.

5.9. More precisely, we claim that we can choose $\tilde{f}$ in such a way that (P, e) contains $(Q, \tilde{f})$; indeed, since $(\bar{P}, \bar{e})$ contains $(\bar{Q}, \bar{f})$, there is a local point $\bar{\bar{\delta}}$ of $\bar{Q}$ on $k\bar{G}$ such that we have $b_{\bar{\bar{\delta}}} = \bar{f}$ and that $\bar{P}_{\bar{\bar{\gamma}}}$ contains $Q_{\bar{\bar{\delta}}}$; then, it follows easily from Proposition 3.8 and from the obvious commutative diagram

$$\begin{array}{ccc} k\bar{G}^{\bar{P}} & \longrightarrow & k\bar{G}^{\bar{Q}} \\ \uparrow & & \uparrow \\ kG^P & \longrightarrow & kG^Q \end{array} \tag{5.9.1}$$

that there is a point δ of Q on kG such that P_γ contains Q_δ and that the image $\bar{\delta}$ of δ in $k\bar{G}$ is contained in $\bar{\bar{\delta}}$, which forces δ to be local; at this point, it is easily checked that we can choose $\tilde{f} = b_\delta$.

5.10. Now, for any $\bar{x} \in \bar{G}$ such that $(\bar{Q}, \bar{f})^{\bar{x}} \subset (\bar{P}, \bar{e})$, the same argument proves that we have $(Q, \tilde{f})^{cx} \subset (P, e)$ for some $x \in G$ lifting $\bar{x}$ and a suitable element c of C; then, since the block b of G is inertially controlled, there are $n \in N$ and $z \in C_G(Q)$ fulfilling $cx = zn$ (cf. 1.7) and therefore we get $\bar{x} = \bar{c}^{-1}\bar{z}\bar{n}$ where $\bar{c}$, $\bar{z}$ and $\bar{n}$ denote the respective images of c, z and n in $\bar{G}$, $\bar{c}^{-1}\bar{z}$ centralizes $\bar{Q}$ and $\bar{n}$ normalizes $(\bar{P}, \bar{e})$. This proves that the block $\bar{b}$ of $\bar{G}$ is also inertially controlled.

5.11. Moreover, since $(Q, \tilde{f})$ is a Brauer (b, G)-pair contained in (P, e), according to our hypothesis $\tilde{f}$ is a block of $C_G(Q) \cdot N_P(Q)$ with trivial source simple modules; but, since the block b of G is inertially controlled, we have

$$E_G(Q, \tilde{f}) \cong \big(N_P(Q)/Q\big) \rtimes N_{\hat{E}^\circ}(Q) \tag{5.11.1}$$

and therefore $C_{\tilde{f}}$ is contained in $C_G(Q)\cdot N_P(Q)$; hence, since we have [2, Theorem 1.8]

$$C_{\tilde{f}}\cdot N_P(Q)=C_G(Q)\cdot N_P(Q)\quad\text{and}\quad C_{\tilde{f}}\cap N_P(Q)=C\cap N_P(Q),\tag{5.11.2}$$

we clearly have [13, 2.6.4]

$$k\big(C\cdot N_P(Q)\big)f\cong \operatorname{Ind}_{C_G(Q)\cdot N_P(Q)}^{C\cdot N_P(Q)}\big(k\big(C_G(Q)\cdot N_P(Q)\big)\tilde{f}\big)\tag{5.11.3}$$

and therefore f is also a block of $C\cdot N_P(Q)$ with trivial source simple modules. Finally, since the k-algebra $k(C_{\bar{G}}(\bar{Q})\cdot N_{\bar{P}}(\bar{Q}))\bar{f}$ is the image of $k(C\cdot N_P(Q))f$, $\bar{f}$ is a block of $C_{\bar{G}}(\bar{Q})\cdot N_{\bar{P}}(\bar{Q})$ with trivial source simple modules too.

5.12. Consequently, setting $\hat{\bar{L}}=\hat{L}/Z$, it follows from our induction hypothesis that the source algebra of the block $\bar{b}$ of $\bar{G}$ is isomorphic to $k_*\hat{\bar{L}}$ and, in particular, we have

$$\dim\big((k\bar{G})_{\tilde{\bar{\gamma}}}\big)=|L|/|Z|;\tag{5.12.1}$$

but, since the point $\tilde{\bar{\gamma}}$ contains the image of γ, we may assume that $(k\bar{G})_{\tilde{\bar{\gamma}}}$ is the image of $(kG)_\gamma$ or, equivalently, that

$$(k\bar{G})_{\tilde{\bar{\gamma}}}\cong k\otimes_{kZ}(kG)_\gamma\tag{5.12.2}$$

and, in particular, we get

$$\dim\big((kG)_\gamma\big)=|Z|\dim\big((k\bar{G})_{\tilde{\bar{\gamma}}}\big)=|L|;\tag{5.12.3}$$

hence, the unitary P-interior algebra homomorphism (2.6.2) is actually an isomorphism

$$k_*\hat{L}\cong(kN)_{\hat{\gamma}}\cong(kG)_\gamma.\tag{5.12.4}$$

5.13. Conversely, assume that the source algebra $(kG)_\gamma$ is isomorphic to $k_*\hat{L}$, so that the unitary P-interior algebra homomorphism (2.6.2) is an isomorphism; then, it follows from equalities (2.7.2) applied to the blocks b of G and $\{1\}$ of $\hat{L}$ that there are no essential pointed groups on kGb (cf. 2.8) and therefore the block b of G is inertially controlled (cf. 2.9).

5.14. For any Brauer (b,G)-pair (Q,f) contained in (P,e), since we have (cf. (1.3.1) and (1.11.1))

$$\begin{aligned}(kG)(Q)\cong kC_G(Q)\quad\text{and}\quad (kG)_\gamma(Q)&\cong(k_*\hat{L})(Q)\cong k_*C_{\hat{L}}(Q)\\ &=k_*\big(C_P(Q)\rtimes C_{\hat{E}}(Q)\big),\end{aligned}\tag{5.14.1}$$

the $C_P(Q)$-interior algebra $(kG)_\gamma(Q)$ is a source algebra of the block f of $C_G(Q)$; then, it follows from Corollary 3.10 that a source algebra of the block f of $C_G(Q)\cdot N_P(Q)$ is isomorphic to the $N_P(Q)$-interior algebra

$$(kG)_\gamma(Q)\otimes_{C_P(Q)}N_P(Q);\tag{5.14.2}$$

finally, according to isomorphisms (5.14.1), this $N_P(Q)$-interior algebra is isomorphic to

$$(k_*\hat{L})(Q) \otimes_{C_P(Q)} N_P(Q) \cong k_* N_{\hat{L}}(Q) = k_*\big(N_P(Q) \rtimes N_{\hat{E}}(Q)\big) \tag{5.14.3}$$

which clearly has trivial source simple modules. We are done. □

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