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A note on Chudnovsky's Fuchsian equations ☆

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ABSTRACT

We show that four exceptional Fuchsian equations, each determined by the four parabolic singularities, known as the Chudnovsky equations, are transformed into each other by algebraic transformations. We describe equivalence of these equations and their counterparts on tori. The latters are the Fuchsian equations on elliptic curves and their equivalence is characterized by transcendental transformations which are represented explicitly in terms of elliptic and theta functions.

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1. Introduction

1.1. Chudnovsky equations

The subject of the present work is the set of four ordinary differential equations

$$x(x-1)(x+1)\Psi'' + \big(3x^2-1\big)\Psi' + (x+0)\Psi = 0, \tag{1}$$

$$x\big(x^2+3x+3\big)\Psi'' + \big(3x^2+6x+3\big)\Psi' + (x+1)\Psi = 0, \tag{2}$$

$$x(x-1)(x+8)\Psi'' + \big(3x^2+14x-8\big)\Psi' + (x+2)\Psi = 0, \tag{3}$$

$$x\big(x^2+11x-1\big)\Psi'' + \big(3x^2+22x-1\big)\Psi' + (x+3)\Psi = 0, \tag{4}$$

reported for the first time by D. Chudnovsky & G. Chudnovsky [3] and considered later more fully in their remarkable work [4]. Once their arising in 1986 it became clear that list (1)–(4) is quite exceptional and one of the features of these equations is the fact that these are the only linear ordinary differential equations (ODEs) of the class

$$p\Psi'' + p'\Psi' + (x+A)\Psi = 0, \qquad p := x(x-\alpha)(x-\beta), \tag{5}$$

solutions of which are known in terms of known special functions. It is interesting also to observe that these equations, solvable as they are, fit no in any currently available algorithmic methods of integration (over ${}_2F_1$-extension fields) known in the differential Picard–Vessiot theory [13].

From the Fuchsian standpoint the equations have the parabolic singularities at each of the points $x = \{0, \alpha, \beta, \infty\}$, i.e., Fuchsian exponent differences are equal to zero there. Smirnov, in his PhD thesis [15] and subsequent work [16], considered equations of the form (5) and the question as to their reducibility to a hypergeometric equation by rational transformations of independent variable $x \mapsto z = R(x)$. He showed that there are finitely many cases of such reductions and found one of them. Solutions to Eqs. (1)–(4) reduce to the hypergeometric functions ${}_2F_1(a, b; c|z)$ indeed. However, transformations are nontrivial and their complete list was written down only recently by F. Beukers [2].[1] Arguments of works [2] and [4] are concerned with integral recurrences and another (simple) explanation is related to the fact revealed by A. Beauville in [1]. He found a complete list of six stable t-families of elliptic curves $F_t(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = 0$ over $\mathbb{P}^1(\mathbb{C})$ with only four singular fibres; these are determined by those t-values that degenerate the curve $F_t = 0$ into a rational curve (zero genus). In the language of linear ODEs this entails existence of Heun's equations, all of whose monodromy groups $\mathfrak{G}_t$ are subgroups of the full modular group $\mathrm{PSL}_2(\mathbb{Z}) =: \boldsymbol{\Gamma}(1)$ and determine the zero genus orbifolds $\mathbb{H}^+/\mathfrak{G}_t$ with four cusps and no elliptic points. This property was also confirmed by a purely group point of view in the classification work [14, Tables 2, 3] and equivalents of Zagier–Beukers three-term

[1] All the cases on pp. 427–428 are correct except for misprint $b(z)^{1/4} \to b(z)^{-1/4}$ and some incorrectness in Case B on p. 428. See also entries (6), (8), and (9) in Tables 12–13 of work [10].

recurrences [2, p. 427] were discovered, shortly after Beauville's list, in Coster's thesis [5] as ones associated with Beauville's curves.

Let us sketch a way of derivation of Beukers' ${}_2F_1$-reduction formulae making use of Beauvilles' results. Consider the original Beauville list [1, p. 658]:

$$\mathrm{X}^3+\mathrm{Y}^3+\mathrm{Z}^3+t\mathrm{XYZ}=0, \tag{I}$$

$$\mathrm{X}\big(\mathrm{X}^2+\mathrm{Z}^2+2\mathrm{ZY}\big)+t\mathrm{Z}\big(\mathrm{X}^2-\mathrm{Y}^2\big)=0, \tag{II}$$

$$\mathrm{X}(\mathrm{X}-\mathrm{Z})(\mathrm{Y}-\mathrm{Z})+t\mathrm{ZY}(\mathrm{X}-\mathrm{Y})=0, \tag{III}$$

$$(\mathrm{X}+\mathrm{Y})(\mathrm{Y}+\mathrm{Z})(\mathrm{Z}+\mathrm{X})+t\mathrm{XYZ}=0, \tag{IV}$$

$$(\mathrm{X}+\mathrm{Y})\big(\mathrm{XY}-\mathrm{Z}^2\big)+t\mathrm{XYZ}=0, \tag{V}$$

$$\mathrm{X}^2\mathrm{Y}+\mathrm{Y}^2\mathrm{Z}+\mathrm{Z}^2\mathrm{X}+t\mathrm{XYZ}=0 \tag{VI}$$

and compute Klein's J-invariants for these elliptic curves. We obtain

$$\begin{aligned}
J_{\mathrm{I}} &= \frac{-1}{12^3}\frac{t^3(t-6)^3(t^2+6t+36)^3}{(t+3)^3(t^2-3t+9)^3}, &\qquad J_{\mathrm{IV}} &= \frac{1}{12^3}\frac{(t+2)^3((t+2)^3-24t)^3}{t^3(t+8)(t-1)^2},\\
J_{\mathrm{II}} &= \frac{4}{27}\frac{(t^4-t^2+1)^3}{t^4(t-1)^2(t+1)^2}, & J_{\mathrm{V}} &= \frac{1}{12^3}\frac{(t^4+16t^2+16)^3}{t^2(t^2+16)},\\
J_{\mathrm{III}} &= \frac{1}{12^3}\frac{((t-3)^4-40(t^2-3t+2))^3}{t^5(t^2-11t-1)}, & J_{\mathrm{VI}} &= \frac{-1}{12^3}\frac{t^3(t^3+24)^3}{(t+3)(t^2-3t+9)}.
\end{aligned} \tag{6}$$

On the other hand, Klein's J is determined by a classical hypergeometric Fuchsian equation of the form (H. Bruns (1875))

$$J(J-1)\Psi_{JJ}+\frac{1}{6}(7J-4)\Psi_J+\frac{1}{144}\Psi=0 \tag{7}$$

whose monodromy group $\mathfrak{G}_J$ is $\boldsymbol{\Gamma}(1)$. We may therefore consider formulae (6) as changes of variables $J \mapsto t$: $J=R(t)$; each such a change substituted in (7) must cause this equation to become the Fuchsian one having monodromy among Beauville's groups [1], namely, group of a certain 4-punctured sphere. Hence, the resulting ODEs $\psi''+p(t)\psi'+q(t)\psi=0$ are solved in terms of ${}_2F_1$-solutions to (7), that is

$$\psi(t)=m(t)\cdot {}_2F_1\left(\frac{1}{12},\frac{1}{12};\frac{2}{3}\middle| J_k(t)\right), \quad k=\mathrm{I},\mathrm{II},\ldots,\mathrm{VI}, \tag{8}$$

where $J_k(t)$ are taken from expressions (6) and $m(t)$ is an easily computable multiplier. All this provides a simple way of getting formulae and results in an equivalent to Beukers' ones [2, pp. 427–428] up to Möbius transformations of variables t and renormalization $\Psi \mapsto \psi=m(t)\Psi$ which has no effect on monodromy representations $\mathfrak{G}_t$. Doing this, we immediately reveal (the known fact [2]) that t-equations for the cases (I), (VI) coincide and case (II) is equivalent to (V) by a trivial scaling $t \mapsto 4it$. In the end this yields the four independent equations which are equivalent to Chudnovsky's list (1)–(4).

1.2. Motivation and results

Transformations between Fuchsian equations of the rational type $x \mapsto z = R(x)$ are the subject of numerous studies and go beyond equations with parabolic singularities, hypergeometric reducibility, or (Heun's) equations with four singular points. Recent results on Heun's equations have been summarized in work [17] (see also references therein) although first examples appeared already in [4]. However rational transformations are a particular case of the general algebraic ones which have not yet been considered in the literature. On the other hand, such a kind substitutions $F(x, z) = 0$ may be thought of as Riemann surfaces and their genera may turn out to be nontrivial in general. In particular, these considerations allow us to obtain their parametrizations (uniformizing Hauptmoduln, i.e., principal moduli in Klein's terminology). These surfaces can split or not split to simpler surfaces but genesis and structure of these reducibilities are presently unknown.

In this note we show that independence of the equations with respect to Beukers' rational transformations reduces to their common algebraic equivalence (Section 3); this is done by certain algebraic substitutions and leads to very nontrivial results concerning Fuchsian equations on tori. Correlating such equations with arising algebraic curves, we obtain Riemann surfaces that admit the transcendental representations in form of (mutual) covers of tori (Section 4). Sections 2.2 and 2.3 contain an additional and more detailed motivation for algebraic/transcendental equivalence of equations under study.

Theorem 1. *Chudnovsky's equations* (1)–(4) *and their counterparts on tori* (*elliptic curves*) *are transformable into each other by algebraic and transcendental changes of independent variables. All the changes are explicitly computable* (*listed below*) *and define the equivalence relations between integrabilities of these equations.*

The exclusive character of the list (1)–(4) tells us that these algebraic curves (Table 1 and Theorem 5) are also exclusive since they realize an equivalence of any of Chudnovsky's equations to any other of them. We also give a treatment to the known Halphen transformation [12,8] as a transcendental (bi-single-valued) analog of birational transformations between polynomial (algebraic) models of an elliptic curve. This allows us to pass explicitly to associated equations on tori. We tabulate these equations and their equivalence which is essentially transcendental and representable in terms of elliptic functions. This is of special interest because implicit algebraic dependencies admissible representations in terms of covers of elliptic tori are very effectively described through Jacobi's theta-functions. Whilst Eqs. (1)–(4) define zero genus orbifolds, they explicitly lead to Riemann surfaces/orbifolds of higher genera being no transformations between Chudnovsky's equations. In particular, the famous Schwarz hyperelliptic curve $y^2 = x^8 + 14x^4 + 1$ appears.

The paper is organized as listed in Contents.

2. Transformations and equivalences

In a nutshell, existence of the above mentioned transitions follows from the fact that each of groups $\mathfrak{G}_t$ in (6) is a subgroup of $\boldsymbol{\Gamma}(1)$ and therefore all of these groups are commensurable each other. Hence it follows that there is a transformation of algebraic form $F(t_1, t_2) = 0$ turning any $\mathfrak{G}_{t_1}$-equation into any other one for $\mathfrak{G}_{t_2}$. These algebraic dependencies are nothing but equalities of J-invariants (6) between themselves. It turns out that the sought-for algebraic changes are not always of complicated form coming from a direct equating J's each other. Below is an example of the most generic case.

Example 1. Denote t's for (III) as $-z$ and x for (IV) and consider equality $J_{\mathrm{III}} = J_{\mathrm{IV}}$:

$$-\frac{((z+3)^4 - 40(z^2+3z+2))^3}{z^5(z^2+11z-1)} = \frac{(x+2)^3((x+2)^3 - 24x)^3}{x^3(x+8)(x-1)^2}. \tag{9}$$

Turning this equation into a polynomial $F(x, z) = 0$, we found that it is irreducible and determines an algebraic curve of genus $g = 5$.

2.1. Substitutions

In order to compare Fuchsian equations it is convenient to pass to their canonical normal form $\psi'' = \mathcal{Q}\psi$ because it is unique as against the generic form $\Psi_{xx} + p\Psi_x + q\Psi = 0$. Corresponding linear transformation $\Psi \rightleftarrows \psi$ is very well known [12,20,13] and may be accompanied by a simultaneous change of independent variable $x \mapsto z = z(x)$:

$$\psi(z) = \sqrt{\frac{dz}{dx}} \mathrm{e}^{\frac{1}{2}\int^x p\,dx} \Psi(x). \tag{10}$$

Then equation for ψ has the form

$$\psi_{zz} = \frac{1}{2}\left\{\frac{z_{xxx}}{z_x^3} - \frac{3}{2}\frac{z_{xx}^2}{z_x^4} + \frac{1}{z_x^2}\left(p_x + \frac{1}{2}p^2 - 2q\right)\right\}\psi. \tag{11}$$

Intermediate transformations $x \mapsto x' \mapsto x'' \mapsto \cdots \mapsto z$ are allowable but the number of such changes and their orders, including inverse transformations, are immaterial for ultimate answer $x \to z$; this formula has an invariant characterization.

In practice, when the change $x \mapsto z$ has been given in form of implicit equation $F(z, x) = 0$, it is useful to have an effective formula for transition to the normal form $\psi'' = \mathcal{Q}(z)\psi$, where primes, as always in the sequel, signify the derivatives with respect to independent variable entering into coefficient of the proper ψ-equation. As usual, when transforming linear ODEs the Schwarz derivative does constantly appear and we use the standard notation for this object:

$$\{f, z\} := \frac{f_{zzz}}{f_z} - \frac{3}{2}\frac{f_{zz}^2}{f_z^2}.$$

With use of this notation we can rewrite the transformation above in form of the following computational rule.

Lemma 2. *Let coefficients of equation*

$$\Psi_{xx} + p\Psi_x + q\Psi = 0 \tag{12}$$

be arbitrary (*rational, algebraic, or transcendental*) *differentiable functions of x. Then linear change* (10) *and the change of variables $x \mapsto z$ defined by the rule $F(z, x) = 0$ transform Eq.* (12) *into the following canonical form*:

$$\psi'' = \frac{1}{2}\mathcal{Q}(z)\psi,$$
$$\mathcal{Q}(z) = \frac{F_z^2}{F_x^2}\left(p_x + \frac{1}{2}p^2 - 2q + \{F, x\}\right) - \frac{F_z}{F_x}p_z - \{F, z\} + 3\frac{F_z}{F_x}\left(\ln\frac{F_z}{F_x}\right)_{xz}, \tag{13}$$

where objects $\{F, x\}$, $\{F, z\}$ are understood as the partial Schwarz derivatives and expression for $\mathcal{Q}(z)$ should be computed modulo $\langle F(z, x)\rangle$.

Proof. Compute the derivatives z_x, z_{xx}, and z_{xx} appearing in (11) according to the rules like

$$z_x = -\frac{F_x}{F_z}, \qquad z_{xx} = -\left(\frac{F_x}{F_z}\right)_x - \left(\frac{F_x}{F_z}\right)_z z_x = -\frac{F_{xx}}{F_z} + 2\frac{F_{xz}F_x}{F_z^2} - \frac{F_{zz}F_x^2}{F_z^3}, \qquad \ldots.$$

Express third derivatives F_{xxx} and F_{zzz} via partial Schwarzians $\{F,x\}$, $\{F,z\}$. Taking into account that p may be an algebraic function $p(x,z)$, we replace the complete derivative p_x presented in (11) with the following object:

$$p_x \mapsto p_x - \frac{F_x}{F_z} p_z.$$

Simplifying the result, one arrives at the formula for $\mathcal{Q}(z)$ above. □

The rule (13) is convenient to use because its last term vanishes if the dependence $F(z,x)=0$ has a split form $X(x)=Z(z)$, which is frequently our case. Such form simplifies calculations of genera of curves and reduces considerably computation tasks when the polynomial operation modulo $\langle F(z,x)\rangle$ has been applied to the answer $\mathcal{Q}(z)$. We shall exploit this lemma throughout the work.

2.2. On equivalence of 2nd order linear ODEs

The main motivation for study of transformations between equations under considerations is the fact that the simple or complicated Fuchsian (not necessarily) equations may be transformed into very simple equations with avail of far non-obvious rational/algebraic/transcendental substitutions.

Proposition 3. *Any two linear 2nd order ODEs*

$$\Psi_{xx} = \mathcal{Q}(x)\Psi, \qquad \psi_{zz} = \tilde{\mathcal{Q}}(z)\psi \tag{14}$$

can be transformed into each other by a point transformation $z = \Xi(x)$.

Proof. Linearity and normality of both of Eqs. (14) implies the linear relation between Ψ and ψ, e.g., $\psi = m\Psi$, with $m = \sqrt{z_x}$, where dependence $z = \Xi(x)$ is as yet unknown. Hence

$$\frac{dx}{\Psi^2} = \frac{dz}{\psi^2}. \tag{15}$$

Whatever the solution $\Psi = \Psi(x)$ is chosen, we can construct the second linearly independent one by Liouville's formula

$$\Psi \int \frac{dx}{\Psi^2}.$$

Therefore $\int \Psi^{-2}\,dx$ is always a certain ratio of two linear independent solutions to the Ψ-equation and this ratio will be the same for the ψ-equation; the ratio depends only on point x. We thus have, instead of (15),

$$\frac{\Psi_2(x)}{\Psi_1(x)} = \frac{\psi_2(z)}{\psi_1(z)}$$

and this relation constitutes an implicit form of the sought-for dependence $z = \Xi(x)$. □

As can well be imagined, such a construction is useless in general because it requires the knowledge of integrals. Since ψ and Ψ are chosen to be arbitrary the general equivalence of Eqs. (14) can be rewritten in form of the following bilinear relation:

$$x \rightleftarrows z: \quad \mathrm{A}\Psi_1(x)\psi_1(z) + \mathrm{B}\Psi_1(x)\psi_2(z) + \mathrm{C}\Psi_2(x)\psi_1(z) + \mathrm{D}\Psi_2(x)\psi_2(z) = 0 \tag{16}$$

with free constants $(\mathrm{A} : \mathrm{B} : \mathrm{C} : \mathrm{D})$.

In the majority of cases integrals of linear ODEs belong to differential fields which are different from those to which the coefficients $\mathcal{Q}(x)$, $\tilde{\mathcal{Q}}(z)$ belong. It is not at all obvious a priori, then, that Chudnovsky's equations admit situations when the family (16) has algebraic representatives $F(x,z) = 0$, whereas all the functions Ψ_1, Ψ_2, ψ_1, and ψ_2 are expressed solely in terms of non-algebraic hypergeometric ${}_2F_1$-transcendents (Beukers' list [2]).

Definition 1. We shall call linear ODEs (14) algebraically equivalent if they are transformable into each other by some algebraic dependence $F(x,z) = 0$.

Remark 1. It is not difficult to see that algebraic equivalence defines an equivalence relation since it satisfies the symmetry, reflection, and transitivity properties. We do not use the separate term for rational equivalence, e.g., $z = R(x)$, because inversions of the rational function $R(x)$ and the change $\psi = \sqrt{R'(x)}\Psi$ always lead to algebraic functions. The transcendent equivalence is always available; this is formula (16). However in Section 5 we shall exhibit examples—Chudnovsky's equations on tori—when equivalence is transcendental but it is simpler than the most general one defined by this formula. It may be also mentioned here that algebraic equivalence is a simplest but nontrivial kind of equivalences.

2.3. *Remarks on monodromy groups*

Yet another point that should be mentioned is the fact that Eqs. (1)–(4) provide the next nontrivial (after a hypergeometric equation) examples of what is called presently the monodromy groups of finite genus. Recall that this property implies that function $x = \chi(\tau)$ defined by inversion of the ratio

$$\tau = \frac{\Psi_1(x)}{\Psi_2(x)} \tag{17}$$

is a single-valued analytic function of variable τ everywhere in the domain of its existence on the plane (τ). As usual, the closed paths[2] on the plain (x) entail transformations

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

which form a finitely generated group $\mathfrak{G}$ (the monodromy group) or merely the monodromy for short [11,18] as equation is of Fuchsian class. Then the τ-plane is covered by domains containing $\mathfrak{G}$-nonequivalent points and pairwise equivalent points on boundaries of the domains. If these domains form a set of non-overlapping circle polygons with finitely many number of sides each (Poincaré polygons) then identifications of these sides determine the standard topological characteristics of the polygon—the genus [7]; in doing so, the function $\chi(\tau)$ becomes single-valued by construction. For brevity, we shall use terminological shorthand the monodromy and genus of the monodromy as synonyms to the monodromy group and genus of the Poincaré polygon representing the group. Being a matrix group from $\mathrm{SL}_2(\mathbb{C})$, it has an exact representation by an automorphism group of the (automorphic) function χ and hence we re-denote this group as $\mathfrak{G}_\chi$:

$$\chi\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(\tau) \quad \Longrightarrow \quad \operatorname{Aut}\chi(\tau) =: \mathfrak{G}_\chi.$$

[2] If coefficient is an algebraic function $\mathcal{Q}(x,y)$ belonging to irrationality $F(x,y) = 0$ then the closure of a path is defined by the value of the pair (x,y) coinciding with the initial one (x_o, y_o) [11].

If $\mathcal{Q}$ is a rational function of x then the monodromy has a zero genus [7,11]. If $\mathcal{Q} = \mathcal{Q}(x, y)$ is an algebraic function belonging to irrationality $F(x, y) = 0$ then the genus, by construction, coincides with topological genus of this curve. We shall also meet Fuchsian equations wherein $\mathcal{Q}$ is an elliptic (transcendental) function $\mathcal{Q}(\mathfrak{u})$. In this case, genus of monodromy is, again by construction, equal to unity.

Explicit $\chi(\tau)$-expressions for equations under consideration can be found in works [14,10] and Eqs. (1)–(3) are related to the classical modular equations as particular cases; the most exhaustive literature and systematic lists of results concerning this subject can be found in [10].

The arbitrary substitutions $x \mapsto z$ destroy in general the property of monodromies to have finite genus but it is clear that any single-valued rational/transcendental change $z = R(x)$ will automatically yield equation (13) with the monodromy $\mathfrak{G}_z$ known to be Fuchsian, i.e., of finite genus, if the monodromy $\mathfrak{G}_x$ was of the same kind. However, this is somewhat trivial way to construct new interesting equations because they will have in general the complicated algebraic coefficients $\mathcal{Q}(x, z)$. As we shall see, the theory of Chudnovsky's equations provides a large number of nontrivial situations when rational functions $\mathcal{Q}(x)$ with zero genus monodromies go into rational functions $\tilde{\mathcal{Q}}(z)$ again, whereas the substitutions themselves have nontrivial genera. Similarly, the unity genera pass to the unity ones (punctured tori; Sections 5.1, 5.2). In other words, genus of manifold on which ODE has been defined, genus of its monodromy group, and genus of the substitution are not one and the same. Therein lies an essential feature of algebraic equivalence of Chudnovsky's equations and Fuchsian monodromies at all.

2.4. Genera of substitutions

Turning back to Eqs. (1)–(4), let us tabulate their canonical forms for further reference. We apply the 'linear part' of Lemma 2 (i.e., independent variable is not changed) and derive that normal forms to Eqs. (1)–(3) become respectively

$$\psi'' = -\frac{1}{4}\frac{(x^2+1)^2}{x^2(x-1)^2(x+1)^2}\psi, \tag{1$'$}$$

$$\psi'' = -\frac{1}{4}\frac{(x+1)(x+3)(x^2+3)}{x^2(x^2+3x+3)^2}\psi, \tag{2$'$}$$

and

$$\psi'' = -\frac{1}{4}\frac{x^4+8x^3+72x^2-64x+64}{x^2(x-1)^2(x+8)^2}\psi. \tag{3$'$}$$

The normal form for Eq. (4) can be obtained analogously, however, by way of illustration of the two last sentences in the previous section, we apply Lemma 2 in its full generality and obtain that the change (9) transforms Eq. (3′) (replacing z with x again) into the following equation

$$\psi'' = -\frac{1}{4}\frac{x^4+12x^3+134x^2-12x+1}{x^2(x^2+11x-1)^2}\psi. \tag{4$'$}$$

This is exactly the normal form to equation (4). We shall refer to Eqs. (1′)–(4′) as Chudnovsky's equations as well. To avoid confusion, we also adjust Beauville's t-parameters in (6) in order to make exact correlation of these J-invariants with list (1)–(4):

$$\text{(I)}:\quad t = -3(x+1), \qquad \text{(II)}:\quad t = x, \qquad \text{(III)}:\quad t = -x,$$

$$\text{(IV)}:\quad t = x, \qquad \text{(V)}:\quad t = 4ix, \qquad \text{(VI)}:\quad t = -3(x+1).$$

Table 1
Genera of curves realizing algebraic equivalencies of Chudnovsky's equations.

	$\mathrm{I}(2')_x$	$\mathrm{II}(1')_x$	$\mathrm{III}(4')_x$	$\mathrm{IV}(3')_x$	$\mathrm{V}(1')_x$	$\mathrm{VI}(2')_x$
$\mathrm{I}(2')_z$	$0_{(12)}$	5	5	$0_{(4)}$	5	$\{1,1,1,1\}_{\mathrm{E}}$
$\mathrm{II}(1')_z$		$0_{(5)}$	5	$\{1,1,1\}_{\mathrm{C}}$	$0_{(2)}$, $\{1,1\}_{\mathrm{L}}$	5
$\mathrm{III}(4')_z$			$0_{(4)}$	5	5	5
$\mathrm{IV}(3')_z$				$0_{(3)}$, 1_{E}	1_{C}, 3	0, 4
$\mathrm{V}(1')_z$					$0_{(3)}$, 3	5
$\mathrm{VI}(2')_z$						$0_{(3)}$, 4

Invariants (6) then read

$$J_{\mathrm{I}} = \frac{1}{64}\frac{(x+1)^3(x+3)^3(x^2+3)^3}{x^3(x^2+3x+3)^3}, \qquad J_{\mathrm{IV}} = \frac{1}{12^3}\frac{(x+2)^3((x+2)^3-24x)^3}{x^3(x+8)(x-1)^2},$$
$$J_{\mathrm{II}} = \frac{4}{27}\frac{(x^4-x^2+1)^3}{x^4(x-1)^2(x+1)^2}, \qquad J_{\mathrm{V}} = \frac{1}{108}\frac{(16x^4-16x^2+1)^3}{x^2(x-1)(x+1)},$$
$$J_{\mathrm{III}} = \frac{-1}{12^3}\frac{((x+3)^4-40(x^2+3x+2))^3}{x^5(x^2+11x-1)}, \qquad J_{\mathrm{VI}} = \frac{1}{64}\frac{(x+1)^3(9(x+1)^3-8)^3}{x(x^2+3x+3)} \tag{18}$$

and generate, by Lemma 2 applied to the 'hypergeometry' (7), the list (1′)–(4′) as follows:

$$(1') \leftarrow\!\shortmid \{\mathrm{II}, \mathrm{V}\}, \qquad (2') \leftarrow\!\shortmid \{\mathrm{I}, \mathrm{VI}\}, \qquad (3') \leftarrow\!\shortmid \mathrm{IV}, \qquad (4') \leftarrow\!\shortmid \mathrm{III}.$$

Theorem 4. *Let algebraic equivalence be generated by identifying Klein's J-invariants (18). Then Chudnovsky's equations (1′)–(4′) are algebraically equivalent with respect to substitutions whose genera are presented in Table* 1.

Proof. Consider equalities of J-invariants (18): $J_k(x) = J_n(z)$ and take, e.g., the case $J_{\mathrm{II}}(z) = J_{\mathrm{III}}(x)$. It corresponds to a table record on intersection of line $\mathrm{II}(1')_z$ and column $\mathrm{III}(4')_x$. Converting this equation into a polynomial $F(x,z) = 0$, we establish that it is not reducible over $\mathbb{C}$. Since this polynomial represents the equality of one and the same quantity—Klein's invariant J—it ensures the mutual equivalence of Chudnovsky's Eqs. $(1')_z \rightleftarrows (4')_x$; of course, this can be checked by a straightforward application of Lemma 2. Computation of genus g by the Riemann–Hurwitz formula gives $g = 5$. Such an irreducibility is not a common rule and we take, as a second instance, equation $J_{\mathrm{II}}(z) = J_{\mathrm{V}}(x)$. Corresponding polynomial $F(x,z) = 0$ splits into several components

$$F(x,z) = \big((z-1)^2 + 4x^2 z\big)\big((z+1)^2 - 4x^2 z\big)$$
$$\times \big(16\big(x^4-x^2\big)\big(z^4-z^2\big)-1\big)\big(16\big(x^4-x^2\big)\big(z^2-1\big)+z^4\big) = 0$$

and direct computations (Lemma 2) show that each of them does realize an algebraic equivalence of Eq. (1′) with itself. The first two components are the rational algebraic curves; their genera are equal to zero. This point has been designated in the table as $0_{(2)}$; subscript stands for a number of rational curves and trivial substitution $x = z$ is taken into account for diagonal cases. The two unities $\{1,1\}_{\mathrm{L}}$ in the entry mean that the two remaining polynomials determine curves of genus $g = 1$ and each of the curves is isomorphic to a lemniscate (L); i.e., their Klein's J-invariants are equal to 1. The symbol 1_{E} designates a curve isomorphic to the equi-anharmonic (E) curve ($J = 0$) and 1_{C} does the curve with invariant $J = \frac{13^3}{2^2 3^5}$. Other entries of the table are processed in a similar manner and all the curves are defined over $\mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\mathrm{i}\sqrt{3})$ at most (splitting fields of polynomials $x^2+11x-1$ and x^2+3x+3).

In all the cases curves of the same genus differ from one another and can be rather complicated. The empty boxes are filled by symmetry. □

Remark 2. We do not have an explanation of 'unpredictable' distribution of genera in Table 1 or explanation as to why *each* of irreducible components does indeed represent an algebraic equivalence of the list (1′)–(4′). This is, perhaps, a quite nontrivial task because it touches upon the problem of construction of all the algebraic equivalences. This goes far beyond the scope of the present work and, as mentioned in Introduction, no the general theory of algebraic transformations has yet been developed.

3. Algebraic equivalence of Chudnovsky equations

3.1. Automorphisms and their consequences

Table 1 shows that there are transformations of the same Chudnovsky equation into itself and these are defined not only through the trivial change $z = x$. Non-obvious examples appear even in the class of linear fractional substitutions. For example, the zero genus family of automorphisms $(2') \rightleftarrows (2')$ contains the transformation

$$(\varepsilon - 1)(x + z) = xz + 3, \qquad \varepsilon := \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}.$$

The mere fact that such transformations do exist is not surprising. Well-known examples are the modular Jacobi–Schlæfli–Sohnke relations between Legendre's moduli $x = k^2(\tau)$ and $z = k^2(N\tau)$. Diagonal cases $(1') \rightleftarrows (1')$ are thus particular analogs of this classical family and other diagonal entries provide certain modular equations belonging to their Beauville monodromy groups. For example, the right lower box of the table contains a genus $g = 4$ modular equation for Beauville's VI-group [1, p. 658], which is conjugate to group $\boldsymbol{\Gamma}(3)$ [14,10,5]. It is of more interest that transformations of such a kind lead to other interesting consequences. They are concerned with rational and elliptic automorphisms. We present here consequences of only two illustrative examples.

Example 2. Let us consider one of the zero genus diagonal quadratic automorphisms

$$(3')_x \rightleftarrows (3')_z\colon \quad zx(z + x + 6) = 8.$$

We can parametrize this dependence by rational functions

$$x = \frac{(\mathrm{T}-1)^2}{\mathrm{T}+1}, \qquad z = \frac{8}{\mathrm{T}^2 - 1}$$

and may consider $x = x(\mathrm{T})$ as a change of variable $x \mapsto \mathrm{T}$ in Chudnovsky's equation (3′). Insomuch as rational uniformizer T itself is always a rational function of coordinates (x, z), that is $\mathrm{T} = R(x, z)$, and Fuchsian equation (3′) has a correct accessory parameter [4], the transformed T-equation will be of the same property. Of course, both of these substitutions will yield the same T-equation. It turns out that equations generated by this way become new Fuchsian ones and renormalization of T can impart them better (canonical) form. We therefore replace the last parametrization with this one:

$$x = 2\frac{(\mathrm{T}-1)^3}{1-\mathrm{T}^3}, \qquad z = 6\frac{\mathrm{T} + \varepsilon\mathrm{T} + \varepsilon}{1 - \mathrm{T}^3}\mathrm{T} - 2$$

and derive that T-equation has a very elegant form indeed:

$$\psi'' = \frac{-9\mathrm{T}^4}{(\mathrm{T}^6 - 1)^2}\psi.$$

It is an equation of the same kind as Chudnovsky's ones, with the difference that it has six parabolic singularities at points $\mathrm{T} = \pm\{1, \varepsilon, \varepsilon + 1\}$. One can show, with use of some manipulations by Jacobi's ϑ-constant series (this is not a subject matter of the present work), that the τ-representation for the corresponding Hauptmodul $\mathrm{T} = \mathrm{T}(\tau)$ has the form

$$\mathrm{T}(\tau) = \varepsilon \frac{\mathrm{i}\vartheta_3^2(\tau) + \sqrt{3}\vartheta_3^2(3\tau)}{\mathrm{i}\vartheta_3^2(\tau) - \sqrt{3}\vartheta_3^2(3\tau)},$$

where the standard Jacobi theta-constant $\vartheta_3(\tau)$ is defined by the series [19]

$$\vartheta_3(\tau) := \sum_{k}^{+\infty}_{-\infty} \mathrm{e}^{k^2\pi\mathrm{i}\tau}.$$

As it follows from the last ψ-equation this $\mathrm{T}(\tau)$ satisfies the equation

$$\frac{\{\mathrm{T}, \tau\}}{\dot{\mathrm{T}}^2} = \frac{-18\mathrm{T}^4}{(\mathrm{T}^6 - 1)^2}.$$

Example 3 *(Non-diagonal automorphisms).* Such automorphisms are possible only for Eqs. ($1'$) and ($2'$). Table 1 tells us that these curves are rational or elliptic ones and they are isomorphic as curves under coinciding genera. Choosing the simplest representatives, we obtain in these cases:

$$(2')_x \rightleftarrows (2')_z\colon \quad \big\{(x+1)^3 - 1\big\}\big\{(z+1)^3 - 1\big\} = 1, \quad g = 1 \quad (J = 0)$$

and

$$\begin{aligned} (1')_x \rightleftarrows (1')_z\colon \quad & 16\big(x^4 - x^2\big)\big(z^4 - z^2\big) = 1, \quad && g = 1 \quad (J = 1), \\ & 4x^2 z = (z+1)^2, && g = 0. \end{aligned} \tag{19}$$

The latter case shows that variable z is a perfect square and we can put $z = \mathrm{T}^2$, where T is a uniformizer for this rational curve. Computing Fuchsian T-equation, we get

$$\psi'' = -\frac{1}{4}\frac{\mathrm{T}^8 + 14\mathrm{T}^4 + 1}{\mathrm{T}^2(\mathrm{T}^4 - 1)^2}\psi,$$

that is yet another (known [10]) Fuchsian ODE with six parabolic singularities.

Eq. (19) will appear in Section 5 when considering equations on tori.

Remark 3. Notwithstanding the fact that one has six Beauville's curves and just four Chudnovsky's equations we cannot discard some two of curves (I), (II), (V), (VI) as excessive. For example, if we cut out the left upper (4×4)-box from Table 1, we would lose many transformations: all the non-zero genus automorphisms $(1') \rightleftarrows (1')$, $(2') \rightleftarrows (2')$, genus 3 transformation $(3') \rightleftarrows (1')$, etc. In this respect Beauville's list is independent of Chudnovsky's one.

Remark 4. Automorphisms coming from Table 1 are not the only possible ones. This results from the fact that some of Chudnovsky's Hauptmoduln are expressed via the classical Jacobi theta-constants [10,14] which are algebraically related to Legendre's modulus $k^2(\tau)$. The latter, as is well known, has a lot (infinite) of algebraic dependencies with itself $k^2(q\tau)$ under $q \in \mathbb{Q}$. Thus, all the possible automorphisms will be analogs of the classical modular families mentioned above.

3.2. Equivalences and integrability of the list (1)–(4)

Although automorphisms produce a large number of nontrivial curves, we shall consider further equivalences of only pairwise distinct equations.

Theorem 5. *Algebraic equivalence defined by Table* 1 *is closed if automorphisms are excluded from consideration. This closedness is determined by the family of genus* 5 *irreducible algebraic curves*:

$$\begin{array}{lll}
(1')_z \rightleftarrows (2')_x\colon & J_{\mathrm{II}}(z)=J_{\mathrm{I}}(x), & J_{\mathrm{V}}(z)=J_{\mathrm{I}}(x),\\
 & J_{\mathrm{II}}(z)=J_{\mathrm{VI}}(x), & J_{\mathrm{V}}(z)=J_{\mathrm{VI}}(x),\\
(1')_z \rightleftarrows (4')_x\colon & J_{\mathrm{II}}(z)=J_{\mathrm{III}}(x), & J_{\mathrm{V}}(z)=J_{\mathrm{III}}(x),\\
(2')_z \rightleftarrows (4')_x\colon & J_{\mathrm{I}}(z)=J_{\mathrm{III}}(x), & J_{\mathrm{VI}}(z)=J_{\mathrm{III}}(x),\\
(3')_z \rightleftarrows (4')_x\colon & J_{\mathrm{IV}}(z)=J_{\mathrm{III}}(x), &
\end{array}$$

and two exceptional cases determined by the (*canonical*) *representatives of minimal genera. These are the elliptic curve*

$$(3')_z \rightleftarrows (1')_x\colon\quad x^2-x^4=\frac{16(z-1)}{z^3(z+8)}\quad \left(J=\frac{13^3}{2^2 3^5}\right) \tag{20}$$

and the zero genus one

$$(3')_z \rightleftarrows (2')_x\colon\quad (x+1)^3=\frac{(z+2)^3}{(z+8)(z-1)^2}. \tag{21}$$

Proof. Except for equalities of Klein's J-invariants we should check the transitivity of relations under consideration. All the irreducible relations $J_k(x)=J_n(z)$ are listed in the first family of the curves above. Reducible cases, according to Table 1, are $(3')\rightleftarrows(1')$, $(3')\rightleftarrows(2')$, and each of automorphisms $J_k(x)=J_k(z)$. Consider, e.g., transitivity

$$(2')_x \to (3')_{\boldsymbol z}, \qquad (3')_{\boldsymbol z} \to (4')_z.$$

For each of the $(2')\to(3')$-relations we expect to get the $(2')_x\to(4')_z$-relation coinciding with one of the two table curves of genus 5. There are two sets of the $(2')\to(3')$-transformations: $\{0\}_4$ and $\{0,4\}$. Take a curve from the first set, e.g., the curve (21):

$$(x+1)^3=\frac{(\boldsymbol z+2)^3}{(\boldsymbol z+8)(\boldsymbol z-1)^2}$$

and supplement it with the unique $(3')\to(4')$-curve (see Example 1)

$$\frac{(\boldsymbol z+2)^3((\boldsymbol z+2)^3-24\boldsymbol z)^3}{\boldsymbol z^3(\boldsymbol z+8)(\boldsymbol z-1)^2}=-\frac{((z+3)^4-40(z^2+3z+2))^3}{z^5(z^2+11z-1)}.$$

Elimination of variable $\boldsymbol z$ from these two equations produces not a new relation but irreducible curve $J_{\mathrm{I}}(x)=J_{\mathrm{III}}(z)$; more precisely, cube of the curve $(2')\rightleftarrows(4')$. Other elements of the sets and transitivity $(1')_x\to(3')_{\boldsymbol z}\to(4')_z$ are checked in a similar manner. The simplest representative of the $(1')\rightleftarrows(3')$-equivalence with a minimal genus is the curve (20). By virtue of irreducibility and transitivity we may also leave single representatives for each of the cases in the $g=5$ family above.[3] As for automorphisms, many of them, rather 'exotic' as they are, preserve the closedness of Table 1. However, there are exceptions. It will suffice to point out at least one counterexample. This is equivalence $J_{\mathrm{III}} \to J_{\mathrm{I}}$ followed by application of a $g=4$ automorphism coming from the $(J_{\mathrm{VI}} \rightleftarrows J_{\mathrm{VI}})$-curve. Corresponding transformation $(4') \to (2') \to (2')$ leads to a cumbersome curve $F(\overset{36}{x}, \overset{36}{z})=0$ of genus $g=25$ (computation is very nontrivial; 36 is a degree of the curve in both the variables). $\square$

We observe in passing that case $(4')$ holds an exceptional position among other equations $(1')$–$(4')$ since it is transformed into other ones only by means of the most complicated changes. The zero genus automorphisms of this equation, apart from trivial ones $x=z$ and $xz+1=0$, are rather non-obvious and cumbersome (not displayed here). Hauptmodul for this equation is also very non-standard [14, Table 3]. We also see that genus 5 transformation $(1') \rightleftarrows (2')$ can be represented as a composition of the simple rational (21) and elliptic curve (20).

Let us consider the question on integrability of Chudnovsky's equations. The hypergeometric series converges only in a unite circle, which is why it would be more convenient to have solutions expressed not in terms of Beukers' ${}_2F_1$-list but in terms of special functions associated with the hypergeometric equation. These are Legendre's complete elliptic integrals $K(k)$, $K'(k)$ [20,12] or general Legendre's P, Q-functions solving the equation [20, Section 15•5]

$$\left(1-s^2\right)Y_{ss} - 2sY_s + \left\{\nu(\nu+1) - \mu^2\left(1-s^2\right)^{-1}\right\}Y = 0. \tag{22}$$

Integrability of Eqs. (1)–(4) in terms of the integrals above is obvious because the first Chudnovsky's equation (1) is in effect equation for a square root of the standard Legendre's elliptic modulus $k^2(\tau)=x^2$ defined by the classical equation [20]

$$(1) \quad \Leftrightarrow \quad \frac{d}{dk}\left(k\left(1-k^2\right)\frac{d\psi}{dk}\right) = k\psi, \qquad \psi = \left\{K(x), K'(x)\right\}.$$

It is common knowledge that there are cases when the ${}_2F_1$-series admits the quadratic rational transformations and the generic hypergeometric equation then reduces to the two-parametric equation (22) [6, Sections 3.1–3.2]. This is indeed the case for equations under question and some simple arguments show that one of the reductions is $(\nu, \mu) = (-\frac{1}{3}, 0)$.

Proposition 6. *All the equations (1)–(4) are integrable in terms of Legendre's integrals K, K' or functions $P_{-\frac{1}{3}}$, $Q_{-\frac{1}{3}}$.*

Proof. It will suffice to integrate one equation of the list $(1')$–$(4')$. We take Eq. $(2')$ and derive that it is transformed into Eq. (22) and the 'hypergeometry' (7) as follows

$$J = \frac{1}{4}\frac{(4s-5)^3}{(s^2-1)(s+1)} \quad \Longrightarrow \quad (x+1)^3(1-s) = 2$$

(these substitutions are verified directly by use of Lemma 2). Computing a multiplier of linear transformation between ψ-functions, we obtain finally that functions

$$\psi_{1,2} = \frac{\sqrt{(x+1)^3-1}}{x+1}\left\{P_{-\frac{1}{3}}\left(1-\frac{2}{(x+1)^3}\right), \quad Q_{-\frac{1}{3}}\left(1-\frac{2}{(x+1)^3}\right)\right\}$$

provide a basis of solutions to Eq. $(2')$. $\square$

[3] This rises the question as to a correlation between these curves. In particular, whether they are isomorphic or not?

We conclude this section with one example which will be used in the last section (Section 5.3) when appearing a hyperelliptic curve.

Example 4. Rational parametrization of the zero genus equivalence (21) generates, as before in Examples 2 and 3, Fuchsian equations for uniformizer T. We obtain here the nice equation

$$\psi'' = -\frac{1}{4}\frac{(\mathrm{T}^6 - 20\mathrm{T}^3 - 8)^2}{\mathrm{T}^2(\mathrm{T}^3+8)^2(\mathrm{T}^3-1)^2}\psi \tag{23}$$

defining monodromy of an 8-punctured sphere; the equation comes from the change $\mathrm{T}^3 = x$ in Chudnovsky's equation (3′). That this x is a perfect cube means that Hauptmodul $\mathrm{T} = \mathrm{T}(\tau)$ is certain to have an explicit representation in terms of classical ϑ- or Dedekind's eta-functions. This is so indeed and using some results of work [10], one can derive that

$$\mathrm{T}(\tau) = -2\frac{\eta^3(2\tau)}{\eta^3(\tau)}\frac{\eta(3\tau)}{\eta(6\tau)}, \tag{24}$$

where $\eta(\tau) := \prod_k (1 - e^{ik\tau})$, $\tau \in \mathbb{H}^+$. It follows that this $\mathrm{T}(\tau)$ satisfies the equation

$$\frac{\{\mathrm{T}, \tau\}}{\dot{\mathrm{T}}^2} = -\frac{1}{2}\frac{(\mathrm{T}^6 - 20\mathrm{T}^3 - 8)^2}{\mathrm{T}^2(\mathrm{T}^3+8)^2(\mathrm{T}^3-1)^2}.$$

4. Chudnovsky's equations and punctured tori

4.1. Equations on tori

Recall that differential equation on torus is, by definition, an ODE of the (normal) form

$$\psi_{\mathrm{uu}} = \Xi(\mathrm{u})\psi$$

with some function $\Xi(\mathrm{u})$ being an elliptic (transcendental) one in variable u. In order this equation be of Fuchsian class, the $\Xi(\mathrm{u})$ must have second order poles at most. Hence this equation should be representable in form of a sum over poles $\mathrm{u} = \alpha$ of $\Xi(\mathrm{u})$:

$$\psi_{\mathrm{uu}} = \left\{\sum_\alpha \bigl(C_\alpha \wp(\mathrm{u}-\alpha) + A_\alpha \zeta(\mathrm{u}-\alpha)\bigr) + A_o\right\}\psi, \qquad \sum_\alpha A_\alpha = 0, \tag{25}$$

where $\wp$ and ζ constitute, together with the σ-function, the standard Weierstrassian basis of the elliptic theory [20,8,19] for equation

$$\begin{aligned} \wp'^2 &= 4\wp^3 - a\wp - b \\ &= 4(\wp - e)\bigl(\wp - e'\bigr)\bigl(\wp - e''\bigr). \end{aligned} \tag{26}$$

Along with the preceding sections, we are interested in Eqs. (25) having a Fuchsian monodromy of finite genus. The case of non-punctured tori is the 'well-trod' domain (the theory of elliptic functions [20]) so the torus will be considered to have at least one puncture; one of the coefficients C_α is equal to $-\frac{1}{4}$. The simplest such model is the singly punctured torus considered for the first time in the classical work [9]:

$$\psi'' = -\frac{1}{4}\bigl\{\wp(\mathrm{u}; a, b) + A\bigr\}\psi. \tag{27}$$

On the other hand, finiteness of genus tells us that Riemann surface, whose fundamental group representation is the monodromy $\mathfrak{G}_{\mathfrak{u}}$ to Eq. (25), is always related to a certain *finitely sheeted* cover $\mathfrak{u} \mapsto s$ over this torus (or cover $s \mapsto \mathfrak{u}$ by several copies of the torus). This means that there always exists an equation

$$\Phi(s, \mathfrak{u}) := G\big(s, \wp(\mathfrak{u}), \wp'(\mathfrak{u})\big) = 0 \tag{28}$$

being polynomial in its s, $\wp$, $\wp'$-arguments and realizing this cover. It is a polynomial in s-argument and transcendental function in $\mathfrak{u}$-variable. This is a generic form of covers $\mathfrak{u} \rightleftarrows s$ being an analog of the standard models of algebraic curves given by polynomial dependencies $P(x, s) = 0$.

The minimal possible number of $\mathfrak{u}$-sheets branching over (s)-plane is equal to 2 since two is the minimal order of elliptic function. Therefore simplest covers by tori are the 2-sheeted ones and, hence, the simplest reduction of (28) is $R(s) = \wp(\mathfrak{u})$, where $R(s)$ is any rational function. By the implicit function theorem, branch points $(\mathfrak{u}_k, s_k)$ of the map $s \mapsto \mathfrak{u}$ are solutions of equations $\{\Phi = 0,\ \Phi_{\mathfrak{u}} = 0\}$ plus separate analysis of the point $\wp(\mathfrak{u}) = \infty$. Hence, the high order rational functions $R(s)$ lead to a large number of such points and the simplest of the cases is thus

$$s = \wp(\mathfrak{u}). \tag{29}$$

We may consider this equality as a change $\mathfrak{u} \mapsto s$ in Eq. (27). Then it becomes a particular case of the well-known algebraic form to the famous Lamé equation [20, Section 23·4]

$$\psi'' = -\frac{3}{16}\left\{\frac{1}{(s-e)^2} + \frac{1}{(s-e')^2} + \frac{1}{(s-e'')^2} - \frac{1}{3}\frac{5s - A}{(s-e)(s-e')(s-e'')}\right\}\psi \tag{30}$$

having the signature $(2, 2, 2, \infty)$ and its single puncture is located at point $s = \infty$.

4.2. On Halphen's transformation

Halphen [8] used further the original trick

$$s \mapsto x: \quad \big\{s = \wp(\mathfrak{u})\big\} \dashrightarrow \{\mathfrak{u} = 2u\} \dashrightarrow \big\{\wp(u) = x\big\}$$

to convert Eq. (30) into the form

$$\psi'' = -\frac{1}{4}\left\{\frac{1}{(x-e)^2} + \frac{1}{(x-e')^2} + \frac{1}{(x-e'')^2} - \frac{2x - A}{(x-e)(x-e')(x-e'')}\right\}\psi \tag{31}$$

which is our case because (31) has the signature $(\infty, \infty, \infty, \infty)$. It is known that inverse Halphen's transformation, once applied to algebraic form (5), turns it into equation

$$p\Psi'' + \frac{1}{2}p'\Psi' + \frac{1}{16}(s + \tilde{A})\Psi = 0, \qquad p := (s-e)\big(s-e'\big)\big(s-e''\big),$$

whose normal form is (30) after a simple adjustment of parameters.

All this material is classical [8, p. 471], [12, §37], [4, p. 185], however, exact correlation between Lame's equations mentioned above and Eq. (27) requires more accurate description. It should be noted some ambiguity in work [4] which mentions an equivalence between four punctured sphere (31) and 1-punctured torus (27), whereas their monodromies $\mathfrak{G}_{\mathfrak{u}}$ and $\mathfrak{G}_x$ have even different ranks; 2 and 3 respectively.

The cover (28) is never single-sheeted one $s \mapsto \mathfrak{u}$. Therefore group $\mathfrak{G}_u$ will be either subgroup of $\mathfrak{G}_s$ (e.g., the case (29)[4]) or commensurable with it (general case (28)). This means in particular that if we have a correct A-parameter for punctured torus (27), i.e., $\mathfrak{u}(\tau)$ is single-valued, then the map $\mathfrak{u} \mapsto s$ of the form (29) yields a single-valued function $s = \wp(\mathfrak{u}(\tau))$. We thus obtain a 'good' A-parameter for s-equation (30) from that of $\mathfrak{u}$-equation (27); so $\mathfrak{G}_s$ is a correct monodromy for (30), whereas in the opposite direction $s \mapsto \mathfrak{u}$ we have a $(1 \mapsto 2)$-map. As for the general cover (28), both of the maps $s \rightleftarrows \mathfrak{u}$ are always non-single-valued and mutual equivalence of the A-parameter problems for (30), (31), and (27) is not obvious a priori. Below is a complete and precise formulation.

Theorem 7. *Halphen's transformation is a transcendental version of birational transformations between representations of elliptic curves (26) in form of covers (28). This entails an equivalence of the A-parameter problems for equations (30), (31), and (27) and computability of their A-parameters one through another. The quantities x, s, and $\mathfrak{u}$ as functions of the ratio $\tau = \psi_2/\psi_1$ are single-valued and computable if one of these functions has been known.*

Proof. Let us use the duplication formula for Weierstrass $\wp$-function in order to treat the Halphen formulae above as the bi-single-valued (transcendental) transformations between two models $\Phi_1(x, \mathfrak{u}) = 0$ and $\Phi_2(s, u) = 0$ of the one elliptic curve (26):

$$\Phi_1\colon \quad x = \wp\left(\frac{1}{2}\mathfrak{u}\right), \qquad \Phi_2\colon \quad s = \wp(2u).$$

Indeed, the equality

$$\wp(2u) = -2\wp(u) + \frac{1}{16}\frac{(12\wp^2(u) - a)^2}{\wp'(u)^2}$$

entails the following single-valued transitions $(x, \mathfrak{u}) \rightleftarrows (s, u)$:

$$x = \wp(u), \qquad s = \frac{1}{16}\frac{(4x^2 + a)^2 + 32bx}{4x^3 - ax - b}, \tag{32}$$

$$\mathfrak{u} = 2u, \qquad u = \frac{1}{2}\mathfrak{u}. \tag{33}$$

These, by Lemma 2, realize explicitly transformations between Eqs. (30), (31), and (27). Although function x is an algebraical one of s it is transcendently single-valued of the pair (s, u). Owing to isomorphism (32)–(33), all the monodromies $\{\mathfrak{G}_x, \mathfrak{G}_s\}$ are the correct Fuchsian ones of genus 0 and $\{\mathfrak{G}_\mathfrak{u}, \mathfrak{G}_u\}$ are of genus 1 as soon as one of them has been known to be a correct Fuchsian monodromy. From (32) it also follows that the free group $\mathfrak{G}_x$ is an index 4 subgroup of non-free group $\mathfrak{G}_s$. In a more explicit manner, the proof uses 'Puiseux developments' for $\mathfrak{u} = \mathfrak{u}(x)$ about points $x = \{e, e', e''\}$. Inverting the standard series for $\wp$-function [20], we get a series of the type

$$\frac{1}{2}\mathfrak{u}_\pm = \omega \pm \sqrt[-2]{12e^2 - a} \cdot \left\{2 - \frac{4e}{12e^2 - a}(x - e) + \cdots\right\}\sqrt{x - e}$$

and the similar series for $u = u_\pm(s)$. In both of these cases the square root $\sqrt{x - e}$ is represented by a single-valued function of τ because $x(\tau) - e$ has an exponential behavior in τ (due to puncture). In turn, $s(\tau) - e$ is a perfect square

[4] This exhibits, incidentally, an interesting fact: a non-free rank 3 group $\mathfrak{G}_s$ has a free subgroup $\mathfrak{G}_\mathfrak{u}$ of a smaller rank. Genus of $\mathfrak{G}_\mathfrak{u}$ is however not zero but unity.

$$s - e = \left\{ \frac{(x-e)^2 - (e-e')(e-e'')}{2\wp'(u)} \right\}^2$$

and $s(\tau)$ is an exponent again in the vicinity of $s = \infty$. So $\mathfrak{u}(\tau)$ and $u(\tau)$ are additively automorphic single-valued functions of τ (Abelian integrals) and $x(\tau)$, $s(\tau)$ are purely automorphic single-valued ones. All of them are computable through any other one by means of Halphen's transformation itself, that is (32)–(33). $\square$

Remark 5. From uniqueness of Chudnovsky's list it immediately follows the uniqueness of the four Lamé equations (30) of signature $(2, 2, 2, \infty)$. Correlating substitutions (18) with (32), one can show that all the transitions between Eqs. (30) and (7) are given by the certain *zero genera* transformations $F(s, J) = 0$. We may of course drop these intermediate Lamé equations and then Halphen's transformation becomes just a single-valued transition from the torus coordinate $\mathfrak{u}$ to the 4-punctured one x by the formula $x = \wp(\frac{1}{2}\mathfrak{u})$; this is checked directly by Lemma 2.

4.3. Chudnovsky's equations on tori

In view of exclusive character of Eqs. (1)–(4), it is useful to display the complete list of associated Fuchsian equations on tori in an explicit form including their equivalences between each other. The first two cases are simple and related to Eqs. (1), (2); they were obtained in work [9] based on some symmetry properties. These cases are equations of the form (27) with a zero value of the parameter A. The two other ones (most nontrivial) do not appear in any modern reference.

Since parameters (a, b) and singular points of Eqs. (1)–(4), (31) (and consequently the A-parameter in (31)) are not invariant quantities, we pass from Weierstrass' $\wp(\mathfrak{u}; a, b)$-representation to the invariant object $\wp(\mathbf{u}|\mu)$ defined by unique modulus μ. The rule reads as follows

$$\wp(\mathfrak{u}; a, b) = \wp\big(\mathfrak{u}\big|\omega, \omega'\big) =: \frac{1}{\omega^2}\wp(\mathbf{u}|\mu), \qquad \mathbf{u} := \frac{\mathfrak{u}}{\omega},$$

where μ and half-periods ω, ω' are computed through the standard elliptic modular inversion problem. In generic case its solution is defined by the chain of equations [8,20]

$$J(\mu) = \frac{a^3}{a^3 - 27b^2}, \qquad \omega = \pm\sqrt{\frac{a}{b}\frac{g_3(\mu)}{g_2(\mu)}}, \qquad \omega' = \mu\omega \tag{34}$$

and Weierstrass' modular forms $g_2(\mu)$ and $g_3(\mu)$ have numerous computational formulae. Most convenient of them are representations in terms of theta-constants. If we introduce the second Jacobi's constant

$$\vartheta_2(\tau) := \sum_{-\infty}^{+\infty}{}_k\, e^{(k+\frac{1}{2})^2\pi i\tau}$$

then one can use the following expressions for these forms [8,19]:

$$g_2(\mu) = \frac{\pi^4}{12}\big\{\vartheta_2^8(\mu) + \vartheta_3^8(\mu) - \vartheta_2^4(\mu)\vartheta_3^4(\mu)\big\},$$

$$g_3(\mu) = \frac{\pi^6}{432}\big\{\vartheta_2^4(\mu) + \vartheta_3^4(\mu)\big\}\big\{2\vartheta_3^4(\mu) - \vartheta_2^4(\mu)\big\}\big\{\vartheta_3^4(\mu) - 2\vartheta_2^4(\mu)\big\}.$$

In order to derive equations on tori we shift singularities of Eqs. (1′)–(4′) into the Weierstrass form (31) with $e + e' + e'' = 0$ and then compute corresponding Klein's J-invariants. One arrives at four tori with moduli $\{\mathrm{i}, \varepsilon, \varrho, \varkappa\}$ [4]:

$$J(\mathrm{i}) = 1, \qquad J(\varepsilon) = 0, \qquad J(\varrho) = \frac{73^3}{2^4 3^7}, \qquad J(\varkappa) = \frac{2^8 31^3}{3^3 5^3}.$$

Theorem 8. *Suppose parameters (a, b, A) correspond to equations of the form (27). Then the following equations*

$$J(\mathrm{i}),\ (a,b,A) = (4,0,0), \qquad \psi_{\mathbf{uu}} = -\wp(2\mathbf{u}|\mathrm{i})\psi, \tag{1''}$$

$$J(\varepsilon),\ (a,b,A) = (0,4,0), \qquad \psi_{\mathbf{uu}} = -\wp(2\mathbf{u}|\varepsilon)\psi, \tag{2''}$$

$$J(\varrho),\ (a,b,A) = \left(\frac{292}{3}, -\frac{4760}{27}, \frac{2}{3}\right), \qquad \psi_{\mathbf{uu}} = -\left\{\wp(2\mathbf{u}|\varrho) - \frac{1}{6}\pi^2\vartheta_2^4(\varrho)\right\}\psi, \tag{3''}$$

$$J(\varkappa),\ (a,b,A) = \left(\frac{496}{3}, -\frac{11044}{27}, \frac{4}{3}\right), \qquad \psi_{\mathbf{uu}} = -\left\{\wp(2\mathbf{u}|\varkappa) - \frac{\sqrt{5}}{75}\pi^2\vartheta_3^4(\varkappa)\right\}\psi \tag{4''}$$

are the complete set of Fuchsian equations on tori being pullback of a ${}_2F_1$-equation by rational functions of x; the intermediate Halphen's transformation $x = \omega^{-2}\wp(\mathbf{u}|\omega'/\omega)$ is assumed to be applied.

Proof. Clearly, only two last equations need to be proved. Performing in (31) Halphen's transformation $\omega^2 x = \wp(\mathbf{u}|\mu)$, we impart to Eq. (31) the form

$$\psi'' = -\left\{\wp(2\mathbf{u}|\mu) + \omega^2 A(\mu)\right\}\psi \tag{35}$$

because $\{\wp(z), z\} = -6\wp(2z)$. If Weierstrass' roots (e, e', e'') and their ordering are known, which is our case, then standard formulae of the elliptic theory [19,8]

$$\vartheta_2^4(\mu) = \frac{4}{\pi^2}(e'' - e')\omega^2, \qquad \vartheta_3^4(\mu) = \frac{4}{\pi^2}(e - e')\omega^2, \qquad \vartheta_4^4(\mu) = \frac{4}{\pi^2}(e - e'')\omega^2$$

give linear relations between any pair of ϑ-constants and values of the ω-constant for each case without resorting to rooting of a $g_{2,3}$-ratio in (34). We find that

$$\omega = \frac{1}{2}\pi\mathrm{i}\vartheta_2^2(\varrho) \quad \text{for } J(\varrho), \qquad \omega = \frac{\sqrt[4]{5}}{10}\pi\mathrm{i}\vartheta_3^2(\varkappa) \quad \text{for } J(\varkappa).$$

Substituting this into (35), we get Eqs. (3″)–(4″). Completeness of the list follows from a completeness of the Beukers–Zagier list [2, pp. 427–428]. □

It is interesting to notice that Table 1 contains an elliptic curve that does not appear in this theorem; this is the curve (20). What is its relation to these tori? To answer this question let us consider Eq. (20) and derive Fuchsian equation on torus defined by this curve. It will suffice to use any of x, z-parametrizations of (20):

$$z = \frac{1}{3}\frac{(3\wp(\mathrm{u}) - 5)^2}{3\wp(\mathrm{u}) + 1}, \qquad x = \frac{8}{\wp'(\mathrm{u})}\frac{3\wp(\mathrm{u}) - 2}{3\wp(\mathrm{u}) - 5},$$

where $\wp(\mathrm{u}) := \wp(\mathrm{u}; \frac{52}{3}, -\frac{280}{27})$, that is $\wp'^2 = \frac{4}{27}(3\wp+7)(3\wp-5)(3\wp-2)$. Applying Lemma 2 with this z-change to equation ($3'$) (or this x-change to ($1'$)), we obtain the Fuchsian equation (changing $\mathrm{u} \mapsto \mathrm{u} - \omega''$)

$$\psi_{\mathrm{uu}} = -\frac{1}{4}\left\{\wp(\mathrm{u}) + \frac{4}{\wp(\mathrm{u}) - \frac{5}{3}} + \frac{4}{3}\right\}\psi$$

belonging to the general class (25). This equation has two punctures at points $\{0, \omega'\}$ since $\frac{5}{3} = \wp(\omega')$:

$$\psi_{\mathrm{uu}} = -\frac{1}{4}\left\{\wp(\mathrm{u}) + \wp\big(\mathrm{u} - \omega'\big) - \frac{1}{3}\right\}\psi.$$

It therefore reduces to an equation with one puncture if we make use of formula for division of the half-period ω' by 2:

$$\wp(\mathfrak{u}|\mu) + \wp(\mathfrak{u} - \mu|\mu) = \wp\left(\mathfrak{u}\,\middle|\,\frac{1}{2}\mu\right) + \wp(\mu|\mu).$$

A simple calculation shows that modulus μ of this torus is found to be $\mu = 2\varrho$; thus, the curve (20) does not produce new u-equation.

5. Transcendental equivalence

5.1. Mutual covers of tori. Examples

Just as Eqs. (1)–(4) are equivalent by algebraic transformations, so are equivalent equations ($1''$)–($4''$). Their equivalence will be realized by transcendental changes $\Xi(\mathfrak{u}, \mathfrak{s}) = 0$ coming from Theorem 5 and Halphen's transformations. These changes constitute mutual covers of tori $(\mathfrak{u})$ and $(\mathfrak{s})$ by each other and are very rich in consequences. Because of this, we shall not build the 'transcendental' analog of Table 1 but restrict ourselves to the most interesting branches of the previous machinery. In order to exhibit the way of getting formulae we consider only two exceptional cases of Theorem 5 and, since examples that follow are the first ones along these lines, expound one of them at greater length.

Example 5. As a first instance we derive the transcendental equivalence ($1''$) $\rightleftarrows$ ($3''$). Let us start from the rational (zero genus) counterparts to Eqs. $(1'')_x$ and $(3'')_z$. We may perform Halphen's transformations $x \mapsto \mathrm{u}$ in ($1'$) and $z \mapsto \mathfrak{s}$ in ($3'$) and arrive at a couple of Fuchsian equations on tori $(\mathfrak{u})$ and $(\mathfrak{s})$ whose monodromies, by virtue of Theorem 7, are known to be Fuchsian. Thus, we put

$$x = \wp(\mathrm{u}|4, 0) =: \frac{1}{\omega^2}\wp(\mathfrak{u}|\mathrm{i}), \qquad z + \frac{7}{3} = \wp\left(\mathfrak{s}\,\middle|\,\frac{292}{3}, -\frac{4760}{27}\right) =: \frac{1}{\tilde{\omega}^2}\wp(\mathfrak{s}|\varrho), \tag{36}$$

where constants ω and $\tilde{\omega}$ are the ω-constants for invariants $J(\mathrm{i})$ and $J(\varrho)$ respectively. The second of these tori is, perhaps, not among the exact solvable modular inversion problems[5]: we compute $-\mathrm{i}\varrho \approx 1.563401922\ldots$ and $\mathrm{i}\tilde{\omega} \approx 0.539128911\ldots$. First torus $\wp'^2 = 4\wp^3 - 4\wp$ is isomorphic to the classical lemniscate $y^2 = x^4 - 1$ and its ω-constant (the lemniscatic constant) was obtained by Gauss. In a ϑ-notation, under normalization $(a, b) = (4, 0)$, the constant has the form $\omega = \frac{1}{2}\pi\vartheta_2^2(\mathrm{i}) \approx 1.311028777\ldots$.

[5] We were unable to find out the value ϱ in tables on imaginary quadratic fields; see, e.g., [19].

Now, we consider an algebraic equivalence (1′) ⇄ (3′) determined, say, by formula (20). Substituting there

$$x = \omega^{-2}\wp(\mathfrak{u}|\mathrm{i}), \qquad z = \tilde{\omega}^{-2}\wp(\mathfrak{s}|\varrho) - \frac{7}{3},$$

we get

$$\omega^{-4}\wp^2(\mathfrak{u}|\mathrm{i})\bigl(1 - \omega^{-4}\wp^2(\mathfrak{u}|\mathrm{i})\bigr) = \frac{432(3\tilde{\omega}^{-2}\wp(\mathfrak{s}|\varrho) - 10)}{(3\tilde{\omega}^{-2}\wp(\mathfrak{s}|\varrho) - 7)^3(3\tilde{\omega}^{-2}\wp(\mathfrak{s}|\varrho) + 17)}$$

and, since

$$\wp'^2(\mathfrak{u}|\mathrm{i}) = 4\bigl(\wp^2(\mathfrak{u}|\mathrm{i}) - \omega^4\bigr)\wp(\mathfrak{u}|\mathrm{i}),$$
$$\wp'^2(\mathfrak{s}|\varrho) = \frac{4}{27}\bigl(3\wp(\mathfrak{s}|\varrho) - 7\tilde{\omega}^2\bigr)\bigl(3\wp(\mathfrak{s}|\varrho) - 10\tilde{\omega}^2\bigr)\bigl(3\wp(\mathfrak{s}|\varrho) + 17\tilde{\omega}^2\bigr),$$

one derives that

$$-\wp(\mathfrak{u}|\mathrm{i})\wp'(\mathfrak{u}|\mathrm{i})^2 = \pi^8\vartheta_2^{16}(\mathrm{i})\tilde{\omega}^6\left\{\frac{3\wp(\mathfrak{s}|\varrho) - 10\tilde{\omega}^2}{(3\wp(\mathfrak{s}|\varrho) - 7\tilde{\omega}^2)\wp'(\mathfrak{s}|\varrho)}\right\}^2.$$

We know that Weierstrass' $\wp$-function is a quadratic ratio of Jacobi's theta-functions plus a branch point e [20]. Since $e = 0$ is one of the branch points for lemniscate, the $\wp(\mathfrak{u}|\mathrm{i})$-function on the left hand side of last equation is a perfect square and, therefore, the equation itself is reducible. A simple calculation with theta-functions shows that

$$\pm\sqrt{\wp(\mathfrak{u}|\mathrm{i})} = \frac{1}{2}\pi\vartheta_2^2(\mathrm{i})\frac{\theta_3(\frac{1}{2}\mathfrak{u}|\mathrm{i})}{\theta_1(\frac{1}{2}\mathfrak{u}|\mathrm{i})}$$

under the standard notation [19,8]

$$\theta_1(\mathrm{u}|\mu) := -\mathrm{i}\sum_{k}^{+\infty}{}_{-\infty}(-1)^k \mathrm{e}^{(k+\frac{1}{2})^2\pi\mathrm{i}\mu}\mathrm{e}^{(2k+1)\pi\mathrm{i}\mathrm{u}}, \qquad \theta_3(\mathrm{u}|\mu) := \sum_{k}^{+\infty}{}_{-\infty}\mathrm{e}^{k^2\pi\mathrm{i}\mu}\mathrm{e}^{2k\pi\mathrm{i}\mathrm{u}},$$
$$\theta_2(\mathrm{u}|\mu) := \sum_{k}^{+\infty}{}_{-\infty}\mathrm{e}^{(k+\frac{1}{2})^2\pi\mathrm{i}\mu}\mathrm{e}^{(2k+1)\pi\mathrm{i}\mathrm{u}}, \qquad \theta_4(\mathrm{u}|\mu) := \sum_{k}^{+\infty}{}_{-\infty}(-1)^k\mathrm{e}^{k^2\pi\mathrm{i}\mu}\mathrm{e}^{2k\pi\mathrm{i}\mathrm{u}}.$$

Finally, we obtain the sought-for transcendental equivalence of two ('very simple') equations (1″) ⇄ (3″).

Proposition 9. *The linear transformation $\Psi = \sqrt{\frac{d\mathfrak{u}}{d\mathfrak{s}}}\psi$ and finitely-sheeted mutual cover of the tori $(\mathfrak{u})$ and $(\mathfrak{s})$*

$$\Xi(\mathfrak{u},\mathfrak{s}): \quad \pm 2\frac{\theta_3(\frac{1}{2}\mathfrak{u}|\mathrm{i})}{\theta_1(\frac{1}{2}\mathfrak{u}|\mathrm{i})}\wp'(\mathfrak{u}|\mathrm{i})\wp'(\mathfrak{s}|\varrho) = \pi^6\vartheta_2^6(\mathrm{i})\vartheta_2^6(\varrho)\frac{6\wp(\mathfrak{s}|\varrho) + 5\pi^2\vartheta_2^4(\varrho)}{12\wp(\mathfrak{s}|\varrho) + 7\pi^2\vartheta_2^4(\varrho)} \tag{37}$$

(transcendental change) transform Fuchsian equations

$$\Psi_{\mathfrak{u}\mathfrak{u}} = -\wp(2\mathfrak{u}|\mathrm{i})\Psi, \qquad \psi_{\mathfrak{s}\mathfrak{s}} = -\left\{\wp(2\mathfrak{s}|\varrho) - \frac{1}{6}\pi^2\vartheta_2^4(\varrho)\right\}\psi$$

into each other.

Direct check of this statement is a highly nontrivial exercise even with use of Lemma 2.

Remark 6. Transcendental equivalence (37) differs from the general one given by formula (16) because it does not involve ${}_2F_1$-series appearing in Ψ, ψ-solutions.

Example 6. We choose an equivalence of Eqs. $(2'')_{\mathfrak{u}}$ and $(3'')_{\mathfrak{s}}$ defined by the simplest relation of genus zero, that is (21). In this case we have the equi-anharmonic torus $\wp'^2 = 4\wp^3 - 4$. Its ω-constant and relation between ϑ-constants read as follows

$$\omega = \frac{1}{6}\sqrt[4]{-27}\pi\,\vartheta_2^2(\varepsilon), \qquad \vartheta_3(\varepsilon) = \sqrt[6]{\mathrm{i}}\vartheta_2(\varepsilon)$$

($\omega \approx 1.214325323\ldots$). Applying the same technique as in the previous example, we derive, after a little algebra, one of the equivalences $(2'')_{\mathfrak{u}} \rightleftarrows (3'')_{\mathfrak{s}}$:

$$\pm 3\sqrt[4]{-3}\,\frac{\theta_2\theta_3\theta_4}{\theta_1\theta_1\theta_1}\left(\frac{1}{2}\mathfrak{u}\middle|\varepsilon\right) = \frac{\theta_1^2\theta_3}{\theta_4^2\theta_2}\left(\frac{1}{2}\mathfrak{s}\middle|\varrho\right) \tag{38}$$

(no ϑ-constants here at all) where we performed an additional simplification by converting all the Weierstrassian functions into Jacobi's theta's. This result can also be treated as the fact that relation (38) represents a finitely-sheeted mutual cover of two punctured tori whose global coordinate functions $\mathfrak{u} = \mathfrak{u}(\tau)$ and $\mathfrak{s} = \mathfrak{s}(\tau)$ satisfy the two autonomic ODEs

$$\{\mathfrak{u}, \tau\}\dot{\mathfrak{u}}^{-2} = -2\wp(2\mathfrak{u}|\varepsilon), \qquad \{\mathfrak{s}, \tau\}\dot{\mathfrak{s}}^{-2} = -2\wp(2\mathfrak{s}|\varrho) + \frac{1}{3}\pi^2\vartheta_2^4(\varrho).$$

5.2. *Transcendental automorphism and Abelian integral*

We have shown above that there are nontrivial *algebraic* automorphisms of Eqs. $(1')$–$(4')$. Using Halphen's transformation and Theorem 5, we deduce that there are *transcendental* automorphisms between Eqs. $(1'')$–$(4'')$. Here is one nice example based on the elliptic curve (19). As before, we obtain

$$\wp(\mathfrak{u}; 4, 0)\wp'(\mathfrak{u}; 4, 0)^2 \cdot \wp(\mathfrak{s}; 4, 0)\wp'(\mathfrak{s}; 4, 0)^2 = 1. \tag{39}$$

Further analysis of this example leads a remarkable consequence which we are about to exhibit below.

By Theorem 7 functions $\mathfrak{u} = \mathfrak{u}(\tau)$ and $\mathfrak{s} = \mathfrak{s}(\tau)$ satisfy a common nonlinear 3rd order ODE. Is it possible to get analytic formulae to its solutions?

Inversion $x = \chi(\tau)$ of the ratio (17) for Eq. $(1')$ is known. This is a square of Legendre's modulus $k^2(\tau) = \vartheta_2^4(\tau)/\vartheta_3^4(\tau)$ [20,19,8]. Insomuch as we deal with automorphism, the second function $z(\tau)$ should be the same as $\chi(\tau)$ with the difference that its argument is merely a linear fractional function of the τ-argument for $\chi(\tau)$. Some routine computations with ϑ-constants show that

$$x = \frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}, \qquad z = \frac{\vartheta_2^2(\frac{\tau-1}{\tau+1})}{\vartheta_3^2(\frac{\tau-1}{\tau+1})}, \tag{40}$$

that is a parametrization of the lemniscate (19). The Halphen transformation $x = \wp(\mathfrak{u}; 4, 0)$ tells us that $\mathfrak{u}$ is an everywhere finite quantity for all x:

$$\pm\mathfrak{u} = \int\limits_{\infty}^{x} \frac{ds}{\sqrt{4s^3 - 4s}} = \cdots. \tag{41}$$

So we shall find $\mathfrak{u} = \mathfrak{u}(\tau)$ if we can represent this integral in terms of known functions.

Changing here integration variable $s \mapsto \sqrt[-2]{s}$, we get[6]

$$\cdots = -\frac{1}{4} \int\limits_{0}^{1/x^2} s^{-\frac{3}{4}}(1-s)^{-\frac{1}{2}}\, ds = \sqrt[-2]{x} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{4}; \frac{5}{4} \middle| \frac{1}{x^2}\right), \tag{42}$$

since [6, Section 2.2.2]

$$\int\limits_{0}^{z} s^{\alpha-1}(1-s)^{-\beta}\, ds = \frac{1}{\alpha} z^{\alpha} \cdot {}_2F_1(\beta, \alpha; \alpha+1|z), \qquad \operatorname{Re}(\alpha) > 0. \tag{43}$$

An important point here is the fact that this ${}_2F_1$-representation for indefinite integral (42) should be understood as a complex-valued analytic function being an additive one with respect to periodicity moduli for integral (41). Insomuch as (41) or (42) is an elliptic integral, it has only two independent periods [20,19] and we assign them to integration over segments $s \in [0, 1]$ and $s \in (-\infty, 1]$:

$$\frac{1}{4} \int\limits_{0}^{1} s^{-\frac{3}{4}}(1-s)^{-\frac{1}{2}}\, ds =: \varPi, \qquad \frac{1}{4} \int\limits_{-\infty}^{1} s^{-\frac{3}{4}}(1-s)^{-\frac{1}{2}}\, ds = -\mathrm{i}\varPi.$$

Moreover, the integral is a lemniscatic one; this being so, its periods must be combinations of the ω-constant appearing in Example 5. This is so indeed and we found that [20, Section 22•8]

$$\varPi = \sqrt{\frac{\pi^3}{8}}\, \Gamma\left(\frac{3}{4}\right)^{-2} \approx 1.311028777\ldots,$$

i.e., $\varPi$ coincides with the lemniscatic ω, as it should. Correlating now (40) and (41)–(42) and gathering all the remaining constants, we obtain, upon simplification, the following result.

Proposition 10. *The two additively automorphic functions*

$$\mathfrak{u}(\tau) = \frac{2}{\pi \vartheta_2^2(\mathrm{i})} \frac{\vartheta_3(\tau)}{\vartheta_2(\tau)} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{4}; \frac{5}{4} \middle| \frac{\vartheta_3^4(\tau)}{\vartheta_2^4(\tau)}\right), \qquad \mathfrak{s}(\tau) = \mathfrak{u}\left(\frac{\tau-1}{\tau+1}\right) \tag{44}$$

satisfy the common ODE

$$\{\mathfrak{u}, \tau\} = -2\wp(2\mathfrak{u}|\mathrm{i})\dot{\mathfrak{u}}^2$$

[6] It follows, incidentally, that the nice identity $\wp({}_2F_1(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}|z); 4z, 0) = 1$ holds for all z.

and turn the $(1'')_{\mathfrak{u}} \rightleftarrows (1'')_{\mathfrak{s}}$*-automorphism*

$$\pm 8\frac{\theta_2\theta_3^2\theta_4}{\theta_1^4}\left(\frac{1}{2}\mathfrak{u}\middle|\mathrm{i}\right) = \frac{\theta_1^4}{\theta_2\theta_3^2\theta_4}\left(\frac{1}{2}\mathfrak{s}\middle|\mathrm{i}\right) \tag{45}$$

into identity in variable τ.

Expression (45) was obtained, as in Example 6, by a θ-simplification of reducible equality (39). Complete verification of this statement is also a good exercise.[7] It is worth to be noticed that expression (44) is the first instance of analytic formula for the additively automorphic object (Abelian integral) on a Riemann surface—more precisely, on orbifold—of a negative (non-zero) curvature. To the best of our knowledge no one explicit formula of such a kind was known hitherto.

Let us say a few words concerning computing genera of covers $\Xi(\mathfrak{u},\mathfrak{s}) = 0$ in example of Eq. (45). First of all we establish the common base periods for a θ-ratio on left hand side of this equation and derive that they are the same as for function $\wp(\mathfrak{u}|\mathrm{i})$. Hence variables $\mathfrak{u}$, $\mathfrak{s}$ are assumed to belong to the square formed by vertices $(0, 2, 2\mathrm{i}, 2+2\mathrm{i})$ and (45) defines a transcendental $(4:4)$-cover. We need to determine its branch points $(\mathfrak{u}_k, \mathfrak{s}_k)$ and their ramification indices q_k. Based on the implicit function theorem, form, as usual, equations $\Xi(\mathfrak{u},\mathfrak{s}) = 0$ and $\Xi_{\mathfrak{u}}(\mathfrak{u},\mathfrak{s}) = 0$; their compatibility condition then gives equations defining these points. We have a separation of variables $\mathfrak{U}(\mathfrak{u}) = \mathfrak{S}(\mathfrak{s})$ and this simplifies computation of genus as before in case of pure algebraic equations. An easy calculation yields

$$\theta_1\left(\frac{1}{2}\mathfrak{u}_1\middle|\mathrm{i}\right) = 0, \qquad \theta_3\left(\frac{1}{2}\mathfrak{u}_2\middle|\mathrm{i}\right) = 0, \qquad \pi^2\vartheta_2^4(\mathrm{i})\theta_3^4\left(\frac{1}{2}\mathfrak{u}_3\middle|\mathrm{i}\right) = 2\theta_1^4\left(\frac{1}{2}\mathfrak{u}_3\middle|\mathrm{i}\right),$$
$$\mathfrak{u}_1 = 0, \qquad \mathfrak{u}_2 = \tau + 1.$$

One has three points over $\mathfrak{u}_1$: $\mathfrak{s} = \{1, \tau, \tau+1\}$ and the respective indices q_k are $\{4\}$, $\{2, 2\}$, and $\{4\}$. There are no ramifications over point $\mathfrak{u}_2$ since Eq. (45) has a structure $\theta_3^2 \sim \theta_1^4$ in the vicinity of this place. The four remaining points $\mathfrak{u}_3$, as the local analysis shows, turn out to be just points of regularity: $q_k = \{1, 1, 1, 1\}$ there. Now, using the Riemann–Hurwitz formula $g = \frac{1}{2}\sum(q_k - 1) + N(g' - 1) + 1$ with $N = 4$ and $g' = 1$ (torus being covered), we obtain

$$g = \frac{1}{2}\big\{(2-1)2 + (4-1)2\big\} + 4(1-1) + 1 = 5$$

and arrive again at a Riemann surface of genus five.

Analogs of formulae (41)–(45) for the equi-anharmonic case are derived in a similar manner. Lemniscatic and equi-anharmonic cases are the only ones we were able to obtain explicit analytic formulae.

5.3. *A hyperelliptic curve*

The remarkable fact is that covers and Halphen's transformations provide the independent ways of generation of algebraic curves and integrable Fuchsian equations with finite genus monodromies. All this is obtained by correlating Chudnovsky's curves $F(x, z) = 0$, base covers of tori $\{x = \wp(\mathfrak{u}), z = \tilde{\wp}(\mathfrak{s})\}$, and transcendental covers $\Xi(\mathfrak{u},\mathfrak{s}) = 0$ between themselves. Lack of space prevents us giving an exhaustive analysis and we restrict ourselves to considering a distinguishing example.

[7] To ensure against typos we had tested this proposition numerically.

Example 7. Let us consider transcendental counterpart of curve (21), that is Eq. (38). Its genus is easily counted because it is seen at once that ramifications $\mathfrak{u} = \mathfrak{u}(\mathfrak{s})$ are possible only at place $\theta_1(\frac{1}{2}\mathfrak{u}|\varepsilon) = 0$. Right hand side of (38) tells us that there are only two points over this $\mathfrak{u}$, namely, points determined by equations $\theta_4(\frac{1}{2}\mathfrak{s}|\varrho) = 0$ and $\theta_2(\frac{1}{2}\mathfrak{s}|\varrho) = 0$. Both of their indices are, obviously, $q = \{3\}$. Riemann–Hurwitz formula above shows, thus, that genus of (38), as a $(3:3)$-cover, is $g = 3$. What can we say about algebraic models to this cover?

Replacing variables $x \mapsto x - 1$, $z \mapsto z - \frac{7}{3}$ and introducing the second coordinates of tori as $\wp'(\mathfrak{u}|\varepsilon) =: y$ and $\wp'(\mathfrak{s}|\varrho) =: 4\sqrt{3}w$, we may rewrite (38) as follows:

$$x^3 = \frac{(3z-1)^3}{(3z+17)(3z-10)^2}, \qquad y^2 = 4x^3 - 4, \qquad 324w^2 = (3z-7)(3z-10)(3z+17).$$

This 1-dimensional surface in a 4-space (x, y, z, w) contains the plane $\{(y, z), (y, w), (x, w)\}$-curves of respective genera $g = \{0, 1, 3\}$. Of course, they can not appear in the previous analysis but we can do birational transformations. Doing that, we observe that the genus $g = 1$ curve $F(y, w) = 0$ is isomorphic to the curve (20) with a duplicate modulus $\mu = 2\varrho$ and the genus $g = 3$ curve

$$3^{-3}\big(x^3-1\big)^2\big(\big(x^3-1\big)w^2+9\big)w^4 = 2^{-4}\big(x^6+64x^3+16\big)w^2 + x^3 - 1$$

is isomorphic to the hyperelliptic form $v^2 = (u^3 + 8)(u^3 - 1)u$. We do not display here all these birationalities. One immediately recognizes that the branch u-points are singularities of Eq. (23). On the other hand, Table 1 contains two genus 3 curves and, curiously, we found that one of them is also hyperelliptic; this is the $(3')_z \rightleftarrows (1')_x$ curve. Moreover, both of these hyperelliptic surfaces are isomorphic to one another and can be turned into the famous classical Schwarz form

$$\boldsymbol{z}^2 = \boldsymbol{x}^8 + 14\boldsymbol{x}^4 + 1.$$

The last step is Fuchsian equations. Lemma 2, upon application of the chain of transitions $x \mapsto w \mapsto u \mapsto \boldsymbol{x}$, gives, however, a Fuchsian $\boldsymbol{x}$-equation with singularities located at roots $(\alpha^8 + 14\alpha^4 + 1)(\alpha^5 - \alpha) = 0$ with 'excessive' roots $\alpha^5 - \alpha = 0$. Nevertheless, we can use of (23) because it has also been arisen from the $(3')_z \rightleftarrows (1')_x$-curve (21). Different ways produce different Fuchsian equations but one curve.

Finally, insomuch as the explicit form for Schwarz's Hauptmodul $\boldsymbol{x}(\tau)$ does not appear in the literature, it is pertinent to present here its 'non-excessive' form:

$$\boldsymbol{x}(\tau) = \frac{1+\mathrm{i}}{2}\,\frac{(1+\sqrt{3})\mathrm{T}(\tau)+2}{\mathrm{T}(\tau)-1-\sqrt{3}},$$

where $\mathrm{T}(\tau)$ is defined by formula (24). By this means the transformation $\mathrm{T} \mapsto \boldsymbol{x}$ leads to a Fuchsian equation with eight parabolic singularities $\alpha^8 + 14\alpha^4 + 1 = 0$. The equation is obtained by use of Lemma 2 applied to (23):

$$\psi'' = \frac{48(\boldsymbol{x}^5-\boldsymbol{x})^2}{(\boldsymbol{x}^8+14\boldsymbol{x}^4+1)^2}\,\psi.$$

6. Conclusive remarks

An abundance of Riemann surfaces/orbifolds coming form Chudnovsky's equations is a nontrivial result in its own right and all this material requires development of an independent theory explaining the genesis of the huge variety of surfaces, further classification, and with it unification of getting the formulae. Proposition 10 is not restricted to the elliptic and holomorphic integrals. As it follows from formula (43) any holomorphic or meromorphic Abelian integral

$$\mathfrak{A} = \int^{z} s^{\frac{k}{n}} \big(s^m - 1\big)^{\frac{\ell}{n}}\, ds$$

belonging to the algebraic irrationality $w^n = z^k(z^m - 1)^\ell$ with three branch points $z = \{0, 1, \infty\}$ can be worked out in a similar manner

$$\mathfrak{A} \dashrightarrow \mathfrak{A}(\tau) \sim {}_2F_1\big(\chi(\tau)\big), \qquad \{\mathfrak{A}, \tau\} = \Xi(\mathfrak{A})\dot{\mathfrak{A}}^2$$

if Hauptmodul $z = \chi(\tau)$ is known. This is frequently our case; e.g., automorphisms considered in Section 3.1 lead to curves $w^n = (z^6 - 1)^\ell$ and $w^n = z^k(z^4 - 1)^\ell$.

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