Dirac cohomology and translation functors

S. Mehdia,⁎, P. Pandžicb

a Département de Mathématiques, CNRS-7122, Université de Lorraine – Metz, France
b Department of Mathematics, University of Zagreb, Croatia

Abstract

We study the relationship between the Dirac cohomology of a \((g,K)\)-module \(X\) and the Dirac cohomology of a Jantzen–Zuckerman translate of \(X\). More precisely, we show that if \(X\) is unitary, and if some submodule \(X'\) of a translate of \(X\) has nonzero Dirac cohomology, then \(X\) has nonzero Dirac cohomology. We also show that the space of harmonic spinors (i.e., the kernel of the Dirac operator) related to \(X'\) embeds into a certain product of harmonic spinors for \(X\) and harmonic spinors for the finite-dimensional module used to define the translation. This generalizes, with a simpler proof, results of Mehdı and Parthasarathy (2008) [MP1] and Mehdı and Parthasarathy (2010) [MP2].

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Let \(G\) be a connected real reductive Lie group with Cartan involution \(Θ\) such that \(K = G^Θ\) is a maximal compact subgroup of \(G\). Let \(g = ℱ \oplus p\) be the corresponding Cartan decomposition of the complexified Lie algebra of \(G\). Let \(B\) be an invariant nondegenerate symmetric bilinear form on \(g\), such that \(B(ℱ, p) = 0\). Then \(B\) is nondegenerate on both \(ℱ\) and \(p\). Let \(U(g)\) be the universal enveloping algebra of \(g\) and let \(C(p)\) be the Clifford algebra of \(p\) with respect to \(B\). Let \(D \in U(g) \otimes C(p)\) be the Dirac operator [P1,V2]. \(D\) is defined as

\[
D = \sum_i b_i \otimes d_i,
\]

where \(b_i\) is any basis of \(p\) and \(d_i\) is the dual basis with respect to \(B\). \(D\) is independent of the choice of the basis \(b_i\), and \(K\)-invariant for the tensor product of adjoint actions on the factors.

⁎ Corresponding author.
E-mail addresses: mehdi@univ-metz.fr (S. Mehdı), pandzic@math.hr (P. Pandžic).

0021-8693/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jalgebra.2012.11.022
Let \( X \) be a \((g, K)\)-module, and let \( S \) be a spin module for \( C(p) \). Then \( D \) acts on \( X \otimes S \). The Dirac cohomology of \( X \) is defined as

\[
H_D(X) = \ker D / \ker D \cap \text{Im} D.
\]

It is a module for the spin double cover \( \tilde{R} \) of \( K \).

Dirac cohomology has turned out to be an interesting invariant of \((g, K)\)-modules. First, the modules having nonzero Dirac cohomology are interesting. They include most of the \( A_q(\lambda) \)-modules [HKP], in particular the discrete series representations; finite-dimensional modules [K3,HKP,MZ]; unitary highest weight modules [HPR,HPP]; and many of the unipotent representations [BP]. Furthermore, Dirac cohomology is related to other better known kinds of cohomology: \((g, K)\)-cohomology [HP1, HKP] and, in some cases, \( n \)-cohomology [HPR]. Finally, as we will see in Section 2, unitary representations with Dirac cohomology are in a certain precise sense extremal among all unitary representations. It is an interesting open problem to classify all such representations.

We remark that it is also interesting to study analogues of Dirac operators and cohomology in other settings, in particular the cohomology with respect to Kostant’s cubic Dirac operators. For that setting, see [K2,K3,MP3]. For some further generalizations, see [Ku,AM,KMP].

Another important tool in representation theory is the translation principle, based on the Jantzen–Zuckerman translation functors [JZ]; for a comprehensive treatment, see [KV, Chapter VII]. The definition is as follows. Let \( Z(g) \) be the center of \( \mathfrak{u}(g) \), let \( \mathfrak{h} \) be a Cartan subalgebra of \( g \), and let \( \chi_\lambda : Z(g) \to \mathbb{C} \) be the character corresponding to \( \lambda \in \mathfrak{h}^* \) under the Harish-Chandra isomorphism. Let \( M(g, K)_\lambda \) denote the category of \((g, K)\)-modules with generalized infinitesimal character \( \chi_\lambda \). In other words, a \((g, K)\)-module \( X \) is in \( M(g, K)_\lambda \) if there is a positive integer \( n \) such that \( X \) is annihilated by \((z - \chi_\lambda(z))^n\) for every \( z \in Z(g) \).

Let \( X \in M(g, K)_\lambda \) and let \( F_\nu \) be the irreducible finite-dimensional \((g, K)\)-module with extremal weight \( \nu \in \mathfrak{h}^* \). By a theorem of Kostant [K1], the module \( X \otimes F_\nu \) is \( Z(g) \)-finite. The Jantzen–Zuckerman translate of \( X \) by \( \nu \) is the summand of \( X \otimes F_\nu \) with generalized infinitesimal character \( \chi_{\lambda + \nu} \). In this way one obtains an exact covariant functor

\[
\Psi_{\lambda + \nu}^{\lambda + \nu} : M(g, K)_\lambda \to M(g, K)_{\lambda + \nu}.
\]

It turns out that this functor is an equivalence of categories if \( \lambda \) and \( \lambda + \nu \) are in the same (integral) Weyl chamber, and if they are “equisingular”, i.e., their stabilizers in the Weyl group are the same. In particular, under these assumptions the functor \( \Psi_{\lambda + \nu}^{\lambda + \nu} \) takes an irreducible (nonzero) \((g, K)\)-module with infinitesimal character \( \chi_\lambda \) to an irreducible (nonzero) \((g, K)\)-module with infinitesimal character \( \chi_{\lambda + \nu} \).

If \( \lambda + \nu \) is more singular than \( \lambda \), then \( \Psi_{\lambda + \nu}^{\lambda + \nu} \) takes an irreducible module either to an irreducible module or to zero. Other cases are increasingly more complicated.

This paper is devoted to studying the relation between the Dirac cohomology of a module \( X \in M(g, K)_\lambda \) and the Dirac cohomology of its translate \( \Psi_{\lambda + \nu}^{\lambda + \nu}(X) \). Mehdi and Parthasarathy obtained some results in this direction in the case of discrete series representations and cohomologically induced representations [MP1,MP2]. In the cases they studied, Dirac cohomology is the same as the space of harmonic spinors, \( \ker D \). They relate the harmonic spinors for \( X, \Psi_{\lambda + \nu}^{\lambda + \nu}(X) \) and \( F_\nu \) by a certain “product” (Theorem 4.2 in [MP1], Theorem 1 in [MP2]). In this paper we generalize, with a much simpler proof, their results to the case when \( X \) is a \((g, K)\)-module satisfying the Dirac inequality (see Lemma 3.2 and Proposition 5.2). This condition will be automatically satisfied for unitary modules (see Proposition 2.2). We also prove the following theorem relating Dirac cohomology of a module with the Dirac cohomology of its translates. As we shall see, this theorem does not require the results about harmonic spinors. However, Proposition 5.2 establishes a translation principle for harmonic spinors. More precisely, the conclusion of the theorem below remains valid if one replaces Dirac cohomology by harmonic spinors, without the assumption of unitarity on \( X_\lambda \), only the Dirac inequality is required.

**Theorem 1.3.** Let \( \mathfrak{h} \) be a Cartan subalgebra of \( g \), and let \( \lambda \in \mathfrak{h}^* \). Let \( X_\lambda \) be a unitary \((g, K)\)-module with infinitesimal character \( \chi_\lambda \). Let \( F_\nu \) be the irreducible finite-dimensional \((g, K)\)-module with extremal weight...
Let $X_{λ+ν}$ be a (not necessarily unitary) $(g, K)$-module with infinitesimal character $χ_{λ+ν}$. Suppose that there is an embedding

$$X_{λ+ν} \hookrightarrow X_λ \otimes F_ν.$$  

In other words, $X_{λ+ν}$ is a submodule of $Ψ_{λ+ν}^-(X_λ)$. Suppose also that the kernel of the Dirac operator on $X_{λ+ν} \otimes S$ is nonzero. Then the Dirac cohomology of $X_λ$ is nonzero. In particular, if the Dirac cohomology of $X_{λ+ν}$ is nonzero, then the Dirac cohomology of $X_λ$ is nonzero.

As the notation suggests, $X_λ$ and $X_{λ+ν}$ in the above result can be members of a coherent family (see [V1, Chapter 7]). However, it is not possible to consider this approach to the full extent, since Dirac cohomology is not defined for virtual $(g, K)$-modules. This problem disappears when Dirac cohomology is replaced by its Euler characteristic, the Dirac index. In that case one can obtain much more precise results. This is the subject of our forthcoming joint paper with David Vogan [MPV].

The referee for this paper raised the question of extending Theorem 1.3 to include the case of non-unitary modules $X_λ$. In view of this, we have relaxed the conditions so that Lemma 3.2 and Proposition 5.2 now require only the Dirac inequality to hold on a part of the module. Except in some very simple examples (e.g. $G = SL(2, \mathbb{R})$), modules satisfying the Dirac inequality are not necessarily unitary.

Prompted by this question of the referee, we provide a counterexample in Section 4, with the assertion of Theorem 1.3 failing for certain non-unitary $X_λ$. (This $X_λ$ is however reducible, and it remains open whether the theorem holds for irreducible non-unitary $X_λ$.) We remark that in the setting of [MPV], when the Dirac cohomology is replaced by its Euler characteristic, this problem disappears and unitarity plays no role in the translation principle.

The suggestions of the referee also led us to a simplification in the proof of Theorem 1.3; in particular, the present proof does not use Proposition 5.2. We thank the referee for pointing out this direction to us.

2. Preliminaries on Dirac cohomology

We keep the notation from the introduction. In particular, the Dirac operator $D ∈ U(g) \otimes C(p)$ is given by (1.1), and for a $(g, K)$-module $X$, its Dirac cohomology $H_D(X)$ is defined by (1.2). The following facts can be found in [HP2].

An important property of $D$ is the fact that its square is given by the following formula due to Parthasarathy [P1]:

$$D^2 = -(\text{Cas}_g \otimes 1 + ∥ρ_\|)^2 + (\text{Cas}_λ + ∥ρ_λ∥^2).$$  \hspace{1cm} (2.1)

where $\text{Cas}_g$ (resp. $\text{Cas}_λ$) is the Casimir element of $U(g)$ (resp. $U(λ, λ)$), and $λ$ is the diagonal copy of $λ$ in $U(g) \otimes C(p)$, defined using the obvious embedding $λ \hookrightarrow U(g)$ and the usual map $λ \mapsto so(p) \mapsto C(p)$. This property immediately implies that $D^2$ is a scalar on every $K$-type of $X \otimes S$, and that the eigenspaces of $D^2$ are finite-dimensional whenever $X$ is admissible. In particular, it follows that $H_D(X)$ is finite-dimensional for admissible $X$.

If $X$ is unitary, then we can combine the corresponding inner product on $X$ with the usual inner product on the spin module $S$, and get an inner product on $X \otimes S$ such that $D$ is self-adjoint. Similarly, if $F$ is a finite-dimensional $(g, K)$-module, one can use the inner product on $F$, invariant for a compact form of $g$, and combine it with the same usual inner product on $S$ to conclude that $D$ is skew self-adjoint. This leads to

**Proposition 2.2.** If $X$ is a unitary $(g, K)$-module, then there is an inner product on $X \otimes S$ such that $D$ is self-adjoint. In particular, $D^2 \geq 0$ on $X \otimes S$, and

$$H_D(X) = \text{Ker} D = \text{Ker} D^2.$$  \hspace{1cm} (2.3)
If \( F \) is a finite-dimensional \((g, K)\)-module, then there is an inner product on \( F \otimes S \) such that \( D \) is skew self-adjoint. In particular, \( D^2 \leq 0 \) on \( F \otimes S \), and (2.3) holds for \( F \).

The “Dirac inequality” \( D^2 \geq 0 \) is a very useful necessary condition for unitarity due to Parthasarathy [P2]. It can be written out more explicitly using (2.1). We can now explain the claim from the introduction about unitary modules with nonzero Dirac cohomology being extremal among all unitary modules. Namely, \( D^2 \geq 0 \) for all unitary modules, and modules with Dirac cohomology are exactly those for which the equality is attained on at least one \( \lambda \) corresponding to \( \lambda \in \mathfrak{h}^* \).

As we shall see, this lemma implies Theorem 1.3.

3. Proof of Theorem 1.3

Let \( X_\lambda, X_{\lambda+\nu} \) and \( F_\nu \) be as in the statement of Theorem 1.3. In particular, \( X_\lambda \) has infinitesimal character corresponding to \( \lambda \in \mathfrak{h}^* \), \( X_{\lambda+\nu} \) has infinitesimal character corresponding to \( \lambda + \nu \), and \( F_\nu \) has extremal weight \( \nu \).

Furthermore, there is an embedding

\[ \varphi : X_{\lambda+\nu} \hookrightarrow X_\lambda \otimes F_\nu. \]

The embedding \( \varphi \) gives rise to

\[ \varphi = \varphi \otimes \text{id} : X_{\lambda+\nu} \otimes S \hookrightarrow X_\lambda \otimes F_\nu \otimes S \quad (3.1) \]

where \( S \) is the spin module for \( C(p) \).

The Dirac operator \( D \) acts on each of the modules \( X_{\lambda+\nu} \otimes S, X_\lambda \otimes S, F_\nu \otimes S \) and \( X_\lambda \otimes F_\nu \otimes S \). To avoid any ambiguity, we denote this operator by \( D_{X_{\lambda+\nu}}, D_{X_\lambda}, D_{F_\nu} \) and \( D_{X_\lambda \otimes F_\nu} \) respectively. Moreover, we denote by \( D_1 \) respectively \( D_2 \) the operators on \( X_\lambda \otimes F_\nu \otimes S \) given by tensoring \( D_{X_\lambda} \) with the identity on \( F_\nu \), respectively \( D_{F_\nu} \) with the identity on \( X_\lambda \). By definition one has

\[ D_{X_\lambda \otimes F_\nu} = D_1 + D_2. \]

In this setting, we prove the following lemma. As we shall see, this lemma implies Theorem 1.3.

Lemma 3.2. Let \( X_\lambda, X_{\lambda+\nu} \) and \( F_\nu \) be \((g, K)\)-modules as above. Assume that:

\[ D_1^2 \geq 0 \quad \text{on} \quad \varphi(\text{Ker}(D_{X_{\lambda+\nu}})). \quad (3.3) \]

Then one has

\[ \varphi(\text{Ker}(D_{X_{\lambda+\nu}})) \subseteq \text{Ker} D_1^2 \cap \text{Ker} D_2. \]

**Proof.** Since \( \varphi \) is \( g \)-equivariant, we see that the map (3.1) satisfies the property \( \varphi \circ D_{X_{\lambda+\nu}} = D_{X_\lambda \otimes F_\nu} \circ \varphi \). It follows that \( D_1 + D_2 = 0 \) on \( \varphi(\text{Ker}(D_{X_{\lambda+\nu}})) \), and hence

\[ D_1^2 = D_2^2 \quad \text{on} \quad \varphi(\text{Ker}(D_{X_{\lambda+\nu}})). \]
Since $F_v$ is finite-dimensional, Proposition 2.2 says that $D_{F_v}^2$ is non-positive, and $\text{Ker}(D_{F_v}^2) = \text{Ker}(D_{F_v})$; the same then holds for $D_2$. The claim now follows from condition (3.3). □

**Proof of Theorem 1.3.** Since $\varphi$ is injective, $\text{Ker}(D_{X_{\lambda+v}}) \neq 0$ implies that $\varphi(\text{Ker}(D_{X_{\lambda+v}})) \neq 0$. Since $X_\lambda$ is unitary, Proposition 2.2 implies that $D_{X_{\lambda}}^2 \geq 0$, and hence $D_1^2 \geq 0$. So we can apply Lemma 3.2 and conclude that $\text{Ker}(D_1^2)$ cannot be zero, because it contains $\varphi(\text{Ker}(D_{X_{\lambda+v}})) \neq 0$.

On the other hand, $\text{Ker}(D_1) = \text{Ker}(D_{X_{\lambda}}) \otimes F_v$ and $\text{Ker}(D_1^2) = \text{Ker}(D_{X_{\lambda}}^2) \otimes F_v$. Since $X_\lambda$ is unitary, Proposition 2.2 implies that

$$\text{Ker}(D_1^2) = \text{Ker}(D_1) = \text{Ker}(D_{X_{\lambda}}) \otimes F_v = H_D(X_{\lambda}) \otimes F_v \neq 0.$$ 

The theorem follows. □

4. A counterexample

In this section we give an example of a non-unitary module $X_{\lambda}$ for which the conclusion of Theorem 1.3 fails.

Let $G$ be the group $\text{SL}(2, \mathbb{R})$. We use the notation of [HP2, Section 1.3.10] (adapted from [V1, Chapter 1]). In particular, we denote by $V_{\lambda, \epsilon}$ the principal series representation with parameters $\lambda \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$. This representation has a basis $v_n$, $n \equiv \epsilon \mod 2$, and the action of the basic elements

$$W = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad X = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}; \quad Y = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

of $\mathfrak{sl}(2, \mathbb{C})$ is given in this basis by

$$W \cdot v_n = n v_n; \quad X \cdot v_n = \frac{1}{2}(\lambda + (n+1))v_{n+2};$$

$$Y \cdot v_n = \frac{1}{2}(\lambda - (n-1))v_{n-2}. \quad (4.1)$$

We take for $X_{\lambda}$ the principal series representation $V_{1,0}$. In particular, $X_{\lambda}$ has even $K$-types and infinitesimal character $\lambda = \rho_0 = 1$. It contains the trivial module as the unique irreducible quotient and two discrete series representations as submodules.

We take $\nu = -1$, so $F_v$ is the standard (two-dimensional) module.

The module $X_{\lambda+1}$ is the principal series representation $V_{0,1}$, with odd $K$-types and infinitesimal character zero. It is a direct sum of two limits of discrete series representations.

It is well known that

$$V_{1,0} \otimes F_v = V_{0,1} \oplus V_{2,1}. \quad (4.2)$$

This fact can be deduced from general principles, but it is also not difficult to check explicitly. To see this, let us denote by $f_1$ and $f_{-1}$ the standard weight vectors in $F_v$, and by $v_n$, $n \in 2\mathbb{Z}$, the basis of $V_{1,0}$ from (4.1). Then $v_{-2} \otimes f_1$ respectively $v_2 \otimes f_{-1}$ are highest respectively lowest weight vectors in $V_{1,0} \otimes F_v$ (because $v_{-2}$ respectively $v_2$ are highest respectively lowest weight vectors in $V_{1,0}$). These two vectors generate a copy of $V_{0,1}$. On the other hand, a calculation shows that the vector $v_0 \otimes f_1 + v_2 \otimes f_{-1}$ generates a copy of $V_{2,1}$.

It follows from (4.2) that $X_{\lambda+1} = V_{0,1}$ is the Jantzen–Zuckerman translate of $X_{\lambda} = V_{1,0}$. In particular, the assumption $X_{\lambda+1} \leftrightarrow X_{\lambda} \otimes F_v$ of Theorem 1.3 holds. However, the assumption that $X_{\lambda}$ is unitary does not hold.
Now we recall that by [HP2, 9.6.5], Dirac cohomology for any \((g, K)\)-module \(V\) for \(SL(2, \mathbb{R})\) can be calculated from the formula

\[
H_D(V) = \left( \text{Ker} Y / (\text{Im} X \cap \text{Ker} Y) \otimes s_{-1} \right) \oplus \left( \text{Ker} X / (\text{Im} Y \cap \text{Ker} X) \otimes s_1 \right),
\]

where \(s_{\pm 1}\) denote the basic elements of the spin module \(S\) of weights \(-1\) and \(1\) respectively. This formula combined with (4.1) implies that \(H_D(X_\lambda)\) is 0, while \(H_D(X_{\lambda + \nu})\) consists of two copies of the trivial \(\tilde{K}\)-module. Thus the conclusion of Theorem 1.3 fails.

We finish this section by remarking that the two copies of the trivial module actually cancel in the index, i.e., the Dirac index of \(X_{\lambda + \nu} = V_{0, 1}\) is zero. This illustrates the fact that Dirac index translates better than Dirac cohomology, as mentioned in the introduction.

5. Translation and harmonic spinors

We keep the notation of Section 3. In particular, we again consider the embedding

\[
\varphi : X_{\lambda + \nu} \otimes S \hookrightarrow X_\lambda \otimes F_\nu \otimes S,
\]

and Dirac operators \(D_{X_{\lambda + \nu}}, D_{X_\lambda, D_{F_\nu}, D_{X_\lambda \otimes F_\nu}}, D_1\) and \(D_2\). Recall that \(D_{X_\lambda \otimes F_\nu} = D_1 + D_2\).

We will also need the map

\[
\beta : (X_\lambda \otimes S) \otimes (F_\nu \otimes S) \otimes S^* \to X_\lambda \otimes F_\nu \otimes S,
\]

defined by contracting the second factor \(S\) and the fifth factor \(S^*\). In other words,

\[
\beta(x \otimes s_1 \otimes f \otimes s_2 \otimes \phi) = \phi(s_1)x \otimes f \otimes s_2.
\]

(See Section 4 in [MP1].) In this setting, we prove the following proposition. This proposition generalizes, with a much simpler proof, Theorem 4.2 in [MP1] and Theorem 1 in [MP2]. It expresses harmonic spinors corresponding to \(X_{\lambda + \nu}\), i.e., \(\text{Ker}(D_{X_{\lambda + \nu}})\), as a linear combination of tensor products of spinors in the kernel of \(D_{X_\lambda}^2\) with harmonic spinors corresponding to \(F_\nu\). In particular, the conclusion of Theorem 1.3 remains valid if one replaces Dirac cohomology by harmonic spinors, without the assumption of unitarity on \(X_\lambda\), only the Dirac inequality is required.

**Proposition 5.2.** Let \(X_\lambda, X_{\lambda + \nu}\) and \(F_\nu\) be \((g, K)\)-modules as above. Assume that (3.3) holds, i.e., that

\[
D_1^2 \geq 0 \quad \text{on} \quad \varphi(\text{Ker}(D_{X_{\lambda + \nu}})).
\]

Then one has

\[
\varphi(\text{Ker}(D_{X_{\lambda + \nu}})) \subseteq \beta(\text{Ker}(D_{X_\lambda}^2) \otimes \text{Ker}(D_{F_\nu}) \otimes S^*).
\]

In particular, if \(\text{Ker}(D_{X_{\lambda + \nu}})\) is nonzero then both \(\text{Ker}(D_{X_\lambda})\) and \(\text{Ker}(D_{F_\nu})\) are nonzero.

**Proof.** Recall that by Lemma 3.2, \(\varphi(\text{Ker}(D_{X_{\lambda + \nu}})) \subseteq \text{Ker}(D_1^2) \cap \text{Ker}(D_2)\), so it is enough to prove that

\[
\text{Ker}(D_1^2) \cap \text{Ker}(D_2) \subseteq \beta(\text{Ker}(D_{X_\lambda}^2) \otimes \text{Ker}(D_{F_\nu}) \otimes S^*).
\]

We write an element of \(\text{Ker}(D_1^2) \cap \text{Ker}(D_2)\) in the form
\[
\sum_{i=1}^{r} \sum_{j=1}^{s} x_i \otimes f_j \otimes s_{ij},
\]
(5.4)

for some linearly independent \(x_i \in X_\lambda\), some linearly independent \(f_j \in F_\nu\), and some \(s_{ij} \in S\), such that

\[
D_{X_\lambda}^2 \left( \sum_{i=1}^{r} x_i \otimes s_{ij} \right) = 0 \text{ for any } j \in \{1, \ldots, s\}
\]

and

\[
D_{F_\nu} \left( \sum_{j=1}^{s} f_j \otimes s_{ij} \right) = 0 \text{ for any } i \in \{1, \ldots, r\}.
\]

(5.3) will be proved if we show that the element (5.4) is equal to

\[
\beta \left( \sum_{k=1}^{s} \sum_{l=1}^{r} \left( \sum_{i=1}^{r} x_i \otimes s_{ik} \right) \otimes \left( \sum_{j=1}^{s} f_j \otimes s_{lj} \right) \otimes \phi_{kl} \right)
\]

for some choice of \(\phi_{kl} \in S^*\). To see this, it is enough to show that for any choice of \(s_{ij} \in S\), there is a choice of \(\phi_{kl} \in S^*\), such that

\[
\sum_{k=1}^{s} \sum_{l=1}^{r} \phi_{kl}(s_{ik})s_{lj} = s_{ij}, \text{ for any } (i, j) \in \{1, \ldots, r\} \times \{1, \ldots, s\}.
\]

(5.5)

We will prove this as part (d) of the following lemma, which contains some easy but not so familiar facts from linear algebra. The rest of the proposition is immediate. \(\square\)

**Lemma 5.6.**

(a) Let \(A\) be an \(n \times m\) complex matrix. Then there is an \(m \times n\) complex matrix \(B\) such that

\[
ABA = A.
\]

(b) Let \(S\) be a finite-dimensional complex vector space with a bilinear inner product \((\cdot | \cdot)\). Let \(A\) be an \(n \times m\) matrix with entries from \(S\). Then there is an \(m \times n\) matrix \(C\) with entries from \(S\) such that

\[
(A|C)A = A.
\]

Here the inner product \((A|C)\) is defined as the ordinary matrix product, except that in place of multiplication of the (scalar) matrix elements we perform the inner product of the (vector) matrix elements. Similarly, the product of the scalar matrix \((A|C)\) with the vector matrix \(A\) is defined as the ordinary matrix product, with product of the matrix elements being the scalar multiplication.

(c) Let \(s_{ij}, i = 1, \ldots, r, j = 1, \ldots, s\), be any choice of vectors in \(S\). Then there are vectors \(t_{kl} \in S, l = 1, \ldots, r, k = 1, \ldots, s\), such that

\[
\sum_{l=1}^{r} \sum_{k=1}^{s} (s_{ik} | t_{kl})s_{lj} = s_{ij}, \text{ for any } (i, j) \in \{1, \ldots, r\} \times \{1, \ldots, s\}.
\]
(d) Let $s_{ij}, i = 1, \ldots, r, j = 1, \ldots, s$, be any choice of vectors in $S$. Then there are linear functionals $\phi_{kl} \in S^*$, $k = 1, \ldots, s, l = 1, \ldots, r$, such that (5.5) holds.

**Proof.** (a) Using the standard basis, we can think of $A$ as a linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$. Let $v_1, \ldots, v_k$ be a basis of $\text{Im} A$, and let $W$ be a complement to $\text{Im} A$ in $\mathbb{C}^n$. Pick $v'_i \in \mathbb{C}^m$ such that $A v'_i = v_i$. Then $v'_i$ form a basis for a complement of $\text{Ker} A$ in $\mathbb{C}^m$. Define $B : \mathbb{C}^m \to \mathbb{C}^m$ by setting $B v_i = v'_i$ and $B|_W = 0$. We now see that $ABA = A$, since $ABAv'_i = v_i = Av'_i$ and $ABA|_{\text{Ker} A} = 0 = A|_{\text{Ker} A}$.

(b) Let $\dim S = d$. Let $\chi : M_{n,m}(S) \to M_{d,m}(C)$ be the isomorphism obtained by expanding every matrix element into a row-vector, using a fixed basis of $S$. Likewise, let $\eta : M_{d,l}(S) \to M_{d,m,l}(C)$ be the isomorphism obtained by expanding every matrix element into a column-vector. If now $X \in M_{n,m}(S)$ and $Y \in M_{n,m}(S)$, then

$$\chi(X | Y) = \chi(X) \eta(Y), \quad (5.7)$$

where the last product is the ordinary matrix product of scalar matrices. Also, if $Z \in M_{n,m}(C)$, then

$$Z Y = \chi^{-1}(Z \chi(Y)) \quad (5.8)$$

where $Z \chi(Y)$ is the ordinary matrix product of $Z$ and $\chi(Y)$. Let now $A \in M_{n,m}(S)$. By (a), for the matrix $\chi(A) \in M_{n,m}(C)$ there is a matrix $B \in M_{d,m,n}(C)$ such that

$$\chi(A)B \chi(A) = \chi(A). \quad (5.9)$$

Let $C = \eta^{-1}(B)$. Then using (5.8) and (5.7), along with the definition of $C$ and (5.9), we see that

$$(A|C)A = \chi^{-1}(\chi(A)) \eta(C) \chi(A) = \chi^{-1}(\chi(A)B \chi(A)) = \chi^{-1}(\chi(A)) = A. \quad (5.10)$$

(c) Setting $A = (s_{ij})$ and $C = (t_{ij})$, we see that the claim is equivalent to (b).

(d) The claim follows from (c), by setting $\phi_{kl}(s) = (s|t_{kl})$ for $s \in S$. $\square$

**References**


[V2] D.A. Vogan, Dirac operators and unitary representations, 3 talks at MIT Lie groups seminar, Fall 1997.