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Periodic Solutions of Arbitrarily Long Periods in Hamiltonian Systems*

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INTRODUCTION

In 1912, Poincaré announced [*1*] his celebrated conjecture concerning the existence of fixed points of an area preserving transformation of a ring. After some effort he had succeeded in treating only a variety of special cases but because of the significance of this result—in particular, as it pertained to the restricted three-body problem—he released it for the consideration of other mathematicians. A proof of this so-called last geometric theorem of Poincaré was published [*2*] the following year by Birkhoff. In 1926, Birkhoff also published [*3*] certain extensions of this result and then, in 1927, gave [*4*] the following modification of Poincaré's theorem:

> Let T be a one-to-one continuous transformation of a simply connected region R of the plane that leaves a point 0 of R fixed. Assume further that T conserves area and does not rotate about 0 any point of a closed curve C enclosing 0 and that C intersects only once each radial line from 0. Then there exist at least two invariant points of T on the curve C.

Application of this result to dynamical systems is limited to the case of two degrees-of-freedom (see [*4*]) and it was not until 1931 that Birkhoff was able to extend [*5*] this theorem to higher dimensions. The application of this latter result, however, demanded a detailed study of conservative transformations. Taking advantage of earlier work [*6*] on surface transformations, Birkhoff and Lewis presented [*7*] the first application in 1933; they established the existence of infinitely many periodic solutions of a conservative dynamical system in the neighborhood of a given periodic solution of general stable type. Then, in 1934, Lewis [*8*] established the existence

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of an infinite number of periodic solutions of a conservative Hamiltonian system in the neighborhood of an equilibrium point of general stable type.

It is the aim of this paper to present a modest generalization of the theorems of Birkhoff and Lewis and at the same time to establish their results using simpler, though similar, techniques. This paper is also written in the hope that these interesting results might find a wider scope of application. Moser has used [9] the two-dimensional version of Birkhoff's fixed-point theorem to establish periodic motions for the restricted three-body problem and the results of the present paper may be used to establish the existence of periodic motions, for example, in various n-dimensional spring–mass systems [10].

We will be dealing with a conservative n-dimensional Hamiltonian $H(p, q)$ with a stable equilibrium point at the origin, $p = q = 0$. We will further assume that H is analytic in a neighborhood of $p = q = 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental real frequencies of the (oscillatory) linear variational equations associated with H and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Birkhoff has shown [4] that, in general, for any positive integer μ, the Hamiltonian may be reduced by a canonical transformation to the normal form

$$H = H_0(\tau) + \tilde{H}(p, q), \qquad 2\tau_i = p_i^2 + q_i^2, \tag{1}$$

where

$$H_0 = \sum_{i=1}^{n} \lambda_i \tau_i + \sum_{1 \leqslant i \leqslant j \leqslant n} \lambda_{ij} \tau_i \tau_j + \cdots$$

is a polynomial in $\tau_1, \tau_2, \ldots, \tau_n$ of, at most, degree μ and $\tilde{H}$ represents a convergent power series in powers of p_i, q_i which begins with terms of at least degree $2\mu + 2$. The coefficients of H_0 are invariants of the Hamiltonian under canonical transformations. This reduction requires that $\langle \lambda, k \rangle = \sum_{i=1}^{n} \lambda_i k_i \neq 0$ for every integral vector k with

$$| k | = | k_1 | + \cdots + | k_n | \leqslant 2\mu + 1.$$

We shall employ the following norms throughout this paper:

(i) If $x = (x_1, x_2, \ldots, x_n)$, then

$$\| x \| = \sup_{1 \leqslant i \leqslant n} | x_i |.$$

(ii) If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\| A \| = \sup_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | a_{ij} |.$$

All vectors should be interpreted as column vectors and (i) and (ii) necessarily imply that $\| Ax \| \leqslant \| A \| \cdot \| x \|$. Finally, we will use the symbol η to denote a positive scalar quantity and the symbol $\bar{\eta}$ to denote an n-vector each of whose components is the scalar η. Lower-case c's with or without subscripts will denote absolute positive constants.

Representation of Solutions

Assuming that the Hamiltonian $H(p, q)$ has been reduced to normal form through terms of order μ (μ an integer), we may introduce polar canonical coordinates by the formulas

$$p_i = (2\tau_i)^{1/2} \cos \phi_i , \qquad q_i = (2\tau_i)^{1/2} \sin \phi_i .$$

Then the Hamiltonian (1) becomes

$$H = H_0(\tau) + \tilde{H}(\tau, \phi), \tag{2}$$

where

$$H_0 = \langle \lambda, \tau \rangle + \sum_{1 \leqslant i \leqslant j \leqslant n} \lambda_{ij} \tau_i \tau_j + \cdots, \left(\langle \lambda, \tau \rangle = \sum_{i=1}^{n} \lambda_i \tau_i \right)$$

is a polynomial in $\tau_1 ,..., \tau_n$ with real coefficients, and $\tilde{H}$ is periodic of period 2π in the ϕ_i and analytic with respect to τ, ϕ in the domain

$$D : \| \tau - \bar{\eta}_0 \| < \eta_0 < 1, \| \phi \| < \infty.$$

Also in D we have $| \tilde{H} | \leqslant c_0 \| \tau \|^{\mu+1}$.

In vector form, the canonical equations of motion associated with the Hamiltonian become

$$\begin{aligned} \frac{d\tau}{dt} &= -\frac{\partial H}{\partial \phi} = -\frac{\partial \tilde{H}}{\partial \phi}, \\ \frac{d\phi}{dt} &= \frac{\partial H}{\partial \tau} = P(\tau) + \frac{\partial \tilde{H}}{\partial \tau}, \end{aligned} \tag{3}$$

where

$$P(\tau) = \frac{\partial H_0}{\partial \tau} = \lambda + \Lambda \tau + \cdots,$$

and where $\Lambda = (2^{\delta_{ij}} \lambda_{ij})$ with $\lambda_{ij} = \lambda_{ji}$. The solution of Eq. (3) with initial values (τ_0 , ϕ_0) at $t = 0$ may be represented in the form

$$\begin{aligned} \tau &= \tau_0 + R(\tau_0 , \phi_0 , t), \\ \phi &= \phi_0 + tP(\tau_0) + \Phi(\tau_0 , \phi_0 , t), \end{aligned} \tag{4}$$

where R and Φ are periodic of period 2π in the ϕ_{i0} and are analytic in τ_0 , ϕ_0 , t as long as the trajectories remain in D.

Preliminary Lemmas

In this section we will state and prove several preliminary lemmas before proceeding with the proof of a fundamental lemma from which the main theorems follow.

Lemma 1. *For $0 < \eta < \eta_0$ and τ in the domain $\| \tau - \bar{\eta} \| \leqslant \frac{3}{4}\eta$, we have*

(i) $$\left\| \frac{\partial \tilde{H}}{\partial \phi} \right\| \leqslant c_2 \eta^{\mu+1},$$

(ii) $$\left\| \frac{\partial \tilde{H}}{\partial \tau} \right\| \leqslant c_3 \eta^{\mu}.$$

Proof. Since

$$\left\| \frac{\partial \tilde{H}}{\partial \phi} \right\| \leqslant c_1 \| \tau \|^{\mu+1} \quad \text{in} \quad D,$$

we have

$$\left\| \frac{\partial \tilde{H}}{\partial \phi} \right\| \leqslant c_1 2^{\mu+1} \eta^{\mu+1} = c_2 \eta^{\mu+1}$$

for τ in the domain $\| \tau - \bar{\eta} \| < \eta$ and, hence, for $\| \tau - \bar{\eta} \| \leqslant 3\eta/4$. In order to prove (ii), we observe that $| \tilde{H} | \leqslant c_0 2^{\mu+1} \eta^{\mu+1}$ for τ in the domain $\| \tau - \bar{\eta} \| < \eta$. Thus if $\sigma = \tau - \bar{\eta}$, we have $| \tilde{H} | \leqslant c_0 2^{\mu+1} \eta^{\mu+1}$ for $\| \sigma \| < \eta$. Applying Cauchy's inequality to $\partial \tilde{H} / \partial \sigma_i$ in the "disk" $\| \sigma \| \leqslant \frac{3}{4}\eta$ yields

$$\left| \frac{\partial \tilde{H}}{\partial \sigma_i} \right| \leqslant \frac{c_0 2^{\mu+1} \eta^{\mu+1}}{\frac{1}{4}\eta} = c_0 2^{\mu+3} \eta^{\mu}$$

for $i = 1, 2, \ldots, n$. Therefore,

$$\left\| \frac{\partial \tilde{H}}{\partial \tau} \right\| \leqslant c_3 \eta^{\mu}$$

for $\| \tau - \bar{\eta} \| \leqslant \frac{3}{4}\eta$.

Lemma 2. *For $0 < \eta < \eta_0$ and τ_0 in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$, the inequality $\| \tau - \bar{\eta} \| < 3\eta/4$ is maintained along the trajectory (4) for $0 \leqslant t \leqslant \gamma \eta^{-\mu}$ (γ a fixed positive constant independent of η).*

Proof. Let η satisfy $0 < \eta < \eta_0$. Then for τ_0 satisfying $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$, there exists a number $b > 0$ such that for $0 \leqslant t \leqslant b$, the radial vector τ belongs to the domain $\| \tau - \bar{\eta} \| \leqslant \frac{3}{4}\eta$. Furthermore, for τ in this domain we have, from Lemma 1, $\| \partial \tilde{H} / \partial \phi \| \leqslant c_2 \eta^{\mu+1}$; so from Eq. (3), it follows that

$$\| \tau - \tau_0 \| \leqslant \int_0^t \left\| \frac{\partial \tilde{H}}{\partial \phi} (\tau(s), \phi(s)) \right\| ds \leqslant c_2 \eta^{\mu+1} t \leqslant \tfrac{1}{8}\eta, \tag{5}$$

provided $0 \leqslant t \leqslant b$ and $0 \leqslant t \leqslant \gamma\eta^{-\mu}$ wehre $\gamma = 1/5c_2$. This in turn implies that the inequality

$$\| \tau - \bar{\eta} \| \leqslant \| \tau - \tau_0 \| + \| \tau_0 - \bar{\eta} \| < \tfrac{3}{4}\eta$$

is satisfied for all such t and, hence, there does not exist a smallest t in the interval $0 \leqslant t \leqslant \gamma\eta^{-\mu}$ for which the inequality is violated.

The next lemma follows immediately from inequality (5) for $\mu \geqslant 2$.

LEMMA 3. *Let $\mu \geqslant 2$. Then for $0 < \eta < \eta_0$ and τ_0 in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$, along the trajectory* (4) *we have*

$$\| \tau - \tau_0 \| \leqslant c_4 \eta^{\mu-1},$$

for $0 \leqslant t \leqslant \gamma\eta^{-2}$.

LEMMA 4. *For $\mu \geqslant 2$, $0 < \eta < \eta_0$ and τ_0 in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$. we have*

$$\| \Phi(\tau_0, \phi_0, t) \| \leqslant c_8 \eta^{\mu-3},$$

for $0 \leqslant t \leqslant \gamma\eta^{-2}$.

Proof. It follows from Eq. (3) that

$$\phi - \phi_0 = \int_0^t P(\tau(s))\, ds + \int_0^t \frac{\partial \tilde{H}}{\partial \tau} (\tau(s), \phi(s))\, ds. \tag{6}$$

Now, from the mean value theorem, we have

$$P(\tau) = P(\tau_0) + (\partial P/\partial \tau)^* (\tau - \tau_0),$$

where $(\partial P/\partial \tau)^*$ depends upon t, but for $0 < \eta < \eta_0$, $0 \leqslant t \leqslant \gamma\eta^{-2}$, $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$, and $\| \tau - \bar{\eta} \| < \frac{3}{4}\eta$, we have $\| (\partial P/\partial \tau)^* \| \leqslant c_5$. Thus Eq. (6) may be written

$$\phi - \phi_0 = tP(\tau_0) + \int_0^t \left(\frac{\partial P}{\partial \tau} \right)^* (\tau(s) - \tau_0)\, ds + \int_0^t \frac{\partial \tilde{H}}{\partial \tau} (\tau(s), \phi(s))\, ds,$$

and from Lemma 3, it follows that

$$\left\| \int_0^t \left(\frac{\partial P}{\partial \tau} \right)^* (\tau(s) - \tau_0)\, ds \right\| \leqslant c_5 \int_0^t \| \tau(s) - \tau_0 \|\, ds \leqslant c_5 c_4 \eta^{\mu-1} t \leqslant c_6 \eta^{\mu-3}$$

for $0 \leqslant t \leqslant \gamma\eta^{-2}$. Further, we have from Lemma 1,

$$\left\| \int_0^t \frac{\partial \tilde{H}}{\partial \tau} (\tau(s), \phi(s))\, ds \right\| \leqslant c_3 \eta^{\mu} t$$

so long as $\| \tau - \bar{\eta} \| < \frac{3}{4}\eta$. By Lemma 2, this latter inequality is maintained for $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{2}\eta$ and $0 \leqslant t \leqslant \gamma\eta^{-\mu}$. Hence,

$$\left\| \int_0^t \frac{\partial \tilde{H}}{\partial \tau} (\tau(s), \phi(s))\, ds \right\| \leqslant c_7 \eta^{\mu-2},$$

provided $0 \leqslant t \leqslant \gamma\eta^{-2}$. Combining these results with the representation used in Eq. (4) yields the following bound on Φ:

$$\| \Phi \| \leqslant c_6 \eta^{\mu-3} + c_7 \eta^{\mu-2} \leqslant c_8 \eta^{\mu-3}.$$

LEMMA 5. *For $\mu \geqslant 2$, $0 < \eta < \eta_0$ and τ_0 in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{3}\eta$, along the trajectory* (4) *we have*

$$\| \partial\Phi / \partial\tau_0 \| \leqslant c_9 \eta^{\mu-4},$$

for $0 \leqslant t \leqslant \gamma\eta^{-2}$.

Proof. Let $\sigma_0 = \tau_0 - \bar{\eta}$. Then by Lemma 4 for $\| \sigma_0 \| \leqslant \frac{1}{2}\eta$ and $0 \leqslant t \leqslant \gamma\eta^{-2}$, we have $\| \Phi \| \leqslant c_8 \eta^{\mu-3}$. Applying Cauchy's inequality to $\partial\Phi_i / \partial\sigma_{j0}$ in the "disk" $\| \sigma_0 \| \leqslant \frac{1}{3}\eta$, we have

$$\left| \frac{\partial \Phi_i}{\partial \sigma_{j0}} \right| \leqslant \frac{c_8 \eta^{\mu-3}}{\eta/6} = 6 c_8 \eta^{\mu-4},$$

for $i, j = 1, 2, \ldots, n$. Therefore, for $\| \tau_0 - \eta \| \leqslant \frac{1}{3}\eta$ and $0 \leqslant t \leqslant \gamma\eta^{-2}$, we have

$$\| \partial\Phi / \partial\tau_0 \| \leqslant 6 n c_8 \eta^{\mu-4} = c_9 \eta^{\mu-4}.$$

Now finally we will consider the following simple arithmetic result.

LEMMA 6. *For $0 < \eta < \gamma^2$, it is possible to choose a positive number t^* and an integral vector $k = (k_1, \ldots, k_n)$ such that the following inequalities hold:*

(i) $\eta^{-3/2} \leqslant t^* \leqslant \gamma\eta^{-2}$;

(ii) $\|(2\pi/t^*)\, k - (\lambda + v)\| \leqslant 2\pi\eta^{3/2}$,

where

$$v = \Lambda\bar{\eta}. \tag{7}$$

Proof. Consider $\eta > 0$ fixed. Then v is fixed. Now choose t^* such that $\eta^{-3/2} \leqslant t^* \leqslant \gamma\eta^{-2}$ and consider the n quantities

$$\left| \frac{2k_i\pi}{t^*} - (\lambda_i + v_i) \right| = \frac{2\pi}{t^*} \left| k_i - \frac{(\lambda_i + v_i)}{2\pi} t^* \right|.$$

It is always possible to choose integers $k_1, \dots, k_n$ such that

$$\left| k_i - \frac{(\lambda_i + v_i)}{2\pi} t^* \right| \leqslant 1,$$

and for such integers we have

$$\left| \frac{2k_i\pi}{t^*} - (\lambda_i + v_i) \right| \leqslant \frac{2\pi}{t^*} \leqslant 2\pi\eta^{3/2}, \qquad i = 1, 2, \dots, n.$$

Fundamental Lemma

Let $\mu > \frac{5}{2}$, $0 < \eta < \eta_1 = \min(\eta_0, \gamma^2)$ and consider the domain $\tilde{D} : \| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{3}\eta$, $\| \phi_0 \| < \infty$, $\eta^{-3/2} \leqslant t \leqslant \gamma\eta^{-2}$. Then selecting $t^* (\eta^{-3/2} \leqslant t^* \leqslant \gamma\eta^{-2})$ and integers $k_1, \dots, k_n$ in accordance with Lemma 6, we consider the equation

$$\phi - \phi_0 = t^* P(\tau_0) + \Phi(\tau_0, \phi_0, t^*) = 2\pi k, \tag{8}$$

obtained from the second equation in (4) with $t = t^*$. We wish to solve for τ_0 in terms of ϕ_0.

Now, $P(\tau_0)$ may be expressed in the form

$$P(\tau_0) = \lambda + \Lambda\tau_0 + P_1(\tau_0),$$

where P_1 is a matrix polynomial in $\tau_{10}, \dots, \tau_{n0}$ lacking constant and linear terms. Hence, using Lemma 6, Eq. (8) may be rewritten as

$$\begin{aligned} \Lambda\tau_0 &= (2\pi/t^*)\, k - \lambda - P_1(\tau_0) - (1/t^*)\, \Phi(\tau_0, \phi_0, t^*) \\ &= v + \epsilon_1 - P_1(\tau_0) - (1/t^*)\, \Phi(\tau_0, \phi_0, t^*), \end{aligned} \tag{9}$$

where $\| \epsilon_1(\eta) \| \leqslant 2\pi\eta^{3/2}$. We now assume that the determinant $| \Lambda | \neq 0$. Then, using (7), Eq. (9) may be written in the form

$$\begin{aligned} \tau_0 &= \Lambda^{-1}(v + \epsilon_1) - \Lambda^{-1}P_1(\tau_0) - (1/t^*)\, \Lambda^{-1}\Phi(\tau_0, \phi_0, t^*) \\ &= \bar{\eta} + \epsilon_2(\eta) + \tilde{P}(\tau_0) + (1/t^*)\, \tilde{\Phi}(\tau_0, \phi_0, t^*), \end{aligned} \tag{10}$$

where $\epsilon_2 = \Lambda^{-1}\epsilon_1$, $\tilde{P} = -\Lambda^{-1}P_1$ and $\tilde{\Phi} = -\Lambda^{-1}\Phi$. Further, in the domain $\tilde{D}$,

$$\| \tilde{P} \| \leqslant c_{10} \| \tau_0 \|^2 \leqslant (\tfrac{4}{3})^2\, c_{10}\eta^2 = c_{11}\eta^2,$$

and by Lemma 4,

$$\| \tilde{\Phi}/t^* \| \leqslant c_{12}\eta^{\mu-3}/t^* \leqslant c_{12}\eta^{\mu-3}\eta^{3/2} = c_{12}\eta^{\mu-3/2}.$$

Also, we have $\| \epsilon_2 \| \leqslant c_{13}\eta^{3/2}$.

We will now show that if η is sufficiently small the right-hand member of Eq. (10) defines a contraction mapping for τ_0 in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{3}\eta$. A fixed point of this mapping will then be the desired solution of Eq. (8). The unique solution τ_0^* so obtained may be regarded as a single-valued function $\tau_0^* = \theta(\phi_0)$ of ϕ_0 and will be analytic and periodic of period 2π in the ϕ_{i0} since $\tilde{\Phi}$ is analytic in ϕ_0, τ_0 and periodic in the ϕ_{i0}.

Let $G(\tau_0)$ denote the right-hand member of Eq. (10) where we consider η and ϕ_0 as fixed. Now

$$\frac{\partial G}{\partial \tau_0} = \frac{\partial \tilde{P}}{\partial \tau_0} + \frac{1}{t^*} \frac{\partial \tilde{\Phi}}{\partial \tau_0}$$

and since $\tilde{P}$ is at least of degree two in the τ_{i0}, we have $\| \partial \tilde{P} / \partial \tau_0 \| \leqslant c_{14}\eta$ in $\tilde{D}$. Also, by Lemma 5,

$$\left\| \frac{1}{t^*} \frac{\partial \tilde{\Phi}}{\partial \tau_0} \right\| \leqslant \frac{1}{t^*} \| A^{-1} \| \left\| \frac{\partial \Phi}{\partial \tau_0} \right\| \leqslant \eta^{3/2} c_9 \eta^{\mu-4} \| A^{-1} \| = c_{15}\eta^{\mu-5/2}$$

in $\tilde{D}$. Hence, $\| \partial G / \partial \tau_0 \| \leqslant c_{16}\eta^{\kappa_1}$, where $\kappa_1 = \min(1, \mu - \frac{5}{2}) > 0$. Now, employing the mean value theorem, for $\tau_0^{(1)}$ and $\tau_0^{(2)}$ in the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{3}\eta$, we obtain

$$G(\tau_0^{(2)}) - G(\tau_0^{(1)}) = (\partial G / \partial \tau_0)^* (\tau_0^{(2)} - \tau_0^{(1)}),$$

and therefore,

$$\| G(\tau_0^{(2)}) - G(\tau_0^{(1)}) \| \leqslant c_{16}\eta^{\kappa_1} \| \tau_0^{(2)} - \tau_0^{(1)} \| \leqslant \alpha \| \tau_0^{(2)} - \tau_0^{(1)} \|,$$

where $\alpha < 1$ for $0 < \eta < 1/c_{16}^{1/\kappa_1}$ If we now consider a sequence $\tau_0^{(k)}$ of successive approximations of the fixed point with $\bar{\eta}$ as the initial approximation, then using Eq. (10) we have

$$\| \tau_0^{(k)} - \bar{\eta} \| \leqslant \| \tau_0^{(1)} - \bar{\eta} \| \sum_{j=1}^{n} \alpha^{j-1} \leqslant \frac{\| \tau_0^{(1)} - \bar{\eta} \|}{1 - \alpha} \leqslant c_{17}\eta^{\kappa_2},$$

where $\kappa_2 = \min(\frac{3}{2}, \mu - \frac{3}{2}) > 1$. Thus, if $\eta > 0$ is chosen sufficiently small, the last quantity above will be less than $\frac{1}{3}\eta$ and all approximations $\tau_0^{(k)}$ and, hence, the (unique) fixed point τ_0^* belong to the domain $\| \tau_0 - \bar{\eta} \| \leqslant \frac{1}{3}\eta$. In fact, we note from the above inequality that

$$\| \tau_0^* - \bar{\eta} \| \leqslant c_{17}\eta^{\kappa_2}.$$

We have established, therefore, the following lemma.

FUNDAMENTAL LEMMA. *Let the integer $\mu \geqslant 3$ and assume that the determinant $|\Lambda| \neq 0$. If $\eta > 0$ is chosen sufficiently small and t^* is fixed in the interval $\eta^{-3/2} \leqslant t^* \leqslant \gamma\eta^{-2}$, then there exists a manifold M of initial values in phase space, defined by equations of the type*

$$\tau_{i0}^* = \theta_i(\phi_{10}, \dots, \phi_{n0}) \qquad (i = 1, 2, \dots, n)$$

for which the corresponding $\phi_i(t^*)$ of (4) differ from the ϕ_{i0} by integral multiples of 2π. The θ_i are analytic single-valued functions which are periodic of period 2π in the ϕ_{i0} and satisfy $\| \theta - \bar{\eta} \| \leqslant c\eta^{\kappa}$, with $c > 0$ and $\kappa > 1$.

PRINCIPAL RESULTS

Regarding t as a fixed parameter, the transformation (4) that maps the point (τ_0, ϕ_0) to $(\tau(t), \phi(t))$ in $2n$ space is a canonical transformation. Therefore

$$dJ = \sum_{i=1}^{n} (\tau_i \, d\phi_i - \tau_{i0} \, d\phi_{i0}) \tag{11}$$

is an exact differential [*11*] when expressed in terms of τ_0, ϕ_0 and the parameter t. However, along a manifold M of the type described in the Fundamental Lemma, ϕ_i differs from ϕ_{i0} by the constant $2k_i\pi$ when $t = t^*$, and hence $d\phi_i = d\phi_{i0}$, $i = 1, 2, \dots, n$. Thus, for $t = t^*$, Eq. (11) becomes

$$dJ = \sum_{i=1}^{n} (\tau_i - \tau_{i0}) \, d\phi_{i0} \tag{12}$$

along M. Integrating Eq. (12) over the manifold we get a single-valued analytic function J which is periodic of period 2π in the ϕ_{i0}. It is clear that J possesses critical points where $dJ = 0$, which implies from Eq. (12) that $\tau_i = \tau_{i0}$, $i = 1, 2, \dots, n$. Hence, these critical points correspond to invariant points of the transformation (4) which in turn correspond to periodic solutions of Eq. (3) of period t^*. Thus, we have established the following theorem.

THEOREM 1. *Let the integer $\mu \geqslant 3$ and assume that $\langle \lambda, k \rangle \neq 0$ for every integral vector with $|k| = |k_1| + \cdots + |k_n| \leqslant 2\mu + 1$. Then a conservative holomorphic Hamiltonian system of the form (1) with the determinant $|\Lambda| = |2^{\delta_{ij}}\lambda_{ij}| \neq 0$ possesses infinitely many distinct periodic solutions of arbitrarily long (minimum) periods in the neighborhood of $\tau = 0$.*

Theorem 1 solves the same problem as that treated by Lewis [*8*]. However, in the mechanics of his proof, it is necessary to assume that $\mu \geqslant 8n + 4$. Thus, it is possible to significantly reduce the commensurability restrictions on the linear frequencies and establish the same result.

At this point we will consider an analytic Hamiltonian $H(p, q, t)$ which is periodic of period 2π in t and has an equilibrium point at $p = q = 0$. An analogous reduction to normal form may be made provided certain restrictions are made on the commensurability of the linear frequencies $\lambda_1, \lambda_2, ..., \lambda_n$ and 1. In particular, we assume that $\langle \lambda, k \rangle \neq l$ for $| k | \leqslant 2\mu + 1, l = 0, \pm 1, \pm 2, ...$. The Hamiltonian may then be reduced [4] to the form

$$H(\tau, \phi, t) = H_0(\tau) + \tilde{H}(\tau, \phi, t), \tag{13}$$

where H_0 has the same form as in Eq. (2) and $\tilde{H}$ is analytic in τ, ϕ, t and periodic of period 2π in the ϕ_i and in t. Furthermore, $| \tilde{H} | \leqslant c \| \tau \|^{\mu+1}$, where c is a positive constant. This reduction may be accomplished for any integer $\mu > 1$.

The various lemmas stated in this paper are readily extended to the nonautonomous Hamiltonian (13) and, further, the differential form (11) remains exact. Then with $\eta > 0$ sufficiently small we may choose t^* ($\eta^{-3/2} \leqslant t^* \leqslant \gamma\eta^{-2}$) such that $t^* = 2\pi m$, where m is a positive integer and consider the canonical transformation T which maps points along the trajectories from $t = 0$ to $t = 2\pi m$. Any invariant points of T correspond to periodic solutions of the Hamiltonian system of period $2m\pi$. In particular, if one chooses m as a prime and if at least one of the λ_i is not an integer (eliminating 2π as a possible period), then for η sufficiently small, $2\pi m$ is necessarily the least period of these periodic solutions. The proof of the next theorem, therefore, is analogous to that of Theorem 1.

THEOREM 2. *Let $\mu \geqslant 3$ and assume that $\langle \lambda, k \rangle \neq l$* for $| k \leqslant 2\mu + 1$, $l = 0, \pm 1, \pm 2, ...$. *Further assume that at least one of the λ_i is not an integer. Then a holomorphic Hamiltonian system of the form* (13) *with determinant $| \Lambda | \neq 0$ possesses infinitely many distinct* (*subharmonic*) *periodic solutions of arbitrarily long* (*minimum*) *periods in the neighborhood of $\tau = 0$.*

Now suppose we have a conservative dynamical system with $n + 1$ degrees of freedom and a given periodic motion of general stable type. It is well-known that the study of the motion near a periodic solution may be reduced [4] to the consideration of a Hamiltonian system with n degrees of freedom in which H is analytic in p, q, and t and periodic of period 2π in the latter. In this setting the periodic motion appears as a "generalized" equilibrium point, $p = q = 0$. Periodic motions of this latter system then correspond to periodic solutions of the original $(2n + 2)$th-order system near the given periodic motion. By a periodic motion of general stable type we mean one for which the frequencies $\lambda_1, \lambda_2, ..., \lambda_n$ and 1 are incommensurable and the determinant $| A | \neq 0$. Hence we may state the following theorem which was first proven by Birkhoff and Lewis [6] and follows immediately from Theorem 2.

THEOREM 3 (Birkhoff and Lewis). *In a conservative dynamical system there exist infinitely many distinct periodic solutions of arbitrarily long (minimum) periods in the vicinity of every periodic solution of general stable type.*

For a periodic motion of general stable type, the associated Hamiltonian can be expressed in normal form through terms of arbitrarily high order, but for purposes of the proof of Theorem 3 it is only necessary to effect this through terms of order μ. In the proof given by Birkhoff and Lewis it is assumed that $\mu \geqslant 8n + 4$ while in the above proof it is assumed that $\mu > \frac{5}{2}$ (independent of n). Thus, in Theorem 3, we may reduce the incommensurability requirements on the frequencies $\lambda_1 ,..., \lambda_n$ (see the statement of Theorem 2) and this reduction is quite substantial in the higher-dimensional cases.

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