On rearrangeable multirate three-stage Clos networks

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Received 20 June 2006; received in revised form 30 November 2006; accepted 7 December 2006

Communicated by D.-Z. Du

Abstract

Since 1989 when Melen and Turner introduced an elegant model for interconnection networks that carry multirate traffic, the theory and applications of the three-stage Clos network has been extended from circuit switching to the multirate environment. Chung and Ross conjectured that $C(n, 2n-1, r)$ is rearrangeable if each call has weight chosen from a given set of $k$ weights. Lin et al. confirmed the conjecture for a restricted discrete bandwidth case only. In this paper we show that the conjecture of Chung and Ross holds not only in the discrete bandwidth case but also in the continuous bandwidth case for $r \leq \frac{2n}{3} - \frac{23}{5}$.

Keywords: Three-stage Clos network; Multirate network; Rearrangeable

1. Introduction

The symmetric three-stage Clos network $C(n, m, r)$ is considered the most basic and popular multistage interconnection network and has been widely used in the design of telecommunication networks. $C(n, m, r)$ consists of $r$ $(n \times m)$ crossbars (switches) in the first stage (or input stage), $m$ $(r \times r)$ crossbars in the second stage (or central stage), $r$ $(m \times n)$ crossbars in the third stage (or output stage). The $n$ inlets (outlets) on each input (output) crossbar are the inputs (outputs) of the network. There exists exactly one link between every center crossbar and every input (output) crossbar. These links are the internal links while the inputs and outputs are the external links of the network.

In the classical circuit switching, a call between an idle pair (input, output) is routable if there exists a path connecting them such that no link on the path is used by any other connection path. A call is often referred to as a request before it is connected, and connection after it is connected. A network is rearrangeably nonblocking, or simply rearrangeable, if a new request is always routable given that we can reroute existing connections. The problem is to determine the least $m$ such that $C(n, m, r)$ is rearrangeable. A well-known theorem shows that $C(n, m, r)$ is rearrangeable if and only if $m \geq n$.

In 1989 Melen and Turner [7] introduced an elegant model for interconnection networks that carry multirate traffic. The impetus for producing such a model comes from the interest in designing telecommunication switches that handle traffic with a wide range of bandwidth requirements (voice, facsimile, video, etc.). In the multirate environment, a

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connection is a triple \((x, y, w)\) where \(x\) is an inlet, \(y\) an outlet, and \(w\) a weight which can be thought of as the bandwidth requirement (rate) of that connection. In the uniform model, each internal and external link is assumed to have the same capacity, which is normalized to be 1. An external link can generate many requests, while an internal link can carry many connections, as long as the sum of rates does not exceed capacity 1. In applications, the number of distinct rates is often confined to being a small number \(k\). We call this a \(k\)-rate environment. A request frame is a collection of requests such that the total weight of all requests in the frame involving a fixed inlet or outlet does not exceed unity.

Let \(m(n, r)\) denote the minimum value of \(m\) such that \(C(n, m, r)\) is multirate rearrangeable. Chung and Ross [2] conjectured that \(m(n, r) \leq 2n - 1\) if each call has weight chosen from a given set of \(k\) weights. Lin et al. [6] confirmed the conjecture for a restricted discrete bandwidth case where each connection has a weight chosen from a set \([1 \geq p_1 > p_2 > \cdots > p_h > 1/2 \geq p_{h+1} > \cdots > p_k]\) which satisfies that \(p_i\) is an integer multiple of \(p_{i+1}\) for \(i = h + 1, \ldots, k - 1\).

Du et al. [3] show that

\[
\lfloor 11n/9 \rfloor \leq m(n, r) \leq 41n/16 + O(1).
\]

Hu et al. [4] showed that under the monotone routing strategy

\[
m(n, r) \leq 2n + 1 \quad \text{for} \quad n = 2, 3, 4,
\]

\[
m(n, r) \leq 2n + 3 \quad \text{for} \quad n = 5, 6.
\]

Hung [5] proposed a routing algorithm and showed that

\[
m(n, r) \leq 2n + \lceil (n - 1)/2^k \rceil
\]

where \(k\) is any positive integer and \(r \leq n/(2^k - 1)\).

In this paper we show that \(C(n, m, r)\) is rearrangeable if \(m \geq \min\{\lceil \frac{5}{8}n + \frac{5}{18} \rceil, \lceil \frac{5}{8}n + \frac{5}{24} + \frac{1}{6} \min(n, r) \rceil\}\). Accordingly the conjecture of Chung and Ross on rearrangeability of multirate Clos networks is proved true not only in the discrete bandwidth case but also for arbitrary rates for \(r \leq \frac{2n}{5} - \frac{23}{24}\).

2. The general multirate case

Define a bipartite graph \(G\) with the input switches as one part, the output switches as the other part, and an edge with weight \(w\) between vertices \(I\) and \(J\) for each call \((x, y, w)\) where \(x\) (\(y\)) is an inlet (outlet) of the switch \(I\) (\(J\)). The routing problem for \(C(n, m, r)\) can be formulated as an edge-coloring problem on \(G\) where the requirement is that the sum of weights of all edges of the same color at a vertex cannot exceed 1.

In the following we first describe a grouping algorithm to route all requests, and derive several consequences of the algorithm.

Let \(\mathcal{F} = \{(x, y, w)\}\) be a set of connection requests. We use \(\mathcal{I}\) and \(\mathcal{J}\) to denote the set of input switches and output switches, respectively. Obviously, \(|\mathcal{I}| = |\mathcal{J}| = r\). Let \((x, y, w) \in \mathcal{F}\) be a request. If \(x\) is an inlet of input switch \(I \in \mathcal{I}\) and \(y\) an outlet of output switch \(J \in \mathcal{J}\), then we refer to the request \((x, y, w)\) as an \((I, J)\)-request of weight \(w\).

For each \(I \in \mathcal{I}\) and \(J \in \mathcal{J}\), let \(R(I, J)\) be the set of \((I, J)\)-requests in \(\mathcal{F}\). We implement the grouping algorithm on every \(R(I, J)\).

**Algorithm 2.1:** (The grouping algorithm)

1. Let \(i = 1, j = 1\).
2. If \(R(I, J) \neq \emptyset\), order the elements of \(R(I, J)\) into a non-increasing sequence \(r_1, r_2, \ldots, r_k\) according to their weights. Let \(R_i(I, J) = \emptyset\), \(w_i(I, J)\) and \(w(r_i)\) be the total weight of the requests of \(R_i(I, J)\) and the weight of \(r_i\), respectively.
3. For \(j = 1\) to \(k\) do
   - if \(w_i(I, J) + w(r_j) \leq 1\), then \(R_i(I, J) := R_i(I, J) \cup r_j, w_i(I, J) := w_i(I, J) + w(r_j)\).
4. \(R(I, J) := R(I, J) \setminus R_i(I, J), i = i + 1\). Return to step 2.

For each \(I \in \mathcal{I}\) and \(J \in \mathcal{J}\) by the grouping algorithm we group the set \(R(I, J)\) into several subsets, which be denoted as \(R_1(I, J), R_2(I, J), \ldots, R_l(I, J)\), where \(l\) is a function of \(I\) and \(J\). Obviously, we have the following consequence.
Lemma 2.1. For each $I \in \mathcal{I}$, $J \in \mathcal{J}$ and $1 \leq i \leq l - 1$, we have that $W_i(I, J) > \frac{1}{2}$.

Lemma 2.2. For each $I \in \mathcal{I}$, $J \in \mathcal{J}$ and $1 \leq i \leq l - 1$, $W_i(I, J)$ has the following two possibilities:

1. $W_i(I, J) > \frac{2}{3}$;
2. if $W_i(I, J) \leq \frac{2}{3}$, then there is just one request in $R_i(I, J)$ whose weight is larger than $\frac{1}{2}$.

Proof. If there exists $R_i(I, J)$ such that case 1 does not hold, then $W_i(I, J) \leq \frac{2}{3}$. Next we prove that $R_i(I, J)$ satisfies the second case. Assume that $s$ is the largest index such that $r_s$ does not belong to $R_i(I, J)$. Let $r_{i_1} \geq r_{i_2} \geq \cdots \geq r_i$ denote the requests in $R_i(I, J)$ with weights larger than $w(r_s)$. If $t \geq 2$, by

$$w(r_{i_1}) + w(r_{i_2}) + \cdots + w(r_{i_t}) \leq \frac{2}{3}$$

we have that $w(r_s) \leq w(r_{i_t}) \leq \frac{2}{3t}$. Hence,

$$w(r_{i_1}) + w(r_{i_2}) + \cdots + w(r_{i_t}) + w(r_s) \leq \frac{2}{3} + \frac{2}{3t} \leq \frac{2}{3} + \frac{1}{3} = 1.$$

Therefore, according to the grouping algorithm $r_s$ can be put into $R_i(I, J)$, which is a contradiction. So, $t = 1$. Obviously, $\frac{2}{3} \geq w(r_{i_1}) > \frac{1}{2}$, since if $w(r_{i_1}) \leq \frac{1}{2}$ then $r_s$ can be put into $R_i(I, J)$, which is a contradiction. ■

Lemma 2.3. For each $I \in \mathcal{I}$ and $J \in \mathcal{J}$ if there exists $W_i(I, J)$ satisfying the second case in Lemma 2.2, then $\frac{1}{2} < W_i(I, J) \leq \frac{1}{2}$.

Proof. In the proof of Lemma 2.2 we know that once there exists $R_i(I, J)$ satisfying the second case, then all the remaining requests in $R(I, J)$ are larger than $w(r_s) > 1 - w(r_{i_1}) \geq 1 - \frac{2}{3} = \frac{1}{3}$. So, the subsets generated after this satisfy the two cases in Lemma 2.2, except at most one subset. If there exists a subset that does not satisfy the two cases, then it must be $R_i(I, J)$ and $\frac{1}{3} < R_i(I, J) \leq \frac{1}{2}$.

Theorem 2.4. $C(n, m, r)$ is rearrangeable if $m \geq \left\lceil \frac{7}{4}n + \frac{r+1}{2} + \frac{1}{4} \min\{n, r\} \right\rceil$.

Proof. Let $G'$ be a bipartite graph whose vertices are the same as those of $G$, and let there be an edge with weight $W_i(I, J)$ between vertices $I$ and $J$ for each request set $R_i(I, J)$. Note that an edge-coloring of $G'$ induces an edge-coloring of $G$.

Partition the weights into large: $w > 1/2$, and small otherwise. Let $L$ denote the subgraph of $G'$ consisting of only edges of large weights and $S$ denote that consisting of small weights. Since the maximum degree of $S$ is not larger than $r$ and each edge in $S$ has weight at most 1/2, we can obtain that $S$ can be $\lceil \frac{r}{2} \rceil$-colored.

Let $R(I)$ be the set of edges adjacent to the vertex $I$ in $L$. Assume that there are $x$ edges with weight larger than $\frac{2}{3}$, $y$ edges with weight satisfying the second case in Lemma 2.2, and $z$ edges with weight not satisfying the above two cases. Then $y \leq n, z \leq r, \frac{1}{2} (y + z) + \frac{2}{3} x < n$. So, $y + z < 2n$ and $y + z \leq n + r$. Hence, $y + z \leq \min\{2n, n + r\}$. Therefore, $|R(I)| = x + y + z \leq \left\lceil \frac{n-(y+z)/2 + y + z}{2/3} \right\rceil = \left\lceil \frac{3}{2}n + \frac{1}{4}(y + z) \right\rceil \leq \left\lceil \frac{3}{2}n + \frac{1}{4} \min\{2n, n + r\} \right\rceil$. The same result can be obtained for the output switch $J$. So, $L$ can be $\left\lceil \frac{3}{2}n + \frac{1}{4} \min\{2n, n + r\} \right\rceil$-colored. Hence the total number of colors needed is

$$\left\lfloor \frac{3}{2}n + \frac{1}{4} \min\{2n, n + r\} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor \leq \left\lfloor \frac{7}{4}n + \frac{r+1}{2} + \frac{1}{4} \min\{n, r\} \right\rfloor.$$ ■

Corollary 2.5. $C(n, 2n - 1, r)$ is rearrangeable for $r \leq \frac{n}{3} - 2$.

Let $d_G(v)$ denote the degree of a vertex $v$ in $G$. A spanning subgraph of $G$ is a subgraph with the same vertex-set as $G$ (although a vertex can have zero degree). We quote a result from [3].

Lemma 2.6. Let $G$ be any bipartite graph and suppose $k \geq 1$. Then $G$ is the union of $k$ edge-disjoint spanning subgraphs $G_1, \ldots, G_k$ such that

$$\left\lfloor \frac{d_G(v)}{k} \right\rfloor \leq d_{G_i}(v) \leq \left\lceil \frac{d_G(v)}{k} \right\rceil$$

for each $v \in G$. 

2.2 Lemma

Let \( r \) be any positive integer. In this paper we improved Hung’s algorithm and combined the two one-rate and two-rate models are the simplest models of a multirate environment, we can derive one from the more general case: the continuous bandwidth case.

In the continuous bandwidth case when weights are restricted to the interval \([b, B]\), then using the method of edge-coloring we can obtain that \( M(n, r) \leq n - \frac{B}{r} \). In particular, if \( B = \frac{1}{2} \) then \( C(n, 2n - 1, r) \) is rearrangeable. In the general weight case, Hung proposed a routing algorithm and proved that \( m(n, r) \leq 2n + \lceil (n - 1)/2k \rceil \) for \( r \leq n/(2^k - 1) \), where \( k \) is any positive integer. In this paper we improved Hung’s algorithm and combined the two different methods to obtain a better result for the general weight case. The result is more general for the restriction of \( r \) than the improvement made in [5].

References