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RINGS WITH MONOMIAL RELATIONS HAVING LINEAR RESOLUTIONS

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A ring with monomial relations will in this note mean an algebra with a presentation $k[X_1, \ldots, X_n]/(M_1, \ldots, M_r)$, where k is a field and the M_i's are monominals in the X_j's. The aim of this note is to give a necessary and sufficient condition for a ring with monomial relations to have a linear resolution. The crucial points will be a reduction of the problem to the case of rings with squarefree monomial relations, and then a connection between the homology of the Koszul complex for such a ring and the simplicial homology for some associated simplicial complexes. This connection also gives some results of combinatorial nature for simplicial complexes. It also yields a precise condition for the ring to be Cohen–Macaulay.

Notation. In this note A will always denote the polynomial ring $k[X_1, \ldots, X_n]$.

Definition. A graded algebra $A/(f_1, \ldots, f_r)$ is said to have a *t-linear resolution* if $(\mathrm{Tor}_i^A((f_1, \ldots, f_r), k))_j = 0$ for $j \neq i+t$. (The grading of $A/(f_1, \ldots, f_r)$ induces a grading on $\mathrm{Tor}_*^A((f_1, \ldots, f_r), k) = \bigoplus_{j \geq 0}(\mathrm{Tor}_*^A((f_1, \ldots, f_r), k))_j$.) This is equivalent to the statement that $(f_1, \ldots, f_r)$ has a minimal graded A-resolution with maps of degree zero of the form

$$0 \to A^{b_k}[-t-k+1] \to \cdots \to A^{b_2}[-t-1] \to A^{b_1}[-t] \to (f_1, \ldots, f_r) \to 0$$

where $(A[n])_m = A_{n+m}$. The ring R is said to have a linear resolution if it has a t-linear resolution for some t.

Note. It is shown in [1, Theorem 7 and corollary] that an algebra with a linear resolution is a Golod algebra and that an algebra has a 2-linear resolution if and only if it is a Golod algebra and a Koszul algebra.

First we reduce our investigations to the case when the relations consist of squarefree monomials by means of the following two lemmas. A ring with squarefree monomial relations will henceforth be called a face ring.

Lemma 1 [e.g. 3, (e) sec. 4]. *If R is a graded algebra and $x_1, \dots, x_s$ is an R-sequence of elements of degree one, then R has a t-linear resolution if and only if $R/(x_1, \dots, x_s)$ has a t-linear resolution.*

Lemma 2 [2, Proposition]. *If R is a ring with monomial relations there exists a face ring R' and an R'-sequence $y_1, \dots, y_s$ of forms of degree one such that $R \simeq R'/(y_1, \dots, y_s)$.*

Since the construction of R' from R is completely algorithmic (cf. [2]), this justifies the claim that the problem of finding all rings with monomial relations having linear resolutions is reduced to the case of finding all face rings having linear resolutions.

A ring R with monomial relations is $\mathbb{N}^n$-graded, where n is the number of variables. This induces an $\mathbb{N}^n$-grading on $\operatorname{Tor}_*^A(R,k) = H_*(K_R)$ where K_R is the Koszul complex $R\langle T_1, \dots, T_n;\, dT_i = x_i\rangle$. We call the degree $(d_1, \dots, d_n)$ in $\mathbb{N}^n$ squarefree if $d_i \leq 1$ for $i = 1, 2, \dots, n$.

Lemma 3 [2, Lemma 2]. *If R is a face ring, then $H_*(K_R)$ is different from zero only in squarefree degrees.*

To a face ring $R = A/I$ we associate its Stanley–Reisner simplicial complex $\Delta(R)$. This is defined by

$$(i_1, i_2, \dots, i_k) \in \Delta(R) \quad \text{if and only if} \quad X_{i_1} \cdot X_{i_2} \cdot \cdots \cdot X_{i_k} \notin I.$$

Besides $\Delta(R)$ we need to consider all maximal subcomplexes of $\Delta(R)$, i.e. for each $I \subseteq \{1, 2, \dots, n\}$ we consider the largest subcomplex $\Delta_I(R)$ of $\Delta(R)$ which contains all vertices in I. Next we associate $H_*(K_R)$ to the homology of the $\Delta_I(R)$'s. The following lemma is a specification of [4, Theorem 5.1].

Lemma 4. *For $I \subseteq \{1, 2, \dots, n\}$ let $\delta(I) = (d_1, d_2, \dots, d_n)$ where $d_i = 1$ if $i \in I$ and $d_i = 0$ otherwise. let $K_{R(I)}$ be the part of K_R which is of degree $\delta(I)$. Then*

$$H_i(K_{R(I)}) \simeq \tilde{H}_{|I|-i-1}(\Delta_I),$$

where $|I|$ denotes the number of elements in I and $\tilde{H}$ denotes reduced homology.

Proof. We will define a map of k-spaces $f: K_{R(I)} \to \Delta_I^*$, Δ_I^* denoting the dual of Δ_I, such that $f((K_{R(I)})_i) = \Delta_I^{|I|-i-1}$. Let $I = D \cup E = \{d_1, \dots, d_r\} \cup \{e_1, \dots, e_s\}$ with $D \cap E = \emptyset$ and suppose $d_1 < d_2 < \cdots < d_r$ and $e_1 < e_2 < \cdots < e_s$. On the Koszul monomial $M = x_{d_1} \cdot \cdots \cdot x_{d_r} \cdot T_{e_1} \cdot \cdots \cdot T_{e_s}$ we define

$$f(M) = (-1)^{\alpha(D)} \{d_1, \dots, d_r\}^*$$

where

$$\alpha(D)=\binom{r+1}{2}+|\{i\in I;\, i<d_1\}|+\cdots+|\{i\in I;\, i<d_r\}|.$$

Then f is an isomorphism, and it is easy to check that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (\Delta_I)^*_{|I|-i-1} & \longrightarrow & (\Delta_I)^*_{|I|-i} & \longrightarrow & \cdots \\ & & \uparrow f & & \uparrow f & & \\ \cdots & \longrightarrow & (K_{R(I)})_i & \longrightarrow & (K_{R(I)})_{i-1} & \longrightarrow & \cdots \end{array}$$

commutes. Since $\tilde{H}^i(\Delta_I;k)=\tilde{H}_i(\Delta_I;k)$ the lemma follows.

For a graded algebra $R=\bigoplus_{i\ge 0}R_i$ we define its Hilbert series by $R(Z)=\sum_{i\ge 0}\dim_k R_i\cdot Z^i$. The following lemma will be needed later.

Lemma 5. *Let R be a face ring of embedding dimension n and let $\bar{R}=R/(x_1^2,x_2^2,\dots,x_n^2)$. Then $R(Z)=\bar{R}(Z/(1-Z))$.*

Proof. Easy verification.

For a simplicial complex Δ of dimension $d-1$ we define its **f**-vector by $\mathbf{f}(\Delta)=(f_0,\dots,f_d)$, where f_i is the number of faces of dimension $i-1$ in Δ. In particular $f_0=1$ (counting the empty set). The next lemma is well known, but since our proof is so short we give it.

Lemma 6. *Let R be a face ring with associated simplicial complex Δ. Then $R(Z)=f_0+f_1Z/(1-Z)+\cdots+f_dZ^d/(1-Z)^d$. In particular* $\dim\Delta=\mathrm{Kr}\dim R-1$.

Proof. It is clear that $\bar{R}(Z)=f_0+f_1Z+\cdots+f_dZ^d$. The formula for $R(Z)$ follows from Lemma 5. Since Kr dim R is the order of the pole at $Z=-1$ for $R(Z)$, we have the second statement.

The next lemma is a little more precise than [4, Corollary 5.3].

Lemma 7. *Let R be a face ring of Krull dimension d and embedding dimension n. Then R is a Cohen–Macaulay ring if and only if for $i=0,1,\dots,d-2$ we have $H_i(\Delta_I)=0$ for all I with $|I|=n-d+i+2$.*

Proof. The ring R is Cohen–Macaulay if and only if depth $R=d$ and depth $R=\max\{g;\, H_{n-g}(K_r)\neq 0\}$ which is less than or equal to d. The condition for $H_{n-d+1}(K_R)$ to be zero is that $H_{n-d+1}(K_{R(J)})=0$ for all J with $|J|>n-d+1$. Lemma 4 gives that this is equivalent to $\tilde{H}_{|J|-n+d-2}(\Delta_J)=0$ for all J with $|J|>n-d+1$, i.e. to $\tilde{H}_i(\Delta_I)=0$ for all i and I with $|I|=n-d+i+2$.

Hence a simplicial complex of dimension 0 always corresponds to a Cohen–Macaulay ring, a one-dimensional complex corresponds to a Cohen–Macaulay ring if and only if it is connected, a two-dimensional complex corresponds to a Cohen–Macaulay ring if and only if $H_1(\Delta)=0$ and $H_0(\Delta_I)=0$ for all I's with one point removed a.s.o.

We now state some combinatorial-topological consequences from the connection between face rings and simplicial complexes.

Proposition 8. *Let Δ be a simplicial complex of dimension $d-1$ with n vertices. Let k and m be fixed non-negative integers.*

(i) *If for all i and I with $|I|-i=k$ we have $\tilde{H}_i(\Delta_I)=0$, then necessarily $k\geq n-d+2$ and we have $\tilde{H}_i(\Delta_I)=0$ for all i and I with $|I|-i>k$.*

(ii) *If for $i=0,1,\dots,m$ and all I with $|I|-i=k$ we have $\tilde{H}_i(\Delta_I)=0$ for $i=0,1,\dots,m$ and all I with $|I|-i>k$.*

(iii) *If for all i and I with $|I|-i=n-d+2$ we have $\tilde{H}_I(\Delta_i)=0$, then all maximal faces of Δ are of the same dimension.*

Proof. Let R be the face ring associated to Δ. Then (i) follows from the fact that $\operatorname{depth} R>d$ and from the fact that if $H_c(K_R)=0$ then $H_i(K_R)=0$ for $i>c$. Since $H_*(K_R)=\operatorname{Tor}_*^A(R,k)=\operatorname{Hom}(\mathbb{F}_*\otimes k)$ where $\mathbb{F}_*$ is a minimal graded A-resolution of R, we see that if $(\operatorname{Tor}_r^A(R,k))_j=0$ for $j\leq s$, then $(\operatorname{Tor}_{r+1}^A(R,k))_j=0$ for $j\leq s+1$, cf. [3, (j) sec. 4]. This gives that if $H_r(K_{R(I)})=0$ for all I with $|I|\leq s$, then $H_{r+1}(K_{R(I)})=0$ for all I with $|I|\leq s+1$ which gives (ii). To prove (iii) we observe that if $R=A/J$ and

$$J=(x_{i_1(1)},\dots,x_{i_1(K_1)})\cap(x_{i_2(1)},\dots,x_{i_2(K_2)})\cap\cdots\cap(x_{i_m(1)},\dots,x_{i_m(K_m)}),$$

then the maximal faces of Δ are $\{1,2,\dots,n\}\setminus\{i_1(1),\dots,i_1(K_1)\},\dots,\{1,2,\dots,n\}\setminus\{i_m(1),\dots,i_m(K_m)\}$ and conversely. The condition in (iii) is equivalent to R being Cohen–Macaulay, hence all minimal primes have the same dimension which is equivalent to that all maximal faces of Δ are of the same dimension.

We exemplify Proposition 8 with some concrete examples.

Example 1. If Δ is a two-dimensional complex with 5 points with all subsets of three points connected, then $H_1(\Delta_I)\neq 0$ for some I with four elements or $H_2(\Delta)\neq 0$.

Example 2. If Δ is a three-dimensional complex with 6 points and with $\tilde{H}_0(\Delta_I)=0$ for all I with $|I|=4$, $\tilde{H}_1(\Delta_I)=0$ for all I with $|I|=5$ and with $\tilde{H}_2(\Delta)=0$, then all maximal faces of Δ have the same dimension and also $\tilde{H}_0(\Delta_I)=0$ for all I with $|I|>4$ and $\tilde{H}_1(\Delta)=0$.

We now state the main result.

Theorem 9. *Let $R=A/I$ be a face ring. Then R has a t-linear resolution if and only if for all $I \subseteq \{1,2,\dots,n\}$ we have $\tilde{H}_i(\Delta_I(R))=0$ for $i \neq t-2$.*

Proof. Follows directly from Lemma 4 and the interpretation of $H_i(K_R)$ as $\mathrm{Tor}_i^A(R,k)$.

Remark. The condition in Theorem 9 is not topologically invariant. The face ring associated to a (2-dimensional) triangle has a t-linear resolution for every t, but its barycentric subdivision does not have a t-linear resolution for any t. The face ring of a 1-dimensional triangle yields an example of a non-regular ring with a 3-linear resolution whose barycentric subdivision has no linear resolution.

Example 3. One might hope that there would be a simple condition on the set of monomials $\{M_1,\dots,M_r\}$ for the ring $A/(M_1,\dots,M_r)$ to have a linear resolution. The following example shows that this is not possible. Let R be the ring $k[X_1,X_2,X_3,X_4,X_5,X_6]/(X_1X_2X_3, X_1X_2X_4, X_1X_3X_5, X_1X_4X_6, X_1X_5X_6, X_2X_3X_6, X_2X_4X_5, X_2X_5X_6, X_3X_4X_5, X_3X_4X_6)$ studied in [5]. The ring R corresponds to a triangulation of the projective plane $P_2(k)$. Theorem 9 shows that R has a 3-linear resolution if $\mathrm{char}\, k=0$, but that R has no linear resolution if $\mathrm{char}\, k=2$.

Example 4. Let Δ be the $(k-1)$-dimensional skeleton of an $(n-1)$-dimensional simplex. Then the theorem shows that the corresponding face ring

$$R=A/(\{x_{i_1}x_{i_2}\cdot \cdots \cdot x_{i_{k+1}};\ \text{all } 1\le i_1<i_2<\cdots<i_k<i_{k+1}\le n\})$$

has a $(k+1)$-linear resolution.

We shall now study face rings of dimension at most two with linear resolutions in more detail. Face rings of dimension one corresponds to complexes of dimension zero. These rings, $A/(x_ix_j;\ 1\le i<j\le n)$, have 2-linear resolutions according to Example 4. A face ring of dimension d can not have relations of degree $>d+1$ in a minimal presentation, thus a face ring of dimension d can not have a t-linear resolution for $t>d+1$ unless it is regular. If R has dimension 2 and a 3-linear resolution, then all maximal subcomplexes will be connected according to Theorem 9, in particular all pairs of points will be connected. This means that Δ is the complete graph on its vertices, i.e. Δ is the 1-skeleton of a simplex. Thus R is as in Example 4. It only remains to classify face rings of Krull dimension 2 with 2-linear resolutions.

Theorem 10. *Let R be a face ring of Krull dimension two. Then R has a 2-linear resolution if and only if the associated simplicial complex Δ (which is a graph) is a forest. Moreover, if $n=$ emb. dim R, then $\dim_k \mathrm{Tor}_{n-1}^A(R,k)+1$ is the number of trees in the forest. In particular R is Cohen–Macaulay if and only if Δ is a tree. If R is Cohen–Macaulay, then type $R=n-2$ and is thus independent of the shape of the tree.*

Proof. Theorem 9 gives that R has a 2-linear resolution if and only if $H_i(\Delta_I)=0$ if $i\neq 0$ for all I, i.e. that $H_1(\Delta_I)=0$ for all I since Δ is one-dimensional. This gives that R has a 2-linear resolution if and only if Δ is a forest. Lemma 4 gives that $\dim_k \operatorname{Tor}_{n-1}^A(R,k)=\dim_k \tilde{H}_0(\Delta)$ which is one less than the number of components. The ring R is Cohen–Macaulay if and only if $\operatorname{Tor}_{n-1}^A(R,k)=0$, i.e. if and only if Δ is connected. Lemmas 5 and 6 give that

$$R(Z)=1+nZ/(1-Z)+(n-1)Z^2/(1-Z)^2=(1+(n-2)Z)/(1-Z)^2$$

if Δ is a tree. Now

$$(1-Z)^n R(Z)=1+\sum_{i=1}^{n-2}(-1)^i \dim_k \operatorname{Tor}_i^A(R,k)Z^{i+1}$$

if R has a 2-linear resolution, see e.g. [3, (j) sec. 4]. This gives that

$$\operatorname{type} R=\dim_k \operatorname{Tor}_{n-2}^A(R,k)=n+2.$$

From the proof above it follows, that if R is the face ring of a tree, then all Betti numbers $\dim_k \operatorname{Tor}_i^A(R,k)$ depend only on the number of vertices and not on the shape of the tree. This has the following combinatorial interpretation.

Proposition 11. *Let G be a tree with n vertices. Let C_I be the number of connected components of Δ_I. Then*

$$\sum_{|I|=k} C_I=\binom{n}{k}+(n-2)\binom{n-2}{k-1}-\binom{n-2}{k}$$

and is thus independent of the shape of the tree.

Proof. We use the same expression for $(1-Z)^n R(Z)$ as in the proof of Theorem 10. Then

$$\dim_k H_i(K_R)=\dim_k(H_i(K_R))_{i+1}=\sum_{|I|=i+1}\dim_k \tilde{H}_0(\Delta_I)$$
$$=\sum_{|I|=i+1}\dim_k H_0(\Delta_I)-\binom{n}{i+1}$$

gives the result.

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