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# Bailey's very well-poised ${}_6\psi_6$-series identity

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**Abstract**

By means of Abel's method on summation by parts, some two term recurrence relations on very well-poised ${}_6\psi_6$-series are established. Their iteration yields a ${}_6\psi_6$-series transformation with an extra natural number parameter. Evaluating the limiting series via Jacobi's triple product identity, we are led surprisingly to the celebrated bilateral ${}_6\psi_6$-series identity discovered by Bailey (1936). Then we shall further generalize it to a very well-poised ${}_{10}\psi_{10}$-series identity, which contains Shukla's formula (1959) as special case. Finally, the Abel's method on summation by parts will be employed again to investigate the bibasic hypergeometric series summation, which may be considered as an extension of a "split-poised" transformation on terminating ${}_{10}\phi_9$-series due to Gasper (1989).



*Keywords:* Abel's method on summation by parts; Basic hypergeometric series; Jacobi's triple product identity

## 1. Abel's lemma on summation by parts

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators $\nabla$ and $\triangle$, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle\tau_k = \tau_k - \tau_{k+1} \tag{1.1}$$

where $\triangle$ is adopted for convenience in the present paper, which differs from the usual difference operator $\Delta$ only in the minus sign.

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Then *Abel's lemma* on summation by parts may be reformulated as

$$\sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \sum_{k=-\infty}^{+\infty} B_k \triangle A_k \tag{1.2}$$

provided that the series on both sides are convergent or terminating.

**Proof.** According to the definition of the backward difference, we have

$$\sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \sum_{k=-\infty}^{+\infty} A_k\{B_k - B_{k-1}\} = \sum_{k=-\infty}^{+\infty} A_k B_k - \sum_{k=-\infty}^{+\infty} A_k B_{k-1}.$$

Replacing $k$ by $1+k$ for the last sum, we get the following expression:

$$\sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \sum_{k=-\infty}^{+\infty} B_k\{A_k - A_{k+1}\} = \sum_{k=-\infty}^{+\infty} B_k \triangle A_k$$

which is the formula stated in the Abel lemma. □

## 2. Bailey's very well-poised ${}_6\psi_6$-series identity

For two complex indeterminates $q$ and $x$, the shifted factorial of order $n$ with base $q$ is defined by

$$(x;q)_n = (1-x)(1-xq)\cdots\left(1-xq^{n-1}\right) \tag{2.1a}$$
$$= (x;q)_\infty/(xq^n;q)_\infty, \tag{2.1b}$$

where $n$ is assumed, respectively, to be a natural number for the former and integral for the latter but with the additional condition $|q|<1$ in order for the infinite products to be convergent. It is easy to check that the shifted factorial with negative integer order is given by

$$(x;q)_{-n} = \frac{(-1)^n q^{\binom{1+n}{2}} x^{-n}}{(q/x;q)_n}. \tag{2.2}$$

The product and fractional forms of shifted factorials are abbreviated compactly to

$$(\alpha,\beta,\ldots,\gamma;q)_n = (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n, \tag{2.3a}$$
$$\left[\begin{matrix}\alpha,\beta,\ldots,\gamma\\ A,B,\ldots,C\end{matrix}\Bigg|\, q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n}. \tag{2.3b}$$

The unilateral and bilateral basic hypergeometric series are respectively defined by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0,a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\Bigg|\, q;z\right] = \sum_{n=0}^{+\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix}a_0,a_1,\ldots,a_r\\ q,b_1,\ldots,b_s\end{matrix}\Bigg|\, q\right]_n z^n, \tag{2.4a}$$

$${}_r\psi_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix}\Bigg|\, q;z\right] = \sum_{n=-\infty}^{+\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix}\Bigg|\, q\right]_n z^n \tag{2.4b}$$

where the base $q$ will be restricted to $|q|<1$ for non-terminating $q$-series.

Among the classical hierarchy of basic hypergeometric identities, the most important one perhaps is Bailey's very well-poised bilateral ${}_6\psi_6$-series identity.

**Theorem 1.** (Bailey [4], see also [10, II-33].) *For complex parameters $a, b, c, d, e$ satisfying the condition $|qa^2/bcde| < 1$, there holds the very well-poised non-terminating series identity*:

$$ {}_6\psi_6\left[\begin{matrix} qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\middle| q; \frac{qa^2}{bcde}\right] \tag{2.5a} $$

$$ = \left[\begin{matrix} q, & qa, & q/a, & qa/bc, & qa/bd, & qa/be, & qa/cd, & qa/ce, & qa/de \\ qa/b, & qa/c, & qa/d, & qa/e, & q/b, & q/c, & q/d, & q/e, & qa^2/bcde \end{matrix}\middle| q\right]_\infty . \tag{2.5b} $$

This theorem may be considered as bilateral generalization of Rogers' $q$-Dougall sum, which reads as the following (unilateral) non-terminating very well-poised series identity (cf. [10, II-20]):

$$ {}_6\phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d \\ & a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d \end{matrix}\middle| q; \frac{qa}{bcd}\right] \tag{2.6a} $$

$$ = \left[\begin{matrix} qa, & qa/bc, & qa/bd, & qa/cd \\ qa/b, & qa/c, & qa/d, & qa/bcd \end{matrix}\middle| q\right]_\infty \quad \text{where } |qa/bcd| < 1. \tag{2.6b} $$

For the identity displayed in Theorem 1, there have been eight proofs up to now.

- Bailey [4] discovered this formula via the three term relation (cf. [10, III-37]) between non-terminating very well-poised ${}_8\phi_7$-series.
- Slater [12] reproved it through $q$-analog of the Barnes type integral.
- Lakin [12] verified it by combining $q$-difference equations with Carlson's theorem on entire functions.
- Andrews [1] gave a proof by utilizing $q$-difference equations together with Laurent series expansion.
- Askey and Ismail [3] confirmed it by applying Liouville's analytic continuation argument to the non-terminating very well-poised ${}_6\phi_5$-series (Askey–Ismail thought, see also Gasper [7]).
- Askey [2] deduced it from an integral and functional equation, who commented that: "It is annoying that a sum that is this important has not been obtained from a more elementary special case."
- Schlosser [14] rederived it by means of the very well-poised ${}_6\phi_5$-series identity and series rearrangement.
- Recently, Jouhet and Schlosser [13] proved it by employing the Cauchy method to Bailey's transformation on terminating very well-poised ${}_{10}\phi_9$-series.

Most of these proofs are dependent on the very well-poised ${}_6\phi_5$-series identity just displayed or its superiors, which are far from simple. Therefore they cannot be really considered as direct and elementary as demanded by Askey. In this paper, we apply Abel's method on summation by parts to derive a recurrence relation for ${}_6\psi_6$-series with two parameters being shifted by $q$. Then by iterating this recursion, we establish the transformation on ${}_6\psi_6$-series with an extra natural number parameter $m$. Finally, letting $m \to \infty$ and evaluating the result through Jacobi's triple product identity, we establish Bailey's ${}_6\psi_6$-series identity. The whole process involves only series iteration with the convergence condition being maintained. Now that there is a very elementary proof based on the $q$-binomial theorem due to Cauchy (1843) and Gauss (1866) for the Jacobi triple product identity, we can say that the proof presented in this paper is indeed elementary.

This will be presented in the next section. Then in the fourth section, we shall further generalize (2.5a), (2.5b) to a very well-poised ${}_{10}\psi_{10}$-series identity, which contains Shukla's formula [11] as special case. The Abel's method on summation by parts will be employed again in the fifth and the last section to investigate the bibasic hypergeometric series summation, which generalizes a "split-poised" transformation on terminating ${}_{10}\phi_9$-series due to Gasper [6].

## 3. New proof of Bailey's ${}_6\psi_6$-series identity

In order to shorten lengthy expressions, we introduce the following notation for the very well-poised ${}_6\psi_6$-series

$$\Omega(a;b,c,d,e) := {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \,\Big|\, q; \frac{qa^2}{bcde}\right] \tag{3.1}$$

where the condition $|qa^2/bcde| < 1$ is assumed for convergence.

Applying Abel's method on summation by parts to $\Omega$-series, we will derive four recurrence relations. Their iteration leads us to a transformation on $\Omega$-series with an extra natural number parameter $m$. Then Bailey's very well-poised ${}_6\psi_6$-series identity stated in Theorem 1 will be recovered by the limiting case $m \to \infty$ of this last transformation thanks to the Jacobi triple product identity.

### 3.1. Recurrence relations on ${}_6\psi_6$-series

Define two factorial fractions by

$$A_k = \left[\begin{matrix} b, & c, & d, & q^2a^2/bcd \\ qa/b, & qa/c, & qa/d, & bcd/qa \end{matrix} \,\Big|\, q\right]_k, \tag{3.2a}$$

$$B_k = \left[\begin{matrix} qe, & bcd/a \\ qa/e, & q^2a^2/bcd \end{matrix} \,\Big|\, q\right]_k \left(\frac{qa^2}{bcde}\right)^k. \tag{3.2b}$$

Then it is trivial to compute their finite differences:

$$\triangle A_k = \frac{1-aq^{1+2k}}{1-qa}\left[\begin{matrix} b, & c, & d, & q^2a^2/bcd \\ q^2a/b, & q^2a/c, & q^2a/d, & bcd/a \end{matrix} \,\Big|\, q\right]_k q^k \tag{3.3a}$$

$$\times \frac{(1-qa)(1-qa/bc)(1-qa/bd)(1-qa/cd)}{(1-qa/b)(1-qa/c)(1-qa/d)(1-qa/bcd)}; \tag{3.3b}$$

$$\nabla B_k = \frac{1-aq^{2k}}{1-a}\left[\begin{matrix} e, & bcd/qa \\ qa/e, & q^2a^2/bcd \end{matrix} \,\Big|\, q\right]_k \left(\frac{qa^2}{bcde}\right)^k \tag{3.3c}$$

$$\times \frac{e(1-a)(1-qa^2/bcde)}{a(1-e)(1-qa/bcd)}. \tag{3.3d}$$

Under convergent condition $|qa^2/bcde| < 1$, we write down explicitly $\Omega$-series as an infinite sum:

$$\Omega(a;b,c,d,e) = \sum_{k=-\infty}^{+\infty} \frac{1-aq^{2k}}{1-a} \left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \middle| q\right]_k \left(\frac{qa^2}{bcde}\right)^k.$$

According to Abel's lemma on summation by parts, the last infinite series can be manipulated as follows:

$$\frac{a(1-e)(1-qa/bcd)}{e(1-a)(1-qa^2/bcde)} \sum_{k=-\infty}^{+\infty} A_k \nabla B_k = \frac{a(1-e)(1-qa/bcd)}{e(1-a)(1-qa^2/bcde)} \sum_{k=-\infty}^{+\infty} B_k \vartriangle A_k.$$

In terms of basic hypergeometric series, this can be restated as

$$\begin{aligned}\Omega(a;b,c,d,e) &= \frac{a(1-e)(1-qa)(1-qa/bc)(1-qa/bd)(1-qa/cd)}{e(1-a)(1-qa/b)(1-qa/c)(1-qa/d)(1-qa^2/bcde)} \\ &\quad \times \sum_{k=-\infty}^{+\infty} \frac{1-aq^{1+2k}}{1-qa} \left[\begin{matrix} b, & c, & d, & qe \\ q^2a/b, & q^2a/c, & q^2a/d, & qa/e \end{matrix} \middle| q\right]_k \left(\frac{q^2a^2}{bcde}\right)^k \\ &= {}_6\psi_6 \left[\begin{matrix} q\sqrt{qa}, & -q\sqrt{qa}, & b, & c, & d, & qe \\ \sqrt{qa}, & -\sqrt{qa}, & q^2a/b, & q^2a/c, & q^2a/d, & qa/e \end{matrix} \middle| q; \frac{q^2a^2}{bcde}\right] \\ &\quad \times \frac{a(1-e)(1-qa)(1-qa/bc)(1-qa/bd)(1-qa/cd)}{e(1-a)(1-qa/b)(1-qa/c)(1-qa/d)(1-qa^2/bcde)}.\end{aligned}$$

Therefore, we have established the following recurrence relation:

$$\Omega(a;b,c,d,e) = \Omega(qa;b,c,d,qe) \tag{3.4a}$$
$$\times \frac{a(1-e)(1-qa)(1-qa/bc)(1-qa/bd)(1-qa/cd)}{e(1-a)(1-qa/b)(1-qa/c)(1-qa/d)(1-qa^2/bcde)} \tag{3.4b}$$

which has shifted two parameters $a$ and $e$ by $q$ in $\Omega$-series.

In order to evaluate $\Omega$-series, we state further three variants of (3.4) by the symmetry of $\Omega(a;b,c,d,e)$ in $b,c,d,e$.

Firstly, applying (3.4) to $\Omega(qa;b,c,d,qe)$ with two parameters $a$ and $d$ being shifted, we get the transformation

$$\Omega(qa;b,c,d,qe) = \Omega\big(q^2a;b,c,qd,qe\big) \tag{3.5a}$$
$$\times \frac{qa(1-d)(1-q^2a)(1-q^2a/bc)(1-qa/be)(1-qa/ce)}{d(1-qa)(1-q^2a/b)(1-q^2a/c)(1-qa/e)(1-q^2a^2/bcde)}. \tag{3.5b}$$

Secondly, applying (3.4) to $\Omega(q^2a;b,c,qd,qe)$ with two parameters $a$ and $c$ being shifted, we get the transformation

$$\Omega\big(q^2a;b,c,qd,qe\big) = \Omega\big(q^3a;b,qc,qd,qe\big) \tag{3.6a}$$
$$\times \frac{q^2a(1-c)(1-q^3a)(1-q^2a/bd)(1-q^2a/be)(1-qa/de)}{c(1-q^2a)(1-q^3a/b)(1-q^2a/d)(1-q^2a/e)(1-q^3a^2/bcde)}. \tag{3.6b}$$

Thirdly, applying (3.4) to $\Omega(q^3a;b,qc,qd,qe)$ with two parameters $a$ and $b$ being shifted, we get the transformation

$$\Omega\big(q^3a;b,qc,qd,qe\big)=\Omega\big(q^4a;qb,qc,qd,qe\big) \tag{3.7a}$$

$$\times\frac{q^3a(1-b)(1-q^4a)(1-q^2a/cd)(1-q^2a/ce)(1-q^2/de)}{b(1-q^3a)(1-q^3a/c)(1-q^3a/d)(1-q^3a/e)(1-q^4a^2/bcde)}. \tag{3.7b}$$

### 3.2. Transformation on ${}_6\psi_6$-series

Now multiplying four equations (3.4)–(3.7) and then simplifying the result, we establish the following transformation:

$$\Omega(a;b,c,d,e)=\Omega\big(q^4a;qb,qc,qd,qe\big) \tag{3.8a}$$

$$\times\frac{q^6a^4}{bcde}\cdot\frac{1-q^4a}{1-a}\cdot\frac{(1-b)(1-c)(1-d)(1-e)}{(qa^2/bcde;q)_4} \tag{3.8b}$$

$$\times\frac{(qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de;q)_2}{(qa/b,qa/c,qa/d,qa/e;q)_3}. \tag{3.8c}$$

Iterating this relation for $m$-times and then performing computation

$$\prod_{k=0}^{m-1}\frac{q^6(q^{4k}a)^4}{q^{4k}bcde}\frac{1-q^{4+4k}a}{1-q^{4k}a}\frac{(1-q^kb)(1-q^kc)(1-q^kd)(1-q^ke)}{(q^{1+4k}a^2/bcde;q)_4}$$
$$\times\frac{(q^{1+2k}a/bc,q^{1+2k}a/bd,q^{1+2k}a/be,q^{1+2k}a/cd,q^{1+2k}a/ce,q^{1+2k}a/de;q)_2}{(q^{1+3k}a/b,q^{1+3k}a/c,q^{1+3k}a/d,q^{1+3k}a/e;q)_3}$$
$$=\frac{q^{6m^2}a^{4m}}{(bcde)^m}\frac{1-q^{4m}a}{1-a}\frac{(b;q)_m(c;q)_m(d;q)_m(e;q)_m}{(qa^2/bcde;q)_{4m}}$$
$$\times\frac{(qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de;q)_{2m}}{(qa/b,qa/c,qa/d,qa/e;q)_{3m}}$$

we find that, for any natural number $m$, there holds the following general transformation for $\Omega$-series:

$$\Omega(a;b,c,d,e)=\Omega\big(q^{4m}a;q^mb,q^mc,q^md,q^me\big) \tag{3.9a}$$

$$\times\frac{q^{6m^2}a^{4m}}{(bcde)^m}\frac{1-q^{4m}a}{1-a}\frac{(b,c,d,e;q)_m}{(qa^2/bcde;q)_{4m}} \tag{3.9b}$$

$$\times\frac{(qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de;q)_{2m}}{(qa/b,qa/c,qa/d,qa/e;q)_{3m}}. \tag{3.9c}$$

### 3.3. Further transformation on ${}_6\psi_6$-series

Shifting the summation index by $k\to k-2m$, we can further rewrite $\Omega$-series as follows:

$$\Omega\big(q^{4m}a;q^mb,q^mc,q^md,q^me\big)$$
$$={}_6\psi_6\left[\begin{matrix}q^{1+2m}\sqrt{a}, & -q^{1+2m}\sqrt{a}, & q^mb, & q^mc, & q^md, & q^me\\ q^{2m}\sqrt{a}, & -q^{2m}\sqrt{a}, & \frac{q^{1+3m}a}{b}, & \frac{q^{1+3m}a}{c}, & \frac{q^{1+3m}a}{d}, & \frac{q^{1+3m}a}{e}\end{matrix}\,\middle|\,q;\ \frac{q^{1+4m}a^2}{bcde}\right]$$
$$=\sum_{k=-\infty}^{+\infty}\frac{1-q^{2k+4m}a}{1-q^{4m}a}\left[\begin{matrix}q^mb, & q^mc, & q^md, & q^me\\ \frac{q^{1+3m}a}{b}, & \frac{q^{1+3m}a}{c}, & \frac{q^{1+3m}a}{d}, & \frac{q^{1+3m}a}{e}\end{matrix}\,\middle|\,q\right]_k\left\{\frac{q^{1+4m}a^2}{bcde}\right\}^k$$

$$
\begin{aligned}
&=\sum_{k=-\infty}^{+\infty}\frac{1-q^{2k}a}{1-a}\left[\begin{matrix} q^{-m}b, & q^{-m}c, & q^{-m}d, & q^{-m}e \\ \frac{q^{1+m}a}{b}, & \frac{q^{1+m}a}{c}, & \frac{q^{1+m}a}{d}, & \frac{q^{1+m}a}{e}\end{matrix}\Big| q\right]_k \left\{\frac{q^{1+4m}a^2}{bcde}\right\}^k \\
&\quad\times\frac{1-a}{1-q^{4m}a}\left[\begin{matrix} q^{m}b, & q^{m}c, & q^{m}d, & q^{m}e \\ \frac{q^{1+3m}a}{b}, & \frac{q^{1+3m}a}{c}, & \frac{q^{1+3m}a}{d}, & \frac{q^{1+3m}a}{e}\end{matrix}\Big| q\right]_{-2m} \left\{\frac{q^{1+4m}a^2}{bcde}\right\}^{-2m}.
\end{aligned}
$$

Simplifying the factorial fraction displayed in the last line

$$
\begin{aligned}
&\left[\begin{matrix} q^{m}b, & q^{m}c, & q^{m}d, & q^{m}e \\ q^{1+3m}a/b, & q^{1+3m}a/c, & q^{1+3m}a/d, & q^{1+3m}a/e\end{matrix}\Big| q\right]_{-2m} \left\{\frac{q^{1+4m}a^2}{bcde}\right\}^{-2m} \\
&\quad=\left[\begin{matrix} q^{1+m}a/b, & q^{1+m}a/c, & q^{1+m}a/d, & q^{1+m}a/e \\ q^{-m}b, & q^{-m}c, & q^{-m}d, & q^{-m}e\end{matrix}\Big| q\right]_{2m} \left\{\frac{q^{1+4m}a^2}{bcde}\right\}^{-2m} \\
&\quad=\frac{(q^{1+m}a/b,\ q^{1+m}a/c,\ q^{1+m}a/d,\ q^{1+m}a/e;q)_{2m}}{(b,c,d,e,q/b,q/c,q/d,q/e;q)_m}\left\{\frac{q^{6m}a^4}{bcde}\right\}^{-m}
\end{aligned}
$$

we derive the following bilateral transformation

$$
\begin{aligned}
&\Omega\left(q^{4m}a;q^m b,q^m c,q^m d,q^m e\right)=\Omega\left(a;q^{-m}b,q^{-m}c,q^{-m}d,q^{-m}e\right) \\
&\quad\times\frac{1-a}{1-q^{4m}a}\frac{(q^{1+m}a/b,\ q^{1+m}a/c,\ q^{1+m}a/d,\ q^{1+m}a/e;q)_{2m}}{(b,c,d,e,q/b,q/c,q/d,q/e;q)_m}\left\{\frac{q^{6m}a^4}{bcde}\right\}^{-m}.
\end{aligned}
$$

Substituting the last expression into the right-hand side of (3.9a), we establish finally the following surprising transformation on very well-poised bilateral series:

$$
\Omega(a;b,c,d,e)=\Omega\left(a;q^{-m}b,q^{-m}c,q^{-m}d,q^{-m}e\right) \tag{3.10a}
$$

$$
\times\frac{(qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de;q)_{2m}}{(q/b,q/c,q/d,q/e,qa/b,qa/c,qa/d,qa/e;q)_m(qa^2/bcde;q)_{4m}}. \tag{3.10b}
$$

We remark that all the bilateral series involved up to now are convergent under condition $|qa^2/bcde|<1$. Therefore both Abel's lemmas on summation by parts and the iterating process maintain the series convergence.

### *3.4. Confirmation of Bailey's ${}_6\psi_6$-series identity*

When $m\to\infty$, the factorial fraction displayed in (3.10b) becomes almost (2.5b) except for three factorials of infinite order, which correspond to the triple product of Jacobi.

Letting $m\to\infty$ in (3.10) and then appealing the Weierstrass $M$-test on uniformly convergent series (cf. Stromberg [15, p. 141]), we can simplify the limit of $\Omega$-series

$$
\begin{aligned}
&\lim_{m\to+\infty}\Omega\left(a;q^{-m}b,q^{-m}c,q^{-m}d,q^{-m}e\right) \\
&\quad=\lim_{m\to+\infty}{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & q^{-m}b, & q^{-m}c, & q^{-m}d, & q^{-m}e \\ \sqrt{a}, & -\sqrt{a}, & q^{1+m}a/b, & q^{1+m}a/c, & q^{1+m}a/d, & q^{1+m}a/e\end{matrix}\Big| q;\frac{q^{1+4m}a^2}{bcde}\right] \\
&\quad=\sum_{k=-\infty}^{+\infty}\frac{1-aq^{2k}}{1-a}q^{2k^2-k}a^{2k}=\frac{1}{1-a}\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}}a^k
\end{aligned}
$$

where the last line is justified by the parity of summation index.

Recalling the Jacobi triple product identity (cf. [10, II-28]):

$$(q, x, q/x; q)_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} x^k \quad \text{for } |q| < 1 \tag{3.11}$$

we establish the following limiting relation

$$\lim_{m\to+\infty} \Omega\big(a; q^{-m}b, q^{-m}c, q^{-m}d, q^{-m}e\big) = (q;q)_\infty (qa;q)_\infty (q/a;q)_\infty. \tag{3.12}$$

Substituting this expression into (3.10), we obtain the product form of $\Omega$-series:

$$\Omega(a; b, c, d, e) = \left[\begin{matrix} q, & qa, & q/a, & qa/bc, & qa/bd, & qa/be, & qa/cd, & qa/ce, & qa/de \\ qa/b, & qa/c, & qa/d, & qa/e, & q/b, & q/c, & q/d, & q/e, & qa^2/bcde \end{matrix} \middle| q \right]_\infty \tag{3.13}$$

which is exactly the $q$-factorial fraction of infinite order stated in (2.5b). This completes our proof of Bailey's very well-poised bilateral ${}_6\psi_6$-series identity displayed in Theorem 1.

We remark that the Jacobi triple product identity (3.11) can also be considered as limiting case $b, c, d, e \to \infty$ of Bailey's ${}_6\psi_6$-summation formula in view of what we have done for (3.10b).

### 3.5. The Jacobi triple product identity

Our proof relies on Jacobi's triple product identity, which has at least a dozen proofs up to now. The simplest one is due to Cauchy (1843) and Gauss (1866). To make the paper self-contained, we reproduce this proof as follows.

Recall the $q$-binomial theorem (cf. [10, II-4], for example)

$$(x;q)_m = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} x^k \tag{3.14}$$

which can be easily verified by means of induction principle.

Now performing parameter replacements $m \to m+n$, $x \to xq^{-m}$ and $k \to m+k$ and then simplifying the result through relation

$$\big(q^{-m}x;q\big)_{m+n} = \big(q^{-m}x;q\big)_m (x;q)_n = (-1)^m q^{-\binom{1+m}{2}} x^m (q/x;q)_m (x;q)_n$$

we can reformulate the $q$-binomial theorem as the following finite form of the Jacobi triple product identity

$$(x;q)_n (q/x;q)_m = \sum_{k=-m}^{n} (-1)^k \begin{bmatrix} m+n \\ m+k \end{bmatrix} q^{\binom{k}{2}} x^k. \tag{3.15}$$

Letting $m, n \to \infty$ in (3.15) and applying the limiting relation

$$\begin{bmatrix} m+n \\ m+k \end{bmatrix} = \frac{(q;q)_{m+n}}{(q;q)_{m+k}(q;q)_{n-k}} \xrightarrow{m,n\to\infty} \frac{1}{(q;q)_\infty} \quad \text{where } |q| < 1$$

we recover the famous Jacobi triple product identity (3.11).

## 4. Very well-poised bilateral ${}_{10}\psi_{10}$-series identity

With the λ-function being defined by

$$\lambda_w(x,y)=\frac{(w-y)(1-a/wy)}{(x-y)(1-a/xy)} \tag{4.1}$$

we have the following linear combinations

$$\begin{aligned}\big(1-q^ku\big)\big(1-q^ka/u\big)&=\lambda_u(b,d)\big(1-q^kb\big)\big(1-q^ka/b\big)\\&\quad+\lambda_u(d,b)\big(1-q^kd\big)\big(1-q^ka/d\big),\\\big(1-q^kv\big)\big(1-q^ka/v\big)&=\lambda_v(c,e)\big(1-q^kc\big)\big(1-q^ka/c\big)\\&\quad+\lambda_v(e,c)\big(1-q^ke\big)\big(1-q^ka/e\big).\end{aligned}$$

They enable us to express the bilateral ${}_{10}\psi_{10}$-series in terms of ${}_6\psi_6$-series as follows:

$$\begin{aligned}&{}_{10}\psi_{10}\left[\begin{matrix}q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & qu, & qa/u, & qv, & qa/v\\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & u, & a/u, & v, & a/v\end{matrix}\,\bigg|\,q;\frac{q^{-1}a^2}{bcde}\right]\\
&=\lambda_u(b,d)\lambda_v(c,e)\frac{(1-b)(1-a/b)(1-c)(1-a/c)}{(1-u)(1-a/u)(1-v)(1-a/v)}\Omega(a;qb,qc,d,e)\\
&\quad+\lambda_u(b,d)\lambda_v(e,c)\frac{(1-b)(1-a/b)(1-e)(1-a/e)}{(1-u)(1-a/u)(1-v)(1-a/v)}\Omega(a;qb,c,d,qe)\\
&\quad+\lambda_u(d,b)\lambda_v(c,e)\frac{(1-c)(1-a/c)(1-d)(1-a/d)}{(1-u)(1-a/u)(1-v)(1-a/v)}\Omega(a;b,qc,qd,e)\\
&\quad+\lambda_u(d,b)\lambda_v(e,c)\frac{(1-d)(1-a/d)(1-e)(1-a/e)}{(1-u)(1-a/u)(1-v)(1-a/v)}\Omega(a;b,c,qd,qe)\end{aligned}$$

provided that $|q^{-1}a^2/bcde|<1$ for convergence.

Replacing the $\Omega$-function by (3.13), we can reformulate the last result as

$${}_{10}\psi_{10}\left[\begin{matrix}q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & qu, & qa/u, & qv, & qa/v\\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & u, & a/u, & v, & a/v\end{matrix}\,\bigg|\,q;\frac{q^{-1}a^2}{bcde}\right] \tag{4.2a}$$

$$=\frac{(1-a/bc)(1-a/bd)(1-a/be)(1-a/cd)(1-a/ce)(1-a/de)}{(1-u)(1-a/u)(1-v)(1-a/v)(1-a^2/bcde)(1-a^2/bcdeq)} \tag{4.2b}$$

$$\times\,\Omega(a;b,c,d,e)\left\{\begin{matrix}\lambda_u(b,d)\lambda_v(c,e)\frac{bc-a/q}{1-a/de}+\lambda_u(b,d)\lambda_v(e,c)\frac{be-a/q}{1-a/cd}\\ +\lambda_u(d,b)\lambda_v(c,e)\frac{cd-a/q}{1-a/be}+\lambda_u(d,b)\lambda_v(e,c)\frac{de-a/q}{1-a/bc}\end{matrix}\right\}. \tag{4.2c}$$

For each $k$ with $0\leqslant k\leqslant 4$, let $\sigma_k$ be the $k$th elementary symmetric function in $\{b,c,d,e\}$. Then the expression in the braces just displayed can be written in the following symmetric form:

$$\begin{aligned}\text{Eq. (4.2c)}=&\frac{\Omega(a;b,c,d,e)bcde/quv}{(a-bc)(a-bd)(a-be)(a-cd)(a-ce)(a-de)}\\
&\times\big\{(a+u^2)(a+v^2)(a^2-\sigma_4)(a^2-q\sigma_4)\\
&\quad+auv(1-q)(a^2-\sigma_4)(a^2-a\sigma_2+\sigma_4)\\
&\quad+a(a+uv)(u+v)(a^2-q\sigma_4)(\sigma_3-a\sigma_1)\\
&\quad+a^2uv(\sigma_3-a\sigma_1)(q\sigma_3-a\sigma_1)\big\}.\end{aligned}$$

Substituting this expression into (4.2a)–(4.2c) and then simplifying the result, we obtain the following very well-poised bilateral series identity.

**Theorem 2** *(Very well-poised bilateral ${}_{10}\psi_{10}$-series identity). For complex numbers $\{a,b,c,d,e\}$ satisfying $|a^2/bcdeq|<1$, there holds the identity*:

$$
{}_{10}\psi_{10}\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & qu, & qa/u, & qv, & qa/v \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & u, & a/u, & v, & a/v \end{matrix}\,\middle|\, q;\frac{q^{-1}a^2}{bcde}\right] \tag{4.3a}
$$

$$
\begin{aligned}
&= \Omega(a;b,c,d,e)\big\{(a+u^2)(a+v^2)(a^2-\sigma_4)(a^2-q\sigma_4) \\
&\qquad + auv(1-q)(a^2-\sigma_4)(a^2-a\sigma_2+\sigma_4) \\
&\qquad + a(a+uv)\,(u+v)\,(a^2-q\sigma_4)\,(\sigma_3-a\sigma_1) \\
&\qquad + a^2uv(\sigma_3-a\sigma_1)\,(q\sigma_3-a\sigma_1)\big\} \\
&\quad \times \big\{(1-u)(1-v)(a-u)(a-v)(a^2-bcde)(a^2-bcdeq)\big\}^{-1}.
\end{aligned} \tag{4.3b}
$$

Let $u\to\infty$ in Theorem 2, we derive, after some trivial modification, the following important formula.

**Corollary 3** *(Shukla [11]: Very well-poised bilateral ${}_8\psi_8$-series identity). For complex numbers $\{a,b,c,d,e\}$ satisfying $|a^2/bcde|<1$, there holds the identity*:

$$
{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & qv, & qa/v \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & v, & a/v \end{matrix}\,\middle|\, q;\frac{a^2}{bcde}\right] \tag{4.4a}
$$

$$
= \Omega(a;b,c,d,e)\frac{(1-v/b)(1-bv/a)}{(1-v)(1-v/a)}\left\{1-\frac{(1-a/bc)(1-a/bd)(1-a/be)}{(1-v/b)(1-a/bv)(1-a^2/bcde)}\right\}. \tag{4.4b}
$$

The method of this section can further be extended to deal with the following transformation between very well-poised ${}_8\psi_8$-series and balanced ${}_4\phi_3$-series:

$$
{}_8\psi_8\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^mv, & qa/v \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1-m}a/v, & v \end{matrix}\,\middle|\, q;\frac{q^{1-m}a^2}{bcde}\right] \tag{4.5a}
$$

$$
\begin{aligned}
&= \Omega(a;b,c,d,e)\left[\begin{matrix} v/b, & bv/a \\ v, & v/a \end{matrix}\,\middle|\, q\right]_m \\
&\quad \times {}_4\phi_3\left[\begin{matrix} q^{-m}, & bc/a, & bd/a, & be/a \\ & q^{1-m}b/v, & bv/a, & bcde/a^2 \end{matrix}\,\middle|\, q;q\right].
\end{aligned} \tag{4.5b}
$$

This is the terminating version of the transformation due to Jackson [8, Eq. (2.2)] (see also [5, Proposition 17] and [9, Theorem 1.7]).

We remark that Bailey's very well-poised bilateral ${}_6\psi_6$-series identity (2.5a), (2.5b) results also from the common limiting case $v\to\infty$ of (4.4a), (4.4b) and (4.5a), (4.5b).

## 5. Bibasic transformation and symmetric formulae

Define two sequences by factorial fractions

$$\mathcal{A}_k = \left[\begin{matrix} B, & C, & D, & E \\ A/B, & A/C, & A/D, & A/E \end{matrix}\,\middle|\, q\right]_k,$$

$$\mathcal{B}_k = \left[\begin{matrix} pb, & pc, & pd, & pe \\ pa/b, & pa/c, & pa/d, & pa/e \end{matrix}\,\middle|\, p\right]_k.$$

When $a^2 = bcde$ and $A^2 = BCDE$, we can compute the finite differences:

$$\begin{aligned}\triangle\mathcal{A}_k &= \frac{1-Aq^{2k}}{1-A}\left[\begin{matrix} B, & C, & D, & E \\ qA/B, & qA/C, & qA/D, & qA/E \end{matrix}\,\middle|\, q\right]_k q^k \\ &\quad\times \frac{(1-A)(1-A/BC)(1-A/BD)(1-A/CD)}{(1-A/B)(1-A/C)(1-A/D)(1-A/BCD)};\\ \nabla\mathcal{B}_k &= \frac{1-ap^{2k}}{1-a}\left[\begin{matrix} b, & c, & d, & e \\ pa/b, & pa/c, & pa/d, & pa/e \end{matrix}\,\middle|\, p\right]_k p^k \\ &\quad\times \frac{bcd}{a}\frac{(1-a)(1-a/bc)(1-a/bd)(1-a/cd)}{(1-b)(1-c)(1-d)(1-e)}.\end{aligned}$$

According to Abel's lemma $\sum_{n=-\infty}^{+\infty}\mathcal{A}_n\nabla\mathcal{B}_n = \sum_{k=-\infty}^{+\infty}\mathcal{B}_k\triangle\mathcal{A}_k$, we establish the following bilateral bibasic series transformation.

**Theorem 4** *(Bilateral bibasic series transformation). For the indeterminates $p$ and $q$ with $0<|p|<1$ and with $0<|q|<1$, there holds transformation*

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty}\frac{1-ap^{2n}}{1-a}\left[\begin{matrix} b, & c, & d, & e \\ pa/b, & pa/c, & pa/d, & pa/e \end{matrix}\,\middle|\, p\right]_n p^n \left[\begin{matrix} B, & C, & D, & E \\ A/B, & A/C, & A/D, & A/E \end{matrix}\,\middle|\, q\right]_n \\ &\quad= \sum_{k=-\infty}^{+\infty}\frac{1-Aq^{2k}}{1-A}\left[\begin{matrix} B, & C, & D, & E \\ qA/B, & qA/C, & qA/D, & qA/E \end{matrix}\,\middle|\, q\right]_k q^k \\ &\quad\quad\times\left[\begin{matrix} pb, & pc, & pd, & pe \\ pa/b, & pa/c, & pa/d, & pa/e \end{matrix}\,\middle|\, p\right]_k \\ &\quad\quad\times\frac{a}{bcd}\frac{(1-b)(1-c)(1-d)(1-a^2/bcd)}{(1-a)(1-a/bc)(1-a/bd)(1-a/cd)} \\ &\quad\quad\times\frac{(1-A)(1-A/BC)(1-A/BD)(1-A/CD)}{(1-A/B)(1-A/C)(1-A/D)(1-A/BCD)}\end{aligned}$$

*provided that $a^2 = bcde$ and $A^2 = BCDE$, which imply that the first series is 2-balanced with respect to $p$ and the second with respect to $q$.*

**Proof.** We need only to verify the convergence. For the first series, the truncated part $\sum_{n\geqslant 0}$ along the positive direction is convergent for $|p|<1$ and $|q|<1$, which make the series to be comparable with the geometric series $\sum_{n\geqslant 0} p^n$. By means of transformation

$$\frac{(x;q)_{-n}}{(y;q)_{-n}} = \left(\frac{y}{x}\right)^n\frac{(q/y;q)_n}{(q/x;q)_n}$$

the truncated part $\sum_{n<0}$ of the first series along the negative direction can be expressed under replacement $n \to -n$ as

$$\sum_{n=1}^{+\infty} \frac{1-p^{2n}/a}{1-1/a} \left[\begin{matrix} b/a, & c/a, & d/a, & e/a \\ p/b, & p/c, & p/d, & p/e \end{matrix} \middle| p\right]_n p^n \left[\begin{matrix} qB/A, & qC/A, & qD/A, & qE/A \\ q/B, & q/C, & q/D, & q/E \end{matrix} \middle| q\right]_n.$$

This is again a convergent series for $|p|<1$ and $|q|<1$. Therefore the first bilateral series with respect to $n$ is convergent. Similarly, one can check the convergence for the second bilateral series with respect to $k$. $\square$

In particular, letting $e=a$, we derive the following transformation between bibasic unilateral series.

**Proposition 5** *(Bibasic series transformation). For the indeterminates $p$ and $q$ with $0<|p|<1$ and with $0<|q|<1$, there holds transformation*

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \frac{1-ap^{2n}}{1-a} \left[\begin{matrix} a, & b, & c, & d \\ p, & pa/b, & pa/c, & pa/d \end{matrix} \middle| p\right]_n p^n \left[\begin{matrix} B, & C, & D, & E \\ A/B, & A/C, & A/D, & A/E \end{matrix} \middle| q\right]_n \\
&\quad = \sum_{k=0}^{+\infty} \frac{1-Aq^{2k}}{1-A} \left[\begin{matrix} B, & C, & D, & E \\ qA/B, & qA/C, & qA/D, & qA/E \end{matrix} \middle| q\right]_k q^k \\
&\qquad \times \left[\begin{matrix} pa, & pb, & pc, & pd \\ p, & pa/b, & pa/c, & pa/d \end{matrix} \middle| p\right]_k \\
&\qquad \times \frac{(1-A)(1-A/BC)(1-A/BD)(1-A/CD)}{(1-A/B)(1-A/C)(1-A/D)(1-A/BCD)}
\end{aligned}$$

*provided that $a=bcd$ and $A^2=BCDE$, which imply that the first series is 2-balanced with respect to $p$ and the second with respect to $q$.*

Performing parameter replacements in the last transformation

$$\begin{aligned}
&A \to q^{-2m}/A \\
&B \to q^{-m}B/A \\
&C \to q^{-m}C/A \\
&D \to q^{-m}D/A \\
&E \to q^{-m}
\end{aligned}$$

and then reversing the finite series on the right-hand side by $k \to m-k$, we get the following symmetric transformation.

**Corollary 6** *(Bibasic symmetric transformation). For complex parameters $\{a,b,c,d\}$ and $\{A,B,C,D\}$ satisfying $a=bcd$ and $A=BCD$ respectively, there holds bibasic symmetric transformation*

$$\sum_{n=0}^{m} \frac{1-ap^{2n}}{1-a} \left[\begin{matrix} a, & b, & c, & d \\ p, & pa/b, & pa/c, & pa/d \end{matrix} \middle| p\right]_n p^n$$

$$\times\left[\begin{matrix} q^{-m}, & q^{-m}B/A, & q^{-m}C/A, & q^{-m}D/A \\ q^{-m}/A, & q^{-m}/B, & q^{-m}/C, & q^{-m}/D \end{matrix}\bigg| q\right]_n$$

$$=\sum_{k=0}^{m}\frac{1-Aq^{2k}}{1-A}\left[\begin{matrix} A, & B, & C, & D \\ q, & qA/B, & qA/C, & qA/D \end{matrix}\bigg| q\right]_k q^k$$

$$\times\left[\begin{matrix} p^{-m}, & p^{-m}b/a, & p^{-m}c/a, & p^{-m}d/a \\ p^{-m}/a, & p^{-m}/b, & p^{-m}/c, & p^{-m}/d \end{matrix}\bigg| p\right]_k$$

$$\times\left[\begin{matrix} pa, & pb, & pc, & pd \\ p, & pa/b, & pa/c, & pa/d \end{matrix}\bigg| p\right]_m \left[\begin{matrix} q, & qA/B, & qA/C, & qA/D \\ qA, & qB, & qC, & qD \end{matrix}\bigg| q\right]_m.$$

In fact, if letting

$$\Xi\left[\begin{matrix} p: & a, & b, & c, & d \\ q: & A, & B, & C, & D \end{matrix}\right]=\left[\begin{matrix} p, & pa/b, & pa/c, & pa/d \\ pa, & pb, & pc, & pd \end{matrix}\bigg| p\right]_m$$

$$\times\sum_{n=0}^{m}\frac{1-ap^{2n}}{1-a}\left[\begin{matrix} a, & b, & c, & d \\ p, & pa/b, & pa/c, & pa/d \end{matrix}\bigg| p\right]_n p^n$$

$$\times\left[\begin{matrix} q^{-m}B/A, & q^{-m}C/A, & q^{-m}D/A, & q^{-m} \\ q^{-m}/B, & q^{-m}/C, & q^{-m}/D, & q^{-m}/A \end{matrix}\bigg| q\right]_n$$

then we can reformulate the transformation in Corollary 6 as the following symmetric expression:

$$\Xi\left[\begin{matrix} p: & a, & b, & c, & d \\ q: & A, & B, & C, & D \end{matrix}\right]=\Xi\left[\begin{matrix} q: & A, & B, & C, & D \\ p: & a, & b, & c, & d \end{matrix}\right].$$

When $p=q$, the last relation reduces to the "split-poised" transformation.

**Corollary 7.** (Gasper [6, Eq. (2.11)]) *Under the same condition of Corollary* 6, *there holds transformation*

$${}_{10}\phi_9\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & q^{-m}, & q^{-m}B/A, & q^{-m}C/A, & q^{-m}D/A \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & q^{-m}/A, & q^{-m}/B, & q^{-m}/C, & q^{-m}/D \end{matrix}\bigg| q;q\right]$$

$$={}_{10}\phi_9\left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & q^{-m}, & q^{-m}b/a, & q^{-m}c/a, & q^{-m}d/a \\ & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & q^{-m}/a, & q^{-m}/b, & q^{-m}/c, & q^{-m}/d \end{matrix}\bigg| q;q\right]$$

$$\times\left[\begin{matrix} qa, & qb, & qc, & qd, & qA/B, & qA/C, & qA/D \\ qA, & qB, & qC, & qD, & qa/b, & qa/c, & qa/d \end{matrix}\bigg| q\right]_m.$$

## References


[1] G.E. Andrews, Applications of basic hypergeometric series, SIAM Rev. 16 (1974) 441–484.

[2] R. Askey, The very well-poised ${}_6\psi_6$ II, Proc. Amer. Math. Soc. 90 (1984) 575–579.

[3] R. Askey, M.E.H. Ismail, The very well-poised ${}_6\psi_6$, Proc. Amer. Math. Soc. 77 (1979) 218–222.

[4] W.N. Bailey, Series of hypergeometric type which are infinite in both directions, Quart. J. Math. (Oxford) 7 (1936) 105–115.

[5] W. Chu, The Saalschütz chain reactions and bilateral basic hypergeometric series, Constr. Approx. 18 (4) (2002) 579–597. MR 2003h:33017.

[6] G. Gasper, Summation, transformation, and expansion formulas for bibasic series, Trans. Amer. Math. Soc. 312 (1) (1989) 257–277.

[7] G. Gasper, Elementary derivations of summation and transformation formulas for $q$-series, Fields Inst. Commun. 14 (1997) 55–70.
[8] M. Jackson, On well-poised bilateral hypergeometric series of the type ${}_8\Psi_8$, Quart. J. Math. (Ser. 2, Oxford) 1 (1950) 63–68.
[9] S.C. Milne, A multiple series transformation of the very well poised ${}_{4+2k}\Psi_{2k+4}$, Pacific J. Math. 91 (1980) 419–430.
[10] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge Univ. Press, Cambridge, 2004.
[11] H.S. Shukla, A note on the sums of certain bilateral hypergeometric series, Proc. Cambridge Philos. Soc. 55 (1959) 262–266.
[12] L.J. Slater, L. Lakin, Two proofs of the ${}_6\psi_6$ summation theorem, Proc. Edinburgh Math. Soc. (2) 9 (1953–1957) 116–121.
[13] F. Jouhet, M. Schlosser, Another proof of Bailey's ${}_6\psi_6$ summation, Aequationes Math., in press.
[14] M. Schlosser, A simple proof of Bailey's very well-poised ${}_6\psi_6$ summation, Proc. Amer. Math. Soc. 130 (2002) 1113–1123.
[15] K.R. Stromberg, An Introduction to Classical Real Analysis, Wadsworth, Belmont, CA, 1981.