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Product Integrals II: Contour Integrals

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This paper continues the joint work of the authors begun in the article "On Strong Product Integration" [*J. Functional Analysis*, submitted]. We consider product integrals along contours; the point of view and development is analogous to the usual complex variable theory of ordinary contour integrals. Our main results are Theorem 2.3 (homotopy invariance of product integrals, an analog of Cauchy's integral theorem) and Theorem 3.4 (an analog of Cauchy's integral formula or the residue theorem).

INTRODUCTION

In a previous article [1], the product integral, $\prod_a^b e^{A(s)ds}$ of an integrable function $A(s)$ from $a \leqslant s \leqslant b$ to $\mathscr{B}(\mathscr{X})$, the Banach space of bounded linear operators on a complex Banach space $\mathscr{X}$, was defined and studied. Given $u_0 \in \mathscr{X}$ and setting $u(t) = \prod_a^x e^{A(s)ds} \cdot u_0$, $u(t)$ satisfies the integral equation

$$u(t) = u_0 + \int_a^t A(s)\, u(s)\, ds, \qquad a \leqslant t \leqslant b. \tag{0.1}$$

If $A(s)$ is continuous, then (0.1) is equivalent to

$$\begin{aligned} du/dt &= A(t)\, u(t), \qquad a \leqslant t \leqslant b, \\ u(a) &= u_0 . \end{aligned} \tag{0.2}$$

The utility of the product integral consists primarily in its applications to Eqs. (0.1) and (0.2).

In the present paper we study the notion of product integral along a contour, from a point of view substantially parallel to the ordinary complex variable theory of contour integration. Some of the results of the present article, at least in the case that $\mathscr{X}$ is finite dimensional, have been known for some time in the guise of results concerning systems of differential equations; however, the case of $\mathscr{X}$ infinite dimensional does not seem to have been discussed in the literature, and we feel that the systematic employment of the product

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integral notion may cause some of these results to appear in a new light. Contour product integrals have been investigated by Volterra [3]; however, the cited work apparently contains some inaccuracies.

1. The Definition of Contour Product Integrals

1.1. Definitions. A *contour* is a continuous, piecewise continuously differentiable mapping, $\gamma(t)$, from an interval $a \leqslant t \leqslant b$ of the real numbers to the complex plane $\mathbb{C}$. (Explicitly, this means that $\gamma(t)$ is continuous on $[a, b]$, and there is a partition $a < s_1 < s_2 < \cdots < s_n < b$ of $[a, b]$ such that $\gamma(t)$ is continuously differentiable on each of the subintervals $[a, s_1]$, $[s_1, s_2], \ldots, [s_n, b]$, the endpoint derivatives being right or left derivatives.) If Γ is the image of $\gamma(t)$ we denote the contour by $(\Gamma, \gamma(t))$ or sometimes just by Γ if the particular mapping $\gamma(t)$ has been identified or is unimportant. The points $\gamma(a)$ and $\gamma(b)$ are, respectively, the *initial* and *terminal* points of the contour. If $(\Gamma, \gamma(t))$ is a contour, we denote by Γ^{-1} the contour defined by $\gamma(a + b - t)$, $a \leqslant t \leqslant b$. If $\gamma(a) = \gamma(b)$ and $(\Gamma, \gamma(t))$ has no other self-intersections (so that Γ is a Jordan curve, i.e., a homeomorphic image of a circle) we say Γ has *positive orientation* if for some (hence, every) point z_0 in the bounded component of $\mathbb{C}\backslash\Gamma$ we have $(1/2\pi i) \int_\Gamma dz/(z - z_0) = 1$. Finally, if $\gamma(t)$ is a contour, we define $\dot{\gamma}(t)$ to be the derivative of γ at t if this exists and 0 otherwise.

Now suppose that $D \subseteq \mathbb{C}$ is a domain (nonempty, open, connected subset) and $A(z)$ is a continuous function from D to $\mathscr{B}(\mathscr{X})$, the Banach space of bounded linear operators on a complex Banach space $\mathscr{X}$. (We shall be primarily interested in the case that $A(z)$ is an analytic function.) Let $(\Gamma, \gamma(t))$, $a \leqslant t \leqslant b$ be a contour in D.

1.2. Definition. $\prod_\Gamma e^{A(z)dz} = \prod_a^b e^{A(\gamma(t))\dot{\gamma}(t)dt}$.

1.3. *Remarks.* (1) Since $A(\gamma(t))\,\dot{\gamma}(t)$ is continuous except perhaps at finitely many points, it follows from Theorem 1, Corollary 2 of [1] that

$$\prod_a^b e^{A(\gamma(t))\dot{\gamma}(t)dt} = \lim_{\mu(P)\to 0} \prod_{i=2}^n e^{A(\gamma(t_i))\dot{\gamma}(t_i)(\Delta t)_i}$$

where $b = t_1 > t_2 > \cdots > t_{n-1} > t_n = a$ is a partition P of $[a, b]$, $(\Delta t)_i = t_i - t_{i-1}$, and $\mu(P) = \max_i\{(\Delta t)_i\}$.

(2) $\prod_\Gamma e^{A(z)dz} = \lim_{\mu(P)\to 0} \prod_{k=2}^n e^{A(z_k)(\Delta z)_k}$ where P is as in (1), $z_k = \gamma(t_k)$, $(\Delta z)_k = z_k - z_{k-1}$. This is easily proved using the fact that except at points t_k of nondifferentiability of $\gamma(t)$, $(\Delta z)_k = \dot{\gamma}(t_k)(\Delta t)_k + o((\Delta t)_k)$. This shows in what sense the product integral is independent of the particular form of $\gamma(t)$. However, in what follows, we shall always assume that any contour referred to is defined by some particular $\gamma(t)$.

(3) Since (1.2) defines the product integral in terms of a product integral along a real interval, the results of [1] are applicable, and we shall make use of these. For example, Theorem 2 and Definition 9 of [1] imply that

$$\prod_{\Gamma^{-1}} e^{A(z)dz} = \Big(\prod_{\Gamma} e^{A(z)dz}\Big)^{-1}.$$

2. The Product Integral of an Analytic Function and the Analog of Cauchy's Integral Theorem

2.1. Definition. Let $D \subseteq \mathbb{C}$ be a domain, $A(z)$ a mapping from D to $\mathscr{B}(\mathscr{X})$. $A(z)$ is *analytic* in D if $\lim_{h\to 0} h^{-1}(A(z+h) - A(z))$ exists in $\mathscr{B}(\mathscr{X})$ for every $z \in D$.

Now suppose D is a domain and $A(z)$ is analytic in D with values in $\mathscr{B}(\mathscr{X})$. Let $(\Gamma, \gamma(t))$, $a \leqslant t \leqslant b$, be a contour in D. For $z \in \Gamma$ with $\gamma(t) = z$ and $\gamma(a) = z_0$ we denote $\int_a^t A(\gamma(s))\,\dot\gamma(s)\,ds$ by $I_\Gamma^{(1)}(z)$. We may write $I_\Gamma^{(1)}(z) = \int_{z_0}^z A(\zeta)\,d\zeta$ with the understanding that the integration is performed along Γ. We define inductively

$$I_\Gamma^{(n)}(z) = \int_{z_0}^z A(\zeta)\, I_\Gamma^{(n-1)}(\zeta)\, d\zeta, \qquad n = 2, 3,\ldots$$

where the integration is performed along Γ. By Theorem 1 and Property 6 of [1] we have the formula

$$\prod_{\Gamma} e^{A(z)dz} = I + \sum_{n=1}^{\infty} I_\Gamma^{(n)}(\gamma(b)). \tag{2.2}$$

2.3. Theorem. *Let $D \subseteq \mathbb{C}$ be a domain, $A(z)$ analytic in D with values in $\mathscr{B}(\mathscr{X})$, and $(\Gamma_1, \gamma_1(t))$, $(\Gamma_2, \gamma_2(t))$, $a \leqslant t \leqslant b$ two contours in D with $\gamma_1(a) = \gamma_2(a) = z_0$, $\gamma_1(b) = \gamma_2(b) = z_1$. Suppose Γ_1 and Γ_2 are homotopic with fixed endpoints in D. Then*

$$\prod_{\Gamma_1} e^{A(z)dz} = \prod_{\Gamma_2} e^{A(z)dz}.$$

Proof. We will assume first that D is simply-connected. Define $I^{(1)}(z) = \int_{z_0}^z A(\zeta)\,d\zeta$ where the integration is along any contour, and inductively, $I^{(n)}(z) = \int_{z_0}^z A(\zeta)\, I^{(n-1)}(\zeta)\,d\zeta$. Each $I^{(n)}(z)$ is well defined and analytic in D by the ordinary form of Cauchy's theorem. Now for $z \in \Gamma_1$, $I^{(1)}(z) = I_{\Gamma_1}^{(1)}(z)$. Assume inductively that $I^{(k)}(z) = I_{\Gamma_1}^{(k)}(z)$ for $z \in \Gamma_1$. Then we have $I^{(k+1)}(z) = \int_{z_0}^z A(\zeta)\, I^{(k)}(\zeta)\,d\zeta$ where the integration is along any contour; if we require the contour to be Γ_1, this shows that $I^{(k+1)}(z) = I_{\Gamma_1}^{(k+1)}(z)$. Hence $I^{(n)}(z) = I_{\Gamma_1}^{(n)}(z)$ for $n = 1, 2, 3,\ldots$, $z \in \Gamma_1$. The same is true with Γ_1 replaced by Γ_2, and this fact together with (2.2) proves the theorem in the simply-connected case.

We remark that what has been proved so far together with Theorem 1 of [1] shows that if D is simply-connected, then there is an analytic solution of $B'(z)\,B^{-1}(z) = A(z)$ in D. (The prime denotes differentiation with respect to z.) A solution is given by the product integral of A from any initial point to z. Now we remove the hypothesis that D is simply-connected. Let $h(t, u)$: $[a, b] \times [0, 1] \to D$ be the homotopy between γ_1 and γ_2 ($h(t, 0) = \gamma_1(t)$, $h(t, 1) = \gamma_2(t)$). We claim that there is a continuous $\mathscr{B}(\mathscr{X})$-valued function $\mathscr{B}(t, u)$ in $S = [a, b] \times [0, 1]$ such that for each point $(t_0, u_0) \in S$, there is a neighborhood U of $h(t_0, u_0)$ and a $\mathscr{B}(\mathscr{X})$-valued analytic function B in U with $\mathscr{B}(t, u) = B(h(t, u))$ for $h(t, u) \in U$, and $B'B^{-1} = A$ in U. Once established, this together with Theorem 7 of [1], will finish the proof, for then

$$\prod_{\Gamma_1} e^{A(z)dz} = \mathscr{B}(b, 0)\,\mathscr{B}^{-1}(a, 0) = \mathscr{B}(b, 1)\,\mathscr{B}^{-1}(a, 1) = \prod_{\Gamma_2} e^{A(z)dz}.$$

(The middle equality is true because $h(b, 0) = h(b, 1)$, $h(a, 0) = h(a, 1)$.) To prove the existence of $\mathscr{B}(t, u)$, subdivide $[a, b]$ by points t_i and $[0, 1]$ by points u_i so that for each i, j, $h([t_i, t_{i+1}] \times [u_j, u_{j+1}])$ is contained in an open disk U_{ij} in which there exists an analytic B_{ij} with $B'_{ij}B_{ij}^{-1} = A$. Fix j. Since $U_{ij} \cap U_{i+1,j}$ is nonempty and connected, we can multiply each B_{ij} (j fixed) on the right by an element of $\mathscr{B}(\mathscr{X})$ so that $B_{i,j}$ and $B_{i+1,j}$ agree in $U_{i,j} \cap U_{i+1,j}$; this is possible due to the fact that solutions of $B'B^{-1} = A$ are determined up to right multiplication by an element of $\mathscr{B}(\mathscr{X})$ [1, Theorem 4, Corollary 2]. For $u \in [u_j, u_{j+1}]$, define $\mathscr{B}_j(t, u) = B_{i,j}(h(t, u))$, $t \in [t_i, t_{i+1}]$. $\mathscr{B}_j(t, u)$ is continuous in $[a, b] \times [u_j, u_{j+1}]$. We remark that it may happen that $U_{i,j}$ and $U_{i+2,j}$ may intersect, and $B_{i,j}$, $B_{i+2,j}$ may not agree in the intersection, but $\mathscr{B}_j(t, u)$ is well defined by the above prescription. Finally we can multiply each $\mathscr{B}_j$ on the right by an element of $\mathscr{B}(\mathscr{X})$ so that $\mathscr{B}_j$ and $\mathscr{B}_{j+1}$ agree when $u = u_{j+1}$. Then put $\mathscr{B}(t, u) = \mathscr{B}_j(t, u)$ for $u \in [u_j, u_{j+1}]$.

The following corollary has been proved in the above.

2.4. Corollary. *If D is simply-connected, $z_0 \in D$, and if for all $z \in D$, we define $U(z) = \prod_{z_0}^{z} e^{A(\zeta)d\zeta}$ where the integration is along any contour in D, then $U(z)$ is well defined and $U'(z) = A(z)\,U(z)$.*

2.5. Corollary. *If Γ is null homotopic in D, then*

$$\prod_{\Gamma} e^{A(z)dz} = I.$$

We prove one more corollary which describes the relation between the product integrals along concentric contours of an analytic $A(z)$. Suppose D is a domain, Γ_1, Γ_2 contours in D which are Jordan curves with initial points z_1, z_2, respectively. Suppose that Γ_1, Γ_2 are positively oriented, Γ_2 is contained in the interior of Γ_1, and that the region between Γ_1 and Γ_2 is contained in D.

Suppose $A(z)$ is analytic in D. Then $\int_{\Gamma_1} A(\zeta)\,d\zeta = \int_{\Gamma_2} A(\zeta)\,d\zeta$ by a simple application of Cauchy's theorem. For product integrals, we do not obtain simple equality. We have, in fact:

2.6. COROLLARY. *$P_1 = \prod_{\Gamma_1} e^{A(\zeta)d\zeta}$ and $P_2 = \prod_{\Gamma_2} e^{A(\zeta)d\zeta}$ are similar operators, i.e., there exists an invertible operator, S, with $P_1 = SP_2S^{-1}$.*

Proof. Consider Fig. 2.7. We adopt the following notation: If Γ_i, Γ_j are contours with terminal point of Γ_j = initial point of Γ_i, then $\Gamma_j\Gamma_i$ denotes

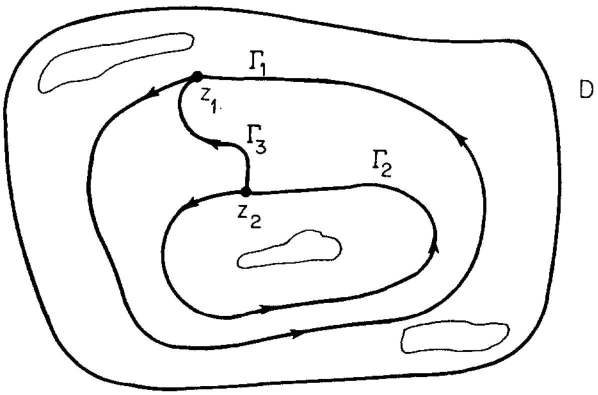

FIGURE 2.7.

the contour consisting of Γ_i followed by Γ_j. Now from the diagram and Theorem 2.3, we have

$$\prod_{\Gamma_1\Gamma_3} e^{A(\zeta)d\zeta} = \prod_{\Gamma_3\Gamma_2} e^{A(\zeta)d\zeta}.$$

Using Theorem 3 of [1] and Remark 3 of 1.3, we can rewrite the last equation as

$$\prod_{\Gamma_1} e^{A(\zeta)d\zeta} = \prod_{\Gamma_3} e^{A(\zeta)d\zeta} \prod_{\Gamma_2} e^{A(\zeta)d\zeta} \left(\prod_{\Gamma_3} e^{A(\zeta)d\zeta}\right)^{-1}.$$

Taking $S = \prod_{\Gamma_3} e^{A(\zeta)d\zeta}$ provides the claimed similarity relation.

3. A CAUCHY INTEGRAL FORMULA FOR PRODUCT INTEGRALS

Throughout the present section the following notation will be in effect: D is a simply-connected domain in $\mathbb{C}$, Γ is a positively oriented contour in D which is a Jordan curve, z_0 is a point of D which lies inside Γ, $A(z)$ denotes an analytic $\mathscr{B}(\mathscr{X})$-valued function in $D\backslash\{z_0\}$ (see Fig. 3.1). In the situation described above, $\int_\Gamma A(\zeta)\,d\zeta$ can be evaluated in the following way: since $\int_\Gamma A(\zeta)\,d\zeta$

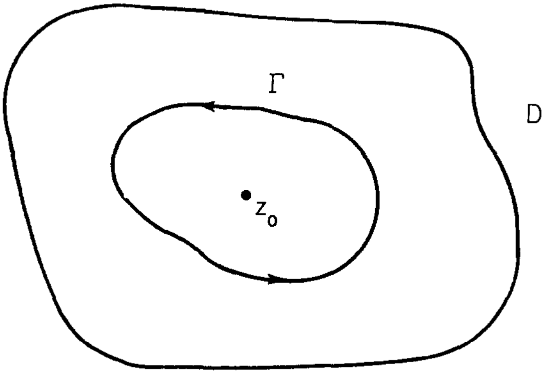

FIGURE 3.1.

is unchanged if the contour Γ is "shrunk" toward the point z_0, we can assume that Γ is contained in an annulus in D in which $A(z)$ has the convergent Laurent expansion $A(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n$; then $\int_\Gamma A(\zeta)\, d\zeta = 2\pi i A_{-1}$. If $\{A(z)\}_{z\in\Gamma}$ is a commutative family, then by Corollary 3 of [1],

$$\prod_\Gamma e^{A(\zeta)d\zeta} = e^{\int_\Gamma A(\zeta)d\zeta} = e^{2\pi i A_{-1}}.$$

If the commutativity assumption is dropped, it is no longer true in general that $\prod_\Gamma e^{A(\zeta)d\zeta}$ and $e^{2\pi i A_{-1}}$ are equal; however, if $A(z)$ has a pole of order at most one at z_0, and a technical condition concerning the spectrum of A_{-1} is satisfied, then $\prod_\Gamma e^{A(\zeta)d\zeta}$ and $e^{2\pi i A_{-1}}$ are similar. We remark that in general it seems quite difficult to compute $\prod_\Gamma e^{A(\zeta)d\zeta}$ or even to determine if this integral is the identity operator, I. The latter question is equivalent to asking when solutions of $\{du/dz = A(z)\, u(z),\ u(z)\colon D\backslash\{z_0\} \to \mathscr{X}\}$ are single valued; this need not be the case even when

$$A(z) = \sum_{n=-1}^{\infty} A_n(z - z_0)^n \qquad \text{and} \qquad e^{2\pi i A_{-1}} = I.$$

We give an example illustrating some of these remarks.

3.2. EXAMPLE. Let

$$S(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix},$$

and let $T(z)$ be the (multivalued in general) function $T(z) = S(z)z^R = S(z)e^{R\log z}$ defined in $\mathbb{C}\backslash\{0\}$. Set

$$A(z) = \frac{dT}{dz} \cdot T^{-1}(z) = z^{-2}\begin{pmatrix} 0 & 0 \\ r_3 & 0 \end{pmatrix} + z^{-1}\begin{pmatrix} 1 + r_1 & 0 \\ 0 & r_4 \end{pmatrix} + \begin{pmatrix} 0 & r_2 \\ 0 & 0 \end{pmatrix}$$

which is analytic in $\mathbb{C}\backslash\{0\}$. Let Γ be the contour given by $\gamma(t) = e^{it}$ for $0 \leqslant t \leqslant 2\pi$. By picking a branch of the logarithm along $\gamma(t)$, $0 < t < 2\pi$ and applying Theorem 7 of [1], we have $\prod_\Gamma e^{A(\zeta)d\zeta} = \prod_\Gamma e^{T'(\zeta)T^{-1}(\zeta)d\zeta} = T(e^{2\pi i})\, T^{-1}(e^0) = e^{2\pi i R}$. We have also

$$A_{-1} = \begin{pmatrix} 1 + r_1 & 0 \\ 0 & r_4 \end{pmatrix}$$

so that

$$e^{2\pi i A_{-1}} = \begin{pmatrix} e^{2\pi i r_1} & 0 \\ 0 & e^{2\pi i r_4} \end{pmatrix}.$$

In general, $e^{2\pi i A_{-1}}$ and $e^{2\pi i R}$ are not equal and are not even similar. For example, if $r_3 = 0$, $r_1 = r_4 = -\frac{1}{2}$, then

$$A_{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad e^{2\pi i A_{-1}} = -I,$$

and

$$e^{2\pi i R} = e^{i\pi\begin{pmatrix} -1 & 2r_2 \\ 0 & -1 \end{pmatrix}} = e^{i\pi\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} e^{\begin{pmatrix} 0 & 2\pi i r_2 \\ 0 & 0 \end{pmatrix}}$$

(the two matrices commute)

$$= -I \cdot \left(I + \begin{pmatrix} 0 & 2\pi i r_2 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & -2\pi i r_2 \\ 0 & -1 \end{pmatrix}$$

which is not similar to $-I$ unless $r_2 = 0$. Notice that in this case, $A(z)$ has only a pole of order 1 at $z = 0$; we also note for later reference that $\| A_{-1} \| = \frac{1}{2}$. If we take $r_3 = 0$, $r_1 = r_4 = 1$, we get an example with $e^{2\pi i A_{-1}} = I$, $A(z)$ has a pole of order 1 at 0 and

$$e^{2\pi i R} = \begin{pmatrix} 1 & 2\pi i r_2 \\ 0 & 1 \end{pmatrix}$$

which is not similar to I unless $r_2 = 0$.

The example indicates that to prove an analog of the Cauchy integral formula for product integrals even for the case $A(z) = A_{-1}/(z - z_0) + B(z)$, $B(z)$ analytic in D, additional assumptions are necessary. One assumption which is sufficient is that the difference set $\sigma(A_{-1}) - \sigma(A_{-1})$ of the spectrum $\sigma(A_{-1})$ of A_{-1} does not contain positive integers. We state a technical result concerning this condition.

3.1. Lemma. *Let* $A \in \mathscr{B}(\mathscr{X})$ *and let* $\operatorname{ad} A: \mathscr{B}(\mathscr{X}) \to \mathscr{B}(\mathscr{X})$ *be the operator defined by* $\operatorname{ad} A(T) = AT - TA$. *Let* $\sigma(A)$ *and* $\sigma(\operatorname{ad} A)$ *denote the spectra of* A, $\operatorname{ad} A$, *respectively. Then* $\sigma(\operatorname{ad} A) = \sigma(A) - \sigma(A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\}$.

Proof. We refer to [4] and [5] for the general proof in the Banach space case. We give a proof for the case $\mathscr{X} = \mathbb{C}^n$ is finite dimensional, in which case the spectra of the operators in question consist of finite sets of eigenvalues. Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_l\}$, and let A^t be the transpose of A. The characteristic equations of A and A^t are the same, so A and A^t have the same spectrum. Choose column vectors $\{X_k\}_{k=1,\ldots,l}$, $\{Y_k\}_{k=1,\ldots,l}$ with $AX_k = \lambda_k X_k$, $A^t Y_k = \lambda_k Y_k$, $(Y_k{}^t A = \lambda_k Y_k{}^t)$. Let Z_{rs} be the nonzero $n \times n$ matrix $X_r Y_s{}^t$ (where $Y_s{}^t$ is the row vector with the same components as Y_s). Then $(\operatorname{ad} A) Z_{rs} = AX_r Y_s{}^t - X_r Y_s{}^t A = (\lambda_r - \lambda_s) Z_{rs}$. Hence $\sigma(A) - \sigma(A) \subseteq \sigma\ (\operatorname{ad} A)$. Next we prove the reverse inclusion. The space $\mathbb{C}^n$ on which A acts can be decomposed as $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_l$ where in each V_k there is a basis $\{f_1, \ldots, f_{r_k}\}$ with

$$\begin{aligned} Af_1 &= \lambda_k f_1 + f_2 \\ Af_2 &= \lambda_k f_2 + f_3 \qquad \text{(Jordan decomposition)} \\ Af_{r_k} &= \lambda_k f_{r_k}. \end{aligned}$$

Let μ be an eigenvalue of $\operatorname{ad} A$ with eigenmatrix Z. Assume $\mu + \lambda_k \notin \sigma(A)$; then $A - \mu - \lambda_k$ is invertible. Now we have $AZ = ZA + \mu Z$ and so $(A - \mu - \lambda_k)Z = Z(A - \lambda_k)$ or

$$Z = (A - \mu - \lambda_k)^{-1} Z(A - \lambda_k). \tag{$*$}$$

Applying Z to vectors on V_k and using $(*)$, we have:

$$\begin{aligned} Zf_1 &= (A - \mu - \lambda_k)^{-1} Zf_2 = (A - \mu - \lambda_k)^{-2} Zf_3 = \cdots \\ &= (A - \mu - \lambda_k)^{-r_k+1} Zf_{r_k} = 0. \end{aligned}$$

Thus Z annihilates V_k. But Z cannot annihilate all the V_k so for some k, $\mu + \lambda_k \in \sigma(A)$. This proves $\sigma(\operatorname{ad} A) \subseteq \sigma(A) - \sigma(A)$ and finishes the proof.

We return to the situation described at the beginning of Section 3.

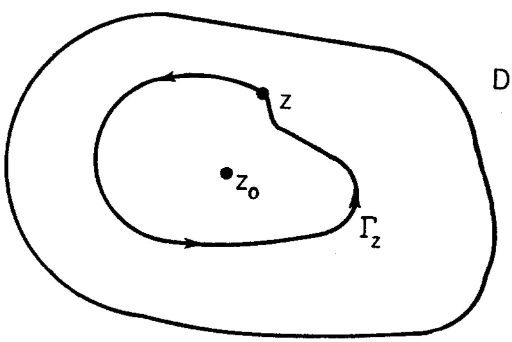

FIGURE 3.3.

3.2. DEFINITION. For $z \neq z_0$, $\Lambda(z) = \prod_{\Gamma_z} e^{A(\zeta)d\zeta}$ where Γ_z is any positively oriented contour in D which is a Jordan curve, starting at z circling z_0 once, and returning to z (see Fig. 3.3). By a simple application of Theorem 2.3, $\Lambda(z)$ is independent of which such contour Γ_z is used, i.e., $\Lambda(z)$ is well defined.

3.3 PROPOSITION. *$\Lambda(z)$ is analytic in $D\backslash\{z_0\}$. If $A(z) = A_{-1}/(z - z_0) + B(z)$, $B(z)$ analytic in D, then $\Lambda(z)$ has a removable singularity at z_0 and* $\lim_{z\to z_0} \Lambda(z) = e^{2\pi i A_{-1}}$.

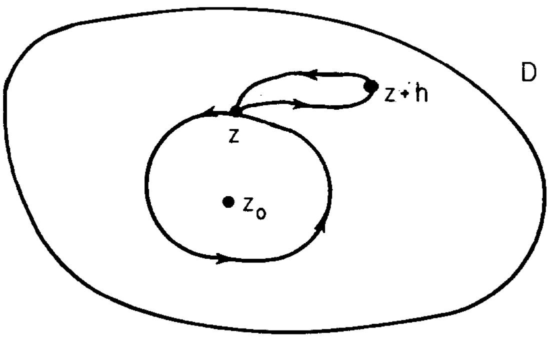

FIGURE 3.4.

Proof. Consider Fig. 3.4. From the figure and Theorem 3 of [1] we have:

$$\Lambda(z+h) = \prod_{z}^{z+h} e^{A(\zeta)d\zeta}\Lambda(z) \prod_{z+h}^{z} e^{A(\zeta)d\zeta} \qquad \text{(along the contours indicated)}$$

so

$$\Lambda(z+h) - \Lambda(z) = \left(\prod_{z}^{z+h} e^{A(\zeta)d\zeta} - I\right)\Lambda(z) \prod_{z+h}^{z} e^{A(\zeta)d\zeta} + \Lambda(z)\left(\prod_{z+h}^{z} e^{A(\zeta)d\zeta} - I\right)$$

hence

$$\frac{d}{dz}\Lambda(z) = [A(z), \Lambda(z)] = A(z)\,\Lambda(z) - \Lambda(z)\,A(z)$$

for $z \neq z_0$. Finally,

$$\prod_{\Gamma=\{z_0+re^{i\theta},\theta_0\leqslant\theta\leqslant\theta_0+2\pi\}} e^{\{A_{-1}/(\zeta-z_0)+B(\zeta)\}d\zeta} = \prod_{\theta_0}^{\theta_0+2\pi} e^{\{iA_{-1}+O(r)\}d\theta}$$

and by Theorem 6 of [1] applied to the last product integral and $\prod_{\theta_0}^{\theta_0+2\pi} e^{iA_{-1}d\theta} = e^{2\pi i A_{-1}}$, we have $\lim_{z\to z_0} \Lambda(z) = e^{2\pi i A_{-1}}$.

We remark that for $z \neq z_0$ and $z' \neq z_0$, $\Lambda(z)$ and $\Lambda(z')$ are similar. This follows from the calculations in the proof of Proposition 3.3. In particular, if $A(z) = I$ for some $z \neq z_0$, then $A(z) \equiv I$, and $e^{2\pi i A_{-1}} = I$; this means that all solutions of $du/dz = A(z)\,u(z)$ are single valued in $D\backslash\{z_0\}$, i.e., have at worst poles or essential singularities at z_0 and not branch points. Note that $e^{2\pi i A_{-1}} = I$ is a necessary but not sufficient condition for single valuedness.

We now prove an analog of Cauchy's integral formula for product integrals.

3.4. THEOREM (Cauchy Integral Formula). *Consider the situation described at the beginning of Section* 3. *Suppose* $A(z) = A_{-1}/(z - z_0) + B(z)$, *with* $B(z)$ *analytic in* D. *Suppose further that* $\sigma(A_{-1}) - \sigma(A_{-1})$ *does not contain positive integers. Then* $\prod_\Gamma{}^{A(\zeta)d\zeta}$ *and* $e^{2\pi i A_{-1}}$ *are similar.*

3.5. *Remarks*

(1) In Volterra [3], the result of this theorem is apparently claimed without the hypothesis on $\sigma(A_{-1}) - \sigma(A_{-1})$. This hypothesis is necessary as Example 3.2 shows.

(2) Later we shall give various explicit formulas for the operator giving the similarity claimed in the theorem. The proof of the theorem gives the similarity in the form of a power series.

(3) The result of the theorem in the finite-dimensional case and in a different guise can be found in [2]; the proof given there differs from the present one.

Proof of Theorem 3.4. We assume without loss of generality that $z_0 = 0$ to simplify the calculation. Since $A(z)$, $A(z')$, $z, z' \neq 0$ are similar, it suffices to show $A(z)$ similar to $e^{2\pi i A_{-1}}$ for $|z|$ sufficiently small. Recall that by a special case of Theorem 9 of [1], if $T(z)$ is analytic and invertible along a contour C in D with initial and terminal points z_1, z_2, and $S(z)$ is continuous along C then

$$\prod_C e^{S(z)dz} = T(z_2) \prod_C e^{\{T^{-1}(z)S(z)T(z) - T^{-1}(z)T'(z)\}dz} T^{-1}(z_1). \tag{3.4.1}$$

Suppose that a single-valued analytic $T(z)$ in a neighborhood of $z = 0$ can be found with:

$$T^{-1}(z)\frac{A_{-1}}{z}T(z) - T^{-1}(z)\,T'(z) = \frac{A_{-1}}{z} + B(z). \tag{3.4.2}$$

Then by (3.4.1) with $S(z) = A_{-1}/z$, $C = \Gamma$ we have

$$e^{2\pi i A_{-1}} = T(z)\,A(z)\,T^{-1}(z) \tag{3.4.3}$$

which would prove the theorem. We thus attempt to solve (3.4.2) which is equivalent to the system:

$$\begin{aligned} T'(z) &= [A_{-1}/z, T(z)] - T(z)\,B(z), \\ &T(z) \text{ invertible near } 0. \end{aligned} \tag{3.4.4}$$

We look for $T(z)$ in the form:

$$T(z) = I + \sum_{n=1}^{\infty} T_n z^n \qquad \text{satisfying (3.4.4).} \tag{3.4.5}$$

The invertibility near 0 is automatic for $T(z)$ of this form, so we substitute (3.4.5) in the first equation of (3.4.4) yielding the formal relation:

$$\sum_{n=1}^{\infty} nT_n z^{n-1} = \sum_{n=1}^{\infty} [A_{-1}, T_n] z^{n-1} + \left(I + \sum_{n=1}^{\infty} T_n z^n\right)\left(\sum_{n=0}^{\infty} B_n z^n\right). \tag{3.4.6}$$

Identifying equal powers of z yields:

$$(n - \operatorname{ad} A_{-1})\, T_n z^{n-1} = (B_0 T_{n-1} + \cdots + B_{n-1}) z^{n-1} \qquad (n \geqslant 1) \tag{3.4.7}$$

and under the hypotheses on $\sigma(A_{-1})$, this is recursively solved by:

$$T_n = (n - \operatorname{ad} A_{-1})^{-1} (B_0 T_{n-1} + \cdots + B_{n-1}). \tag{3.4.8}$$

Now $(n - \operatorname{ad} A_{-1})^{-1} = O(n^{-1})$ as $n \to \infty$, and for $| z |$ sufficiently small, $\| B_n z^n \| = O(1)$ since $\sum_{n=0}^{\infty} B_n z^n$ is convergent. Hence (3.4.8) yields:

$$\| T_n z^n \| = O(n^{-1}) \cdot \sum_{j=0}^{n-1} \| T_j z^j \| \tag{3.4.9}$$

for $| z |$ sufficiently small and where $T_0 = I$. From (3.4.9) we obtain inductively:

$$\| T_n z^n \| = O(1) \qquad \text{as} \quad n \to \infty \qquad (\text{for } | z | \text{ small}). \tag{3.4.10}$$

This proves that (3.4.5) converges in some neighborhood of 0 and finishes the proof of the theorem.

3.6. *Remarks*

(1) The proof of the theorem shows that the result of the theorem is true in the case that $B(z)$ has a zero of order s at 0 and $\sigma(A_{-1}) - \sigma(A_{-1})$ contains no integer $>s$. (The T_n could still be determined from (3.4.7).) This condition on $\sigma(A_{-1}) - \sigma(A_{-1})$ will hold, for example, if $\| A_{-1} \| < (s + 1)/2$ since then $\| \operatorname{ad} A_{-1} \| < s + 1$ and $(n - \operatorname{ad} A_{-1})^{-1}$ exists for $n \geqslant s + 1$. Hence the result of the theorem is always true if $\| A_{-1} \| < \frac{1}{2}$; recall, however, that Example 3.2 showed that the result of the theorem fails to hold in a case where $\| A_{-1} \| = \frac{1}{2}$.

(2) If we are in the finite-dimensional case $\mathscr{X} = \mathbb{C}^n$, the characteristic polynomial of $\Lambda(z)$ is independent of z and thus equals the characteristic polynomial of $e^{2\pi i A_{-1}}$ regardless of the nature of $\sigma(A_{-1})$. Hence the eigenvalues of $\Lambda(z)$ and of $e^{2\pi i A_{-1}}$ are the same, and if these are all distinct then the result of the theorem holds. (Because two $n \times n$ matrices with the same n distinct eigenvalues are similar.)

(3) As remarked at the beginning of Section 3, if $A(z)$ is a commutative family the result of the theorem is always true with "similar" replaced by "equal."

(4) If $\mathscr{X} = \mathbb{C}^n$ and the condition on $\sigma(A_{-1})$ does not hold, then it is possible to show that there is an invertible analytic function $T(z)$ in $\mathbb{C}\backslash\{z_0\}$ such that if $\tilde{A}(z) = T^{-1}(z)\, A(z)\, T(z) - T^{-1}(z)\, T'(z)$, then $\tilde{A}(z)$ has a pole of order one at z_0 and $\tilde{A}_{-1}$ *does* satisfy the spectrum condition of the theorem. (See Theorem 4.2 and the preceding lemma in [2].) Hence in this case $\prod_\Gamma e^{A(\zeta)d\zeta}$ is similar to $e^{2\pi i \tilde{A}_{-1}}$. This result is of some theoretical interest and is useful, for example, in determining the form of solutions of second order ordinary differential equations with regular singular points in the neighborhood of the singularity. However, it seems very difficult to give an explicit formula for $\tilde{A}_{-1}$, since the construction of $T(z)$ involves the transformations which put various successively determined matrices in Jordan normal form.

If the hypotheses of Theorem 3.4 hold, then the operator $T(z)$ giving the similarity between $A(z)$ and $e^{2\pi i A_{-1}}$ is expressed as a power series. In some cases a more explicit integral formula for the similarity can be given. We consider the case $z_0 = 0$, assume $1 \in D$, and consider $A(1)$ for definiteness (see Fig. 3.7).

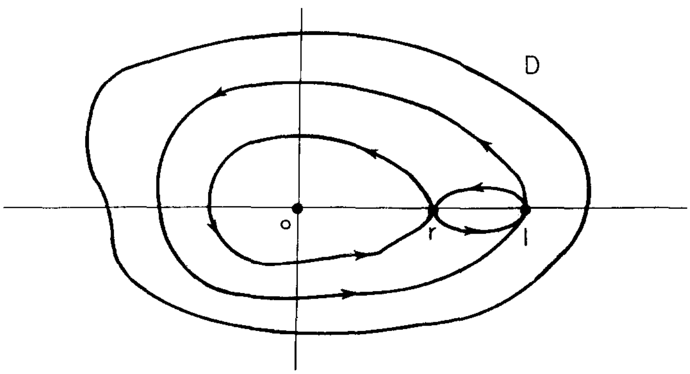

FIGURE 3.7.

3.8. THEOREM. *With notation as in Theorem* 3.4, *assume that* either:

(1) A_{-1} *is skew adjoint*, or

(2) $B(z)$ *has a zero of order* k *at* 0 *and*

$$\| A_{-1} \| < \frac{k+1}{2}.$$

Then $A(1) = \prod_0^1 e^{(s^{-A_{-1}}B(s)s^{A_{-1}})ds} \cdot e^{2\pi i A_{-1}} \cdot \prod_1^0 e^{(s^{-A_{-1}}B(s)s^{A_{-1}})ds}$ *where the product integrals are convergent improper integrals, the path of integration being the line segment from* 0 *to* 1.

Proof. From Fig. 3.7 we have:

$$\Lambda(1) = P(1, r)^{-1} \Lambda(r) P(1, r) \tag{3.8.1}$$

where $P(1, r) = \prod_1^r e^{(A_{-1}/s + B(s))ds}$, the integration being along the line segment from 1 to r, on the x axis. By Theorem 8 of [1] we have:

$$P(1, r) = \prod_1^r e^{(A_{-1}s)/ds} \prod_1^r e^{(\prod_1^s e^{A_{-1}/z\, dz})^{-1} B(s) (\prod_1^s e^{A_{-1}/z\, dz}) ds}$$

$$= r^{A_{-1}} \prod_1^r e^{(s^{-A_{-1}} B(s) s^{A_{-1}}) ds} \qquad \text{where} \quad s^{A_{-1}} = e^{A_{-1} \log s}; \tag{3.8.2}$$

$$\log s = \int_1^s 1/x \, dx \text{ is the natural logarithm.}$$

From (3.8.1), (3.8.2) we have:

$$\Lambda(1) = \prod_r^1 e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds} \cdot e^{2\pi i A_{-1}} \cdot \prod_1^r e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds}$$
$$+ P(1, r)^{-1} (\Lambda(r) - e^{2\pi i A_{-1}}) P(1, r) \tag{3.8.3}$$

(since $r^{\pm A_{-1}}$ and $e^{2\pi i A_{-1}}$ commute). Since $\lim_{r \to 0} (\Lambda(r) - e^{2\pi i A_{-1}}) = 0$ by Proposition 3.3, in order to prove the formula claimed for $\Lambda(1)$ it suffices to have:

(a) $\| P(1, r) \|$ and $\| (P(1, r))^{-1} \|$ bounded as $r \to 0$,

(b) $\lim_{r \to 0} \prod_r^1 e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds}$ and $\lim_{r \to 0} \prod_1^r e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds}$ exist.

If A_{-1} is skew adjoint, i.e., $A_{-1}^* = -A_{-1}$, then $\| s^{\pm A_{-1}} \|^2 = \| s^{\pm A_{-1}^*} s^{\pm A_{-1}} \| = \| s^{\mp A_{-1}} s^{\pm A_{-1}} \| = 1$ so $\| s^{\pm A_{-1}} \|$ is bounded and $\| s^{-A_{-1}} B(s) s^{A_{-1}} \|$ is integrable over $[0, 1]$ so (a) and (b) are true by Theorem 1 of [1]. If $B(z)$ has a zero of order k at 0 and $\| A_{-1} \| < (k+1)/2$ then by Theorem 6 of [1] we have easily $\Lambda(r) - e^{2\pi i A_{-1}} = O(r^{k+1})$ as $r \to 0$, and by Theorem 5 of [1] we have

$$\| P(1, r) \|, \| (P(1, r))^{-1} \| \leqslant r^{-\|A_{-1}\|} e^{(\int_r^1 s^{-2\|A_{-1}\|} \|B(s)\| ds)} \leqslant \text{const}\, r^{-\|A_{-1}\|}$$

as $r \to 0$ since $\int_0^1 s^{-2\|A_{-1}\|} \| B(s) \| \, ds$ is convergent. Thus the second summand in (3.8.3) tends to 0 as $r \to 0$. Furthermore,

$$\| s^{-A_{-1}} B(s) s^{A_{-1}} \| \leqslant s^{-2\|A_{-1}\|} \| B(s) \| \leqslant \text{const}\, s^{\nu}, \qquad \nu > -1,$$

so $\prod_0^1 e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds}$ and $\prod_1^0 e^{s^{-A_{-1}} B(s) s^{A_{-1}} ds}$ exist by Theorem 1 of [1]. This finishes the proof of the theorem.

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