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Perfect codes in direct products of cycles—a complete characterization ☆

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Abstract

Let $G = \times_{i=1}^{n} C_{\ell_i}$ be a direct product of cycles. It is known that for any $r \geqslant 1$, and any $n \geqslant 2$, each connected component of G contains a so-called canonical r-perfect code provided that each ℓ_i is a multiple of $r^n + (r+1)^n$. Here we prove that up to a reasonably defined equivalence, these are the only perfect codes that exist.

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Keywords: Error-correcting codes; Direct product of graphs; Perfect codes; Cycles

1. Introduction

The study of codes in graphs presents a wide generalization of the problem of the existence of (classical) error-correcting codes. In general, for a given graph G we search for a subset X of its vertices such that the r-balls centered at vertices from X form a partition of the vertex set of G. The study of codes in graphs was initiated by Biggs [1]. For more recent results on (perfect) codes see [2,4,7,8,10,12–15]. The paper [11] gives an almost complete description of perfect codes in the direct product of cycles. It is shown that if each ℓ_i is a multiple of $r^n + (r+1)^n$, then $\times_{i=1}^{n} C_{\ell_i}$ has an r-perfect code, and a partial converse is established. The present article

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proves the converse in complete generality. We show that a direct product $\times_{i=1}^{n} C_{\ell_i}$ of cycles has an r-perfect code if and only if every ℓ_i is a multiple of $r^n + (r+1)^n$. Moreover, we prove that up to a reasonably defined equivalence the so-called canonical r-perfect codes are the only r-perfect codes that are possible.

2. Preliminaries

For a graph G the *distance* $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v, is defined as the number of edges on a shortest path from u to v. A set $C \subseteq V(G)$ is an *r-code* in G if $d(u, v) \geqslant 2r + 1$ for any two distinct vertices $u, v \in C$. The code C is called an *r-perfect code* if for any $u \in V(G)$ there is exactly one $v \in C$ such that $d(u, v) \leqslant r$.

Let G be a graph, u a vertex of G, and $r \geqslant 0$. The *r-ball* $B(u, r)$ with center u and diameter r is defined as $B(u, r) = \{x \mid d_G(u, x) \leqslant r\}$. In this terminology $C \subset V(G)$ is an r-perfect code if and only if the r-balls $B(u, r)$, where $u \in C$, form a partition of $V(G)$.

The *direct product* $G \times H$ of graphs G and H is a graph defined on the Cartesian product of the vertex sets of the factors. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1 v_1 \in E(G)$ and $u_2 v_2 \in E(H)$. The direct product of graphs is commutative and associative in a natural way. Hence, for graphs $G_1, \ldots, G_n$ we may write

$$G = G_1 \times \cdots \times G_n = \bigtimes_{i=1}^{n} G_i$$

without parentheses, and the vertices of G can be represented as vectors $\mathbf{v} = (v_1, \ldots, v_n)$, where $v_i \in V(G_i)$, $1 \leqslant i \leqslant n$.

The direct product $G = \times_{i=1}^{n} C_{\ell_i}$ is connected if and only if at most one of the ℓ_i's is even. Otherwise, G consists of 2^{k-1} isomorphic connected components, where k is the number of ℓ_i's that are even, see [9]. Since a direct product of cycles is vertex transitive we can, without loss of generality, assume that $\mathbf{v} = \mathbf{0}$ is a fixed arbitrary vertex of the product considered. For more properties of graph products see [5].

Each $n \times n$ matrix can be associated with a directed graph with weighted directed edges. (Entry a_{ij} corresponds to the weight of the directed edge from i to j and the diagonal entries correspond to the weights of loops.) In particular, the skew-symmetric matrices with nondiagonal entries $|a_{ij}| = 1$ are associated with tournaments. A tournament is a directed graph with an arc between every pair of distinct vertices. Clearly, an arc from i to j can be understood as an arc with weight 1 from u to v and arc from v to u with weight -1. Hence any skew-symmetric matrix with nondiagonal entries $|a_{ij}| = 1$ gives rise to a tournament, and for any tournament T and its adjacency matrix M, the matrix $M - M^T$ is skew-symmetric.

We will use combinatorial definition of determinant [3]

$$\det(A) = \sum_{F \in \mathcal{F}} (-1)^{n+p(F)} C(F)$$

where the sum runs over all factors F of the graph associated to the matrix A, and $C(F)$ is the cost of the factor F, i.e. the product of all weights of directed edges of F, and $p(F)$ is the number of cycles of F. A factor is a spanning subgraph in which all vertices have indegree and outdegree one, i.e., is a union of directed cycles that cover all vertices.

2.1. Construction of the standard codes in products of several cycles

It has been proved in [11] that (each connected component of) the direct product of n cycles contains an r-perfect code, $r \geqslant 1$, if the length of each cycle is a multiple of $r^n + (r+1)^n$. Here we recall the construction from [11] and call the codes constructed this way *canonical codes*.

For a given $r \geqslant 1$ we define $s = 2r + 1$ and use this notation throughout the paper. For description of the canonical perfect codes, we use the following vectors $\mathbf{b}^i \in \mathbb{Z}^n$:

$$\begin{aligned}
\mathbf{b}^1 &= (s,\ 1,\ 1,\ \ldots,\ 1),\\
\mathbf{b}^2 &= (-1,\ s,\ 1,\ \ldots,\ 1),\\
\mathbf{b}^3 &= (-1,\ -1,\ s,\ \ldots,\ 1),\\
&\vdots \qquad\qquad\qquad \ddots\\
\mathbf{b}^n &= (-1,\ -1,\ -1,\ \ldots,\ s).
\end{aligned} \tag{1}$$

Let us call the vectors $\mathbf{b}^1, \ldots, \mathbf{b}^n$ *canonical local vectors*. The corresponding vertices $\mathbf{0}+\mathbf{b}^1, \ldots, \mathbf{0}+\mathbf{b}^n$ are called *canonical local vertices for* $\mathbf{0}$ in [11].

The canonical perfect code is the set

$$Q_n = \left\{ \mathbf{0} + \sum_{i=1}^{n} \alpha_i \mathbf{b}^i \;\middle|\; \alpha_i \in \mathbb{Z} \right\},$$

where the arithmetic in coordinate i is done modulo ℓ_i. Note that due to vertex transitivity, any vertex can be used instead of $\mathbf{0}$. (The resulting code would be Q_n modulo a translation.)

Recall that an r-ball in the product has $r^n + (r+1)^n$ vertices. From the definition of a perfect code, it follows directly that the number of vertices of a perfect code P is

$$|P| = \frac{\ell_1 \cdot \ell_2 \cdots \ell_n}{r^n + (r+1)^n}. \tag{2}$$

It is known [11] that a perfect code P is totally determined by its local structure, i.e. by the set of local vectors

$$\begin{aligned}
\mathbf{b}^1(\mathbf{P}) &= (s,\ a_{12},\ a_{13},\ \ldots,\ a_{1n}),\\
\mathbf{b}^2(\mathbf{P}) &= (a_{21},\ s,\ a_{23},\ \ldots,\ a_{2n}),\\
\mathbf{b}^3(\mathbf{P}) &= (a_{31},\ a_{32},\ s,\ \ldots,\ a_{3n}),\\
&\vdots \qquad\qquad\qquad\quad \ddots\\
\mathbf{b}^n(\mathbf{P}) &= (a_{n1},\ a_{n2},\ a_{n3},\ \ldots,\ s).
\end{aligned} \tag{3}$$

where $|a_{ij}| = 1$ and $a_{ij} = -a_{ji}$.

We can now recall some of the results from [6,11]:

Theorem 1. *(See [11].) Let $r \geqslant 1$, $n \geqslant 2$, and $G = \times_{i=1}^{n} C_{\ell_i}$, where each ℓ_i is a multiple of $r^n + (r+1)^n$. Then each connected component of G contains an r-perfect code.*

Theorem 2. *(See [11].) Let P be an r-perfect code of a connected component of $\times_{i=1}^{n} C_{\ell_i}$, where $r \geqslant 2$, $n \geqslant 2$, and $\ell_i \geqslant 2r+2$. Suppose that P contains $\mathbf{0}$ and the canonical local vertices for $\mathbf{0}$. Then every ℓ_i is a multiple of $r^n + (r+1)^n$.*

Theorem 3. *(See [6,11].) Let $r \geqslant 2$ and $2 \leqslant n \leqslant 4$. Then (a connected component of) $\times_{i=1}^{n} C_{\ell_i}$, $\ell_i \geqslant 2r+2$, contains an r-perfect code if and only if every ℓ_i is a multiple of $r^n + (r+1)^n$.*

Our aim here is to generalize the last result to arbitrary n.

The matrix in which the rows are the coordinates of the local vectors of the code P will be called the *matrix of the code* and denoted by $B(P)$. Note that from the properties of the local structure of the code P we know that the off-diagonal elements form a matrix which is skew-symmetric, i.e. $B^T = -B$, where $B = B(P) - sI$.

$B(Q_n)$ is thus the matrix

$$\begin{bmatrix} s & 1 & 1 & & 1 & & & 1 \\ -1 & s & 1 & & 1 & & & 1 \\ -1 & -1 & s & & 1 & & & 1 \\ \vdots & & & \ddots & & & & \vdots \\ -1 & & & -1 & s & 1 & & 1 \\ \vdots & & & & & \ddots & & \vdots \\ -1 & & & & -1 & & s & 1 \\ -1 & & & & -1 & & -1 & s \end{bmatrix}.$$

The determinant of $B(Q_n)$ is not difficult to compute, for example using the combinatorial definition.

Lemma 4. $\det(B(Q_n)) = \frac{1}{2}((s+1)^n + (s-1)^n) = s^n + \binom{n}{2}s^{n-2} + \binom{n}{4}s^{n-4} + \cdots.$

2.2. Embeddings into $\mathbb{R}^n$

This section develops a formula for $\det(B(P))$. We use the fact that a direct product of cycles can be embedded in a torus, as follows.

The product of n infinite paths has 2^{n-1} components. This product can be embedded in $\mathbb{R}^n$ by letting its vertices be the vectors with integer coordinates, and its edges be the line segments between vertices $(x_1, x_2, \ldots, x_n)$ and $(x_1 \pm 1, x_2 \pm 1, \ldots, x_n \pm 1)$. (Note that this is not technically an embedding, because some edges cross, but it is an embedding of each component.) Now define an equivalence relation $\sim$ on $\mathbb{R}^n$ by declaring $\mathbf{x} \sim \bar{\mathbf{x}}$ if the ith components are congruent modulo ℓ_i. The quotient space $\mathbb{R}^n/\sim$ is an n-dimensional torus, and the direct product of paths under this quotient operation is the product $\times_{i=1}^{n} C_{\ell_i}$ locally embedded in the torus.

For each $\mathbf{x}$ in a perfect code P, there is a parallelepiped $\{\mathbf{x} + \sum \alpha_i \mathbf{b}^i(\mathbf{P}) \mid 0 \leqslant \alpha_i < 1\}$ on the torus. Locally, these $|P|$ parallelepipeds tile the torus, but they cover it 2^{n-1} times. Since the torus has volume $\ell_1 \cdot \ell_2 \cdots \ell_n$, the volume V of each parallelepiped satisfies $|P| \cdot V = \frac{\ell_1 \ell_2 \cdots \ell_n}{r^n + (r+1)^n} V = 2^{n-1} \ell_1 \ell_2 \cdots \ell_n$. Hence

$$\begin{aligned} V &= 2^{n-1}\bigl(r^n + (r+1)^n\bigr) = \frac{1}{2}\bigl(2^n r^n + 2^n (r+1)^n\bigr) \\ &= \frac{\ell_1 \cdot \ell_2 \cdots \ell_n}{2}\bigl((2r)^n + (2r+2)^n\bigr) = \frac{1}{2}\bigl((s-1)^n + (s+1)^n\bigr). \end{aligned} \tag{4}$$

As the determinant of $B(P)$ corresponds to the volume of n-dimensional parallelepiped we conclude

Lemma 5. *For any perfect code P,* $\det(B(P)) = \frac{1}{2}((s-1)^n + (s+1)^n)$.

2.3. Codes, tournaments, determinants

Each perfect code in a direct product of cycles that is embedded into $\mathbb{R}^n/\sim$ can be associated with a skew-symmetric matrix and consequently with a tournament. As a perfect code P is uniquely determined by its local structure, it has a unique matrix $B(P)$ and in turn, properties of $B(P)$ assure that $B(P) - sI$ gives rise to a tournament.

On the other hand, given a tournament T on n vertices, we can obtain a potential local structure as follows. First assign labels $1, \ldots, n$ to vertices of the tournament. The elements of the matrix $A(T)$ are then defined by the rule: $a_{ij} = +1$ if the edge between i and j is directed $i \to j$ and $a_{ij} = -1$ if the edge between i and j is directed $j \to i$. Clearly, the matrix obtained is skew-symmetric and the nondiagonal elements have absolute value one. Matrix $A(T) + sI$ is a potential local structure.

Obvious questions are: how the arbitrary labeling of the vertices of tournament affects the result of the construction, what tournaments can be used to construct a local structure that corresponds to a perfect code, and are there different tournaments that give rise to the same perfect code.

Before we answer these questions we will recall some facts on tournaments, and determinants of the skew-symmetric matrices corresponding to tournaments. We will also define the notions of equivalent codes and equivalent tournaments.

2.4. Determinants of the tournaments

The matrices corresponding to tournaments are skew-symmetric, i.e. $a_{ij} = -a_{ji}$. In particular, they have zero diagonals. In addition, $|a_{ij}| = 1$. In the sequel we will use the following well-known properties of determinants of such skew-symmetric matrices (see [16, pp. 65–77]). Let A be a skew-symmetric matrix with n rows and columns. Then

- $\det(A) = 0$ for odd n;
- $\det(A) = \mathrm{Pf}(A)^2$ for $n = 2k$, where $\mathrm{Pf}(A)$ is the Pfaffian.

The *Pfaffian* is defined as

$$\mathrm{Pf}(A) = \sum \mathrm{sgn}\bigl\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\bigr\} a_{i_1,j_1} \cdot a_{i_2,j_2} \cdots a_{i_k,j_k},$$

where the sum extends over all possible partitions $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\}$ of the set $\{1, 2, \ldots, 2k\}$ and $\mathrm{sgn}\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\}$ is the signature of the corresponding permutation. $\mathrm{Pf}(A)$ is a polynomial of the matrix elements, in particular, it has $\frac{(2k)!}{2^k k!}$ terms of the form $\pm a_{i_1,j_1} \cdot a_{i_2,j_2} \cdots a_{i_k,j_k}$. In our case, all nondiagonal matrix elements have absolute value one, and the number of terms $\frac{(2k)!}{2^k k!} = (2k-1) \cdot (2k-3) \cdots 3 \cdot 1$ is odd. As a sum of odd number of terms of the same absolute value cannot be zero, its square must be positive. Therefore

Lemma 6. *Let A be the adjacency matrix of a tournament on n vertices. Then* $\det(A) = 0$ *for odd n and* $\det(A) \geqslant 1$ *for even n.*

3. Equivalent codes and equivalent tournaments

The perfect code in a graph can correspond to different tournaments depending on different embeddings because we can orient the embedding of each cycle in two ways yielding different coordinates for the local vectors.

Clearly, two codes P_1 and P_2 are equivalent if one can be obtained from another by reversing the orientation of one coordinate. Hence from the matrix $B(P_1)$ one obtains $B(P_2)$ by choosing appropriate index, say i, and reversing the signs of all elements of ith row and ith column. Note that this does not change the value of determinant, i.e. $\det(B(P_1)) = \det(B(P_2))$. On tournaments, this operation means that all directed edges that meet the vertex i change the direction.

The second operation that obviously preserves the equivalence, is a permutation of the labels of vertices of the tournament, which corresponds to a permutation of rows (and columns) of the matrix, and to a reordering of the coordinate axes in the embedding of the code.

Definition 7. The relation $\simeq$ on the set of all tournaments on n vertices is the transitive and reflexive closure of the relation R that is defined by the rule: two tournaments T_1 and T_2 are in relation R, if T_2 is isomorphic to a tournament which is obtained from T_1 by reversing the direction of all directed edges that meet one vertex.

The relation $\simeq$ is clearly symmetric, hence $\simeq$ is an equivalence relation, and we say tournaments T_1 and T_2 are *equivalent* when $T_1 \simeq T_2$.

Recall that two codes embedded in a torus are equivalent if one can be obtained from another by reversing (some) axes and by permuting the coordinates. From the reasoning above it is clear that

Lemma 8. *Equivalent tournaments correspond to equivalent codes.*

Furthermore,

Lemma 9. *The matrices that correspond to equivalent codes have equal determinants.*

We will later need some facts about tournaments on four vertices. There are four nonisomorphic tournaments on four vertices that can be identified with their indegree sequences: $T(0,1,2,3)$, $T(0,2,2,2)$, $T(1,1,1,3)$, $T(1,1,2,2)$ (see Fig. 1). The first tournament, with degree sequence $(0,1,2,3)$, is called the *transitive tournament* because it has no directed cycle.

Applying the transformations above it is easy to see

Lemma 10. $T(0,1,2,3) \simeq T(1,1,2,2)$, $T(0,2,2,2) \simeq T(1,1,1,3)$.

It may be of interest to note that the two equivalence classes are also characterized by the corresponding determinants. More precisely,

Lemma 11. $\det(A(T(0,1,2,3))) = \det(A(T(1,1,2,2))) = 1$, *and* $\det(A(T(0,2,2,2))) = \det(A(T(1,1,1,3))) = 9$.

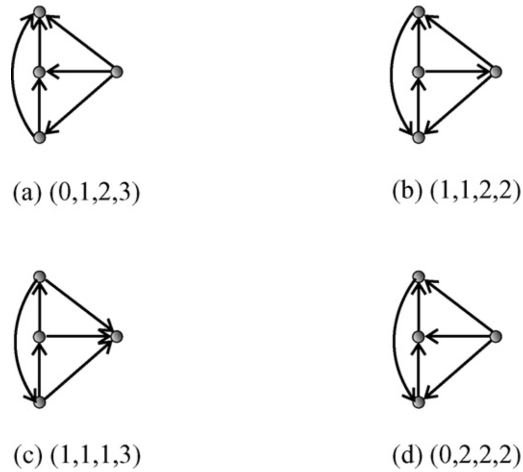

Fig. 1. The four possible tournaments on four vertices and their indegree sequences.

4. The main theorem

Lemma 12. *Any r-perfect code in a product of n cycles is equivalent to the canonical code.*

Proof. Suppose P is an r-perfect code in a product of n cycles, and let $s = 2r + 1$. From Lemmas 5 and 6, $\det(B(Q_n)) = \det(B(P)) = s^n + \binom{n}{2}s^{n-2} + \binom{n}{4}s^{n-4} + \binom{n}{6}s^{n-6} + \cdots$. Let T be the tournament associated to the skew-symmetric matrix $B(P) - sI$, and label the vertices of T as $1, 2, 3, \ldots, n$, where vertex i corresponds to row i of the matrix. By Lemma 9, the proof will be complete if it can be shown that T is equivalent to the transitive tournament. To accomplish this, we first show that any subtournament of T on 4 vertices is equivalent to the transitive tournament on 4 vertices. We begin by examining the s^{n-4} term of $\det(B(P))$. From the combinatorial definition of the determinant, this term is the sum of the factors consisting of exactly $n - 4$ loops. There are $\binom{n}{4}$ ways to choose a subset X of $n - 4$ vertices for the loops. Consider such an X. In extending loops based at X to a factor, we must append either a 4-cycle on the remaining 4 vertices, or a pair of 2-cycles. There are 6 ways to choose a 4-cycle and 3 ways to choose a pair of 2-cycles. The sum of the costs of these 9 factors is easily seen to be $s^{n-4}\det(B_X)$, where B_X is the matrix for the subtournament of T on vertices $\{1, 2, \ldots, n\} - X$. Now, $s^{n-4}\det(B_X) \geqslant s^{n-4}$, by Lemma 12. But as the degree s^{n-4} term of $\det(B(P))$ is simultaneously $\binom{n}{4}s^{n-4}$ and the sum of $\binom{n}{4}$ terms of the form $s^{n-4}\det(B_X)$, we infer that $\det(B_X) = 1$ for any choice of X. Thus, by Lemmas 11 and 12, it follows that any subtournament of T on 4 vertices is equivalent to the transitive tournament on 4 vertices.

We will now prove by induction that P is equivalent to the standard code by proving that the corresponding tournament is equivalent to the transitive tournament. Choose any four vertices and consider the corresponding subtournament. As its determinant must be 1, it is equivalent to the transitive tournament. Now assume a subtournament on k vertices is transitive, the first k vertices are labeled according to the linear order defined by the transitive tournament, and consider a new vertex. We claim that the new tournament is equivalent to the transitive tournament on $k + 1$ vertices.

Observe the directions of directed edges from the new vertex. If there would exist three vertices that are connected to the new vertex as depicted on Fig. 2(c) or (d), then the subtournament

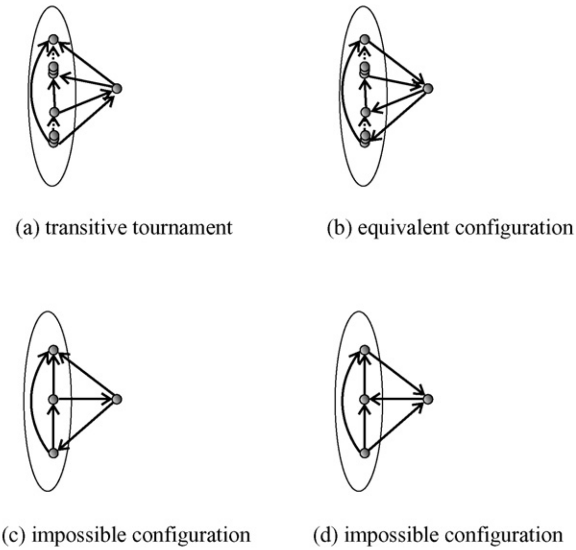

Fig. 2. From the proof of the main theorem.

would have degree sequence $(1, 1, 1, 3)$ or $(0, 2, 2, 2)$ and would have determinant equal to 9 (> 1), which is in contradiction to the assumption that all subtournaments on four vertices have determinant 1. Hence the situation depicted on Fig. 2(c) and (d) is not possible. Therefore, either the labels of all vertices with edges directed towards the new vertex are smaller than the labels of vertices with edges directed from the new vertex or vice versa. But then the tournament is either a transitive tournament (see Fig. 2(a)) or a tournament depicted on Fig. 2(b).

In the second case, we can reverse the directions of the directed edges meeting the new vertex to obtain an equivalent tournament, which is clearly a transitive tournament. It is obvious where in the linear order we have to put the new vertex. The resulting tournament is transitive, hence the subtournament on $k + 1$ vertices is equivalent to the transitive tournament. □

Hence any perfect code is equivalent to the canonical perfect code, therefore from Theorem 2 we have

Theorem 13. *Let $r \geqslant 1$ and $n \geqslant 2$. Then $\times_{i=1}^{n} C_{\ell_i}$, $\ell_i \geqslant 2r + 2$, contains an r-perfect code if and only if every ℓ_i is a multiple of $r^n + (r + 1)^n$.*

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