Integral Semihereditary Orders, Extremality, and Henselization

John S. Kauta*

Department of Mathematical Sciences, University of Malawi, Box 280, Zomba, Malawi

Communicated by Susan Montgomery

Received July 11, 1995

In this paper, we study integral semihereditary orders over a valuation ring in a finite-dimensional simple Artinian ring. In the first section we prove that such orders are extremal. Consequently, in a central division algebra admitting a total valuation ring, the intersection of all the conjugates of the total valuation ring is the unique integral semihereditary order over the center of the total valuation ring. In the second section we characterize, up to conjugacy, integral semihereditary orders over a Henselian valuation ring. In the last section we show that an integral order $R$ over an arbitrary valuation ring $V$ is semihereditary iff its Henselization, $R \otimes V$, where $V$ is the Henselization of $V$, is a semihereditary $V$-order. In this case, there is an inclusion preserving bijective correspondence between semihereditary $V$-orders inside $R$ and semihereditary $V$-orders inside $R \otimes V$.

© 1997 Academic Press

0. Introduction

In this paper, all rings are associative with a multiplicative unit and all modules are unitary. If $R$ is a ring, $J(R)$ will denote its Jacobson radical, $U(R)$ its group of units, $Z(R)$ the center, $R^*$ the set of nonzero divisors, and $M_n(R)$ the ring of $n \times n$ matrices over $R$. The residue ring $R/J(R)$ will be denoted by $\overline{R}$ and if $S$ is an overring of $R$ with $J(S) \subseteq R$ then $R/J(S)$ will be denoted by $\overline{R}$.

Definition 0.1. Let $Q$ be a finite-dimensional $F$-algebra and $V$ a commutative domain with quotient field $F$. A subring $R$ of $Q$ is said to be an order in $Q$ if $RF = Q$. If $V \subseteq Z(R)$ then $R$ is said to be a $V$-order if in

* E-mail address: kauta@unima.wn.apc.org.

© 1997 Academic Press

0021-8693/97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.
addition $R$ is integral over $V$. If a $V$-order $R$ is maximal among the $V$-orders of $Q$ with respect to inclusion then $R$ is called a maximal $V$-order (or just a maximal order if the context is clear).

In this paper, we shall be concerned with $V$-orders in a finite-dimensional central simple $F$-algebra $Q$ when $V$ is a commutative valuation ring of $F$ of arbitrary Krull-dimension. Since a valuation ring is integrally closed, the center of such a $V$-order is precisely $V$. In this case, maximal $V$-orders always exist. Finitely generated maximal $V$-orders need not. We will have more to say about this in due course.

**Definition 0.2.** A ring $R$ is said to be *extremal* if for every overring $S$ such that $J(S) \supseteq J(R)$ we have $S = R$. If $S$ is an overring of $R$, we say that $R$ is extremal *in* $S$ if $R$ is extremal among all subrings of $S$. A $V$-order $R$ is said to be an extremal $V$-order (or just extremal when the context is clear) if it is extremal among all $V$-orders in $Q$.

**Definition 0.3.** A ring $R$ is said to be right semihereditary (resp. right hereditary) if every finitely generated right ideal (resp. every right ideal) is projective as a right $R$-module. A ring is said to be semihereditary (resp. hereditary) if it is both left and right semihereditary (resp. hereditary).

**Definition 0.4.** A ring $R$ is said to be right (resp. left) Bézout if every finitely generated right (resp. left) ideal is principal. It is called Bézout if it is both right and left Bézout.

Examples of extremal $V$-orders are all the maximal $V$-orders (e.g., Azumaya algebras over $V$, integral Dubrovin valuation rings, and Bézout $V$-orders) and semihereditary $V$-orders when $V$ is a DVR see [R, Chap. 9].

**Definition 0.5.** Let $L$ be an additive subgroup of $Q$. Then $O_Q(L) = \{x \in Q \mid xL \subseteq L\}$. $O_Q(L)$ is similarly defined.

In the first section of this paper, we will show that semihereditary $V$-orders are extremal. In the classical case (i.e., when $V$ is a DVR) (see [R, Chap. 9]) a $V$-order $R$ is semihereditary (actually hereditary, since all right (resp. left) ideals are finitely generated when $V$ is DVR) precisely when it is extremal or, equivalently, when $O_Q(J(R)) = R$. This is no longer true when $V$ is not assumed to be Noetherian, as we will see in this section.

In the second section we characterize, up to conjugacy, semihereditary $V$-orders, semihereditary maximal $V$-orders, and Dubrovin valuation rings of $Q$ extending $V$ assuming $V$ is Henselian. In the last section, we show that a $V$-order $R$ is semihereditary iff its Henselization, $R \otimes_V V_h$, where $V_h$ is the Henselization of $V$ (see [E] for definition), is semihereditary. As in [Ha], [R], we show that if $R$ is
semihedrarily, then there is a one-to-one correspondence between semihereditary $V$-orders inside $R$ and semihedrarily $V_h$-orders inside $R \otimes \nu V_h$. We shall also consider the Henselization of semihedrarily maximal $V$-orders and obtain alternate proofs to some of the results in [H M W], [W].

1. EXTREMALITY

Let $R$ be a $V$-order. In this section we show that if $R$ is semihedrarily then it is extremal and if $R$ is extremal then $O(R) = R$. But unlike in the classical theory [R, Chap. 9], the converse of both these facts need not hold in general and finitely generated extremal orders need not exist in a central simple $F$-algebra. We end the section by showing that in a central division algebra admitting a total valuation ring extending $V$, the intersection of all the conjugates of the total valuation ring is the unique semihereditary $V$-order in the division algebra.

Given a $V$-order $R$, $[R/J(V)R : V/J(V)] = [Q : F] < \infty$ since elements of $R$ which are linearly independent mod $J(V)R$ over $V/J(V)$ are linearly independent over $F$. Hence $R/J(V)R$ is Artinian. Further, it follows from Kaplansky’s theorem of PI-algebras [I, Chap. I, Sect. 3] that $J(R) \geq J(V)$ and thus $R/J(R)$ is semisimple Artinian. We begin with the following lemma:

**Lemma 1.1.** Let $L$ be a left ideal of a $V$-order $R$. Then $L^m \subseteq J(V)R$ for some positive integer $m$ if and only if $L \subseteq J(R)$.

**Proof.** By considering the finite-dimensional $V/J(V)$-algebra $R/J(V)R$, we readily see that $J(R)^m \subseteq J(V)R$ for $m$ large enough. Hence if $L \subseteq J(R)$, then $L^m \subseteq J(V)R$ for a large $m$. Conversely, suppose $L^m \subseteq J(V)R$ for some $m$. Then

$$(LR)^m = L(RL) \cdots (RL)R \subseteq L^mR \subseteq J(V)R \subseteq J(R).$$

Hence $LR \subseteq J(R)$. 

We remark here that given a commutative valuation ring $V$, $J(V)$ is principal if and only if $J(V)^2 \neq J(V)$: let $\phi$ be a valuation on $F$ with valuation ring $V$. If $J(V) = \pi V$, then $\phi(\pi)$ (resp. $\phi(\pi^2)$) is the least element of $\phi(J(V))$ (resp. $\phi(J(V)^2)$) and hence $J(V)^2 \neq J(V)$ since $\phi(\pi^2) > \phi(\pi^2)$. If $J(V)$ is not principal, then $\phi(J(V))$ does not have a least element. In this case, let $x \in J(V)$. Then there exists a $y \in J(V)$ with $\phi(y) < \phi(x)$. Hence $x = (xy^{-1})y \in J(V)^2$ and therefore $J(V)^2 = J(V)$. The same argument holds for invariant valuation rings in division algebras.
In fact, it is true for Dubrovin valuation rings [Dz, Lemma 7.8] and it remains true for Bézout $V$-orders as will be shown later in this paper. We are now ready to prove the following proposition:

**Proposition 1.2.** Let $K$ be a subfield of $F$ and let $U$ be a valuation ring of $K$. Let $x \in Q$. If $ux^k$ is integral over $U$ for every positive integer $k$ and every $u \in J(U)$ then $x$ is integral over $U$.

**Proof.** Clearly, $x$ is algebraic over $K$. We will split the proof into two cases:

**Case A.** $J(U)^2 = J(U)$. Let $f(t)$ be the minimal polynomial of $x$ over $K$. Then $f(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_0 \in K[t]$. Let $0 \neq u \in J(U)$ and let $g(t) = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_0 u^m$. Then $g(ux) = 0$, so $g(t)$ is the minimal polynomial of $ux$ over $K$. But $ux$ is integral over $U$ by assumption, so $g(t) \in U[t]$. Hence $\alpha_{m-1}u, \alpha_{m-2}u^2, \ldots, \alpha_0 u^m \in U$ for every $u \in J(U)$. Therefore $\alpha_{m-1}u, \alpha_{m-2}u^2, \ldots, \alpha_0 u^m \subseteq U$ since $U$ is a valuation ring of $K$. But $J(U)^2 = J(U)$. So $\alpha_i J(U) \subseteq U$ and therefore $\alpha_i J(U) \subseteq J(U)$, for all $i$. But this means each $\alpha_i \in U$ and so $f(t) \in U[t]$.

**Case B.** $J(U)^2 \neq J(U)$. By the remark above, $J(U) = \pi U$, a principal ideal. Consider $K[x]$, a finite-dimensional $K$-algebra since $x$ is algebraic over $K$. Let $A = K[x]/J(K[x]) \cong K_1 \oplus K_2 \oplus \cdots \oplus K_n$, where $K_i$ are fields. Since $K[x]$ is Artinian, $J(K[x])$ is nilpotent, hence $J(K[x]) \cap K = 0$.

Let $K_1, K_2, \ldots, K_n$ are all finite field extensions of $K$. Let $\bar{x} = x_1 + x_2 + \cdots + x_n$ be the image of $x$ in $A$, with $x_i \in K_i$. We will show that each $x_i$ is integral over $U$.

Let $C_i$ be the integral closure of $U$ in $K_i$. Then $C_i$ is a Krull ring which coincides with the intersection of all the extensions of $U$ to $K_i$ [E, 13.3(b)]. Let $W_i$ be an extension of $U$ to $K_i$. We will show that $x_i \in W_i$, and hence $x_i \subseteq C_i$. Let $\phi_i$ be a valuation on $K_i$ with valuation ring $W_i$ and suppose $x_i \notin W_i$. Then $x_i^{-1} \notin J(W_i)$. But for $k$ large enough, $J(W_i)^k \subseteq J(U)W_i$. So $x_i^{-k} \in \pi W_i$ which implies $\phi_i(x_i^{-k}) \geq \phi_i(\pi) \Rightarrow -k \phi_i(x_i) \geq \phi_i(\pi) \Rightarrow k \phi_i(x_i) \leq -\phi_i(\pi)$. But $\pi x_i^k$ is integral over $U$ by assumption. Hence $\pi x_i^k \in W_i \Rightarrow \phi_i(\pi x_i^k) \geq 0 \Rightarrow \phi_i(\pi) \geq -k \phi_i(x_i) \Rightarrow k \phi_i(x_i) \geq -\phi_i(\pi)$, hence $k \phi_i(x_i) = -\phi_i(\pi)$ for any $k$ large enough. This is absurd. So $x_i \in W_i$, and hence $x_i \subseteq C_i$. Therefore $x_i$ is integral over $U$. So there exists a monic polynomial $h(t) \in U[t]$ such that $h(x_i) \in J(K[x])$. But $J(K[x])$ is nilpotent. Hence $h'(x_i) = 0$ for some $l$ and so $x$ is integral over $U$.

**Corollary 1.3.** (a) If $L$ is a $U$-submodule of $Q$ integral over $U$, then the ring $O_L(L)$ is integral over $U$.

(b) If $R$ is a $V$-order, then so are $O(J(R))$ and $O_J(V)R$.
Proof. For part (a), let \( x \in O_2(L) \). Then \( x^k \in O_2(L) \) for every positive integer \( k \), since \( O_2(L) \) is a ring. But \( J(U) \subseteq L \). So \( u x^k \in L \) for every \( u \in J(U) \) and every \( k \). The result thus follows from Proposition 1.2. Part (b) is a direct consequence of part (a).

Given two \( V \)-orders \( B \supseteq A \), then \( U(A) = U(B) \cap A \): suppose \( u \in A \) is a unit in \( B \). Let \( f(t) = t^m + \alpha_{m-1} t^{m-1} + \cdots + \alpha_0 \) be the minimal polynomial of \( u \) over \( F \). Then \( f \in V[t] \), since \( u \) is integral over \( V \) and \( V \) is integrally closed. Since \( u \) is invertible in \( B \), a \( V \)-order, \( u^{-1} \) is also integral over \( V \) and hence \( \alpha_0 \) is a unit in \( V \), by considering the reduced norm of \( u \) over \( F \). Thus \( 1 = -\alpha_0^{-1} (u^m - a - \alpha_{m-1} u^{m-2} - \cdots + \alpha_0 u) \) happening inside \( A \), hence \( u \) is a unit in \( A \) as well. Now suppose \( J(B) \subseteq A \) as well. Let \( x \in J(B) \), \( a, b \in A \). Then \( 1 - ab \in U(B) \cap A = U(A) \) and hence \( x \in J(A) \). Therefore \( J(B) \subseteq J(A) \). These facts will be used frequently.

**Proposition 1.4.** Let \( R \) be a \( V \)-order. If \( R \) is an extremal \( V \)-order then \( \text{OJ}_R = R \). If \( B \) is a \( V \)-order containing \( R \) such that \( R \) is extremal in \( B \) then \( J(B) \subseteq J(R) \).

Proof. Let \( S = \text{OJ}_R \). Then \( S \) is an order containing \( R \). By Corollary 13(b), \( S \) is a \( V \)-order. But \( J(R) \) is a left ideal of \( S \) such that \( J(R)^m \subseteq J(V) R \subseteq J(V) S \) for \( m \) large enough. Hence \( J(R) \subseteq J(S) \), by Lemma 1.1. So \( S = R \), since \( R \) is extremal.

Now let \( R' = R + J(B) \). Then \( R \subseteq R' \subseteq B \). If \( a, b \in R' \) and \( x \in J(R) \) then it can easily be established that \( 1 - ab \in U(B) \cap A = U(A) \) and hence \( J(R) \subseteq J(R') \). But \( R \) is extremal in \( B \). So \( R = R' \) and hence \( J(B) \subseteq R \). But this means \( J(B) \subseteq J(R) \).

**Theorem 1.5.** Let \( H \) be a semihereditary \( V \)-order in \( Q \).

(a) If \( B \) is a \( V \)-order containing \( H \) then \( J(B) \subseteq J(H) \).

(b) \( \text{OJ}_H = H \).

So \( H \) is an extremal \( V \)-order.

Proof. (a) It suffices to show that \( J(B) \subseteq H \). So let \( \alpha \in J(B) \). Let \( x = (\begin{smallmatrix} t & r \\ -r t & -r \alpha \end{smallmatrix}) \). Then the right annihilator of \( x \) in \( M_2(H) \),

\[
\text{Rt-Ann}_{M_2(H)}(x) = \left\{ \begin{pmatrix} t & r \\ -r t & -r \alpha \end{pmatrix} : t, r, \alpha t, \alpha r \in H \right\}.
\]

Since \( M_2(H) \) is semihereditary as well, \( x M_2(H) \) is projective. Thus \( \text{Rt-Ann}_{M_2(H)}(x) = e M_2(H) \) for some \( e = e^2 \). Let \( e = (\begin{smallmatrix} a & b \\ -a b & -a \end{smallmatrix}) \), \( a, b, \alpha \).
\(\alpha b \in H\). Since \(e^2 = e\), we have

\[a^2 - baa = a \quad (1)\]
\[ab - b\alpha b = b. \quad (2)\]

From (2) we have \(ab = b + b\alpha b = b(1 + \alpha b) = bu\), \(u\) a unit in \(B\) and hence a unit in \(H\) as well. But

\[\text{Rt-Ann}_{M_2(H)}(x) = eM_2(H) \subseteq \left( \begin{array}{cc} aH + bH & aH + bH \\ -a\alpha H - abH & -a\alpha H - abH \end{array} \right).\]

Since \(bH = buH = abH \subseteq aH\), and \(-abH \subseteq -a\alpha H\), we have

\[\text{Rt-Ann}_{M_2(H)}(x) \subseteq \left( \begin{array}{cc} aH & aH \\ -a\alpha H & -a\alpha H \end{array} \right).\]

Since \(H\) is a \(V\)-order, there exists \(0 \neq v \in V\) such that \(v\alpha \in H\). So \((\begin{array}{c} e \\\ -e \end{array}) \in \text{Rt-Ann}_{M_2(H)}(x)\) and so \(v = ah\) for some \(h \in H\). Hence \(a \in R^*\), therefore (1) implies \(a - b\alpha = 1\) and hence \(a \in U(B)\) since \(\alpha \in J(B)\). So we have \(a \in U(H)\). But \(\alpha a \in H\). So \(\alpha \in H\).

(b) Suppose \(\alpha \in O(J(H)) \setminus H\). Let \(S = O(J(H))\). Then \(S\) is a \(V\)-order, by Corollary 1.3(b), and it contains \(H\). So \(J(S) \subseteq J(H)\) by part (a). But \(J(H)\) is a left ideal of \(S\) such that \(J(H)^m \subseteq J(V)H \subseteq J(V)S\) for some \(m\) large enough. Therefore \(J(S) = J(H)\) by virtue of Lemma 1.1. Now consider \(x = (\begin{array}{c} e \\\ 0 \end{array})\). As above, \(\text{Rt-Ann}_{M_2(H)}(x) = eM_2(H)\) for some \(e = e^2\). We have \(eM_2(S) \subseteq \text{Rt-Ann}_{M_2(S)}(x)\). Let \(y \in \text{Rt-Ann}_{M_2(S)}(x)\). Since \(M_2(H)\) is a \(V\)-order in \(M_2(Q)\), \(\exists \bar{v} \neq v \in V\) such that \(\bar{v}y \in M_2(H)\). So \(\bar{v}y \in \text{Rt-Ann}_{M_2(H)}(x) = eM_2(H)\) and therefore \(\bar{v}y = ez\) for some \(z \in M_2(H)\). Hence \(\bar{v}y = ez = \bar{v}y \Rightarrow y \in eM_2(S)\). Therefore \(\text{Rt-Ann}_{M_2(S)}(x) = eM_2(S)\). Suppose \(e = (\begin{array}{c} a \\
\end{array}\begin{array}{c} b \\
-\alpha a \\
-\alpha b \end{array})\), where \(a, b, \alpha a, \alpha b \in H\) as before. Then we have

\[\text{Rt-Ann}_{M_2(S)}(x) = \left\{ \begin{array}{cc} t \\
-\alpha t \end{array} \begin{array}{cc} r \\
-\alpha r \end{array} \left( \begin{array}{cc} S \\
S \end{array} \right) \right\}\]

Let \(T = aH + bH\), a right ideal of \(H\). Since \(\alpha a\) and \(\alpha b\) are in \(H\), \(\alpha T \subseteq H\). If \(T = H\) then \(\alpha \in H\), a contradiction. So \(T \subseteq H\). Since any element of \(S\) can appear as the first entry in a matrix in \(\text{Rt-Ann}_{M_2(S)}(x)\), \(S \subseteq aS + bS = T \cdot S\) and therefore \(T \cdot S = S\). Thus one can write \(1 = \sum t_is_i\) with \(t_i \in T, s_i \in S\). If \(h \in J(H)\) then \(h = \sum t_is_i\). But \(s_ih \in J(H)\) and
$t, J(H) \subseteq T$. So $h \in T$ and hence $J(H) \subseteq T$. Let $\overline{T} = T/J(H)$, $\overline{S} = S/J(S) = S/J(H)$. Then $\overline{T} = f \cdot \overline{H}$, $f$ an idempotent in $\overline{H}$ since $\overline{H}$ is semisimple Artinian. We have $f \neq 1$ since $T \nsubseteq H$.

Now $T : S = S \Rightarrow \overline{T} : \overline{S} = \overline{S} \Rightarrow (1 - \overline{f}) \cdot \overline{T} : \overline{S} = (1 - \overline{f}) \cdot \overline{S}$. But $(1 - \overline{f}) \cdot \overline{S} = 0$. Therefore $(1 - \overline{f}) \cdot \overline{S} = 0 \Rightarrow 1 - \overline{f} = 0$ and hence $\overline{f} = 1$, a contradiction. So we must have $O(J(H)) = H$. From (a) and (b) it follows that $H$ is an extremal $V$-order. [1]

In [M, Theorem 5.7(3)], P. Morandi produces an example of a maximal $V$-order in $M_2(F)$ which is primary but not Bézout. Such an order cannot be semihereditary, since any primary semihereditary order is a Dubrovin valuation ring [D, Theorem 4] and hence Bézout. But every maximal $V$-order is an extremal $V$-order. So we immediately have the following:

Example 1.6 [Morandi]. Extremal $V$-orders need not be semihereditary.

An example of a non-maximal extremal order which is not semihereditary would be interesting. When $V$ is a DVR, i.e., the classical case, semihereditary $V$-orders are precisely the extremal $V$-orders (see [R, J]). In a subsequent paper by this author, it will be shown that inside integral Dubrovin valuation rings extending $V$, semihereditary $V$-orders are precisely the extremal ones.

The following example was communicated to the author by Morandi:

Example 1.7 [Morandi]. Let $R$ be a $V$-order such that $O(J(R)) = R$. Then $R$ need not be extremal.

Proof. Let $V$ be a valuation ring of rank greater than 1, and $P_1 \nsubseteq P_2$ distinct prime ideals of $V$.

Let $R_1 = \left(\frac{V}{P_1}\right)$, $R_2 = \left(\frac{V}{P_2}\right)$. Then $R_1 \nsubseteq R_2$

$$J(R_1) = \left(\begin{array}{c} J(V) \\ V \end{array} \right) \begin{pmatrix} P_1 \\ J(V) \end{pmatrix} \subseteq \left(\begin{array}{c} J(V) \\ V \end{array} \right) \begin{pmatrix} P_2 \\ J(V) \end{pmatrix} = J(R_2).$$

Hence $R_1$ cannot be extremal. But we have $O(J(R_1)) = R_1$. [1]

We observe here that even inside $M_2(V)$, an integral Dubrovin valuation ring of $M_2(F)$, $O(J(R)) = R \Rightarrow R$ is extremal, let alone semihereditary.

Remark. Bézout $V$-orders always exist in central simple $F$-algebras (see [G, Theorem 6.11; M, Theorem 3.4]) and are maximal orders by [G, Theorem 3.7]. It can be shown that such orders are always semihereditary (e.g., see the argument in the proof of [M, Lemma 4.11]: one only needs to
consider the projectivity of finitely generated regular right (resp. left) ideals to show that an order in a simple Artinian ring is right (resp. left) semihereditary; this is trivial when the order is Bézout).

Now suppose $B$ is a finitely generated Bézout $V$-order. Since it is Bézout, $B_h = B \otimes V_h$, where $V_h$ is the Henselization of $V$, is a maximal $V_h$-order by [HMW, Theorem 17]. Since $B_h$ is also finitely generated over $V_h$ in this case, it is contained in a Bézout $V_h$-order $B$ by [M, Proposition 2.3(a)]. By maximality of $B_h$ we have $B_h = B$, a Bézout $V_h$-order. Thus a Bézout $V$-order $B$ such that $B \otimes V_h$ is not Bézout, e.g., [M, Example 4.14], cannot be finitely generated over $V$. Since Bézout $V$-orders in $Q$ are pairwise conjugate [G, Theorem 6.12], if one is not finitely generated then neither are the rest.

We now can prove the following proposition:

**Proposition 1.8.** A finitely generated extremal $V$-order $R$ need not exist inside a central simple algebra. In fact, such an $R$ exists if and only if all the Dubrovin valuation rings extending $V$ are finitely generated, if and only if $R$ is a finitely generated semihereditary $V$-order.

**Proof.** Let $R$ be such an order. Then $R$ is contained in a Bézout $V$-order $B$, by [M, Proposition 2.3]. Since $[B/J(B): V/J(V)] < \infty$, there exists $a_1, a_2, \ldots, a_n \in B$ such that $B = a_1 V + a_2 V + \cdots + a_m V + J(B)$. But by Proposition 1.4, $J(B) \subseteq R$, since $R$ is extremal. Hence $B = a_1 V + a_2 V + \cdots + a_m V + R$, a finitely generated Bézout $V$-order. But finitely generated Bézout $V$-orders need not exist. So such an $R$ need not exist.

Suppose such an $R$ exists. Then $B$ above is a finitely generated Bézout $V$-order. By the remark before this proposition, we have that $B \otimes V_h$ is Bézout, hence $B$ is a Dubrovin valuation ring of $Q$ by [HMW, Theorem 17]. Hence all Dubrovin valuation rings extending $V$ are finitely generated since they are conjugate. Conversely, any finitely generated Dubrovin valuation ring is a finitely generated maximal $V$-order and hence a finitely generated extremal $V$-order.

As we have seen, if $R$ is a finitely generated extremal $V$-order then $R$ is contained in an integral (actually finitely generated) Dubrovin valuation ring and hence it is semihereditary as was mentioned after Example 1.6. Conversely, any semihereditary $V$-order is extremal, by Theorem 1.5. Therefore by Theorem 1.5, finitely generated semihereditary $V$-orders need not exist inside central simple algebras. But Dubrovin valuation rings and hence Bézout $V$-orders always exist. Further, the proof of Proposition 1.8 shows that finitely generated maximal $V$-orders are precisely the

---

1 The proof that inside an integral Dubrovin valuation ring extending $V$ extremal $V$-orders are semihereditary is independent of this proposition.
finitely generated Dubrovin valuation rings. Hence the Henselization of a finitely generated maximal V-order is maximal, giving a partial answer to an open question in [HMW] namely: Is the Henselization of a maximal V-order maximal?

We will now classify extremal V-orders inside finite-dimensional division algebras admitting total valuation rings.

**Theorem 1.9.** Let D be a division algebra admitting a total valuation ring extending V. Then the integral closure of V in D is the unique extremal V-order (and hence the unique semihereditary V-order) in D.

**Proof.** Let \( B_0 = \text{Int}_V(V) \), the integral closure of V in D. Then by [BG, Theorem 4], \( B_0 \) is a ring and if \( B_1, \ldots, B_n \) are all the conjugates of the total valuation ring then by [BG, Lemma 2 and Theorem 3], \( B_0 = B_1 \cap B_2 \cap \cdots \cap B_n \). Let R be an extremal V-order. Then \( R \subseteq B_0 \), since \( B_0 = \text{Int}_V(V) \) and R is a V-order. But both R and \( J(B_i) \) contain \( J(V) \). Hence for each i, \( R/(J(B_i) \cap R) \) is finite-dimensional over \( V/J(V) \). But one has the embedding

\[
R/(J(B_i) \cap R) \twoheadrightarrow B_i/J(B_i)
\]

and \( B_i/J(B_i) \) is a division algebra finite-dimensional over \( V/J(V) \). It follows that \( R/(J(B_i) \cap R) \) is a division algebra and hence \( J(B_i) \cap R \) is a maximal ideal of R. Hence \( J(R) \subseteq J(B_i) \cap R \). Clearly, \( \cap_i J(B_i) \subseteq J(B_0) \). Let \( x \in \cap_i J(B_i) \) and \( a, b \in B_0 \). Then \( 1 - axb \in U(B_i) \) for all i and thus \( 1 - axb \in U(B_0) \). Therefore \( x \in J(B_0) \). Hence \( J(R) \subseteq \cap_i J(B_i) \subseteq J(B_0) \). Since R is extremal, we must have \( R = B_0 \). But Bézout V-orders always exist in finite-dimensional simple Artinian rings and every such an order is a semihereditary V-order. Thus by Theorem 1.5, \( B_0 \) is the unique semihereditary V-order in D.

A noteworthy special case of this theorem is the

**Corollary 1.10.** If D admits an invariant valuation ring extending V, say \( \Delta \), then \( \Delta \) is the unique extremal (and hence the unique semihereditary) V-order in D.

### 2. OVER HENSELIAN VALUATION RINGS

In this section we restrict our attention to the case when V is Henselian. Strictly speaking, we only need the fact that the division ring D of the central simple algebra \( Q = M_n(D) \) admits an invariant valuation ring \( \Delta \)
extending $V$. Such is always the case when $V$ is Henselian. Over such a valuation ring, semihereditary $V$-orders take a particularly sharp form which lends itself to direct computation. We begin by making the following definition, following [M]:

**Definition 2.1.** An order $R$ is said to be of type $\mathcal{H}$ if $R = (\Delta_{ij})$ where:

1. $\Delta_{ij}$ is a nonzero $D$-bisubmodule of $D$.
2. $\Delta_{ii} = D$ for all $i$.
3. [Morandi’s Condition: MC] $\forall \alpha \neq 0 \in D, \alpha \in \Delta_{ij} \Rightarrow \alpha^{-1} \in \Delta_{ji}$.
4. $\Delta_{kj} \cdot \Delta_{jl} \subseteq \Delta_{ki} \forall j, k, l$.

If $I$ is a $\Delta$-submodule of $D$, we define $I^{-1}$ to be $\{x \in D \mid xI \subseteq \Delta\}$. In [M, Chap. 4], Morandi introduced the notion of $\mathcal{H}$-orders where for each tuple $(i, j)$, $\Delta_{ij}$ or $\Delta_{ji}$ was an (invariant valuation) overring of $\Delta$ and $\Delta_{ij}^{-1} = \Delta_{ji}$ (thus our definition above is weaker than the original one). It was shown in that paper that those types of $\mathcal{H}$-rings are semihereditary maximal $V$-orders. In this section, we will prove a converse of that theorem as well. Condition (iii) of the definition above directly relates to Lemma 4.5 of [M] (and hence called “Morandi’s Condition” or MC in short) and will be critical in proving Theorem 2.4.

The following lemma is the same as [M, Lemma 4.5] and the one after that the same as [M, Lemma 4.6]. Lemma 2.3 will be used to show that an order of type $\mathcal{H}$ is semihereditary. But due to typographical errors in the text of the proof of [M, Lemma 4.6], it has been deemed necessary to reproduce the arguments here. The corrected proof was furnished to this author by the original author.

**Lemma 2.2 [Morandi].** Let $R = (\Delta_{ij})$ be of type $\mathcal{H}$. If $x_1, \ldots, x_n \in D$, not all zero, then there is an $i$ with $x_ix_i^{-1} \in \Delta_{ij}$ for all $j$.

The proof of the lemma above only uses the fact that MC holds. Note that if $x_ix_i^{-1} \in \Delta_{ij}$ then $x_i^{-1}x_j \in \Delta_{ij}$ since $\Delta_{ij}$ is a $\Delta$-submodule of $D$ and $\Delta$ is an invariant valuation ring of $D$.

**Lemma 2.3 [Morandi].** With the hypothesis as above, $xR$ is projective as an $R$-module for all $x \in Q = M_n(D)$.

**Proof [by Morandi].** We first suppose that $xR$ is projective for all $x \in e_{ij}R$ for all $i$ and prove $xR$ is projective for any $x$. We do this by showing $e_jxR$ is projective, where $e_j = e_{11} + e_{22} + \cdots + e_{jj}$. We use induction on $j$, the case $j = 1$ is true by assumption. So suppose $e_{j-1}xR$ is
projective for all \( x \in e_i R \). We have the exact sequence of \( R \)-modules

\[
0 \rightarrow e_j xR \cap (1 - e_j - 1)R \rightarrow e_j xR \rightarrow e_j - 1 e_j xR \rightarrow 0.
\]

Now \( e_j - 1 e_j xR = e_j - 1 xR \) and \( e_j xR \cap (1 - e_j - 1)R \subseteq e_j R \cap (1 - e_j - 1)R = e_j R \).

Since \( e_j - 1 xR \) is projective (by the inductive hypothesis), the sequence splits. So

\[
e_j xR \cong e_j - 1 xR \oplus (e_j xR \cap (1 - e_j - 1)R).
\]

Thus \( e_j xR \cap (1 - e_j - 1)R \) is a cyclic right \( R \)-module and a submodule of \( e_j R \), hence is projective by assumption. Therefore we get \( e_j xR \) is a direct sum of two projective modules, hence projective. Thus by induction \( e_j xR \) is projective for all \( j \). Therefore with \( j = n \) we get \( e_n xR = xR \) is projective.

We now show that \( xR \) is projective for \( x \in e_i M_\alpha(D) \). Recall that \( xR \) is projective if and only if \( \text{Rt-Ann}_R(x) = eR \) for some idempotent \( e \in R \).

(This holds for \( x \in M_\alpha(D) \), not just for \( x \in R \) as \( FR = M_\alpha(D) \) and \( \text{Rt-Ann}_R(x) = \text{Rt-Ann}_R(x\alpha) \) for any \( 0 \neq \alpha \in F \)).

Say \( x = \sum_j x_j e_{ij} \in e_i M_\alpha(D) \) with \( x_j \in D \). If \( x = 0 \) then we are done, else by Lemma 2.2, there is \( i_0 \) with \( x_{i_0 - 1} e_{i_0 j} \in \Delta_{i_0 j} \) for all \( j \) and hence \( x_{i_0} - 1 e_{i_0 j} \in \Delta_{i_0 j} \) for all \( j \). Let \( E \) be the permutation matrix which switches the \( i_0 \)th and \( i \)th rows. Let \( e = I_n - x_{i_0}^{-1} E x \in R \), \( I_n \) the identity matrix. Since \( xx_{i_0}^{-1} E x = x \), we have \( xe = 0 \) (so \( e \in \text{Rt-Ann}_R(x) \)). Let \( c \in \text{Rt-Ann}_R(x) \). Then \( ec = (I_n - x_{i_0}^{-1} E x)c = c - 0 = c \). In particular, \( ee = e \) and \( \text{Rt-Ann}_R(x) = eR \). So \( xR \) is projective for any \( x \). \( \square \)

**Theorem 2.4.** \( R \) is a semihereditary \( V \)-order if and only if \( R \) is conjugate to an order of \( \mathcal{H} \) type.

(See also [M, Sect. 4] for a special case of this theorem.) ·

**Proof.** Suppose \( R \) is a semihereditary \( V \)-order. Then \( R \) contains a full set of primitive orthogonal idempotents by [Go, Theorem 1]. After a conjugation if necessary, we may assume that all the standard idempotents, \( e_{11}, e_{22}, \ldots, e_{nn} \in R \). Then \( e_{ij} R e_{ii} \) is a semihereditary \( V \)-order in \( D \) by [Sa], hence \( e_{ij} R e_{ii} = \Delta \) by Corollary 1.10. Set \( \Delta_{ij} = e_{ij} R e_{ij} \). Then \( \Delta_{ij} \neq 0 \), since \( R \) is an order in \( Q \). Since \( D \subseteq R \), we have \( \Delta e_{ij} R e_{ij} = e_{ij} (\Delta R e_{ij}) = e_{ij} R e_{ij} \Delta \) and so \( \Delta_{ij} \) is a \( \Delta \)-bisubmodule of \( D \). \( R \) is a ring, so \( \Delta_{kj} \cdot \Delta_{jl} = e_{kk} R e_{jj} \subseteq e_{kk} R e_{jj} = \Delta_{kj} \). We only have to show MC holds.

Suppose \( \exists i_0, j_0 \) and an \( 0 \neq \alpha \in D \) such that \( \alpha \notin \Delta_{i_0 j_0} \) and \( \alpha^{-1} \notin \Delta_{j_0 i_0} \). Since \( \Delta \) is a valuation ring, \( i_0 \neq j_0 \). Let

\[
\Gamma = (e_{i_0 j_0} + e_{j_0 i_0}) R (e_{i_0 j_0} + e_{j_0 i_0}) \cong \begin{pmatrix} \Delta & \Delta_{j_0 i_0} \\ \Delta_{i_0 j_0} & \Delta \end{pmatrix}.
\]
Then $\Gamma$ is a semihereditary order in $M_2(D)$, by [Sa]. Consider $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(D)$. As in the proof of Theorem 1.5,

$$\text{Rt}-\text{Ann}_{R}(x) = \left\{ \begin{pmatrix} t & r \\ -\alpha t & -\alpha r \end{pmatrix} \mid t, \alpha r \in \Delta, r \in \Delta_{ij_{0}0}, \alpha t \in \Delta_{ij_{0}1} \right\}.$$

We have $\alpha t \in \Delta_{ij_{0}0}$ and $t \in \Delta$. But $\alpha \notin \Delta_{ij_{0}0}$. So $t \in R(\Delta)$. Since $\Gamma$ is a semihereditary order in $M_2(D)$, $\text{Rt}-\text{Ann}_{R}(x)$ is generated by an idempotent

$$\begin{pmatrix} a & b \\ -aa & -ab \end{pmatrix} = \begin{pmatrix} a & b \\ -aa & -ab \end{pmatrix}^2.$$

So $1 = a - ba$. But $a \in R(\Delta)$. Hence $ab$ is a unit in $\Delta$. But $b \in \Delta_{ij_{0}0} \Rightarrow b\Delta' \supseteq b\Delta \Rightarrow a\Delta_{ij_{0}0} \supseteq ab\Delta = \Delta$ since $ab$ is a unit in $\Delta$. Hence $a^{-1} \in \Delta_{ij_{0}1}$, a contradiction and so MC holds.

On the other hand, let $R = (\Delta_{ij})$ be of type $\mathcal{I}$H. We wish to show that $R$ is a semihereditary $V$-order in $Q = M_n(D)$. Conditions (ii) and (iv) of Definition 2.1 guarantee that $R$ is indeed a ring with the identity element of $Q$. Clearly, $FR = Q$ since $\Delta_{ij}$ are nonzero $\Delta$-submodules of $D$ ($F \cdot \Delta_{ij} = F \cdot \Delta \cdot \Delta_{ij} = D \cdot \Delta_{ij} = D$). By the proof of [M, Proposition 4.3], $R$ is a $V$-order. But $M_n(R)$ is of type $\mathcal{I}$H whenever $R$ is. Hence Lemma 2.3 shows that for each $r$, every principal right ideal of $M_n(R)$ is projective, so $R$ is right semihereditary by [S]. Similarly, $R$ is left semihereditary and hence it is semihereditary.

In short, $\mathcal{I}$H orders are precisely semihereditary $V$-orders containing all the standard idempotents of $Q$. Since all semihereditary $V$-orders contain a full set of primitive orthogonal idempotents, each of them is conjugate to one of type $\mathcal{I}$H.

The following theorems give more precise information about orders $R = (\Delta_{ij})$ of type $\mathcal{I}$H. When $Q = M_n(D)$ is a division algebra, then $\Delta$ is the unique semihereditary $V$-order in $Q$, by Corollary 1.10. So assume $n > 1$.

Fix $1 \leq i_{0} < j_{0} \leq n$ and set $S = \Delta_{ij_{0}0}$, $T = \Delta_{ij_{0}1}$. Because $S \cdot T \subseteq \Delta$, we must have $S \subseteq \Delta$ or $T \subseteq \Delta$. If $S, T \subseteq \Delta$ then MC fails (with $\alpha = 1$). So assume $S \subseteq \Delta$ and $T \supseteq \Delta$. Here and in the rest of this section, we will use the fact that if $I$ is a $\Delta$-bisubmodule of $D$, then so is $xI$ and $Ix$ for any $x$ in $\Delta$ since $\Delta$ is an invariant valuation ring of $D$. Further, $I^{-1}$ is also a $\Delta$-submodule of $D$ and coincides with $\{x \in D \mid xI \subseteq \Delta\}$. We also know that all $\Delta$-submodules in $D$ are linearly ordered by inclusion.

**Theorem 2.5.** Let $R$ be as above. Then we can only have the following (possibly overlapping) cases:

1. **Case A.** $S = T = \Delta$. 


Case B. \( T = \Delta, S = J(\Delta) \).

Case C. \( S = J(\Delta), T \supsetneq \Delta \), and \( T = \Delta^{-1} \) (only when \( J(V) \) is principal).

Case D. \( T \supsetneq \Delta, S \subseteq J(\Delta), \) and \( T = S^{-1} \).

Case E. \( T \supsetneq \Delta, S \subseteq J(\Delta), T \neq \Delta^{-1}, S = \alpha^{-1} \Delta, \) and \( T = \alpha J(\Delta) \) for any \( \alpha \in S^{-1} \setminus T \).

When \( n = 2 \), Case E is conjugate to Case B.

**Proof.** Suppose \( S = \Delta \). Then \( T = \Delta \) since \( S \cdot T \subseteq \Delta \). Hence \( S = T = \Delta \), so we are in Case A. Suppose \( T = \Delta \). Then \( S \supsetneq J(\Delta) \), else MC fails. It follows that \( S = \Delta \) or \( S = J(\Delta) \), so we are in Case A or Case B. If \( S = J(\Delta) \) and \( T \supsetneq \Delta \) then we claim \( T = S^{-1} \) and \( J(V) \) is principal (so we are in Case C). Clearly, \( T \subseteq S^{-1} \). If \( \beta \in S^{-1} \) and \( \alpha \in T \setminus \Delta \), then \( \alpha^{-1} \in J(\Delta) \), so \( \beta \alpha^{-1} \in \Delta \). Hence \( \Delta \beta \alpha^{-1} \subseteq \Delta \), so \( \Delta \beta \subseteq \Delta \alpha \). In particular, \( \beta \in \Delta \alpha \subseteq T \).

Now suppose \( J(V) \) is not principal. Then \( J(\Delta) \) is not principal by [D3, Theorem 1(2); D2, Lemmas 7 and 8] and hence \( J(\Delta)^2 = J(\Delta) \) by [D2, Lemmas 7 and 8]. Since \( T \cdot S \subseteq \Delta \), we obtain \( T \cdot J(\Delta) \subseteq \Delta \Rightarrow T \cdot J(\Delta)^2 \subseteq J(\Delta) \Rightarrow T \subseteq O_1(J(\Delta)) = \Delta \), a contradiction. So \( J(V) \) is principal.

At this point we are reduced to \( T \supsetneq \Delta, S \subseteq J(\Delta) \). We know \( T \subseteq S^{-1} \). Suppose \( T \supsetneq S^{-1} \). Let \( \alpha \in S^{-1} \setminus T \). Then \( \alpha^{-1} \in S \) since \( R \) is of type \( \mathcal{SP} \).

But \( \alpha S \subseteq \Delta \), so \( \alpha S \subseteq J(\Delta) \) or \( J(\Delta) \subseteq \alpha S \subseteq \Delta \). If \( \alpha S \subseteq J(\Delta) \subseteq \Delta \) then \( S \subseteq \alpha^{-1} J(\Delta) \subseteq \alpha^{-1} \Delta \subseteq S \) (since \( \alpha^{-1} \in S \), an ideal of \( \Delta \)). Therefore \( \alpha^{-1} J(\Delta) = \alpha^{-1} \Delta \), a contradiction. Thus \( J(\Delta) \subseteq \alpha S \subseteq \Delta \Rightarrow \alpha S = \Delta \Rightarrow S = \alpha^{-1} \Delta \) for any \( \alpha \in S^{-1} \setminus T \). Let \( \phi \) be a valuation on \( D \) with valuation ring \( \Delta \). Then \( \phi(\alpha) = \phi(\alpha') \forall \alpha, \alpha' \in S^{-1} \setminus T \), since \( S = \alpha^{-1} \Delta = \alpha'^{-1} \Delta \).

Therefore \( S^{-1} = \{ x \in D \mid \phi(x) \geq \phi(\alpha) \} \) for any \( \alpha \in S^{-1} \setminus T \). Now fix such an \( \alpha \) and let \( x \in D \) with \( \phi(x) > \phi(\alpha) \). Then \( x \in S^{-1} \) but \( x \notin S^{-1} \setminus T \) and hence \( x \in T \). This shows \( T = \{ x \in D \mid \phi(x) > \phi(\alpha) \} \) for every \( \alpha \in S^{-1} \setminus T \). Hence \( T = \{ x \in D \mid \phi(x) > 0 \} \) for any \( \alpha \in S^{-1} \setminus T \) and therefore \( T = \{ x \in D \mid \alpha^{-1} x \in J(\Delta) = \alpha J(\Delta) \} \).

For \( n = 2 \) we have \( R = \left( \begin{array}{cc} \alpha^{-1} & \Delta \\ \alpha J(\Delta) & \Delta \end{array} \right) \) and

\[
\left( \begin{array}{cc} 0 & \alpha^{-1} \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} \Delta & \alpha^{-1} \Delta \\ \alpha J(\Delta) & \Delta \end{array} \right) \left( \begin{array}{cc} 0 & \alpha^{-1} \\ 1 & 0 \end{array} \right)^{-1} = \left( \begin{array}{cc} \Delta & J(\Delta) \\ \Delta & \Delta \end{array} \right).
\]

We shall now characterize semihereditary maximal \( V \)-orders and Dubrovin valuation rings—always assuming that \( V \) is Henselian. We will, of course, also assume that the rings are of type \( \mathcal{SP} \), by virtue of Theorem 2.4. The following theorem is embedded in the proofs of [H M W, Proposition 6; M, Proposition 4.3], but not stated in such explicit terms.
THEOREM 2.6. Let $R$ be of type $\mathcal{H}$. Then $R$ is a semihereditary maximal $V$-order if and only if $\Delta_{ij}^2 = \Delta_j \forall i, j$.

Proof. Suppose $R$ is maximal. Condition (iv) of Definition 2.1 implies that $\Delta_{ij} \subseteq \Delta_{ij}^{-1}$. Suppose for some pair $r, s$ we have $\Delta_{sr} \subseteq \Delta_{rs}^{-1}$. By setting $\Delta_{sr} = \Delta_{sr}^{-1}, \Delta_{ij} = \Delta_{ui} = \Delta$ for each $i$ and for any other pair $i, j \Delta_{ij} = \max(\Delta_{ij}, \Delta_{ui} \cdot \Delta_{ui}^{-1} \cdot \Delta_{uj})$, we get a larger $V$-order (see [HMW], Proof of Proposition 6]). This is a contradiction.

For the other direction, suppose $R' \supsetneq R$ is another $V$-order. Then $R'$ also contains all the standard idempotents, hence $R' = (\Delta_{ij}), \Delta_{ij}$ a $\Delta$-bi-submodule of $D$ containing $\Delta_{ij}$. So $\Delta_{ij} \supsetneq \Delta_{ij}, \forall i \supsetneq \Delta_{ij}$. Let $x \in \Delta_{ij}$. Then $x \Delta_{ij} \subseteq \Delta_{ij}$ is integral over $V$, since $R'$ is a $V$-order. Hence $x \Delta_{ij} \subseteq \Delta_{ij} \Rightarrow x \Delta_{ij} \subseteq \Delta_{ij}$ since $\Delta_{ij} \supsetneq \Delta_{ij}$. Hence $x \in \Delta_{ij}^{-1} = \Delta_{ij}$. So $R = R'$.

PROPOSITION 2.7. A semihereditary $V$-order $R$ is maximal if and only if $J(R) = J(\Delta) R$.

Proof. By Theorem 2.4 and the fact that $\Delta$ is an invariant valuation ring, we may assume that $R$ is of type $\mathcal{H}$. Set $R = (\Delta_{ij})$ as usual.

There is a positive integer $k$ such that $J(\Delta)^k \subseteq J(V) \Delta$. Hence $(J(\Delta) R)^k = J(\Delta)^k R \subseteq J(V) R \subseteq J(R)$. So $J(\Delta) R \subseteq J(R)$.

To prove the reverse containment, observe that since $R$ contains all the standard idempotents, the ideal $J(R) = (\Omega_{ij})$ and $J(\Delta) R = (\Omega'_{ij})$, where $\Omega_{ij}, \Omega'_{ij}$ are $\Delta$-submodules of $D$. We want to show that $\Omega_{ij} \supsetneq \Omega'_{ij} \forall i, j$. Note that since $e_{ii} Re_{ii} = \Delta$, we have

$$\Omega_{ij} = e_{ii} J(R) e_{ii} = J(e_{ii} J Re_{ii}) = J(\Delta) = J(\Delta) \Delta_{ii} = \Omega'_{ii}.$$

Now suppose $i \neq j$ and let $e = e_{ii} + \epsilon_{jj}$, an idempotent of $R$. We have

$$e Re = \left( \begin{array}{cc} \Delta & \Delta_{ij} \\ \Delta_{ij} & \Delta \end{array} \right),$$

and henceforth the two rings shall be identified with each other. If $\Delta_{ij}$ is cyclic, say $\Delta_{ij} = a\Delta$ for some $a \in D$, then it follows from Theorem 2.6 that $\Delta_{ij} = a^{-1}\Delta$. The argument of Theorem 2.5(E) then shows that $eRe$ is conjugate to $M_2(\Lambda)$.

If $\Delta_{ij}$ is not cyclic, then again by Theorem 2.6, $\Delta_{ij}$ is not cyclic. Let $\phi$ be a valuation on $D$ with valuation ring $\Delta$. Then $\phi(\Delta_{ij})$ (and $\phi(\Delta_{ij})$) does not have a smallest element. Let $x \in \Delta_{ij}$. Then $\exists y \in \Delta_{ij} \ni \phi(y) < \phi(x)$. So $xy^{-1} \in J(\Delta) \Rightarrow x = (xy^{-1})y \in J(\Delta) \Delta_{ij} \Rightarrow \Delta_{ij} = J(\Delta) \Delta_{ij}, \Delta_{ij} \Rightarrow J(\Delta)e Re = (e, \Delta_{ij}^{-1}).$

In either case, $eRe/J(\Delta)e Re$ is a semisimple ring. We have thus shown that for each pair of subscripts $i, j, i \neq j$, $J(\Delta)e Re \supsetneq eJ(R)e$ where
\( e = e_{ii} + e_{jj}. \) Thus
\[
\Omega'_{ij} = e_{ii}(J(\Delta)eRe)e_{jj} \geq e_{ii}(eJ(R)e)e_{jj} = \Omega_{ij}
\]
and hence \( J(\Delta)R = J(R). \)

Conversely, suppose \( J(R) = J(\Delta)R. \) Let \( R' \supseteq R \) be another \( V \)-order. Then \( J(R') \subseteq J(R), \) since \( R \) is a semihereditary \( V \)-order, by Theorem 1.5. Hence we have
\[
J(R') \subseteq J(R) = J(\Delta)R \subseteq J(\Delta)R'.
\]
There exists a positive integer \( k \) such that \( (J(\Delta)R')^k = J(\Delta)^k R' \subseteq J(V)R' \subseteq J(R'). \) Hence \( J(\Delta)^k R' \subseteq J(R'). \) Therefore we have \( J(R') = J(R) \) and hence \( R' = R \) since \( R \) is an extremal \( V \)-order. \( \square \)

**Corollary 2.8.** Given a semihereditary maximal \( V \)-order \( R \) there exists a positive integer \( k \) such that \( J(R)^k = J(V)R. \)

This assertion follows immediately from Proposition 2.7 and [D, Theorem 1(6)], since \( \Delta \) is a Dubrovin valuation ring.

**Corollary 2.9.** Let \( R \) be a semihereditary \( V \)-order. If \( J(R) = J(V)R, \) then \( R \) is a maximal \( V \)-order. If \( J(V) \) is not principal and \( R \) is maximal, then \( J(R) = J(V)R. \)

**Proof.** Suppose \( J(R) = J(V)R \) and let \( B \) be another \( V \)-order containing \( R. \) By Theorem 1.5, \( J(B) \subseteq J(R) \) and hence we have
\[
J(V)B \subseteq J(B) \subseteq J(R) = J(V)R \subseteq J(V)B.
\]
Hence \( J(R) = J(B) \) and so \( R = B, \) since \( R \) is extremal. So \( R \) is maximal. Now suppose \( R \) is maximal. Proposition 2.7 shows that \( J(R) = J(\Delta)R. \) If \( J(V) \) is not principal, then \( J(\Delta) \) is not principal by [D, Theorem 1(2)] and hence \( J(\Delta)^2 = J(\Delta). \) But since \( \Delta \) is a \( V \)-order, there exists a positive integer \( k \) such that \( J(\Delta)^k \subseteq J(V)\Delta. \) So we have \( J(V)\Delta \subseteq J(\Delta) = J(\Delta)^k \subseteq J(V)\Delta. \) Hence \( J(\Delta) = J(V)\Delta, \) so \( J(R) = J(V)R. \) \( \square \)

We end this section by characterizing Dubrovin valuation rings of type \( \mathcal{S}, \) i.e., those that contain all the standard idempotents. We note that, over Henselian valuation rings, all Dubrovin valuation rings are integral over their center and hence conjugate to one of type \( \mathcal{S}. \) In a central simple algebra, a semihereditary order whose center is a Henselian valuation ring need not be integral as the following example shows: suppose \( \text{rank}(V') > 1 \) and let \( W \supseteq V' \) be another valuation ring. Then \( R = (V')^{(W)} \) is an order in \( M_r(F) \) with center \( V. \) This order is semihereditary but not a \( V \)-order; it is clearly not a \( V \)-order since \( V \) is integrally closed and hence \( W \)
cannot be integral over $V$. It contains the $V$-order $(V^{W}_{W,J(W)})$ which is semihereditary by Theorem 2.4. Hence $R$ is semihereditary by [M, Lemma 4.10].

**Theorem 2.10.** Let $R = (\Delta_{ij})$ be an order of type $\mathcal{J}\mathcal{F}$. Then $R$ is a Dubrovin valuation ring if and only if $R$ is a maximal $V$-order and for each pair $(i, j)$, $\Delta_{ij}$ is a cyclic $\Delta$-bisubmodule of $D$.

**Proof.** Assume $R$ is a Dubrovin valuation ring. Clearly, $R$ has to be a maximal $V$-order. Fix $1 \leq r < t \leq n$. Then by [D2, Theorem 7(2)]

$$R' = \left( \begin{array}{cc} \Delta & \Delta_{rt} \\ \Delta_{tr} & \Delta \end{array} \right)$$

is a Dubrovin valuation ring of $M_n(D)$. We know that $\Delta_{rt} \cdot \Delta_{tr} \subseteq \Delta$. Suppose $\Delta_{rt} \cdot \Delta_{tr} = \Delta$. Then $\exists a_1, \ldots, a_n \in \Delta_{rt}$, $b_1, \ldots, b_n \in \Delta_{tr} \ni \sum_i a_i b_i = 1$. Clearly, $\Delta_{rt} = a_1 \Delta + \cdots + a_k \Delta$, and $\Delta_{tr} = b_1 \Delta + \cdots + b_k \Delta$. Hence both of them are cyclic, since $\Delta$ is a valuation ring of $D$. If $\Delta_{rt}$ (or $\Delta_{tr}$) is not cyclic, then $\Delta_{rt} \cdot \Delta_{tr} \subseteq J(\Delta)$ and hence

$$I = \left( \begin{array}{cc} J(\Delta) & \Delta_{rt} \\ \Delta_{tr} & J(\Delta) \end{array} \right)$$

is an ideal of $R'$. But $R'/I \cong \Delta/J(\Delta) \oplus \Delta/J(\Delta)$. Hence $R'$ has more than one maximal ideal so it cannot be a Dubrovin valuation ring. This is a contradiction. Hence $\Delta_{ij}$ is cyclic $\forall i, j$.

On the other hand, suppose $\Delta_{ij} = a_i \Delta$ and $R$ is a maximal $V$-order. Hence $\Delta_{ij}^{-1} = \Delta_{ji}$ by Theorem 2.6. Let $\phi$ be a valuation on $D$ with valuation ring $\Delta$. Then $\phi(a_{ij}) = -\phi(a_{ji}) \forall i, j$. Condition (iv) of Definition 2.1 implies that $\phi(a_{ij}) + \phi(a_{jk}) \geq \phi(a_{ik})$. Similarly, $\phi(a_{ji}) + \phi(a_{jk}) \geq \phi(a_{ik})$ and hence

$$\phi(a_{ij}) + \phi(a_{jk}) = \phi(a_{ik}). \quad (\ast)$$

Since $R$ contains all the standard idempotents, the ideal $J(\Delta)R (= R/J(\Delta)) = (\Omega_{ij})$, where $\Omega_{ij} = a_i \cdot J(\Delta)$. Suppose $(\Omega_{ij}) \not\subseteq (\Omega'_{ij})$ is another ideal of $R$. Since $a_i J(\Delta)$ is a maximal $\Delta$-bisubmodule of $a_i \Delta$, we must have $\Omega_{ij} = a_i J(\Delta)$ or $\Omega_{ij} = a_i \Delta$. Suppose $\exists r, s$ such that $\Omega_{rs} = a_i \Delta$. We have $\Omega_{rs} a_{sk} \Delta \subseteq \Omega_{rk}$ for $k = 1, \ldots, n$, since $(\Omega_{ij})$ is an ideal of $R$. By $(\ast)$ above, $a_{sk} \Delta = a_{sk} \Delta a_{sk} \Delta \subseteq \Omega_{rk}$, $k = 1, \ldots, n$. Given any $i, j, \Delta_{ij} \Omega_{ij} \subseteq \Omega_{ij}$. Thus $\Delta_{aij} \Delta \subseteq \Omega_{ij}$ and therefore by $(\ast)$ above $a_{ij} \Delta \subseteq \Omega_{ij}$. Thus $(\Omega_{ij}) = R$ and hence $J(\Delta)R$ is a maximal ideal. But by Proposition 2.7, $J(R) = J(\Delta)R$. Hence $R$ is a primary semihereditary $V$-order and therefore a Dubrovin valuation ring, by [D2, Theorem 4(1)].
We summarize these results as follows: if $R = (\Delta_{ij})$ is a ring with $\Delta_{ij}$ nonzero $\Delta$-bisubmodules of $D$ and $\Delta_{ii} = \Delta$ then $R$ is a semihereditary maximal $V$-order if and only if $\Delta_{ij}^{-1} = \Delta_{ji}$, $\forall i, j$. If in addition the $\Delta_{ij}$ are cyclic, then it is a Dubrovin valuation ring. Such orders are automatically of $\mathcal{S} \mathcal{M}$ type and hence the results follow from Theorems 2.6 and 2.10.

3. THE HENSELIZATION OF A SEMIHEREDITARY $V$-ORDER

Given any order $H$ with $Z(H) = V$ we will denote its Henselization, $H \otimes_{V} V_h$, where $V_h$ is the Henselization of $V$, by $H_h$. Similarly $Q \otimes_{F} F_h$, where $F_h$ is the field of fractions of $V_h$, will be denoted by $Q_h$. In [H a, R] it was proved that when $V$ is a DVR, a $V$-order $H$ is semihereditary (necessarily hereditary) if and only if its completion, $H \otimes_{V} \hat{V}$, is a semihereditary (necessarily hereditary) $\hat{V}$-order, where $\hat{V}$ is the $J(V)$-adic completion of $V$. Further, if $H$ is hereditary then there is a one-to-one inclusion-preserving correspondence between hereditary $V$-orders inside $H$ and hereditary $\hat{V}$-orders inside $H \otimes_{V} \hat{V}$. In this section, we will prove analogous results by making use of Henselization instead of completion. As usual, $V$ has arbitrary Krull-dimension in our case.

The statements contained in the following lemma will be used frequently:

**Lemma 3.1.** Let $A$ be $V$-order. Then we have:

1. $Q \cap A_h = A$
2. $A + J(A_h) = A + J(V)A_h = A_h$
3. $J(A_h) = J(A) \otimes_{V} V_h$
4. $J(A_h) \cap Q = J(A)$
5. $A/J(A) \cong A_h/J(A_h)$.

**Proof.** (1) See [D 1, Lemma 2].

(2) If $x \in A_h$ then $x = \Sigma a_i \tilde{v}_i$, with $a_i \in A$ and $\tilde{v}_i \in V_h$. But $V_h = V + J(V_h) = V + J(V)V_h$. So the result follows.

(3) See [HM W, Lemma 12].

(4) This follows from (3) and [D 1, Lemma 2].

(5) This is an easy consequence of (2) and (4).

The following proposition was motivated by the work of D. Miller [see M I] in which it was proved that the Henselization of a semihereditary order finitely generated over its center, $V$, is semihereditary. In this paper, the $V$-orders need not be finitely generated over their centers.
**Proposition 3.2.** If $H$ is a semihereditary $V$-order then $H_h$ is a semihereditary $V_h$-order.

**Proof.** Let $I$ be a finitely generated right ideal of $H$, say $I = x_1 H + \cdots + x_n H$. Let $x_i = \sum_j a_{ij} \tilde{v}_{ij}$, $a_{ij} \in H$, $\tilde{v}_{ij} \in V_h$. We want to prove that $I$ is a projective $H_h$-module. We first claim $I$ is finitely presented as an $H_h$-module.

By [Na, 43.9] there exists an $s \in J(V_h)$ integral over $V$ such that $V' = V[s]_{J(V')}$ contains $\tilde{v}_{ij}$ and $V_h$ is the Henselization of $V'$. (Notice that $V/J(V) \cong V[s]/(J(V) + sV[s])$, hence $J(V) + sV[s]$ is a maximal ideal of $V[s]$.) We thus can write $\tilde{v}_{ij} = v_{ij} u^{-1}$, $v_{ij} \in V[s]$ and $u \in V[s] \setminus (J(V) + sV[s])$. Let $K = F[s]$, $Q' = Q \otimes K$, $H' = H \otimes V[s]$, $I' = x_1 u H' + \cdots + x_n u H' \subseteq H'$. Thus we have

$$H' = H \cdot V[s] \subseteq H \cdot V_h = H_h.$$

And so we obtain the inclusions

$$H \hookrightarrow H' \hookrightarrow H_h.$$

Since $H_h$ is torsion-free over $H$ (in the sense of [Lv]; no regular element of $H$ annihilates any nonzero element of $H_h$) and $I' \subseteq H_h, I'$ is a torsion-free right $H$-module. Since $I'$ is finitely generated over $H'$ which in turn is finitely generated over $H$ (actually a finite free $H$-module, since $s$ is integral over $V$, a valuation ring), $I'$ is finitely generated over $H$. Because $Q$, the two-sided quotient ring of $H$, is a central simple algebra, $I'$ can be embedded in a free $H$-module, by [Lv, Theorem 5.2]. Hence by [CE, Proposition 6.2], $I'$ is $H$-projective, since $H$ is semihereditary. We have the following exact sequence of $H'$-modules:

$$0 \to L \to (H')^k \to I' \to 0.$$

But $I'$ is projective, finitely generated over $H$, and $H'$ is finitely generated over $H$. Hence the sequence splits over $H$ and so $L$ is finitely generated over $H$.

Since $V'$ is a localization of $V[s]$, it is a flat left $V[s]$-module. Since $V_h$ is a Henselization of $V'$, it is a flat left $V'$-module. Hence $V_h$ is a flat left $V[s]$-module and we have the following exact sequence of $H' \otimes V[s] V_h = H \otimes V[s] \otimes V[s] V_h = H_h$-modules:

$$0 \to L \otimes V[s] V_h \to (H')^k \otimes V[s] V_h \to I' \otimes V[s] V_h \to 0. \quad (*)$$

Note that

$$Q' \otimes V[s] V_h = Q' \otimes K F_h = Q' F_h = Q' V_h.$$
Therefore tensoring any $V[s]$-submodule of $Q'$ by the ring $V_h$ is the same thing as multiplying the module by $V_h$ (everything happening inside $Q' \otimes_k KF_h = Q \otimes_k K \otimes_k KF_h = Q_h$). So $I' \otimes V[s]V_h = I'V_h$. But $V[s] \subseteq W := K \cap V_h$. Hence $J(W) = J(V_h) \cap W \supseteq J(V) + sV[s]$. Since $J(V) + sV[s]$ is a maximal ideal of $V[s]$, $J(V_h) \cap V[s] = J(V) + sV[s]$ and hence $u$ is a unit in $V_h$. Therefore $I' \otimes V[s]V_h = I'V_h = I$ and the sequence (*) above shows that $I$ is a finitely presented $H_h$-module.

We now proceed to show $I$ is projective over $H_h$.

Since $H_h$ is semiperfect (i.e., $\overline{H}_h$ is semisimple Artinian and, by a result of Azumaya [Az, Theorem 24], idempotents of $\overline{H}_h$ can be lifted to $H_h$), by [Lam, Proposition 24.12], there exists an exact sequence of $H_h$-modules,

$$0 \to M \to P \xrightarrow{\theta} I \to 0,$$

where $M \subseteq P \cdot J(H_h)$ and $P$ is a finitely generated projective $H_h$-module (i.e., $P$ is a projective cover of $I$). But $I$ is finitely presented. Hence by Schanuel’s lemma, $M$ is finitely generated. We wish to show that $M = 0$.

Let $J = J(H_h)$ and $\psi : P \to I/J$ be the natural map. Let $x \in \ker \psi$. Then $\phi(x) \in J \Rightarrow \theta(x) = \sum y_jh_i, h_i \in J, y_j \in I$. So for each $i$ there exist $p_i \in P$ such that $\theta(p_i) = y_i$ and hence $\theta(x - \sum p_ih_i) = 0$ and therefore $x - \sum p_ih_i \in M$ which implies $x \in PJ + M$ and thus $\ker \psi = PJ + M = PJ$. Hence

$$P/JP \cong I/JI.$$  

The Tor-$\otimes$ sequence yields

$$\Tor_1(P, H_h/J) \xrightarrow{\beta_1} \Tor_1(I, H_h/J) \xrightarrow{\beta_2} M \otimes_{H_h} H_h/J \xrightarrow{\beta_3} P \otimes_{H_h} H_h/J,$$

$$\xrightarrow{\beta_4} I \otimes_{H_h} H_h/J.$$

But $P \otimes_{H_h} H_h/J \cong P/PJ \cong I/JI \cong I \otimes_{H_h} H_h/J$. So $\beta_4$ is an isomorphism (induced by the one in (* *)), and hence $\beta_3 = 0$. Moreover $P$ is projective, hence $\Tor_1(P, H_h/J) = 0$. So we have

$$\Tor_1(I, H_h/J) \cong M \otimes_{H_h} H_h/J \cong M/MJ.$$  

The exact sequence

$$0 \to I \to H_h \to H_h/I \to 0$$
gives rise to
\[ \text{Tor}_2(H_v, H_v/J) \longrightarrow \text{Tor}_2(H_v/I, H_v/J) \longrightarrow \text{Tor}_2(I, H_v/J) \longrightarrow \text{Tor}_2(H_v, H_v/J) \]
\[ \text{Tor}_2(I, H_v/J) \longrightarrow \text{Tor}_2(H_v, H_v/J) \]
since \( H_v \) is free
\[ 0 \longrightarrow 0 \]
Therefore \( M/MJ \cong \text{Tor}_2(I, H_v/J) \cong \text{Tor}_2(H_v/I, H_v/J) \). But by Lemma 3.1 we have
\[ 0 \longrightarrow J(H) \longrightarrow H \longrightarrow H_v \longrightarrow 0, \]
a sequence of \( H \)-modules. Since \( V_h \) is a flat \( V \)-module, we get a sequence of \( H_v \)-modules:
\[ 0 \longrightarrow J(H) \otimes_V V_h \longrightarrow H \otimes_V V_h \longrightarrow (H_v/J) \otimes_V V_h \longrightarrow 0. \]
But
\[ (H_v/J) \otimes_V V_h = (H_v/J) \otimes_V (V + J(V)V_h) \]
\[ = (H_v/J) \otimes_V V + J(V)(H_v/J) \otimes_V V_h = H_v/J \]
since \( J(V)(H_v/J) = 0 \). This sequence thus gives rise to the \( \text{Tor} \)-sequence of \( H_v \)-modules:
\[ \text{Tor}_2(H_v/I, H_v) \longrightarrow \text{Tor}_2(H_v/I, H_v/J) \longrightarrow \text{Tor}_2(H_v/I, J(H) \otimes_V V_h) \longrightarrow \text{Tor}_2(H_v/J, H_v). \]
\[ 0 \longrightarrow 0 \]
And so \( \text{Tor}_2(H_v/I, H_v/J) \cong \text{Tor}_2(H_v/I, J(H) \otimes_V V_h) \).

Since \( H \) is semihereditary, \( J(H) \) is a flat \( H \)-module. Since \( V \) is commutative, \( J(H) \otimes_V V_h \) is a flat left \( H_v \)-module as well.

Hence \( \text{Tor}_2(H_v/I, J(H) \otimes_V V_h) = 0 \), which implies \( M/MJ = 0 \) and hence \( M = MJ \). But \( M \) is a finitely generated \( H_v \)-module. So \( M = 0 \), by the Nakayama–Azumaya lemma and hence \( I \cong P \), a projective \( H_v \)-module.

So \( H_v \) is right semihereditary. It can similarly be proved that \( H_v \) is left semihereditary, and hence it is semihereditary. It is obviously a \( V_h \)-order in \( Q_h \).

Let \((L, W)\) be an extension of \((F, V)\), i.e., \( L \) is a field extension of \( F \) and \( W \) is a valuation ring of \( L \) with \( W \cap F = V \). Then \( W \) is a flat \( V \)-module, since \( V \) is semihereditary and \( W \) is torsion-free over \( V \). Further, since \( J(V)W \subseteq J(W) \neq W \), \( W \) is actually a faithfully flat \( V \)-module.

The following proposition does not assume that \( H \) is a \( V \)-order:

**Proposition 3.3.** Let \( H \) be any algebra over \( V \) and \((L, W)\) an extension of \((F, V)\). If \( H \otimes_V W \) is semihereditary, then so is \( H \).

**Proof.** We first show that principal right ideals of \( H \) are projective. Let \( a \in H \). Let
\[ P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow aH \longrightarrow 0 \]
be an $H$-projective resolution of $aH$. Then

$$
\to P_i \otimes_V W \to P_{i-1} \otimes_V W \to
$$

is an $H \otimes_V W$-projective resolution of $aH \otimes_V W$. Let $C$ be an arbitrary left $H$-module and consider the complex

$$
\to P_i \otimes_H C \xrightarrow{d_i} P_{i-1} \otimes_H C \to .
$$

Then we also have

$$
\to (P_i \otimes_H C) \otimes_V W \xrightarrow{c_i} (P_{i-1} \otimes_H C) \otimes_V W \to .
$$

And so $\text{Image}(c_i) = (\text{Image}(d_i)) \otimes_V W$ and, since $\otimes_V W$ is exact, $\text{Ker}(c_i) = (\text{Ker}(d_i)) \otimes_V W$. Since $\otimes_V W$ is an exact functor, the sequence

$$
0 \to \text{Image}(d_{i+1}) \to \text{Ker}(d_i) \to \text{Tor}_i^H(aH, C) \to 0
$$

yields

$$
0 \to \text{Image}(d_{i+1}) \otimes_V W \to \text{Ker}(d_i) \otimes_V W \to \text{Tor}_i^H(aH, C) \otimes_V W \to 0.
$$

So $\text{Tor}_i^H(aH, C) \otimes_V W = \text{Tor}_i^{H \otimes_V W}(aH \otimes_V W, C \otimes_V W)$ for any left $H$-module $C$. In particular,

$$
\text{Tor}_i^H(aH, C) \otimes_V W = \text{Tor}_1^{H \otimes_V W}(aH \otimes_V W, C \otimes_V W) = 0
$$

for every $H$-module $C$ since $aH \otimes_V W = (a \otimes 1)H \otimes_V W$ is $H \otimes_V W$-projective by assumption [Rt, Theorem 8.4]. Hence $\text{Tor}_i^H(aH, C) = 0$ for every left $H$-module $C$, since $\otimes_V W$ is faithfully flat. Therefore $aH$ is a flat right $H$-module, by [Rt, Theorem 8.9]. To show that $aH$ is $H$-projective, we now only need to show that it is finitely presented. So consider the exact sequence of $H$-modules,

$$
0 \to \text{Rt-Ann}_H(a) \to H \xrightarrow{\times a} aH \to 0,
$$

where $\times a$ is left multiplication by $a$. Since $\otimes_V W$ is an exact functor, we obtain the following exact sequence,

$$
0 \to \text{Rt-Ann}_H(a) \otimes_V W \xrightarrow{a \otimes 1} H \otimes_V W \to aH \otimes_V W \to 0,
$$
where \( \times (a \otimes 1) \) is just left multiplication by \( a \otimes 1 \). Hence \( \text{Rt-Ann}_{H} (a) \otimes_{V} W = \text{Rt-Ann}_{H \otimes_{V} W} (a \otimes 1) \). But \( H \otimes_{V} W \) is semihereditary by assumption. So \( \text{Rt-Ann}_{H \otimes_{V} W} (a \otimes 1) = e (H \otimes_{V} W) \), \( e^{2} = e \in H \otimes_{V} W \). So \( e \in \text{Rt-Ann}_{H} (a) \otimes_{V} W \), by the observation above. Hence there exists \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{k} \in \text{Rt-Ann}_{H} (a) \), \( w_{1}, w_{2}, \ldots, w_{k} \in W \) such that \( e = \alpha_{1} \otimes w_{1} + \alpha_{2} \otimes w_{2} + \cdots + \alpha_{k} \otimes w_{k} \).

Now consider \( N = \alpha_{1} H + \cdots + \alpha_{k} H \subseteq \text{Rt-Ann}_{H} (a) \). Then

\[
e (H \otimes_{V} W) \subseteq N \otimes_{V} W \subseteq \text{Rt-Ann}_{H} (a) \otimes_{V} W = e (H \otimes_{V} W).
\]

Hence \( N \otimes_{V} W = \text{Rt-Ann}_{H} (a) \otimes_{V} W \). Since \( \otimes_{V} W \) is a faithfully flat functor, we have the equality \( \text{Rt-Ann}_{H} (a) = N \). So \( \text{Rt-Ann}_{H} (a) \) is a finitely generated \( H \)-module. Therefore \( aH \) is finitely presented and hence projective, since it is flat.

Notice that \( M_{l} (H \otimes_{V} W) \) is semihereditary for every \( l \), since \( H \otimes_{V} W \) is. But \( M_{l} (H \otimes_{V} W) \cong M_{l} (H) \otimes_{V} W \). Hence for every \( l \), every principal right ideal of \( M_{l} (H) \) is projective, by the argument above and therefore \( H \) is right semihereditary, by [S]. By analogous arguments, \( H \) is also left semihereditary and hence is semihereditary. \( \blacksquare \)

We are now ready for the main theorem of this section.

**Theorem 3.4.** A \( V \)-order \( H \) is semihereditary if and only if its Henseliza-
\( \bar{H} \), is a semihereditary \( V_{\bar{H}} \)-order. In this case, there is a one-to-one inclusion-preserving correspondence between semihereditary \( V \)-orders inside \( H \) and semihereditary \( V_{\bar{H}} \)-orders inside \( \bar{H} \) given by

\[
A \mapsto A_{\bar{H}}, \quad \mathcal{A} \cap Q \leftrightarrow \mathcal{A}.
\]

**Proof.** The first assertion is an immediate consequence of Propositions 3.2 and 3.3. Let \( H \) be a semihereditary \( V \)-order. By virtue of Lemma 3.1,

\[
H / J (H) \cong H_{\bar{H}} / J (H_{\bar{H}}).
\]

Hence there is a one-to-one inclusion-preserving correspondence between subrings of \( H \) that contain \( J (H) \) and subrings of \( H_{\bar{H}} \) that contain \( J (H_{\bar{H}}) \). Under this correspondence, \( V \)-orders clearly correspond to \( V_{\bar{H}} \)-orders. Any semihereditary \( V \)- (resp. \( V_{\bar{H}} \)) order inside \( H \) (resp. \( H_{\bar{H}} \)) contains \( J (H) \) (resp. \( J (H_{\bar{H}}) \)) by Theorem 1.5. Let \( A \) be a semihereditary \( V \)-order inside \( H \) and let \( \mathcal{A} \) be the \( V_{\bar{H}} \)-order inside \( H_{\bar{H}} \) obtained by the correspondence just described. so

\[
\mathcal{A} = A + J (H_{\bar{H}}).
\]

By Lemma 3.1, \( A_{\bar{H}} = A + J (A_{\bar{H}}) \supseteq \mathcal{A} \). But \( A_{\bar{H}} = A \cdot V_{\bar{H}} \subseteq \mathcal{A} \). Thus \( \mathcal{A} = A_{\bar{H}} \) and \( \mathcal{A} \) is semihereditary, by Proposition 3.2. Conversely, let \( \mathcal{A} \) be a semihereditary \( V_{\bar{H}} \)-order inside \( H_{\bar{H}} \) and consider the corresponding subring
A of \( H \) that contains \( J(H) \) and satisfies
\[
A + J(H_h) = \mathscr{A}.
\]

We have \( A_h = A \cdot V_h \subseteq \mathscr{A} \). On the other hand, \( A_h = A + J(A_p) \) by Lemma 3.1. Since \( J(A) \supseteq J(H) \), by Lemma 3.1 we have \( J(A_h) = J(A) \otimes_{V_h} J(H) \otimes_{V_h} J(V_h) = J(H_h) \) and thus \( A_h \supseteq A + J(H_h) = \mathscr{A} \). So \( A_h = \mathscr{A} \) and since \( \mathscr{A} \) is semihereditary, \( A \) must be semihereditary by Proposition 3.3.

So in the language of [R, Chap. 9], a semihereditary \( V \)-order \( H \) is *radically covered* by the semihereditary \( V_h \)-order \( H_h \) by Lemma 3.1: \( H_h \cap Q \supseteq H \) and \( J(H_h) \cap Q \supseteq J(H) \).

We end this section by considering the Henselization of a semihereditary maximal \( V \)-order. In [W], it was shown that the Henselization of an integral Dubrovin valuation ring (which is necessarily a semihereditary maximal \( V \)-order) is maximal. In fact, it was shown that a Dubrovin valuation ring is integral if and only if its Henselization is a Dubrovin valuation ring. In [HMW] it was shown that the Henselization of a Bézout \( V \)-order (a semihereditary maximal \( V \)-order as well) is a semihereditary maximal \( V \)-order which need not be a Bézout \( V \)-order. We will prove these theorems using different methods.

We note that if \( A \) is a \( V \)-order such that \( A_h \) is a maximal \( V_h \)-order, then \( A \) must be a maximal \( V \)-order: if \( B \supseteq A \) is another \( V \)-order, then \( A_h = B_h \), since \( A_h \) is maximal. Hence \( A = B \), since \( V_h \) is faithfully flat.

**Proposition 3.5.** Let \( A \) be a semihereditary \( V \)-order and suppose \( J(V) = \pi V \), a principal ideal of \( V \). Then \( A \) is a maximal \( V \)-order if and only if \( A_h \) is a maximal \( V_h \)-order.

**Proof.** Suppose \( A \) is maximal. By Lemma 3.1, \( A_h = A + \pi A_h \) and hence
\[
A_h/\pi A_h \cong A/(A \cap \pi A_h).
\]

But \( A \cap \pi A_h = \pi A \): it is clear that \( A \cap \pi A_h \supseteq \pi A \). Now let \( x \in A \cap \pi A_h \). Then \( x = \pi a \) for some \( a \in A_h \). Hence \( \pi^{-1} x = a \in Q \cap A_h = A \), by Lemma 3.1. Therefore \( x \in \pi A \) and the equality holds. So
\[
A_h/\pi A_h \cong A/\pi A. \quad (\ast)
\]
Let $\mathcal{B}$ be a $V_h$-order containing $A_h$. Since $A_h$ is semihereditary by Proposition 3.2, $J(\mathcal{B}) \subseteq J(A_h)$ by Theorem 1.5 and hence we have

$$\pi A_h \subseteq \pi \mathcal{B} \subseteq J(\mathcal{B}) \subseteq J(A_h) \subseteq A_h.$$ 

Because of $(\ast)$ above, there exists an ideal $I$ of $A$, containing $\pi A$, such that

$$I + \pi A_h = \pi \mathcal{B}.$$ 

Hence $Q \cap (I + \pi A_h) = Q \cap (\pi \mathcal{B}) = \pi (Q \cap \mathcal{B}) = \pi A$ since $A$ is maximal. But by the modular law, $Q \cap (I + \pi A_h) = I + \pi A$. So $I = \pi A$ since $I \supseteq \pi A$ and hence $\pi A + \pi A_h = \pi \mathcal{B}$ and therefore $\pi A_h = \pi \mathcal{B}$ which implies $A_h = \mathcal{B}$ and so $A_h$ is maximal. 

**Lemma 3.6.** Let $A$ be a semihereditary $V$-order. If $J(A) = J(V)A$ then $A$ is a maximal $V$-order and $A_h$ is a semihereditary maximal $V_h$-order.

**Proof.** The proof that $A$ is a maximal $V$-order is the same as that of the first part of the proof of Corollary 2.9, even though in this case $V$ need not be Henselian.

Notice that by Lemma 3.1,

$$J(A_h) = J(A) \otimes V_h = J(A) \cdot V_h = J(V) \cdot A \cdot V_h = J(V_h)A_h.$$ 

The second result therefore follows from the first. 

The following lemma relies heavily on the theory of Bézout orders in simple Artinian rings as formulated in [G, G].

**Lemma 3.7.** Let $R$ be a Bézout $V$-order. Then $J(R)^2 \neq J(R)$ if and only if $J(R)$ is a principal right (and left) ideal of $R$, if and only if $J(V)$ is a principal ideal of $V$.

**Proof.** Let $R$ be a Bézout $V$-order. Then by [M, Theorem 3.4], $R = B_1 \cap B_2 \cap \cdots \cap B_k$, where the $B_i$ are Dubrovin valuation rings with center $V$ satisfying the “Intersection Property” described in [G, G]. We also have $J(R) = \bigcap_i J(B_i)$. It is known that $J(B_i) \cap R$ is a prime ideal of $R$, $(r \in R | r + J(B_i) \cap R$ is regular in $R/(J(B_i) \cap R)$) is a left and right $\mathcal{B}$-re
set in $R$, and the localization of $R$ at $J(B_j) \cap R$ coincides with $B_j$. Thus localizing $J(R)$ at $J(B_j) \cap R$ one gets

$$J(R)B_j = J(B_j)$$

and therefore

$$J(R)^2 = \bigcap_i \left( J(R)^2 B_j \right) = \bigcap_i J(R)J(B_j) = \bigcap_i J(R)B_j J(B_j) = \bigcap_i J(B_j)^2.$$ 

Therefore if $J(B_j)^2 = J(B_j)$ for each $j$ then $J(R)^2 = J(R)$. On the other hand, if $J(R)^2 = J(R)$ then $\bigcap_i J(B_j)^2 = \bigcap_i J(B_j)$ and after localizing at $J(B_j) \cap R$ we see that $J(B_j)^2 = J(B_j)$ for each $i$. By [D$_2$, Lemma 7.8], $J(B_j)^2 = J(B_j)$ for each $i$ iff $J(B_j)$ is not a principal right ideal and, by [D$_3$, Theorem 1(2)], this is true iff $J(V)$ is not a principal ideal of $V$. Thus $J(R)^2 = J(R)$ iff $J(V)$ is not a principal ideal of $V$.

Now assume $J(V)$ is not a principal ideal. Then $J(R)^2 = J(R)$. If $J(R)$ is a principal right ideal, say $J(R) = cR$, then

$$J(B_j) = J(R)B_j = cB_j$$

and hence $J(B_j)$ is a principal right ideal, and so $J(V)$ is a principal ideal, a contradiction. Thus if $J(V)$ is not principal then neither is $J(R)$.

Suppose $J(V)$ is principal. Since

$$[J(R)/J(V)^2 R : V/J(V)] \leq [R/J(R)^2 R : V/J(V)] < \infty,$$

there exists $a_1, a_2, \ldots, a_i \in J(R)$ such that $J(R) = a_1 V + a_2 V + \cdots + a_i V + J(V)R$, a finitely generated right ideal of $R$ since $J(V)$ is a principal ideal of $V$. Thus $J(R)$ is a principal ideal of $R$, since $R$ is Bézout.

We now prove the main result in [HMW] using a different method.

**Proposition 3.8.** Let $B$ be a Bézout $V$-order. Then $B_h$ is a maximal semihereditary $V_h$-order.

**Proof.** By [G$_1$, Theorem 3.7; M, Theorem 3.4], $B$ is a maximal $V$-order. It is obviously semihereditary, being Bézout. Hence $B_h$ is a semihereditary $V_h$-order, by Proposition 3.2. If $J(V)$ is principal, then $B_h$ is maximal, by Proposition 3.5. If $J(V)$ is not principal, then $J(B)^2 = J(B)$, by Lemma 3.7. But there exists an $m$ such that $J(B)^m \subseteq J(V)B$. Since $J(B)^2 = J(B)$, we must have

$$J(V)J(B) \subseteq J(B) = J(B)^m \subseteq J(V)J(B)$$

and hence $J(B) = J(V)B$. Therefore $B_h$ is maximal, by Lemma 3.6.
We now prove a result by Wadsworth in [W] using a different method.

**Proposition 3.9.** Let A be a Dubrovin valuation ring with center V. Then A is integral if and only if \( A \otimes_V V_h \) is a Dubrovin valuation ring.

**Proof.** Suppose A is integral. Then it is a semihereditary V-order by [D₂, Theorem 4] and hence \( A_h \) is a semihereditary \( V_h \)-order by Proposition 3.2. But by Lemma 3.1, \( \mathcal{A} \cong \mathcal{A}_h \). Hence \( \mathcal{A}_h \) is a simple Artinian ring since \( \mathcal{A} \) is by [D₂, Theorem 4]. Hence \( A_h \) is a matrix local semihereditary order, hence a Dubrovin valuation ring by the same theorem.

Now suppose \( A_h \) is a Dubrovin valuation ring. Since \( V_h \) is Henselian, \( A_h \) is integral and since \( A \subseteq Q \cap A_h \), A is integral over \( V \).

**Proposition 3.10.** Let A be a semihereditary maximal V-order with \( J(V) \) not principal. Then \( A_h \) is maximal if and only if \( J(A) = J(V)A \).

**Proof.** Suppose \( A_h \) is maximal. Then \( J(A_h) = J(V_h)A_h \), by Corollary 2.9. Since \( J(V)A \subseteq J(A) \) and

\[
(J(V)A) \otimes_V V_h = J(V_h)A_h = J(A_h) = J(A) \otimes_V V_h,
\]

\( J(V)A = J(A) \), because \( V(V_h) \) is faithfully flat. The converse follows from Lemma 3.6.

We conjecture that if A is a semihereditary maximal V-order with \( J(V) \) not principal, then \( J(A) = J(V)A \). This is true for integral Dubrovin and Bézout V-orders. By our results it would follow that the Henselization of every semihereditary maximal V-order is a semihereditary maximal \( V_h \)-order.

**ACKNOWLEDGMENTS**

Most of this work was part of the author’s Ph.D. thesis at Indiana University, submitted in 1993. The author acknowledges the assistance received from Darrell Haile, the advisor, and Patrick Morandi during that time. In addition, the author thanks the referee for his/her useful suggestions.

**REFERENCES**


