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Optimal decay rates for the compressible fluid models of Korteweg type ☆

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ABSTRACT

We consider the compressible Navier–Stokes–Korteweg system that models the motions of the compressible isothermal viscous capillary fluids. We prove the optimal L^2 and L^p, $p \geqslant 2$ decay rates for the classical solutions and their spatial derivatives. In particular, the optimal L^2 decay rate of the second-order spatial derivatives is obtained. The proof is based on the detailed study of the linear decay estimates and nonlinear energy estimates.

1. Introduction

The compressible Navier–Stokes–Korteweg system that governs the motions of the compressible isothermal viscous capillary fluids can be formulated as

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - \nu \nabla \operatorname{div} u = \kappa \rho \nabla \Delta \rho, \tag{1.2}$$

for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. The unknown variables ρ, u represent the density, velocity of the fluid respectively and $p = p(\rho)$ is the pressure satisfying $p'(\rho) > 0$ for $\rho > 0$. The constants $\mu > 0$, $\nu \geqslant 0$ are the viscosity coefficients and $\kappa > 0$ is the capillary coefficient. We supplement the system with initial data

$$\rho|_{t=0} = \rho_0, \qquad u|_{t=0} = u_0. \tag{1.3}$$

The formulation of the theory of capillarity with diffuse interfaces was first introduced by Korteweg [19], and derived rigorously by Dunn and Serrin [5]. Recently, there are some mathematical theory concerning the existence of the solutions to the compressible Navier–Stokes–Korteweg system. We refer to [1,8] for the global existence of weak solutions, [2] for the existence of solutions in the critical Besov spaces, [20] for the local existence of strong solutions and [9,10] for the existence of classical solutions. In particular, using a method of [24], Hattori and Li [10] proved the following theorem for the global existence of the solutions to the system (1.1)–(1.3):

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Theorem 1.1. *Let $s \geqslant 3$ be an integer and $\bar{\rho}$ be a positive constant. Assume that $\rho_0 - \bar{\rho} \in H^{s+1}$, $u_0 \in H^s$. There exists a constant δ such that if*

$$\|\rho_0 - \bar{\rho}\|_{H^{s+1}} + \|u_0\|_{H^s} \leqslant \delta, \tag{1.4}$$

then there is a unique global solution (ρ, u) of the Cauchy problem (1.1)–(1.3) *satisfying*

$$\big\|(\rho - \bar{\rho})(t)\big\|_{H^{s+1}}^2 + \big\|u(t)\big\|_{H^s}^2 + \int_0^t \big\|\nabla\rho(s)\big\|_{H^{s+1}}^2 + \big\|\nabla u(s)\big\|_{H^s}^2\,ds \leqslant C\big(\|\rho_0 - \bar{\rho}\|_{H^{s+1}}^2 + \|u_0\|_{H^s}^2\big), \quad t \geqslant 0. \tag{1.5}$$

However, up to our best knowledge, there is no result of the decay rates for the solutions to the system (1.1)–(1.3). We will establish the various optimal L^2 and L^p, $p \geqslant 2$ decay rates of the solutions and also their spatial derivatives, depending on the regularity of initial data. The main results are stated in the following theorem:

Theorem 1.2. *Assume that the assumptions of Theorem* 1.1 *are satisfied and that the L^1 norm of $(\rho_0 - \bar{\rho}, u_0)$ is finite. Let (ρ, u) be the solution of the system* (1.1)–(1.3) *obtained in Theorem* 1.1*, then*

$$\big\|\nabla(\rho - \bar{\rho})(t)\big\|_{H^s} + \big\|\nabla u(t)\big\|_{H^{s-1}} \leqslant C_0(1+t)^{-\frac{5}{4}}, \tag{1.6}$$

$$\big\|(\rho - \bar{\rho}, u)(t)\big\|_{L^p} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad \forall p \in [2, 6]. \tag{1.7}$$

If in addition $s \geqslant 4$, then we have

$$\big\|\nabla^2(\rho - \bar{\rho})(t)\big\|_{H^{s-1}} + \big\|\nabla^2 u(t)\big\|_{H^{s-2}} \leqslant C_0(1+t)^{-\frac{7}{4}}, \tag{1.8}$$

$$\big\|\big(\nabla(\rho - \bar{\rho}), \nabla u\big)(t)\big\|_{L^p} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad \forall p \in [2, 6], \tag{1.9}$$

$$\big\|(\rho - \bar{\rho}, u)(t)\big\|_{L^p} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad \forall p \in [2, \infty]. \tag{1.10}$$

The optimal decay rates of the lowest-order derivatives in each of (1.6)–(1.10) are consistent with the following linear optimal decay estimates:

Theorem 1.3. *Let $s \geqslant 0$ be an integer. Assume that (ϱ, v) is the solution of the linearized Navier–Stokes–Korteweg system* (2.1)–(2.3) *with the initial data $\varrho_0 \in H^{s+1} \cap L^1$ and $v_0 \in H^s \cap L^1$, then*

$$\big\|\varrho(t)\big\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}}\big(\big\|(\varrho_0, v_0)\big\|_{L^1} + \big\|(\varrho_0, v_0)\big\|_{L^2}\big), \tag{1.11}$$

$$\big\|\nabla^{k+1}\varrho(t)\big\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}-\frac{k+1}{2}}\big(\big\|(\varrho_0, v_0)\big\|_{L^1} + \big\|\big(\nabla^{k+1}\varrho_0, \nabla^k v_0\big)\big\|_{L^2}\big), \tag{1.12}$$

$$\big\|\nabla^k v(t)\big\|_{L^2} \leqslant C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\big(\big\|(\varrho_0, v_0)\big\|_{L^1} + \big\|\big(\nabla^{k+1}\varrho_0, \nabla^k v_0\big)\big\|_{L^2}\big), \tag{1.13}$$

for $0 \leqslant k \leqslant s$.

For the compressible isentropic (or non-isentropic) Navier–Stokes equations, *i.e.*, $\kappa = 0$, many important progresses have been made on the convergence rates of the solutions, see [3,4,6,7,11–18,21–23,25–27] and the references therein for instance. Among them, when there is no force, Matsumura and Nishida [23] obtained the optimal L^2 convergence rate for the compressible viscous and heat-conductive fluid in $\mathbb{R}^3$

$$\big\|(\rho - \bar{\rho}, u, \theta - \bar{\theta})(t)\big\|_{L^2} \leqslant C_0(1+t)^{-\frac{3}{4}}, \quad t \geqslant 0,$$

if the small initial disturbance belongs to $H^3 \cap L^1$, and Ponce [25] gave the optimal L^p convergence rate

$$\big\|\nabla^k(\rho - \bar{\rho}, u, \theta - \bar{\theta})(t)\big\|_{L^2} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{k}{2}}, \quad t \geqslant 0,$$

for $2 \leqslant p \leqslant \infty$ and $0 \leqslant k \leqslant 2$ if the small initial disturbance belongs to $H^s \cap W^{s,1}$ with sufficiently large integer s. The optimal L^p, $1 \leqslant p < 2$ convergence rates were also obtained by the detailed study of the Green function [11,12,17,22]. When there is an external potential force, the optimal L^p, $2 \leqslant p \leqslant 6$ decay rate of the solutions and the optimal L^2 decay rate of the first-order derivatives were obtained in a series of papers [6,7,27]. When there is a self-consist electric potential force, *i.e.*, for the compressible Navier–Stokes–Poisson equations, the global existence and optimal decay rates of the solutions were proved recently by Li et al. [21]. It was observed that the rotating effect of electric field makes the momentum of the compressible Navier–Stokes–Poisson equations decay at a slower rate. Finally, concerning the long time decay rates of global solutions for half space and exterior domain or for the general external force, we refer to the papers [3,4,13–16,18,26] for instance.

The rest of this paper is organized as follows. In Section 2, we use the Fourier analysis to study carefully the linear L^2 decay estimates for the linearized Navier–Stokes–Korteweg system and prove Theorem 1.3. In Section 3, we will do some crucial energy estimates for both the first-order derivatives and higher-order derivatives. In Section 4, as in [27,6,7], we will use the energy estimates derived in Section 3 to deduce the Lyapunov-type energy inequalities, then combining them with the linear L^2 decay estimates Theorem 1.3 to prove Theorem 1.2.

In this paper, we use the standard notations L^p, H^s to denote the L^p and Sobolev space on $\mathbb{R}^3$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$ respectively. We use C to denote a constant depending only on the physical coefficients but may vary at different formulas and denote C_0 to be constants depending additionally on the initial data.

2. Linear decay estimates

In this section, we consider the Cauchy problem for the linearized Navier–Stokes–Korteweg system of the reformulated nonlinear problem (4.3)–(4.5):

$$\varrho_t + \gamma \operatorname{div} v = 0, \tag{2.1}$$

$$v_t - \mu' \Delta v - \nu' \nabla \operatorname{div} v + \gamma \nabla \varrho - \kappa' \nabla \Delta \varrho = 0, \tag{2.2}$$

$$\varrho|_{t=0} = \varrho_0, \qquad v|_{t=0} = v_0. \tag{2.3}$$

Here the positive constants μ', ν', γ, k' are defined in (4.1).

The solutions (ϱ, v) of the linear problem (2.1)–(2.3) can be expressed as

$$(\varrho, v)^t(t) = G(t) * (\varrho_0, v_0)^t, \quad t \geqslant 0. \tag{2.4}$$

Here $G(t) := G(x,t)$ is the Green's matrix for the system (2.1)–(2.2).

To derive the large time behavior of the solutions, we first give an explicit expression for the Fourier transform $\hat{G}(\xi, t)$ of the Green's matrix $G(x,t)$.

Lemma 2.1. *The Fourier transform $\hat{G}$ of the Green's matrix for the linear system* (2.1)–(2.2) *is given by*

$$\begin{aligned} \hat{G}(\xi,t) &= \hat{G}_1(\xi,t) + \hat{G}_2(\xi,t) \\ &:= \begin{bmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -i\gamma\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\xi^t \\ -i\gamma\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\xi & e^{-\mu'|\xi|^2 t} I + \left(\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-\mu'|\xi|^2 t}\right)\frac{\xi\xi^t}{|\xi|^2} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ -i\kappa'\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)|\xi|^2\xi & 0 \end{bmatrix}, \end{aligned} \tag{2.5}$$

where $\eta = \mu' + \nu'$ and

$$\lambda_\pm(\xi) = -\frac{\eta}{2}|\xi|^2 \pm \frac{1}{2}\sqrt{\left(\eta^2 - 4\kappa'\gamma\right)|\xi|^4 - 4\gamma^2|\xi|^2}. \tag{2.6}$$

The representation above holds for $|\xi| \neq 0$, $\frac{2\gamma}{\sqrt{\eta^2 - 4\kappa'\gamma}}$ when $\kappa' < \kappa'^*$; and for $|\xi| \neq 0$ when $\kappa' \geqslant \kappa'^*$, where

$$\kappa'^* = \frac{\eta^2}{4\gamma} \equiv \frac{(\mu' + \nu')^2}{4\gamma}. \tag{2.7}$$

Proof. The proof is in spirit of Hoff and Zumbrun [11]. Applying the Fourier transform to the system (2.1)–(2.2), we have

$$\hat{\varrho}_t = -i\gamma\xi \cdot \hat{v}, \tag{2.8}$$

$$\hat{v}_t = \left(-\mu'|\xi|^2 I - \nu'\xi\xi^t\right)\hat{v} - i\gamma\xi\hat{\varrho} - i\kappa'|\xi|^2\xi\hat{\varrho} = 0. \tag{2.9}$$

From (2.8)–(2.9), we obtain the following initial value problem for $\hat{\varrho}(\xi, t)$:

$$\begin{cases} \hat{\varrho}_{tt} + \eta|\xi|^2\hat{\varrho}_t + \left(\gamma^2|\xi|^2 + \kappa'\gamma|\xi|^4\right)\hat{\varrho} = 0, \\ \hat{\varrho}(\xi,0) = \hat{\varrho}_0(\xi), \quad \hat{\varrho}_t(\xi,0) = -i\gamma\xi \cdot \hat{v}_0(\xi). \end{cases} \tag{2.10}$$

The $\lambda_\pm(\xi)$ defined in (2.6) are exactly the eigenvalues of the ODE (2.10), hence for $\lambda_- \neq \lambda_+$, the initial conditions in (2.10) give that

$$\hat{\varrho}(\xi,t) = \left(\frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right)\hat{\varrho}_0(\xi) - i\gamma\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)\xi \cdot \hat{v}_0(\xi). \tag{2.11}$$

Substituting (2.11) into (2.9), by a straightforward computation, we obtain

$$\hat{v}(\xi,t) = -i\big(\gamma+\kappa'|\xi|^2\big)\left(\frac{e^{\lambda_+ t}-e^{\lambda_- t}}{\lambda_+-\lambda_-}\right)\xi\hat{\varrho}_0(\xi) + \left[e^{-\mu'|\xi|^2 t}I + \left(\frac{\lambda_+ e^{\lambda_+ t}-\lambda_- e^{\lambda_- t}}{\lambda_+-\lambda_-} - e^{-\mu'|\xi|^2 t}\right)\frac{\xi\xi^t}{|\xi|^2}\right]\hat{v}_0(\xi). \tag{2.12}$$

Hence (2.11)–(2.12) give (2.5) and the proof of Lemma 2.1 is completed. □

We denote

$$b = \frac{1}{2}\sqrt{4\gamma^2|\xi|^2 - \big(\eta^2-4\kappa'\gamma\big)|\xi|^4}. \tag{2.13}$$

For $\kappa' \geqslant \kappa'^*$, $b>0$ is a real number for any fixed $|\xi|\neq 0$, hence

$$\lambda_\pm(\xi) = -\frac{\eta}{2}|\xi|^2 \pm bi, \tag{2.14}$$

that is, the eigenvalues of the linear system in Fourier transform are not real. Hence in this case, we have

$$\frac{e^{\lambda_+ t}-e^{\lambda_- t}}{\lambda_+-\lambda_-} = \frac{\sin(bt)}{b}e^{-\frac{\eta}{2}|\xi|^2 t}, \tag{2.15}$$

$$\frac{\lambda_+ e^{\lambda_+ t}-\lambda_- e^{\lambda_- t}}{\lambda_+-\lambda_-} = \left[\cos(bt)+\frac{\eta}{2}\frac{\sin(bt)}{b}|\xi|^2\right]e^{-\frac{\eta}{2}|\xi|^2 t}, \tag{2.16}$$

$$\frac{\lambda_+ e^{\lambda_- t}-\lambda_- e^{\lambda_+ t}}{\lambda_+-\lambda_-} = \left[\cos(bt)-\frac{\eta}{2}\frac{\sin(bt)}{b}|\xi|^2\right]e^{-\frac{\eta}{2}|\xi|^2 t}. \tag{2.17}$$

Thus we can show the pointwise estimates for $\hat{G}(\xi,t)$ explicitly. While for $0<\kappa'<\kappa'^*$, (2.15)–(2.17) still hold for $|\xi|\ll 1$ but do not hold globally. For the high frequencies $|\xi|\gg 1$, b is a pure imaginary number and then the eigenvalues are real.

Now we study the behavior of $\hat{G}(\xi,t)$ for both the low frequency and high frequency. The estimates will be different for $\hat{G}_1(\xi,t)$, $\hat{G}_2(\xi,t)$ which are defined in (2.5). It turns out that $\hat{G}_2(\xi,t)$ deceases faster at low frequency but decay slower at high frequency. This results that the decay rates of the linearized Navier–Stokes–Korteweg system are same to those of the linearized compressible Navier–Stokes equations [11] and faster than those of the linearized compressible Navier–Stokes–Poisson equations [21], but it requires the higher regularity of the initial data.

We simply denote $\hat{g}_{11}$, $\hat{g}_{12}$, $\hat{g}_{21}$, $\hat{g}_{22}$ be the four components of $\hat{G}_1$ and $\hat{g}$ be the nonzero component of $\hat{G}_2$. We shall estimate them term by term and we divide the arguments into two cases in terms of the value of the capillary coefficient κ'. Let $R>0$ be any fixed constant.

Case 1: $\kappa'\geqslant\kappa'^*$. For $|\xi|\geqslant R$, using the fact that

$$\left|\frac{\sin(bt)}{b}|\xi|^2 e^{-\frac{\eta}{2}|\xi|^2 t}\right| \leqslant t|\xi|^2 e^{-\frac{\eta}{2}|\xi|^2 t} \leqslant 4e^{-\frac{\eta}{4}|\xi|^2 t}, \tag{2.18}$$

and substituting (2.15)–(2.17) into (2.5), we deduce that, setting $\eta' = \min\{\frac{\eta}{4},\mu'\}$,

$$\big|\hat{g}_{11}(\xi,t),\hat{g}_{22}(\xi,t)\big| \leqslant Ce^{-\eta'|\xi|^2 t}, \qquad \big|\hat{g}_{12}(\xi,t),\hat{g}_{21}(\xi,t)\big| \leqslant C|\xi|^{-1}e^{-\eta'|\xi|^2 t},$$
$$\big|\hat{g}(\xi,t)\big| \leqslant C|\xi|e^{-\eta'|\xi|^2 t}, \quad |\xi|\geqslant R. \tag{2.19}$$

While for $|\xi|\leqslant R$, noticing that $b=O(|\xi|)$, we easily obtain

$$\big|\hat{g}_{11}(\xi,t),\hat{g}_{22}(\xi,t),\hat{g}_{12}(\xi,t),\hat{g}_{21}(\xi,t)\big| \leqslant Ce^{-\eta'|\xi|^2 t},$$
$$\big|\hat{g}(\xi,t)\big| \leqslant C|\xi|^2 e^{-\eta'|\xi|^2 t}, \quad |\xi|\leqslant R. \tag{2.20}$$

Case 2: $0<\kappa'<\kappa'^*$. Observe that if we define $a=\eta-\sqrt{\eta^2-4\kappa'\gamma}$, then $a>0$ and

$$\mathrm{Re}\big(\lambda_\pm(\xi)\big) \leqslant -\frac{a}{2}|\xi|^2, \quad \forall\xi\in\mathbb{R}^3. \tag{2.21}$$

Now for $|\xi|\geqslant R$, since $|\lambda_\pm|$, $|\lambda_+-\lambda_-| = O(|\xi|^2)$, we derive from the expression (2.5) and (2.21) that, setting $a' = \min\{\frac{a}{2},\mu'\}$,

$$\big|\hat{g}_{11}(\xi,t),\hat{g}_{22}(\xi,t)\big| \leqslant Ce^{-a'|\xi|^2 t}, \qquad \big|\hat{g}_{12}(\xi,t),\hat{g}_{21}(\xi,t)\big| \leqslant C|\xi|^{-1}e^{-a'|\xi|^2 t},$$
$$\big|\hat{g}(\xi,t)\big| \leqslant C|\xi|e^{-a'|\xi|^2 t}, \quad |\xi|\geqslant R. \tag{2.22}$$

While for $|\xi| \leqslant R$, since $|\lambda_\pm|,\ |\lambda_+ - \lambda_-| = O(|\xi|)$, we derive from the expression (2.5) and (2.21) that

$$\left|\hat{g}_{11}(\xi,t), \hat{g}_{22}(\xi,t), \hat{g}_{12}(\xi,t), \hat{g}_{21}(\xi,t)\right| \leqslant Ce^{-a'|\xi|^2 t},$$
$$\left|\hat{g}(\xi,t)\right| \leqslant C|\xi|^2 e^{-a'|\xi|^2 t}, \quad |\xi| \leqslant R. \tag{2.23}$$

Now we are in a position to prove the linear decay estimates in Theorem 1.3.

Proof of Theorem 1.3. We may conclude from (2.19), (2.20), (2.22), (2.23) that for any $\kappa' > 0$ there exists some $\beta := \beta(\kappa') > 0$ such that

$$\left|\hat{g}_{11}(\xi,t), \hat{g}_{22}(\xi,t), \hat{g}_{12}(\xi,t), \hat{g}_{21}(\xi,t), \hat{g}(\xi,t)\right| \leqslant Ce^{-\beta|\xi|^2 t}, \quad |\xi| \leqslant R, \tag{2.24}$$

and

$$\left|\hat{g}_{11}(\xi,t), \hat{g}_{22}(\xi,t)\right| \leqslant Ce^{-\beta|\xi|^2 t}, \qquad \left|\hat{g}_{12}(\xi,t), \hat{g}_{21}(\xi,t)\right| \leqslant C|\xi|^{-1} e^{-\beta|\xi|^2 t},$$
$$\left|\hat{g}(\xi,t)\right| \leqslant C|\xi| e^{-\beta|\xi|^2 t}, \quad |\xi| \geqslant R. \tag{2.25}$$

Now by the expression (2.5), we have

$$\hat{\varrho}(\xi,t) = \hat{g}_{11}\hat{\varrho}_0 + \hat{g}_{12}\hat{v}_0, \qquad \hat{v}(\xi,t) = \hat{g}_{21}\hat{\varrho}_0 + \hat{g}_{22}\hat{v}_0 + \hat{g}\hat{\varrho}_0. \tag{2.26}$$

Let $R > 0$ be a fixed constant as before. By the pointwise estimates (2.24), (2.25), together with the Parsevel theorem and Hausdroff–Young's inequality, we obtain

$$\begin{aligned}
\left\|\varrho(t)\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} \left|\hat{g}_{11}(\xi,t)\hat{\varrho}_0(\xi) + \hat{g}_{12}(\xi,t)\hat{v}_0(\xi)\right|^2 d\xi \\
&\leqslant C \int_{|\xi|\leqslant R} e^{-\beta|\xi|^2 t}\left(\left|\hat{\varrho}_0(\xi)\right|^2 + \left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\quad + C \int_{|\xi|\geqslant R} e^{-\beta|\xi|^2 t}\left(\left|\hat{\varrho}_0(\xi)\right|^2 + |\xi|^{-2}\left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\leqslant C(1+t)^{-\frac{3}{2}}\left\|(\hat{\varrho}_0, \hat{v}_0)\right\|_{L^\infty}^2 + Ce^{-\beta R^2 t}\left\|(\hat{\varrho}_0, \hat{v}_0)\right\|_{L^2}^2 \\
&\leqslant C(1+t)^{-\frac{3}{2}}\left(\left\|(\varrho_0, v_0)\right\|_{L^1}^2 + \left\|(\varrho_0, v_0)\right\|_{L^2}^2\right),
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
\left\|\nabla^{k+1}\varrho(t)\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^{2(k+1)}\left|\hat{g}_{11}(\xi,t)\hat{\varrho}_0(\xi) + \hat{g}_{12}(\xi,t)\hat{v}_0(\xi)\right|^2 d\xi \\
&\leqslant C \int_{|\xi|\leqslant R} e^{-\beta|\xi|^2 t}|\xi|^{2(k+1)}\left(\left|\hat{\varrho}_0(\xi)\right|^2 + \left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\quad + C \int_{|\xi|\geqslant R} e^{-\beta|\xi|^2 t}\left(|\xi|^{2(k+1)}\left|\hat{\varrho}_0(\xi)\right|^2 + |\xi|^{2k}\left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\leqslant C(1+t)^{-\frac{3}{2}-k-1}\left\|(\hat{\varrho}_0, \hat{v}_0)\right\|_{L^\infty}^2 + Ce^{-\beta R^2 t}\left\|\left(|\xi|^{k+1}\hat{\varrho}_0, |\xi|^k\hat{v}_0\right)\right\|_{L^2}^2 \\
&\leqslant C(1+t)^{-\frac{3}{2}-k-1}\left(\left\|(\varrho_0, v_0)\right\|_{L^1}^2 + \left\|\left(\nabla^{k+1}\varrho_0, \nabla^k v_0\right)\right\|_{L^2}^2\right),
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
\left\|\nabla^k v(t)\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^{2k}\left|\hat{g}_{21}(\xi,t)\hat{\varrho}_0(\xi) + \hat{g}_{22}(\xi,t)\hat{v}_0(\xi) + \hat{g}(\xi,t)\hat{\varrho}_0(\xi)\right|^2 d\xi \\
&\leqslant C \int_{|\xi|\leqslant R} e^{-\beta|\xi|^2 t}|\xi|^{2k}\left(\left|\hat{\varrho}_0(\xi)\right|^2 + \left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\quad + C \int_{|\xi|\geqslant R} e^{-\beta|\xi|^2 t}\left(|\xi|^{2(k+1)}\left|\hat{\varrho}_0(\xi)\right|^2 + |\xi|^{2k}\left|\hat{v}_0(\xi)\right|^2\right) d\xi \\
&\leqslant C(1+t)^{-\frac{3}{2}-k}\left(\left\|(\varrho_0, v_0)\right\|_{L^1}^2 + \left\|\left(\nabla^{k+1}\varrho_0, \nabla^k v_0\right)\right\|_{L^2}^2\right),
\end{aligned} \tag{2.29}$$

for $0 \leqslant k \leqslant s$. The proof of Theorem 1.3 is completed. □

3. Energy estimates

In this section we will derive the energy estimates for the spatial derivatives of the solutions to the system (1.1)–(1.2). We recall from Theorem 1.1 that $s \geqslant 3$ and

$$\left\|(\rho-\bar{\rho})(t)\right\|_{H^{s+1}}+\left\|u(t)\right\|_{H^s} \leqslant C\delta. \tag{3.1}$$

This together with the Sobolev's inequality and the continuity equation (1.1) imply

$$\sup_x\left|\left(\rho-\bar{\rho}, \nabla\rho, \nabla^2\rho, \rho_t, u, \nabla u\right)(t)\right| \leqslant C\delta. \tag{3.2}$$

In particular, $\bar{\rho}/2 \leqslant \rho \leqslant 2\bar{\rho}$. These will be kept in mind in the rest of this paper. We simply denote by ∇^k the operator vector with components consisting of all the spatially differential operators ∂^α with multi-index $|\alpha|=k$. We also denote $(\cdot,\cdot)$ by the L^2 inner product on $\mathbb{R}^3$. It seems more convenient to rewrite the system as

$$\rho_t+\partial_{x_j}\left(\rho u^j\right)=0, \tag{3.3}$$

$$u_t^i+u^j\partial_{x_j}u^i+\partial_{x_i}h(\rho)-\rho^{-1}\left(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u\right)-\kappa\partial_{x_i}\Delta\rho=0 \tag{3.4}$$

where we have used the Einstein's summation convection, $1 \leqslant i, j \leqslant 3$ and

$$h(\rho)=\int_{\bar{\rho}}^{\rho}\frac{p'(s)}{s}\,ds.$$

We first establish the energy estimates for the first-order derivatives.

Lemma 3.1. *Under the assumptions of Theorem* 1.1, *we have*

$$\frac{d}{dt}E_1(t)+\left\|\nabla^2\rho(t)\right\|_{H^1}^2+\left\|\nabla^2u(t)\right\|_{L^2}^2 \leqslant C\left(\left\|\nabla\rho(t)\right\|_{L^2}^2+\left\|\nabla u(t)\right\|_{L^2}^2\right), \tag{3.5}$$

where the energy functional $E_1(t)$ is equivalent to $\|\nabla\rho(t)\|_{H^1}^2+\|\nabla u(t)\|_{L^2}^2$, that is, there exists a constant $C>0$ such that

$$C^{-1}\left(\left\|\nabla\rho(t)\right\|_{H^1}^2+\left\|\nabla u(t)\right\|_{L^2}^2\right) \leqslant E_1(t) \leqslant C\left(\left\|\nabla\rho(t)\right\|_{H^1}^2+\left\|\nabla u(t)\right\|_{L^2}^2\right). \tag{3.6}$$

Proof. First applying the operator $\rho\nabla$ to the momentum equations (3.4) and then taking the L^2 inner product with ∇u^i, we have

$$\begin{aligned}&\left(\rho\nabla\left(u_t^i+u^j\partial_{x_j}u^i\right),\nabla u^i\right)+\left(\rho\nabla\partial_{x_i}h(\rho),\nabla u^i\right)-\left(\rho\nabla\left[\rho^{-1}\left(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u\right)\right],\nabla u^i\right)-\kappa\left(\rho\nabla\partial_{x_i}\Delta\rho,\nabla u^i\right)\\&\quad\equiv I_1+I_2+I_3+I_4=0.\end{aligned} \tag{3.7}$$

Integrating by parts and using the continuity equation (3.3), and by (3.2), we can estimate these four terms above as follows

$$\begin{aligned}I_1&=\left(\rho,\frac{1}{2}\partial_t\left|\nabla u^i\right|^2\right)-\left(\partial_{x_j}\left(\rho u^j\right),\frac{1}{2}\left|\nabla u^i\right|^2\right)+\left(\rho\nabla u^j\partial_{x_j}u^i,\nabla u^i\right)\\&\geqslant\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho|\nabla u|^2\,dx-C\delta\|\nabla u\|_{L^2}^2,\end{aligned} \tag{3.8}$$

$$\begin{aligned}I_2&=\left(\nabla\partial_{x_i}h(\rho),\nabla\left(\rho u^i\right)\right)-\left(\nabla\partial_{x_i}h(\rho),\nabla\rho u^i\right)\\&=\left(\nabla h(\rho),\nabla\rho_t\right)-\left(\nabla h(\rho),\partial_{x_i}\left(\nabla\rho u^i\right)\right)\\&=\left(h'(\rho),\frac{1}{2}\partial_t|\nabla\rho|^2\right)-\left(\nabla h(\rho),\partial_{x_i}\left(\nabla\rho u^i\right)\right)\\&\geqslant\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}h'(\rho)|\nabla\rho|^2\,dx-C\delta\|\nabla\rho\|_{H^1}^2,\end{aligned} \tag{3.9}$$

$$\begin{aligned}I_3&=-\left(\nabla\left(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u\right),\nabla u^i\right)-\left(\rho\nabla\left(\rho^{-1}\right)\left(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u\right),\nabla u^i\right)\\&\geqslant\mu\left\|\nabla^2u^i\right\|_{L^2}^2-C\delta\left(\|\nabla\rho\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right),\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
I_4 &= \kappa\big(\nabla\Delta\rho, \nabla\partial_{x_i}\big(\rho u^i\big)\big) - \kappa\big(\nabla\Delta\rho, \partial_{x_i}\big(\nabla\rho u^i\big)\big) \\
&\geqslant \frac{\kappa}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big|\nabla^2\rho\big|^2\,dx - C\delta\|\nabla\rho\|_{H^2}^2. \qquad (3.11)
\end{aligned}
$$

Summing up (3.7)–(3.11), we arrive at

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big(\rho|\nabla u|^2 + h'(\rho)|\nabla\rho|^2 + \kappa\big|\nabla^2\rho\big|^2\big)\,dx + \mu\big\|\nabla^2 u\big\|_{L^2}^2 \leqslant C\delta\big(\|\nabla\rho\|_{H^2}^2 + \|\nabla u\|_{L^2}^2\big). \qquad (3.12)
$$

Next applying the operator $\rho\nabla$ again to Eqs. (3.4) and then taking the L^2 inner product with $\nabla\partial_{x_i}\rho$, we have

$$
\begin{aligned}
&\big(\rho\nabla u_t^i, \nabla\partial_{x_i}\rho\big) + \big(\rho\nabla\big(u^j\partial_{x_j}u^i\big), \nabla\partial_{x_i}\rho\big) + \big(\rho\nabla\partial_{x_i}h(\rho), \nabla\partial_{x_i}\rho\big) \\
&\quad - \big(\rho\nabla\big[\rho^{-1}\big(\mu\Delta u^i + \nu\partial_{x_i}\operatorname{div}u\big)\big], \nabla\partial_{x_i}\rho\big) - \kappa(\rho\nabla\partial_{x_i}\Delta\rho, \nabla\partial_{x_i}\rho) \\
&\equiv J_1 + J_2 + J_3 + J_4 + J_5 = 0. \qquad (3.13)
\end{aligned}
$$

The first term J_1 that involves the time derivative of u can be estimated as

$$
\begin{aligned}
J_1 &= \frac{d}{dt}\big(\rho\nabla u^i, \nabla\partial_{x_i}\rho\big) - \big(\nabla u^i, \rho_t\nabla\partial_{x_i}\rho\big) + \big(\partial_{x_i}\big(\rho\nabla u^i\big), \nabla\rho_t\big) \\
&\geqslant \frac{d}{dt}\int_{\mathbb{R}^3}\rho\nabla u^i\nabla\partial_{x_i}\rho\,dx - C\delta\|\nabla\rho\|_{H^1}^2 - C\|\nabla u\|_{H^1}^2. \qquad (3.14)
\end{aligned}
$$

The other terms $J_2 \sim J_5$ can be estimated as follows

$$
\begin{aligned}
J_2 &\geqslant -C\delta\big(\|\nabla\rho\|_{H^1}^2 + \|\nabla u\|_{H^1}^2\big), \qquad (3.15)\\
J_3 &= \big(\rho h'(\rho)\nabla\partial_{x_i}\rho, \nabla\partial_{x_i}\rho\big) + \big(\rho h''(\rho)\nabla\rho\partial_{x_i}\rho, \nabla\partial_{x_i}\rho\big) \\
&\geqslant C\|\nabla\partial_{x_i}\rho\|_{L^2}^2 - C\delta\|\nabla\rho\|_{H^1}^2, \qquad (3.16)\\
J_4 &= -\big(\rho^{-1}\big(\mu\Delta u^i + \nu\partial_{x_i}\operatorname{div}u\big), \nabla(\rho\nabla\partial_{x_i}\rho)\big) \\
&\geqslant -C_\delta\big\|\nabla^2 u^i\big\|_{L^2}^2 - \delta\|\nabla\rho\|_{H^2}^2, \qquad (3.17)\\
J_5 &= \kappa\big(\nabla^2\partial_{x_i}\rho, \nabla(\rho\nabla\partial_{x_i}\rho)\big) \\
&\geqslant C\big\|\nabla^2\partial_{x_i}\rho\big\|_{L^2}^2 - \delta\big\|\nabla^2\rho\big\|_{L^2}^2. \qquad (3.18)
\end{aligned}
$$

Summing up (3.13)–(3.18), we obtain

$$
\frac{d}{dt}\int_{\mathbb{R}^3}\rho\nabla u^i\nabla\partial_{x_i}\rho\,dx + C\big\|\nabla^2\rho\big\|_{H^1}^2 \leqslant C_\delta\|\nabla u\|_{H^1}^2 + C\delta\|\nabla\rho\|_{L^2}^2. \qquad (3.19)
$$

Consequently, multiplying (3.19) by an appropriate small constant β_1 and then adding it with (3.12), by the smallness of δ, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\big(\rho|\nabla u|^2 + h'(\rho)|\nabla\rho|^2 + \kappa\big|\nabla^2\rho\big|^2 + 2\beta_1\rho\nabla u^i\nabla\partial_{x_i}\rho\big)\,dx + C_{\beta_1}\big(\big\|\nabla^2 u\big\|_{L^2}^2 + \big\|\nabla^2\rho\big\|_{H^1}^2\big) \\
&\quad \leqslant C\big(\|\nabla\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big). \qquad (3.20)
\end{aligned}
$$

We conclude our lemma by defining $E_1(t)$ to be the $C_{\beta_1}^{-1}$ times the integral in (3.20) since β_1 is small. $\square$

Now we derive the energy estimates for the higher-order derivatives. The strategy is similar to the proof of Lemma 3.1, but the argument is much more dedicate since we want to bound all the higher-order derivatives only in terms of the second-order derivatives. It is worth to mention here that we will frequently use the fact

$$
\|f\|_{L^\infty} \leqslant \|\nabla f\|_{H^1}, \quad \forall f \in H^2. \qquad (3.21)
$$

For simply, we define the commutator $[\partial, f]g = \partial(fg) - f\partial g$, where ∂ representing any differential operator and f, g are functions.

We shall establish the following lemma.

Lemma 3.2. *Under the assumptions of Theorem* 1.1, *we have*

$$\frac{d}{dt}E_2(t)+\left\|\nabla^3\rho(t)\right\|_{H^{s-1}}^2+\left\|\nabla^3 u(t)\right\|_{H^{s-2}}^2\leqslant C\left(\left\|\nabla^2\rho(t)\right\|_{L^2}^2+\left\|\nabla^2 u(t)\right\|_{L^2}^2\right),\tag{3.22}$$

where the energy functional $E_2(t)$ is equivalent to $\|\nabla^2\rho(t)\|_{H^{s-1}}^2+\|\nabla^2 u(t)\|_{H^{s-2}}^2$, that is, there exists a constant $C>0$ such that

$$C^{-1}\left(\left\|\nabla^2\rho(t)\right\|_{H^{s-1}}^2+\left\|\nabla^2 u(t)\right\|_{H^{s-2}}^2\right)\leqslant E_2(t)\leqslant C\left(\left\|\nabla^2\rho(t)\right\|_{H^{s-1}}^2+\left\|\nabla^2 u(t)\right\|_{H^{s-2}}^2\right).\tag{3.23}$$

Proof. We fix $2\leqslant k\leqslant s$. As we did for the estimates of the first-order derivative, we first applying the operator $\rho\nabla^k$ to Eqs. (3.4) and then taking the L^2 inner product with $\nabla^k u^i$, we have

$$\begin{aligned}&\left(\rho\nabla^k\left(u_t^i+u^j\partial_{x_j}u^i\right),\nabla^k u^i\right)+\left(\rho\nabla^k\partial_{x_i}h(\rho),\nabla^k u^i\right)\\&\quad-\left(\rho\nabla^k\left[\rho^{-1}\left(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u\right)\right],\nabla^k u^i\right)-\kappa\left(\rho\nabla^k\partial_{x_i}\Delta\rho,\nabla^k u^i\right)\\&\quad\equiv I_{k1}+I_{k2}+I_{k3}+I_{k4}=0.\end{aligned}\tag{3.24}$$

Integrating by parts and using Eq. (3.3), and by (3.1), (3.2), (3.21), we can estimate the four terms above as follows. First we estimate I_{k1} as

$$\begin{aligned}I_{k1}&=\left(\rho,\frac{1}{2}\partial_t\left|\nabla^k u^i\right|^2\right)-\left(\partial_{x_j}\left(\rho u^j\right),\frac{1}{2}\left|\nabla^k u^i\right|^2\right)+\left(\rho\left[\nabla^k,u^j\right]\partial_{x_j}u^i,\nabla^k u^i\right)\\&\geqslant\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho\left|\nabla^k u\right|^2dx-C\left\|\nabla^k u\right\|_{L^2}\sum_{\substack{|\alpha|\geqslant 1\\|\alpha|+|\beta|=k}}\left\|\partial^\alpha u^j\partial^\beta\partial_{x_j}u^i\right\|_{L^2}.\end{aligned}\tag{3.25}$$

Separating the case of $|\alpha|=1$ from the other cases, we bound the summation in (3.25) by

$$\left\|\partial u^j\right\|_{L^\infty}\left\|\partial^{k-1}\partial_{x_j}u^i\right\|_{L^2}+\sum_{\substack{|\alpha|\geqslant 2\\|\alpha|+|\beta|=k}}\left\|\partial^\alpha u^j\right\|_{L^2}\left\|\partial^\beta\partial_{x_j}u^i\right\|_{L^\infty}\leqslant C\delta\left\|\nabla^2 u\right\|_{H^{k-1}}.\tag{3.26}$$

Hence we have

$$I_{k1}\geqslant\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho\left|\nabla^k u\right|^2dx-C\delta\left\|\nabla^2 u\right\|_{H^{k-1}}^2.\tag{3.27}$$

Next we rewrite I_{k2} as follows

$$\begin{aligned}I_{k2}&=-\left(\nabla^k h(\rho),\partial_{x_i}\left(\rho\nabla^k u^i\right)\right)\\&=-\left(\nabla^k h(\rho),\nabla^k\partial_{x_i}\left(\rho u^i\right)\right)+\left(\nabla^k h(\rho),\partial_{x_i}\left[\nabla^k,\rho\right]u^i\right)\\&=\left(h'(\rho)\nabla^k\rho,\nabla^k\rho_t\right)-\left(\left[\nabla^{k-1},h'(\rho)\right]\nabla\rho,\nabla^k\partial_{x_i}\left(\rho u^i\right)\right)+\left(\nabla^k h(\rho),\partial_{x_i}\left[\nabla^k,\rho\right]u^i\right).\end{aligned}\tag{3.28}$$

The first term is easily estimated as

$$\left(h'(\rho)\nabla^k\rho,\nabla^k\rho_t\right)\geqslant\frac{d}{dt}\int_{\mathbb{R}^3}\frac{h'(\rho)}{2}\left|\nabla^k\rho\right|^2dx-C\delta\left|\nabla^k\rho\right|_{L^2}^2.\tag{3.29}$$

To estimate the last two terms involving the spatial derivatives of $h(\rho)$, $h'(\rho)$ in (3.28), we notice that for any function $f(\rho)$ and multi-index α

$$\partial^\alpha f(\rho)=\text{a sum of products } f^{\gamma_1,\dots,\gamma_l}(\rho)\partial^{\gamma_1}\rho\cdots\partial^{\gamma_l}\rho,\tag{3.30}$$

where $f^{\gamma_1,\dots,\gamma_l}(\rho)$ are some derivatives of $f(\rho)$ and $1\leqslant|\gamma_i|\leqslant|\alpha|$, $i=1,\dots,l$. Hence the second term in (3.28) can be bounded by

$$\begin{aligned}\left|\left(\left[\nabla^{k-1},h'(\rho)\right]\nabla\rho,\nabla^k\partial_{x_i}\left(\rho u^i\right)\right)\right|&\leqslant C\sum_{\substack{|\alpha|\geqslant 1\\|\alpha|+|\beta|=k-1}}\left\|\partial^\alpha h'(\rho)\right\|_{L^\infty}\left\|\partial^\beta\nabla\rho\right\|_{L^2}\left(\left\|\nabla^{k+1}u\right\|_{L^2}+\left\|\nabla^k\partial_{x_i}\left((\rho-\bar\rho)u^i\right)\right\|_{L^2}\right)\\&\leqslant C\delta\left\|\nabla^2\rho\right\|_{H^{k-1}}\left\{\left\|\nabla^{k+1}u\right\|_{L^2}+\sum_{|\alpha|+|\beta|=k+1}\left\|\partial^\alpha(\rho-\bar\rho)\partial^\beta u\right\|_{L^2}\right\}\\&\leqslant C\delta\left(\left\|\nabla^2\rho\right\|_{H^k}^2+\left\|\nabla^{k+1}u\right\|_{L^2}^2\right),\end{aligned}\tag{3.31}$$

where we have bounded the summation in the second inequality above by, for the case of $|\alpha|=0$ or $|\alpha|=k+1$,

$$C\delta\big(\|\partial^{k+1}\rho\|_{L^2}+\|\partial^{k+1}u\|_{L^2}\big),$$

while for the other case $1\leqslant|\alpha|\leqslant k$,

$$C\|\partial^\alpha\rho\|_{L^\infty}\|\partial^\beta u\|_{L^2}\leqslant C\delta\|\nabla^2\rho\|_{H^k}.$$

Similarly, we bound the last term in (3.28) by

$$\begin{aligned}\big|\big(\nabla^k h(\rho),\partial_{x_i}[\nabla^k,\rho]u^i\big)\big| &\leqslant C\|\nabla^k h(\rho)\|_{L^\infty}\sum_{\substack{|\alpha|\geqslant1\\|\alpha|+|\beta|=k}}\big(\|\partial^\alpha\partial_{x_i}\rho\|_{L^2}\|\partial^\beta u\|_{L^2}+\|\partial^\alpha\rho\|_{L^2}\|\partial^\beta\partial_{x_i}u^i\|_{L^2}\big)\\ &\leqslant C\delta\big(\|\nabla^2\rho\|_{H^k}^2+\|\nabla^2u\|_{H^{k-1}}^2\big).\end{aligned}\tag{3.32}$$

Collecting (3.28), (3.29), (3.31), (3.32), we obtain

$$I_{k2}\geqslant\frac{d}{dt}\int_{\mathbb{R}^3}\frac{h'(\rho)}{2}|\nabla^k\rho|^2\,dx-C\delta\big(\|\nabla^2\rho\|_{H^k}^2+\|\nabla^2u\|_{H^{k-1}}^2\big).\tag{3.33}$$

Now we rewrite and estimate the term I_{k3} as

$$\begin{aligned}I_{k3}&=-\big(\nabla^k(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u),\nabla^k u^i\big)-\big(\rho[\nabla^k,\rho^{-1}](\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u),\nabla^k u^i\big)\\ &\geqslant\mu\|\nabla^{k+1}u\|_{L^2}^2-C\sum_{\substack{|\alpha|\geqslant1\\|\alpha|+|\beta|=k}}\|\partial^\alpha(\rho^{-1})\|_{L^\infty}\|\partial^\beta(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u)\|_{L^2}\|\nabla^k u^i\|_{L^2}\\ &\geqslant\mu\|\nabla^{k+1}u\|_{L^2}^2-C\delta\big(\|\nabla^2\rho\|_{H^k}^2+\|\nabla^2u\|_{H^{k-1}}^2\big).\end{aligned}\tag{3.34}$$

Finally, the last term I_{k4} is estimated as

$$\begin{aligned}I_{k4}&=\kappa\big(\nabla^k\Delta\rho,\nabla^k\partial_{x_i}(\rho u^i)\big)-\kappa\big(\nabla^k\Delta\rho,\partial_{x_i}[\nabla^k,\rho]u^i\big)\\ &\geqslant-\kappa\big(\nabla^k\Delta\rho,\nabla^k\rho_t\big)-C\|\nabla^k\Delta\rho\|_{L^2}\sum_{\substack{|\alpha|\geqslant1\\|\alpha|+|\beta|=k}}\|\partial^\alpha\partial_{x_i}\rho\partial^\beta u^i\|_{L^2}+\|\partial^\alpha\rho\partial^\beta\partial_{x_i}u^i\|_{L^2}\\ &\geqslant\frac{\kappa}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla^{k+1}\rho|^2\,dx-C\delta\|\nabla^2\rho\|_{H^k}^2,\end{aligned}\tag{3.35}$$

where we have bounded the summation in the first inequality above by, for $|\alpha|=k$

$$\|\nabla^{k+1}\rho\|_{L^2}\|u\|_{L^\infty}+\|\nabla^k\rho\|_{L^2}\|\nabla u\|_{L^\infty}\leqslant C\delta\|\nabla^2\rho\|_{H^{k-1}},$$

while for $1\leqslant|\alpha|\leqslant k-1$

$$\|\partial^\alpha\partial_{x_i}\rho\|_{L^\infty}\|\partial^\beta u\|_{L^2}+\|\partial^\alpha\rho\|_{L^\infty}\|\partial^\beta\partial_{x_i}u\|_{L^2}\leqslant C\delta\|\nabla^2\rho\|_{H^k}.$$

Consequently, summing up (3.24), (3.27), (3.33)–(3.35), we arrive at

$$\frac12\frac{d}{dt}\int_{\mathbb{R}^3}\big(\rho|\nabla^k u|^2+h'(\rho)|\nabla^k\rho|^2+\kappa|\nabla^{k+1}\rho|^2\big)\,dx+\mu\|\nabla^{k+1}u\|_{L^2}^2\leqslant C\delta\big(\|\nabla^2\rho\|_{H^k}^2+\|\nabla^2u\|_{H^{k-1}}^2\big).\tag{3.36}$$

Next applying the operator $\rho\nabla^k$ again to (3.4) and then taking the L^2 inner product with $\nabla^k\partial_{x_i}\rho$, we have

$$\begin{aligned}&\big(\rho\nabla^k u_t^i,\nabla^k\partial_{x_i}\rho\big)+\big(\rho\nabla^k(u^j\partial_{x_j}u^i),\nabla^k\partial_{x_i}\rho\big)+\big(\rho\nabla^k\partial_{x_i}h(\rho),\nabla^k\partial_{x_i}\rho\big)\\ &\quad-\big(\rho\nabla^k[\rho^{-1}(\mu\Delta u^i+\nu\partial_{x_i}\operatorname{div}u)],\nabla^k\partial_{x_i}\rho\big)-\kappa\big(\rho\nabla^k\partial_{x_i}\Delta\rho,\nabla^k\partial_{x_i}\rho\big)\\ &\equiv J_{k1}+J_{k2}+J_{k3}+J_{k4}+J_{k5}=0.\end{aligned}\tag{3.37}$$

The first term J_{k1} that involves the time derivative of u can be estimated as

$$\begin{aligned}J_{k1}&=\frac{d}{dt}\big(\rho\nabla^k u^i,\nabla^k\partial_{x_i}\rho\big)-\big(\nabla^k u^i,\rho_t\nabla^k\partial_{x_i}\rho\big)+\big(\partial_{x_i}(\rho\nabla^k u^i),\nabla^k\rho_t\big)\\ &\geqslant\frac{d}{dt}\int_{\mathbb{R}^3}\rho\nabla^k u^i\nabla^k\partial_{x_i}\rho\,dx-C\delta\big(\|\nabla^{k+1}\rho\|_{L^2}^2-\|\nabla^k u\|_{L^2}^2\big)+A,\end{aligned}$$

where

$$\begin{aligned}|A| &= \left|\left(\partial_{x_i}\left(\rho\nabla^k u^i\right), \nabla^k\rho_t\right)\right| \\ &= \left|\left(\rho\nabla^k\partial_{x_i}u^i + \partial_{x_i}\rho\nabla^k u^i, \bar\rho\nabla^k \operatorname{div} u + \nabla^k\left[(\rho-\bar\rho)\operatorname{div} u + \nabla\rho\cdot u\right]\right)\right| \\ &\leqslant C\left\|\nabla^2 u\right\|_{H^{k-1}}\left(\left\|\nabla^{k+1}u\right\|_{L^2} + \left\|\nabla^k\left[(\rho-\bar\rho)\operatorname{div} u + \nabla\rho\cdot u\right]\right\|_{L^2}\right) \\ &\leqslant C\left(\left\|\nabla^2\rho\right\|_{H^{k-1}}^2 + \left\|\nabla^2 u\right\|_{H^{k-1}}^2\right).\end{aligned}$$

Hence

$$J_{k1} \geqslant \frac{d}{dt}\int_{\mathbb{R}^3}\rho\nabla^k u^i\nabla^k\partial_{x_i}\rho\,dx - C\left(\left\|\nabla^2\rho\right\|_{H^{k-1}}^2 + \left\|\nabla^2 u\right\|_{H^{k-1}}^2\right). \tag{3.38}$$

The second term J_{k2} is easy,

$$J_{k2} \geqslant -C\delta\left(\left\|\nabla^2\rho\right\|_{H^{k-1}}^2 + \left\|\nabla^2 u\right\|_{H^{k-1}}^2\right). \tag{3.39}$$

The other terms $J_{k3} \sim J_{k5}$ can be estimated as follows

$$\begin{aligned}J_{k3} &= \left(\rho h'(\rho)\nabla^k\partial_{x_i}\rho, \nabla^k\partial_{x_i}\rho\right) + \left(\rho\left[\nabla^k, h'(\rho)\right]\partial_{x_i}\rho, \nabla^k\partial_{x_i}\rho\right) \\ &\geqslant C\left\|\nabla^k\partial_{x_i}\rho\right\|_{L^2}^2 - C\sum_{\substack{|\alpha|\geqslant 1\\ |\alpha|+|\beta|=k}}\left\|\partial^\alpha h'(\rho)\right\|_{L^\infty}\left\|\partial^\beta\partial_{x_i}\rho\right\|_{L^2}\left\|\nabla^k\partial_{x_i}\rho\right\|_{L^2} \\ &\geqslant C\left\|\nabla^k\partial_{x_i}\rho\right\|_{L^2}^2 - C\delta\left\|\nabla^2\rho\right\|_{H^k}^2, \qquad (3.40) \\ J_{k4} &= \left(\nabla^{k-1}\left[\rho^{-1}\left(\mu\Delta u^i + \nu\partial_{x_i}\operatorname{div} u\right)\right], \nabla\left(\rho\nabla^k\partial_{x_i}\rho\right)\right) \\ &\geqslant -C\sum_{|\alpha|+|\beta|=k-1}\left\|\partial^\alpha\left(\rho^{-1}\right)\right\|_{L^\infty}\left\|\partial^\beta\nabla^2 u\right\|_{L^2}\left\|\nabla^2\rho\right\|_{H^{k-1}} \\ &\geqslant -C\left(\left\|\nabla^2 u\right\|_{H^{k-1}}^2 + \left\|\nabla^2\rho\right\|_{H^{k-1}}\right), \qquad (3.41) \\ J_{k5} &= \kappa\left(\nabla^{k+1}\partial_{x_i}\rho, \nabla\left(\rho\nabla^k\partial_{x_i}\rho\right)\right) \\ &\geqslant \frac{\kappa\bar\rho}{2}\left\|\nabla^{k+1}\partial_{x_i}\rho\right\|_{L^2}^2 - C\left\|\nabla^{k+1}\partial_{x_i}\rho\right\|_{L^2}\|\nabla\rho\|_{L^\infty}\left\|\nabla^k\partial_{x_i}\rho\right\|_{L^2} \\ &\geqslant \frac{\kappa\bar\rho}{4}\left\|\nabla^{k+1}\partial_{x_i}\rho\right\|_{L^2}^2 - C\delta\left\|\nabla^{k+1}\rho\right\|_{L^2}^2. \qquad (3.42)\end{aligned}$$

Summing up (3.37)–(3.42), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho\nabla^k u^i\nabla^k\partial_{x_i}\rho\,dx + C\left\|\nabla^{k+1}\rho\right\|_{H^1}^2 \leqslant C\left(\left\|\nabla^2\rho\right\|_{H^{k-1}}^2 + \left\|\nabla^2 u\right\|_{H^{k-1}}^2\right). \tag{3.43}$$

Consequently, multiplying (3.43) by a small constant β_k and then adding it with (3.36), summing up for $2 \leqslant k \leqslant s$, by using the interpolation inequality

$$\left\|\nabla^2\rho\right\|_{H^{k-1}}^2 + \left\|\nabla^2 u\right\|_{H^{k-2}} \leqslant \varepsilon\left\|\left(\nabla^{k+2}\rho, \nabla^{k+1}u\right)\right\|_{L^2} + C_\varepsilon\left\|\left(\nabla^2\rho, \nabla^2 u\right)\right\|_{L^2}, \tag{3.44}$$

and the smallness of δ, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\sum_{k=2}^{s}\left(\rho\left|\nabla^k u\right|^2 + h'(\rho)\left|\nabla^k\rho\right|^2 + \kappa\left|\nabla^{k+1}\rho\right|^2 + 2\beta_k\rho\nabla^k u^i\nabla^k\partial_{x_i}\rho\right)dx + C\left(\left\|\nabla^2 u\right\|_{H^{s-1}}^2 + \left\|\nabla^2\rho\right\|_{H^s}^2\right) \\ &\quad\leqslant C\left(\left\|\nabla^2\rho\right\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2\right).\end{aligned} \tag{3.45}$$

The lemma follows by defining $E_2(t)$ to be the C^{-1} times the integral in (3.45) provided that β_k are small. □

4. Proof of Theorem 1.2

We will prove Theorem 1.2 in this section. At first, we reformulate the nonlinear problem (1.1)–(1.3). Setting

$$\gamma = \sqrt{p'(\bar{\rho})}, \qquad \kappa' = \frac{\bar{\rho}}{\gamma}\kappa, \qquad \mu' = \frac{\mu}{\bar{\rho}}, \qquad \nu' = \frac{\nu}{\bar{\rho}}, \tag{4.1}$$

and introducing the new variables

$$\varrho = \rho - \bar{\rho}, \qquad v = \frac{\bar{\rho}}{\gamma}u, \tag{4.2}$$

then the system (1.1)–(1.3) is reformulated as

$$\varrho_t + \gamma \operatorname{div} v = S_1, \tag{4.3}$$

$$v_t - \mu'\Delta v - \nu'\nabla \operatorname{div} v + \gamma\nabla\varrho - \kappa'\nabla\Delta\varrho = S_2, \tag{4.4}$$

$$\varrho|_{t=0} = \varrho_0 = \rho_0 - \bar{\rho}, \qquad v|_{t=0} = v_0 = \frac{\bar{\rho}}{\gamma}u_0, \tag{4.5}$$

where

$$S_1 = -\frac{\gamma}{\bar{\varrho}} \operatorname{div}(\varrho v), \tag{4.6}$$

$$S_2 = -\frac{\varrho}{(\varrho+\bar{\rho})\bar{\rho}}\big(\mu'\Delta v + \nu'\nabla \operatorname{div} v\big) - \frac{\bar{\rho}}{\gamma}\left[\frac{p'(\varrho+\bar{\rho})}{\varrho+\bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}}\right]\nabla\varrho - \frac{\gamma}{\bar{\rho}}v\cdot\nabla v. \tag{4.7}$$

According to (2.7) and (4.1), the critical value of capillary coefficient is defined by

$$\kappa^* = \frac{(\mu+\nu)^2}{4\bar{\rho}^3}. \tag{4.8}$$

By Duhamel's principle, we represent the solutions to the system (4.3)–(4.5) as

$$U(t) = G(t) * U_0 + \int_0^t G(t-s) * F\big(U(s)\big)\,ds, \tag{4.9}$$

where $U = (\varrho, v)^t$, $U_0 = (\varrho_0, v_0)^t$, $F(U) = (S_1, S_2)^t$. Now we shall combine the linear decay estimates in Theorem 1.3 and the energy estimates derived in Section 3 to prove the nonlinear decay rates of solutions to the Navier–Stokes–Korteweg system.

First we notice from (4.6)–(4.7) that, roughly speaking,

$$\begin{aligned}
&S_1 \sim \nabla U\cdot U,\\
&\nabla S_1 \sim \nabla^2 U\cdot U + \nabla U\cdot\nabla U,\\
&\nabla^2 S_1 \sim \nabla^3 U\cdot U + \nabla^2 U\cdot\nabla U,\\
&\nabla^3 S_1 \sim \nabla^4 U\cdot U + \nabla^3 U\cdot\nabla U + \nabla^2 U\cdot\nabla^2 U,\\
&\nabla^4 S_1 \sim \nabla^5 U\cdot U + \nabla^4 U\cdot\nabla U + \nabla^3 U\cdot\nabla^2 U,\\
&S_2 \sim \nabla^2 U\cdot U + \nabla U\cdot U,\\
&\nabla S_2 \sim \nabla^3 U\cdot U + \nabla^2 U\cdot\nabla U + \nabla^2 U\cdot U + \nabla U\cdot\nabla U,\\
&\nabla^2 S_2 \sim \nabla^4 U\cdot U + \nabla^3 U\cdot\nabla U + \nabla^2 U\cdot\nabla^2 U + \nabla^3 U\cdot U + \nabla^2 U\cdot\nabla U,\\
&\nabla^3 S_2 \sim \nabla^5 U\cdot U + \nabla^4 U\cdot\nabla U + \nabla^3 U\cdot\nabla^2 U + \nabla^4 U\cdot U + \nabla^3 U\cdot\nabla U + \nabla^2 U\cdot\nabla^2 U.
\end{aligned} \tag{4.10}$$

We also need the following elementary inequality:

Lemma 4.1. *Let $r_1, r_2 > 0$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2} \leqslant C(r_1,r_2)(1+t)^{-\min\{r_1,r_2,r_1+r_2-1-\varepsilon\}}, \tag{4.11}$$

for an arbitrarily small $\varepsilon > 0$.

We first let $s \geqslant 3$ to derive (1.6)–(1.7). For this, we estimate the time-decay rate for the first-order derivatives of the solutions.

Lemma 4.2. *Under the assumption of Theorem* 1.1, *then we have*

$$\left\|\nabla U(t)\right\|_{L^2} \leqslant CK_0(1+t)^{-\frac{5}{4}} + C\delta\int_0^t (1+t-s)^{-\frac{5}{4}}\left\|\nabla U(s)\right\|_{H^{s-1}}\,ds, \tag{4.12}$$

where $K_0 = \|\varrho_0\|_{H^{s+1}\cap L^1} + \|v_0\|_{H^s\cap L^1}$.

Proof. Applying (1.12) with $k=0$ and (1.13) with $k=1$ in Theorem 1.3 to the integral formulation (4.9), we obtain

$$\begin{aligned}\left\|\nabla\varrho(t)\right\|_{L^2} &\leqslant C(1+t)^{-\frac{5}{4}}\left(\left\|(\varrho_0, v_0)\right\|_{L^1} + \left\|(\nabla\varrho_0, v_0)\right\|_{L^2}\right)\\ &\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}}\left(\left\|(S_1,S_2)(s)\right\|_{L^1} + \left\|(\nabla S_1, S_2)(s)\right\|_{L^2}\right)ds, \end{aligned}\tag{4.13}$$

$$\begin{aligned}\left\|\nabla v(t)\right\|_{L^2} &\leqslant C(1+t)^{-\frac{5}{4}}\left(\left\|(\varrho_0, v_0)\right\|_{L^1} + \left\|\left(\nabla^2\varrho_0, \nabla v_0\right)\right\|_{L^2}\right)\\ &\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}}\left(\left\|(S_1,S_2)(s)\right\|_{L^1} + \left\|\left(\nabla^2 S_1, \nabla S_2\right)(s)\right\|_{L^2}\right)ds. \end{aligned}\tag{4.14}$$

By (4.10) and (1.5), we easily deduce that

$$\left\|(S_1,S_2)(s)\right\|_{L^1} \leqslant C\left\|U(s)\right\|_{L^2}\left\|\nabla U(s)\right\|_{H^1} \leqslant C\delta\left\|\nabla U(s)\right\|_{H^1}, \tag{4.15}$$

$$\left\|\nabla S_1(s)\right\|_{H^1} + \left\|S_2(s)\right\|_{H^1} \leqslant C\left\|\nabla U(s)\right\|_{H^1}\left\|\nabla U(s)\right\|_{H^2} \leqslant C\delta\left\|\nabla U(s)\right\|_{H^2}. \tag{4.16}$$

Since $s \geqslant 3$, plugging (4.15)–(4.16) into (4.13)–(4.14), we obtain (4.12), and this completes the proof of Lemma 4.2. □

Now multiplying (3.22) by a small constant α_1 and then adding it with (3.5), adding $\|(\nabla\varrho, \nabla v)(t)\|_{L^2}^2$ to the both sides, we find that

$$\frac{d}{dt}M_1(t) + C\left\|\nabla\varrho(t)\right\|_{H^{s+1}}^2 + C\left\|\nabla v(t)\right\|_{H^s}^2 \leqslant C\left\|\nabla U(t)\right\|_{L^2}^2, \tag{4.17}$$

where $M_1(t) = E_1(t) + \alpha_1 E_2(t)$, by (3.6) and (3.23), is equivalent to $\|\nabla\varrho(t)\|_{H^s}^2 + \|\nabla v(t)\|_{H^{s-1}}^2$. Hence (4.17) immediately implies that for some $D_1 > 0$

$$\frac{d}{dt}M_1(t) + D_1 M_1(t) \leqslant C\left\|\nabla U(t)\right\|_{L^2}^2. \tag{4.18}$$

If we define

$$N_1(t) = \sup_{0\leqslant s\leqslant t}(1+s)^{\frac{5}{2}} M_1(s), \tag{4.19}$$

then

$$\left\|\nabla\varrho(s)\right\|_{H^s} + \left\|\nabla v(s)\right\|_{H^{s-1}} \leqslant C\sqrt{M_1(s)} \leqslant C(1+t)^{-\frac{5}{4}}\sqrt{N_1(t)}, \quad 0 \leqslant s \leqslant t, \tag{4.20}$$

and from (4.12), (4.20), by Lemma 4.1, we have

$$\begin{aligned}\left\|\nabla U(t)\right\|_{L^2} &\leqslant CK_0(1+t)^{-\frac{5}{4}} + C\delta\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{4}}\,ds\,\sqrt{N_1(t)}\\ &\leqslant C(1+t)^{-\frac{5}{4}}\left(K_0 + \delta\sqrt{N_1(t)}\right). \end{aligned}\tag{4.21}$$

Applying the Gronwall's lemma to (4.17), together with (4.21), we obtain

$$\begin{aligned} M_1(t) &\leqslant M_1(0)e^{-D_1 t} + C\int_0^t e^{-D_1(t-s)}\left\|\nabla U(s)\right\|_{L^2}^2\,ds \\ &\leqslant M_1(0)e^{-D_1 t} + C\int_0^t e^{-D_1(t-s)}(1+s)^{-\frac{5}{2}}\big(K_0^2+\delta^2 N_1(t)\big)\,ds \\ &\leqslant C(1+t)^{-\frac{5}{2}}\big(M_1(0)+K_0^2+\delta^2 N_1(t)\big). \end{aligned} \tag{4.22}$$

In view of $N_1(t)$, we have

$$N_1(t) \leqslant C\big(M_1(0)+K_0^2+\delta^2 N_1(t)\big), \tag{4.23}$$

which implies that if $\delta > 0$ is small enough, then

$$N_1(t) \leqslant C\big(M_1(0)+K_0^2\big) \leqslant CK_0^2. \tag{4.24}$$

This in turn gives

$$\left\|\nabla\varrho(t)\right\|_{H^s} + \left\|\nabla v(t)\right\|_{H^{s-1}} \leqslant CK_0(1+t)^{-\frac{5}{4}}. \tag{4.25}$$

This proves (1.6).

For (1.7), first by Sobolev's inequality and (1.6), we have

$$\left\|(\varrho,v)(t)\right\|_{L^6} \leqslant C\left\|(\nabla\varrho,\nabla v)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{5}{4}}. \tag{4.26}$$

On the other hand, applying (1.11) and (1.13) with $k=0$ in Theorem 1.3 to the integral formulation (4.9) again, by (4.10), (1.5) and (1.6), we obtain

$$\begin{aligned} \left\|(\varrho,v)(t)\right\|_{L^2} &\leqslant C(1+t)^{-\frac{3}{4}}\big(\left\|(\varrho_0,v_0)\right\|_{L^1} + \|\varrho_0\|_{H^1} + \|v_0\|_{L^2}\big) \\ &\quad + C\int_0^t (1+t-s)^{-\frac{3}{4}}\big(\left\|(S_1,S_2)(s)\right\|_{L^1} + \left\|S_1(s)\right\|_{H^1} + \left\|S_2(s)\right\|_{L^2}\big)\,ds \\ &\leqslant CK_0(1+t)^{-\frac{3}{4}} + C\int_0^t (1+t-s)^{-\frac{3}{4}}\delta\left\|(\nabla\varrho,\nabla u)(s)\right\|_{H^1}\,ds \\ &\leqslant CK_0(1+t)^{-\frac{3}{4}} + C_0\delta\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{5}{4}}\,ds \\ &\leqslant C_0(1+t)^{-\frac{3}{4}}. \end{aligned} \tag{4.27}$$

Hence, by the interpolation, it follows from (4.26)–(4.27) that for any $2 \leqslant p \leqslant 6$

$$\left\|(\varrho,v)(t)\right\|_{L^p} \leqslant \left\|(\varrho,v)(t)\right\|_{L^2}^{\theta}\left\|(\varrho,v)(t)\right\|_{L^6}^{1-\theta} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{4.28}$$

where $\theta = \frac{6-p}{2p}$. This proves (1.7).

We now turn to prove (1.8)–(1.10) by requiring that $s \geqslant 4$. We estimate the time-decay rate for the second-order derivatives of the solutions.

Lemma 4.3. *Under the assumption of Theorem* 1.1 *and letting* $s \geqslant 4$, *then we have*

$$\left\|\nabla^2 U(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{7}{4}} + C\delta\int_0^t (1+t-s)^{-\frac{7}{4}}\left\|\nabla^2 U(s)\right\|_{H^{s-2}}\,ds. \tag{4.29}$$

Proof. Applying (1.12) with $k = 1$ and (1.13) with $k = 2$ in Theorem 1.3 to (4.9), we obtain

$$\|\nabla^2\varrho(t)\|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}}\big(\|(\varrho_0, v_0)\|_{L^1} + \|(\nabla^2\varrho_0, \nabla v_0)\|_{L^2}\big) + C\int_0^t (1+t-s)^{-\frac{7}{4}}\big(\|(S_1, S_2)(s)\|_{L^1} + \|(\nabla^2 S_1, \nabla S_2)(s)\|_{L^2}\big)\,ds, \tag{4.30}$$

$$\|\nabla^2 v(t)\|_{L^2} \leqslant C(1+t)^{-\frac{7}{4}}\big(\|(\varrho_0, v_0)\|_{L^1} + \|(\nabla^3\varrho_0, \nabla^2 v_0)\|_{L^2}\big) + C\int_0^t (1+t-s)^{-\frac{7}{4}}\big(\|(S_1, S_2)(s)\|_{L^1} + \|(\nabla^3 S_1, \nabla^2 S_2)(s)\|_{L^2}\big)\,ds. \tag{4.31}$$

By (4.10), we have

$$\|(\nabla^3 S_1, \nabla^2 S_2)(s)\|_{L^2} \leqslant C\|\nabla U(s)\|_{H^2}\|\nabla^2 U(s)\|_{H^2} \leqslant C\delta\|\nabla^2 U(s)\|_{H^2}. \tag{4.32}$$

Hence by (4.15), (4.16), (4.32) and (1.6)–(1.7), from (4.30)–(4.31) we have

$$\begin{aligned}
\|\nabla^2 U(t)\|_{L^2} &\leqslant CK_0(1+t)^{-\frac{7}{4}} + C\int_0^t (1+t-s)^{-\frac{7}{4}}\|U(s)\|_{H^2}\|\nabla U(s)\|_{H^2}\,ds \\
&\quad + C\delta\int_0^t (1+t-s)^{-\frac{7}{4}}\|\nabla^2 U(s)\|_{H^2}\,ds \\
&\leqslant CK_0(1+t)^{-\frac{7}{4}} + C_0\int_0^t (1+t-s)^{-\frac{7}{4}}(1+s)^{-2}\,ds \\
&\quad + C\delta\int_0^t (1+t-s)^{-\frac{7}{4}}\|\nabla^2 U(s)\|_{H^2}\,ds \\
&\leqslant C_0(1+t)^{-\frac{7}{4}} + C\delta\int_0^t (1+t-s)^{-\frac{7}{4}}\|\nabla^2 U(s)\|_{H^2}\,ds.
\end{aligned} \tag{4.33}$$

Since $s \geqslant 4$, (4.33) yields (4.29), and the proof of Lemma 4.3 is completed. □

Now adding $\|\nabla^2\varrho(t)\|_{L^2}^2 + \|\nabla^2 v(t)\|_{L^2}^2$ to both sides of (3.22), we have

$$\frac{d}{dt}E_2(t) + \|\nabla^2\varrho(t)\|_{H^s}^2 + \|\nabla^2 v(t)\|_{H^{s-1}}^2 \leqslant C\|\nabla^2 U(t)\|_{L^2}^2. \tag{4.34}$$

By (3.23), $E_2(t)$ is equivalent to $\|\nabla^2\varrho(t)\|_{H^{s-1}}^2 + \|\nabla^2 v(t)\|_{H^{s-2}}^2$. Hence (4.34) implies that for some $D_2 > 0$

$$\frac{d}{dt}E_2(t) + D_2 E_2(t) \leqslant C\|\nabla^2 U(t)\|_{L^2}^2. \tag{4.35}$$

If we define

$$N_2(t) = \sup_{0\leqslant s\leqslant t}(1+s)^{\frac{7}{2}}E_2(s), \tag{4.36}$$

then

$$\|\nabla^2\varrho(s)\|_{H^{s-1}} + \|\nabla^2 v(s)\|_{H^{s-2}} \leqslant C\sqrt{E_2(s)} \leqslant C(1+t)^{-\frac{7}{4}}\sqrt{N_2(t)}, \quad 0 \leqslant s \leqslant t, \tag{4.37}$$

and from (4.29),

$$\begin{aligned}
\|\nabla^2 U(t)\|_{L^2} &\leqslant C_0(1+t)^{-\frac{7}{4}} + C\delta\int_0^t (1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{7}{4}}\,ds\sqrt{N_2(t)} \\
&\leqslant C(1+t)^{-\frac{7}{4}}\big(C_0 + \delta\sqrt{N_2(t)}\big).
\end{aligned} \tag{4.38}$$

An application of the Gronwall lemma on (4.35), together with (4.38), yields

$$\begin{aligned} E_2(t) &\leqslant E_2(0)e^{-D_2t} + C\int_0^t e^{-D_2(t-s)}\left\|\nabla^2 U(s)\right\|_{L^2}^2 ds \\ &\leqslant E_2(0)e^{-D_2t} + C\int_0^t e^{-D_2(t-s)}(1+s)^{-\frac{7}{2}}\big(C_0+\delta^2 N_2(t)\big)\, ds \\ &\leqslant C(1+t)^{-\frac{7}{2}}\big(C_0+\delta^2 N_2(t)\big). \end{aligned} \tag{4.39}$$

In view of $N_2(t)$, we have

$$N_2(t) \leqslant C\big(C_0+\delta^2 N_2(t)\big), \tag{4.40}$$

which implies that if $\delta > 0$ is small enough, then

$$N_2(t) \leqslant C_0. \tag{4.41}$$

This gives

$$\left\|\nabla^2\varrho(t)\right\|_{H^{s-1}} + \left\|\nabla^2 v(s)\right\|_{H^{s-2}} \leqslant C_0(1+t)^{-\frac{7}{4}}. \tag{4.42}$$

This proves (1.8).

For (1.9), by Sobolev's inequality, we obtain from (1.8) that

$$\left\|(\nabla\varrho,\nabla v)(t)\right\|_{L^6} \leqslant C\left\|\big(\nabla^2\varrho,\nabla^2 v\big)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{7}{4}}. \tag{4.43}$$

Hence, by the interpolation, it follows from (1.6) and (4.43) that for any $2 \leqslant p \leqslant 6$

$$\begin{aligned} \left\|(\nabla\varrho,\nabla v)(t)\right\|_{L^p} &\leqslant \left\|(\nabla\varrho,\nabla v)(t)\right\|_{L^2}^{\theta}\left\|(\nabla\varrho,\nabla v)(t)\right\|_{L^6}^{1-\theta} \\ &\leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}, \end{aligned} \tag{4.44}$$

where $\theta = \frac{6-p}{2p}$. This proves (1.9).

Finally, to prove (1.10), by the Sobolev interpolation inequality and (1.7)–(1.8), we have

$$\left\|(\varrho,v)(t)\right\|_{L^\infty} \leqslant C\left\|(\varrho,v)(t)\right\|_{L^2}^{\frac{1}{4}}\left\|\big(\nabla^2\varrho,\nabla^2 v\big)(t)\right\|_{L^2}^{\frac{3}{4}} \leqslant C_0(1+t)^{-\frac{3}{2}}. \tag{4.45}$$

Hence, by the interpolation, it follows from (4.45) and (1.7) that for any $2 \leqslant p \leqslant \infty$

$$\left\|(\varrho,v)(t)\right\|_{L^p} \leqslant \left\|(\varrho,v)(t)\right\|_{L^2}^{\frac{2}{p}}\left\|(\varrho,v)(t)\right\|_{L^\infty}^{1-\frac{2}{p}} \leqslant C_0(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}. \tag{4.46}$$

This proves (1.10) and the proof of Theorem 1.2 is completed.

5. Conclusion

In this paper we have proved the first result on the decay rates of solutions to the compressible Navier–Stokes–Korteweg system. It is important that the application of our method is of wide range: the strategy for proving the optimal L^2 decay rate of the second-order derivatives of solutions in Theorem 1.2 can be applied to the other equations of compressible fluids such as the compressible Navier–Stokes equations, the Navier–Stokes–Poisson equations, MHD, etc.

We shall generalize our results in this paper to the following cases in the future study: (i) to include a potential external force; (ii) to consider the general non-isothermal system in which an equation of energy conservation for the temperature θ is added; (iii) to consider the case when the capillary coefficient depends on the density ρ. On the other hand, we may modify the energy estimates combining with the linear estimates in Theorem 1.3 to prove the global existence and the optimal L^2 decay rate of strong solutions to the Cauchy problem (1.1)–(1.3) with initial data $\rho_0 - \bar\rho \in H^2$ and $u_0 \in H^1$, while the local existence was proved recently in [20].

At last, we point out that the key point in this paper is the absence of vacuum. When there is vacuum, even the local existence of strong solutions (or classical solutions) remains open due to the third-order term $\nabla\Delta\rho$ multiplying by ρ. Only the global weak solutions with vacuum can be found in [1,8]. We leave this to study in the future.

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