J. Differential Equations 246 (2009) 1744–1753

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Hyperbolicity and exponential convergence of the Lax–Oleinik semigroup

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ARTICLE INFO

Article history:
Received 8 August 2006
Revised 23 May 2008
Available online 24 January 2009

MSC:
37J50
49L25
70H20

Keywords:
Hamilton–Jacobi equation
Viscosity solutions
Aubry set

ABSTRACT

For a convex superlinear Lagrangian $L:TM\to\mathbb{R}$ on a compact manifold M it is known that there is a unique number c such that the Lax–Oleinik semigroup $\mathcal{L}_t+ct:C(M,\mathbb{R})\to C(M,\mathbb{R})$ has a fixed point. Moreover for any $u\in C(M,\mathbb{R})$ the uniform limit $\tilde u=\lim_{t\to\infty}\mathcal{L}_t u+ct$ exists.
In this paper we assume that the Aubry set consists in a finite number of periodic orbits or critical points and study the relation of the hyperbolicity of the Aubry set to the exponential rate of convergence of the Lax–Oleinik semigroup.

1. Introduction

Consider a convex superlinear Lagrangian $L:TM\to\mathbb{R}$ on a d-dimensional compact manifold M. For $t\geqslant 0$ define the (backward) Lax–Oleinik semigroup $\mathcal{L}_t:C(M,\mathbb{R})\to C(M,\mathbb{R})$ by

$$\mathcal{L}_t u(x)=\inf\left\{u\bigl(\gamma(0)\bigr)+\int_0^t L(\gamma,\dot\gamma):\ \gamma:[0,t]\to M \text{ is piecewise } C^1,\ \gamma(t)=x\right\}.$$

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[1] Partially supported by grant CONACYT 46467-F.
[2] Partially supported by DGAPA-UNAM 100505.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.12.012

The function $S : M \times \mathbb{R}^+ \to \mathbb{R}$ given by $S(x,t) = \mathcal{L}_t u(x)$ is a viscosity solution of the Hamilton–Jacobi initial value problem

$$S_t + H(x, S_x) = 0, \qquad S(x,0) = u(x). \tag{1}$$

It was shown in [1,2] that there is a unique number $c = c(L)$ such that $\mathcal{L}_t + ct$ has a fixed point for any $t > 0$. Any fixed point u is a backward viscosity solution of

$$H\big(x, Du(x)\big) = c. \tag{2}$$

Moreover for any $u \in C(M, \mathbb{R})$ the uniform limit

$$\tilde{u} = \lim_{t\to\infty} \mathcal{L}_t u + ct$$

exists. One can also define the forward Lax–Oleinik semigroup $\mathcal{L}_t^*$ by

$$\mathcal{L}_t^* u(x) = \sup\left\{ u\big(\gamma(t)\big) - \int_0^t L(\gamma, \dot{\gamma}):\ \gamma : [0,t] \to M \text{ is piecewise } C^1,\ \gamma(0) = x \right\}.$$

Again $\mathcal{L}_t^* - ct$ has a fixed point for any $t > 0$ and any such fixed point u is a forward viscosity solution of (2). The semigroup $\mathcal{L}_t^*$ gives the solution to a Hamilton–Jacobi final value problem.

Our goal in this paper is to establish a relation of the hyperbolicity of the Aubry set to the exponential rate of convergence of the semigroup $\mathcal{L}_t + ct$.

Theorem 1. *Assume that the Aubry set consists in a finite number of hyperbolic periodic orbits or critical points of the Euler–Lagrange flow. Then, there is $\mu > 0$ such that for any $u \in C(M, \mathbb{R})$ there is $K > 0$ such that*

$$\|\mathcal{L}_t u + ct - \bar{u}\|_0 \leqslant K e^{-\mu t} \quad \forall t \geqslant 0. \tag{3}$$

Theorem 2. *Let $L : TM \to \mathbb{R}$ be given by $L(x,v) = \frac{1}{2}v^2 - V(x)$ with*

$$\max_x V(x) = c, \qquad V^{-1}(c) = \{x_1, \dots, x_m\}.$$

Suppose that there is $\mu > 0$ such that for any $u \in C(M, \mathbb{R})$ there is $K > 0$ such that (3) *holds. Then* $(x_i, 0)$, $i = 1, \dots, m$, *is a hyperbolic critical point of the Euler–Lagrange flow.*

Remark 1. For Theorem 2, we only need that (3) holds for the function $u \equiv 0$.

2. Aubry set and static classes

We recall the definition of the Peierls Barrier [3] and Mañé's action potential [4]. Define the action of a piecewise C^1 curve $\gamma : [0,T] \to M$ as

$$A(\gamma) = \int_0^T L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds.$$

Given a constant $k \in \mathbb{R}$ and $x_1, x_2 \in M$ let

$$h_T^k(x_1, x_2) = \inf\big\{ A(\gamma) + kT \ \big|\ \gamma : [0,T] \to M \text{ joins } x_1 \text{ and } x_2 \big\},$$

and

$$h^k(x_1,x_2) = \liminf_{T\to\infty} h^k_T(x_1,x_2),$$
$$\Phi^k(x_1,x_2) = \inf_T h^k_T(x_1,x_2).$$

Since time T is not bounded, there is only one possible value of k that will make the function h^k different from being identically $-\infty$ or ∞, this is again $c = c(L)$. We define $\Phi_T = h^c_T$ and the Peierls Barrier $h = h^c$. Mañé's action potential Φ^k is identically $-\infty$ for $k < c(L)$ and finite for $k \geqslant c(L)$. We will also define $\Phi = \Phi^c$. In [3], it is shown that Φ_T actually converges uniformly to h.

Given a fixed $y \in M$, the function $x \mapsto -h(x,y)$ is a forward viscosity solution of (2), whereas $x \mapsto h(y,x)$ is a backward viscosity solution.

We now define as in [3] the Aubry set $\mathcal{A} \subset M$:

$$\mathcal{A} = \{x \in M,\ h(x,x) = 0\}$$

(in Ref. [3] it was called the Peierls set.)

In close relation to Mather's graph theorem [5], it is shown in [6], that the set $\mathcal{A}$ can be lifted, in a unique way, to a set $\tilde{\mathcal{A}} \subset TM$ that is an invariant set of minimizing orbits of the Euler–Lagrange flow. This set projects homeomorphically to $\mathcal{A}$ through the usual projection from TM to M. We also call the set $\tilde{\mathcal{A}}$ "Aubry set."

The "static classes" form a partition of $\mathcal{A}$, defined by the equivalence relation on $\mathcal{A}$: $x \sim y$ if and only if

$$h(x,y) + h(y,x) = 0.$$

If the Aubry set $\tilde{\mathcal{A}}$ is made up of a finite union of periodic orbits or critical points of the Euler–Lagrange flow, each static class is a periodic orbit or a critical point.

3. Proof of Theorem 1

Adding a constant to L we may take $c(L) = 0$. We assume that the Aubry set consists in a finite number of hyperbolic periodic orbits or critical points Γ_i: $\varphi_t(x_i, v_i) = (\boldsymbol{\gamma}_i(t), \boldsymbol{\gamma}'_i(t))$, $t \in \mathbb{R}$, $1 \leqslant i \leqslant m$. In the case of a periodic orbit we denote by T_i its minimal period and in the case of a critical point we put $T_i = 1$.

Let $\lambda_{i,j}$, $j = 1, \ldots, d^*$, be the positive Lyapunov exponents of γ_i where $d^* = d$ if γ_i is a critical point and $d^* = d-1$ if γ_i is a periodic orbit. Set $\lambda = \min_{i,j} \lambda_{ij}$, $\mathbf{T}_S = T_1 + \cdots + T_m$, $T = \min_{i\in[1,m]} T_i$.

Fix V_i a tubular neighborhood of Γ_i in TM, where the flow is orbit equivalent to its linearization. According to a result of Belitskii [7] there is $0 < \alpha < 1$ such that the linearizing map $F_i : B_i \to V_i$ is α-Hölder. We define

$$V = \bigcup_{i=1}^m V_i.$$

In [8] it was proved that for any backward viscosity solution v of (2)

$$v(x) = \min_{i\in[1,m]} v(x_i) + h(x_i,x). \tag{4}$$

Closely related to this fact we have the following

Proposition 1. *For $u \in C(M,\mathbb{R})$ let $\bar u := \lim_{t\to\infty} \mathcal{L}_t u$. Then*

$$\bar u(x) = \min_{z\in M} u(z) + h(z,x) \tag{5}$$

$$= \min\bigl\{u(z) + h(z,x_i) + h(x_i,x)\colon\ i\in[1,m],\ z\in M\bigr\}. \tag{6}$$

Proof. For any $x\in M$ and $t>0$ there is $y_t(x)$ such that

$$\mathcal{L}_t u(x) = u\bigl(y_t(x)\bigr) + \Phi_t\bigl(y_t(x),x\bigr) \leqslant u(z) + \Phi_t(z,x) \quad \forall z.$$

Choose $t_n\to\infty$ such that $(y_{t_n}(x))$ converges to some $Y(x)$, then $(\Phi_{t_n}(y_{t_n}(x),x))$ converges to $h(Y(x),x)$ and so

$$\bar u(x) = u\bigl(Y(x)\bigr) + h\bigl(Y(x),x\bigr) = \min_{z\in M} u(z) + h(z,x).$$

In particular, for $x=x_i$ there is $y_i\in M$ such that

$$\bar u(x_i) = u(y_i) + h(y_i,x_i) = \min_{z\in M} u(z) + h(z,x_i),$$

and then

$$\begin{aligned}\bar u(x) &= \min_{i\in[1,m]} \bar u(x_i) + h(x_i,x)\\ &= \min\bigl\{u(z) + h(z,x_i) + h(x_i,x)\colon\ i\in[1,m],\ z\in M\bigr\}. \qquad \Box\end{aligned}$$

Letting $u\in C(M,\mathbb{R})$, to prove Theorem 1 we have to establish two inequalities. We first prove that there is $K>0$ such that

$$\mathcal{L}_t u - \bar u \leqslant K\exp\left(-\frac{\lambda T}{2\mathbf{T}}t\right). \tag{7}$$

Given $x\in M$, for every piecewise C^1 curve $\gamma:[0,t]\to M$ with $\gamma(0)=x$

$$\mathcal{L}_t u(x) \leqslant u\bigl(\gamma(0)\bigr) + \int_0^t L(\gamma,\dot\gamma).$$

For some $j\in[1,m]$ we have that

$$\bar u(x) = \bar u(y_j) + h(y_j,x),$$

and to prove inequality (7) we will construct curves joining y_i and x with action approximating $h(y_i,x)$.

For $x\in M$ let $i\in[1,m]$ such that

$$\bar u(x) = \bar u(x_i) + h(x_i,x).$$

Since $z \mapsto h(x_i, z)$ is a backward viscosity solution of (2), there is a semistatic curve $\beta_x :]-\infty, 0] \to M$ with $\beta_x(0) = x$ such that

$$\int_t^0 L(\beta_x, \beta_x') = h(x_i, x) - h(x_i, \beta_x(t)), \quad t < 0.$$

We may assume that Γ_i is the α-limit of $\{(\beta_x, \beta_x')\}$. In fact, let Γ_j be the α-limit of $\{(\beta_x, \beta_x')\}$, then we have

$$h(x_i, x) = h(x_i, x_j) + h(x_j, x).$$

Since $\bar{u}(x_j) \leqslant \bar{u}(x_i) + h(x_i, x_j)$ we have that

$$\bar{u}(x) \leqslant \bar{u}(x_j) + h(x_j, x) \leqslant \bar{u}(x_i) + h(x_i, x) = \bar{u}(x)$$

and then $\bar{u}(x) = u(y_j) + h(y_j, x_j) + h(x_j, x)$.

Since $y \mapsto -h(y, x_j)$ is a forward viscosity solution of (2), there is a semistatic curve $\omega_j : [0, \infty[\to M$ such that $\omega_j(0) = y_j$ and

$$\int_0^t L(\omega_j, \omega_j') = h(y_j, x_j) - h(\omega_j(t), x_j), \quad t > 0.$$

Let Γ_k be the ω-limit of $\{(\omega_j, \omega_j')\}$, then we have

$$\begin{aligned} h(y_j, x_j) &= h(y_j, x_k) + h(x_k, x_j), \\ d\big((\omega_j(t), \omega_j'(t)), \varphi_{t+d_1}(x_k, v_k)\big) &\leqslant C_1 e^{-\lambda t}, \quad t > \tau(V), \\ d\big((\beta_x(t), \beta_x'(t)), \varphi_{t-d}(x_j, v_j)\big) &\leqslant C_1 e^{\lambda t}, \quad t < -\tau(V), \end{aligned}$$

with $0 < d_1 < T_k$, $0 < d < T_j$.

According to Theorem 3-11.1 in [9] there are $i_1 = k, \dots, i_l = j$ and semistatic curves $\beta_r : \mathbb{R} \to M$, $r = 2, \dots, l$, such that $\Gamma_{i_{r-1}}$ and Γ_{i_r} are the α and ω limits of $\{(\beta_r(t), \beta_r'(t)): t \in \mathbb{R}\}$ respectively. Since all orbits Γ_i are hyperbolic and the semistatic curves β_r are in fact heteroclinic connections we may assume that

$$\begin{aligned} d\big((\beta_r(t), \beta_r'(t)), \varphi_t(x_{i_{r-1}}, v_{i_{r-1}})\big) &\leqslant C_1 e^{\lambda t}, \quad t < -\tau(V), \\ d\big((\beta_r(t), \beta_r'(t)), \varphi_{t+d_r}(x_{i_r}, v_{i_r})\big) &\leqslant C_1 e^{-\lambda t}, \quad t > \tau(V), \end{aligned}$$

with $0 < d_r < T_{i_r}$. We have

$$\int_s^t L(\beta_r, \beta_r') = h(x_{i_{r-1}}, x_{i_r}) - h(x_{i_-}, \beta_r(s)) - h(\beta_r(t), x_{i_r}).$$

We now define a curve whose action approximates $h(y_j, x)$ that is made of pieces of the heteroclinic connections β_r and some transition curves c_r exponentially close to Γ_{i_r}. (See Fig. 1.)

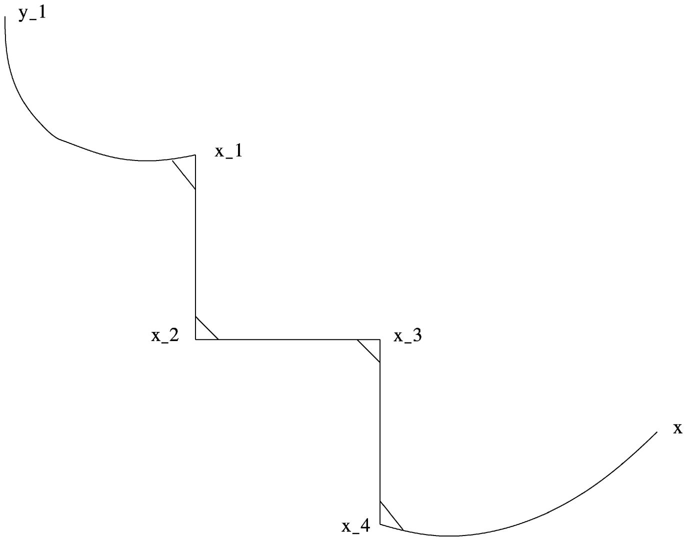

Fig. 1.

Let $\beta_1 = \omega_j$, $\beta_{l+1} = \beta_x(t+d)$. For $1 < r \leqslant l+1$ let

$$\mathbf{d}_r = d_1 + \cdots + d_{r-1}, \qquad \mathbf{T}_r = T_{i_1} + \cdots + T_{i_{r-1}},$$
$$a_r(n) = \begin{cases} nT_k - d_1, & r = 1, \\ (2n+1)\mathbf{T}_r + nT_{i_r} - \mathbf{d}_{r+1}, & 1 < r \leqslant l, \\ (2n+1)\mathbf{T}_{l+1} - \mathbf{d}_{l+1} - d, & r = l+1. \end{cases}$$

Note that $a_{l+1}(n) \leqslant (2n+1)\mathbf{T}$.

There is $\bar{\tau}(V) > 0$ such that for any $x \in M$, $t \geqslant \bar{\tau}(V) - 2\max_i T_i$, we have $\beta_r(t) \in V$, $r = 0, \ldots, l+1$.

Consider the curve $\alpha_n : [0, a_{l+1}(n)] \to M$, defined by

$$\alpha_n(s) = \begin{cases} \beta_1(s), & s \in [0, a_1(n)], \\ \beta_r(s - (2n+1)\mathbf{T}_r + \mathbf{d}_r), & s \in [a_{r-1}(n) + T_{i_{r-1}}, a_r(n)],\ r > 1, \\ c_r(s), & s \in [a_r(n), a_r(n) + T_{i_r}], \end{cases}$$

where $c_r : [a_r(n), a_r(n) + T_{i_r}] \to M$ is defined using tubular coordinates

$$\psi_r : U_r \to \mathbb{S}^1 \times \mathbb{R}^{d-1}, \qquad \psi_r(z) = \big(\exp\big(i\eta_1(z)\big), \eta_2(z)\big)$$

around γ_{i_r} by the expression

$$\begin{aligned}(\eta_1, \eta_2) \circ c_r(s) = {} & \left(1 - \frac{s - a_r(n)}{T_{i_r}}\right)(\eta_1, \eta_2) \circ \beta_r\big(s - (2n+1)\mathbf{T}_r + \mathbf{d}_r\big) \\ & + \left(\frac{s - a_r(n)}{T_{i_r}}\right)(\eta_1, \eta_2) \circ \beta_{r+1}\big(s - (2n+1)\mathbf{T}_{r+1} + \mathbf{d}_{r+1}\big),\end{aligned}$$

$$\int_0^{a_{l+1}(n)} L\big(\alpha_n, \alpha_n'\big) = \int_0^{a_1(n)} L\big(\beta_1, \beta_1'\big) + \sum_{r=2}^{l} \int_{-nT_{i_{r-1}}}^{nT_{i_r} - d_r} L\big(\beta_r, \beta_r'\big)$$

$$
\begin{aligned}
&+\sum_{r=1}^{l}\int_{a_r(n)}^{a_r(n)+T_{i_r}} L\big(c_r,c_r'\big)+\int_{-nT_j}^{-d} L\big(\beta_{l+1},\beta_{l+1}'\big)\\
&=h(y_j,x_k)-h\big(a_1(n),x_k\big)+\sum_{r=1}^{l}h(x_{i_r},x_{i_{r+1}})\\
&-\sum_{r=2}^{l}h\big(x_{i_{r-1}},\beta_r(-nT_{i_{r-1}})\big)+h\big(\beta_r(nT_{i_r}-d_r),x_{i_r}\big)\\
&+\sum_{r=1}^{l}\int_{a_r(n)}^{a_r(n)+T_{i_r}} L\big(c_r,c_r'\big)+h(x_j,x)-h\big(x_j,\beta_{l+1}(-nT_j)\big).
\end{aligned}
$$

Since $\int_0^{T_j} L(\boldsymbol{\gamma}_j,\boldsymbol{\gamma}_j')=0$ and

$$
d\big(c_r(s),\boldsymbol{\gamma}_{i_r}\big(s-(2n+1)\mathbf{T}_r+\mathbf{d}_{r+1}\big)\big)+\big|c'(s)-\boldsymbol{\gamma}_{i_r}'\big(s-(2n+1)\mathbf{T}_r+\mathbf{d}_{r+1}\big)\big|\leqslant C_2e^{-\lambda nT_{i_r}},
$$

we have

$$
\begin{aligned}
\mathcal{L}_{a_{l+1}(n)}u(x)-\bar{u}(x)\leqslant{}&\sum_{r=1}^{l}\int_{a_r(n)}^{a_r(n)+T_{i_r}} L\big(c_r,c_r'\big)\\
&-\sum_{r=2}^{l}h\big(x_{i_{r-1}},\beta_r(-nT_{i_{r-1}})\big)+h\big(\beta_r(nT_{i_r}-d_r),x_{i_r}\big)\\
&-h\big(a_1(n),x_k\big)-h\big(x_j,\beta_{l+1}(-nT_j)\big)\\
\leqslant{}& C_3e^{-\lambda nT}\leqslant K\exp\bigg(-\frac{\lambda T}{2\mathbf{T}}a_{l+1}(n)\bigg).
\end{aligned}
$$

Now we establish the other inequality.

For $x\in M$, $t>0$ let $\gamma_t:[-t,0]\to M$ be a curve such that $\gamma_t(0)=x$ and

$$
\mathcal{L}_tu(x)=u\big(\gamma_t(-t)\big)+\int_{-t}^{0}L\big(\gamma_t,\gamma_t'\big)=u\big(\gamma_t(-t)\big)+\Phi_t\big(\gamma_t(-t),x\big).
$$

For any $s\in[-t,0]$, $i\in[1,m]$ we have

$$
\begin{aligned}
\bar{u}(x)&\leqslant u\big(\gamma_t(-t)\big)+h\big(\gamma_t(-t),x_i\big)+h(x_i,x) &(8)\\
&\leqslant u\big(\gamma_t(-t)\big)+\Phi\big(\gamma_t(-t),\gamma_t(s)\big)+h\big(\gamma_t(s),x_i\big)\\
&\quad+h\big(x_i,\gamma_t(s)\big)+\Phi\big(\gamma_t(s),x\big) &(9)\\
&\leqslant u\big(\gamma_t(-t)\big)+\int_{-t}^{0}L\big(\gamma_t,\gamma_t'\big)+h\big(\gamma_t(s),x_i\big)+h\big(x_i,\gamma_t(s)\big) &(10)\\
&=\mathcal{L}_tu(x)+h\big(\gamma_t(s),x_i\big)+h\big(x_i,\gamma_t(s)\big). &(11)
\end{aligned}
$$

Inequality (8) follows from (6), inequality (9) is twice triangle inequality, inequality (10) follows from the definition of Φ.

The idea of the proof is to choose s for each t sufficiently large such that the last two terms in (11) are $O(e^{-\mu t})$. Since h is Lipschitz and $h(x_i, x_i) = 0$, this reduces to choose s such that $d(\gamma_t(s), x_i)$ is $O(e^{-\mu t})$. In fact we will show that they are exponentially close in the tangent bundle. The main idea is that if an orbit remains a long time say of order T in the neighborhood of a hyperbolic saddle, then there is some point that is at distance of order $e^{-\mu T}$ of the saddle. This is trivial for a linear system and the general case follows from the α-Hölder linearization.

We need the following lemma.

Lemma 1. *Let $W = \bigcup_{i=1}^m W_i$ be a neighborhood of the Aubry set in TM. Then, there exist $T, C > 0$ such that if $\gamma : [-t, 0] \to M$, $t > T$, is a minimizer, then the time that $(\gamma(\tau), \gamma'(\tau))$ remains outside W is less than C.*

Proof. Suppose that L has the special property that is greater or equal to zero, and that is equal to zero only in the Aubry set. In this case the Lagrangian is bounded from below by δ outside the neighborhood W. Since the action of the minimizers is bounded independently of t, the lemma follows easily.

To prove the general case we use a theorem of Fathi and Siconolfi [10] that claims the existence of f, a C^1 strict critical subsolution of the Hamilton–Jacobi equation, which means that the Lagrangian $L - df$ has the property described above. Moreover, according to a result of P. Bernard [11] in the case when the Aubry set is a collection of hyperbolic periodic orbits, the function f may be chosen C^∞.

The curves γ realizing the minimum in the Lax transformation for the function u and the Lagrangian L are the same curves realizing the minimum for the Lagrangian $L - df$ and the function $u + f$. So we obtain the lemma. □

Recall that V_i are neighborhoods of the orbits of the Aubry set in TM where we can linearize the flow and $V = \bigcup_i V_i$. Since the velocity of any minimizer is bounded by the same constant, and the time it can remain outside V is bounded, the number of times it can go from one V_i to other V_j is bounded by say N, we conclude that for $t > T^*$, any minimizer $\gamma : [-t, 0] \to M$ stays in at least one V_i a time interval larger than $\frac{t}{N}$.

As we explained, we then have the following proposition.

Proposition 2. *There are positive constants C, T and μ such that for any $\gamma : [-t, 0] \to M$, $t \geqslant T$, minimizing curve, there is $\tau_\gamma \in [-t, 0]$ such that*

$$d\big(\big(\gamma(\tau_\gamma), \gamma'(\tau_\gamma)\big), (x_i, v_i)\big) \leqslant C \exp(-\mu t)$$

for some $i \in [1, m]$.

This finishes the proof of Theorem 1.

4. Proof of Theorem 2

Lemma 2. *Let $L : TM \to \mathbb{R}$ be given by $L(x, v) = \frac{1}{2} v^2 - V(x)$ with*

$$\max_x V(x) = 0, \qquad V^{-1}(0) = \{x_1, \ldots, x_m\}.$$

Suppose that there is $\mu > 0$ such that for $u \equiv 0$ there is $K > 0$ such that

$$\|\mathcal{L}_t u - \bar{u}\|_0 \leqslant K e^{-\mu t} \quad \forall t \geqslant 0. \tag{12}$$

Then $(x_i, 0)$, $i = 1, \ldots, m$, is a hyperbolic critical point of the Euler–Lagrange flow.

Proof. For any x_j the function $h_j(x) = h(x_j, x)$ is a viscosity solution of the Hamilton–Jacobi equation

$$\frac{1}{2}\left|D\phi(x)\right|^2 + V(x) = 0.$$

Suppose $(x_i, 0)$ is not hyperbolic, which means that x_i is a degenerate maximum of V. Let $0, -\lambda_1^2, \dots, -\lambda_k^2$, $\lambda_i > 0$, be the eigenvalues of $\operatorname{Hess} V(x_i)$. By the splitting lemma [12], there are local coordinates (y, z) around x_i such that x_i corresponds to the origin and

$$-2V(y,z) = \psi(y) + \lambda_1^2 z_1^2 + \dots + \lambda_k^2 z_k^2, \tag{13}$$

$$D\psi(0) = 0, \qquad \operatorname{Hess}\psi(0) = 0. \tag{14}$$

Thus, there is $C > 0$ such that

$$\left|D_z\sqrt{-2V(y,z)}\right| \leqslant C, \tag{15}$$

$$\lim_{(y,z)\to 0} D_y\sqrt{-2V(y,z)} = 0. \tag{16}$$

The linearization of the Euler–Lagrange flow at $(x_i, 0)$ has eigenvalues $0, \pm\lambda_1, \dots, \pm\lambda_k$. Denote by W^u, W^s, W^c the unstable, stable, and center manifolds at $(x_i, 0)$ respectively.

Claim 1. *There exists a calibrated curve $\gamma :]-\infty, 0] \to M$ with α-limit x_i such that $(\gamma(t), \dot\gamma(t))$ is not in W^u.*

Indeed, let 2δ be smaller than the minimum of $h(x_i, x_j)$ for all $j \neq i$. Let U be the open set of points p such that $h(x_i, p) < \delta$. For any point p in U take a minimizing curve starting in x_i at time $-T$ and ending in p at time 0. The limit curve, as T tends to infinite exists because the velocities are bounded, and it is in fact a minimizer $\gamma :]-\infty, 0] \to M$ with α-limit x_i and $\gamma(0) = p$. Some of these curves lie on the unstable manifold, but since there are some zero eigenvalues this manifold has positive codimension. This proves the claim.

Let $\gamma :]-\infty, 0] \to M$ be as in the claim, then there is a trajectory $\varphi_t(w)$ of the Euler–Lagrange flow on W^c such that

$$d\big(\varphi_t(w), \big(\gamma(t), \dot\gamma(t)\big)\big) = O\big(e^{\mu t}\big), \quad t \to -\infty.$$

Since $(\gamma(t), \dot\gamma(t))$ is not in W^u then, writing $\gamma(t) = (\gamma_y(t), \gamma_z(t))$ in local coordinates, we have

$$\lim_{t\to-\infty} \frac{\dot\gamma_z(t)}{|\dot\gamma(t)|} = 0.$$

For the function $u \equiv 0$ we have

$$\lim_{t\to\infty} \mathcal{L}_t u(x) = \min_j h_j(x),$$

and there is a neighborhood W_i of x_i such that for $x \in W_i$

$$\lim_{t\to\infty} \mathcal{L}_t u(x) = h_i(x).$$

Since

$$h_i\big(\gamma(0)\big) - h_i\big(\gamma(-t)\big) = \int_{-t}^{0} \frac{1}{2}\dot{\gamma}^2 - V(\gamma) = -\int_{-t}^{0} 2V(\gamma) \geqslant \mathcal{L}_t u\big(\gamma(0)\big),$$

$$\frac{d}{ds}h_i\big(\gamma(s)\big) = -2V\big(\gamma(s)\big) = \dot{\gamma}(s)^2, \qquad \frac{d}{ds}\log h_i\big(\gamma(s)\big) = \frac{-2V(\gamma(s))}{h_i(\gamma(s))}.$$

By L'Hopital rule and (15), (16)

$$\begin{aligned}\lim_{s\to-\infty} \frac{\log h_i(\gamma(s))}{s} &= \lim_{s\to-\infty} \frac{d}{ds}\log h_i\big(\gamma(s)\big)\\ &= \lim_{s\to-\infty} \frac{-2DV(\gamma(s))\dot{\gamma}(s)}{-2V(\gamma(s))}\\ &= -2\lim_{s\to-\infty} \frac{D_yV(\gamma(s))\dot{\gamma}_y(s) + D_zV(\gamma(s))\dot{\gamma}_z(s)}{\sqrt{-2V(\gamma(s))}|\dot{\gamma}(s)|}\\ &= 0. \end{aligned} \tag{17}$$

Assumption (12) gives

$$C\exp(-\mu t) \geqslant h_i\big(\gamma(0)\big) - \mathcal{L}_t u\big(\gamma(0)\big) \geqslant h_i\big(\gamma(-t)\big)$$

so that

$$\begin{aligned} -\log C + \mu t &\leqslant -\log h_i\big(\gamma(-t)\big),\\ \mu &\leqslant \liminf_{t\to\infty} -\frac{\log h_i(\gamma(-t))}{t}, \end{aligned} \tag{18}$$

contradicting (17). □

References

[1] P.L. Lions, G. Papanicolaou, S.R.S. Varadhan, Homogeneization of Hamilton–Jacobi equation, unpublished preprint.
[2] A. Fathi, Sur la convergence du semi-group de Lax–Oleinik, C. R. Acad. Sci. Paris Sér. I 327 (3) (1998) 267–290.
[3] A. Fathi, The Weak KAM Theorem in Lagrangian Dynamics, Cambridge Univ. Press, 2003.
[4] R. Mañé, Lagrangian flows: The dynamics of globally minimizing orbits, in: F. Ledrappier, J. Lewowicz, S. Newhouse (Eds.), International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañé), in: Pitman Res. Notes Math., vol. 362, 1996, pp. 120–131; reprinted in: Bol. Soc. Bras. Mat. 28 (2) (1997) 141–153.
[5] J. Mather, Action minimizing measures for positive definite Lagrangian systems, Math. Z. 207 (1991) 169–207.
[6] A. Fathi, Théorème KAM faible et Théorie de Mather sur les systems Lagrangiens, C. R. Acad. Sci. Paris Sér. I 324 (1997) 1043–1046.
[7] G.R. Belitskii, On the Grobman–Hartman theorem in class C^α, unpublished preprint.
[8] G. Contreras, Action potential and weak KAM solutions, Calc. Var. Partial Differential Equations 13 (4) (2001) 427–458.
[9] G. Contreras, R. Iturriaga, Global minimizers of autonomous Lagrangian, 22 Goloquio Brasileiro, 1999.
[10] A. Fathi, A. Siconolfi, Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation, Invent. Math. 155 (2004) 363–388.
[11] P. Bernard, Smooth critical sub-solutions of the Hamilton–Jacobi equation, Math. Res. Lett. 14 (3) (2007) 503–511.
[12] S. López de Medrano, A splitting lemma for C^r functions, in: Singularity Theory, Trieste, 1991, World Scientific, River Edge, NJ, 1995, pp. 444–450.