All-derivable points of operator algebras

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Abstract

Let \( A \) be an operator subalgebra in \( B(H) \), where \( H \) is a Hilbert space. We say that an element \( Z \in A \) is an all-derivable point of \( A \) for the norm-topology (strongly operator topology, etc.) if, every norm-topology (strongly operator topology, etc.) continuous derivable linear mapping \( \phi \) at \( Z \) (i.e. \( \phi(ST) = \phi(S)T + S\phi(T) \) for any \( S, T \in A \) with \( ST = Z \)) is a derivation. In this paper, we show that every invertible operator in the nest algebra \( \text{alg} \mathcal{N} \) is an all-derivable point of the nest algebra for the strongly operator topology. We also prove that every nonzero element of the algebra of all \( 2 \times 2 \) upper triangular matrixes is an all-derivable point of the algebra.

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1. Introduction and preliminaries

Let \( \mathcal{A} \) be an operator subalgebra in \( B(H) \), and let \( L(\mathcal{A}) \) denote the set of all linear mappings on \( \mathcal{A} \). We say that \( \phi \in L(\mathcal{A}) \) is a derivable mapping at \( Z \) (generalized derivable mapping at \( Z \)) if \( \phi(ST) = \phi(S)T + S\phi(T) \) (\( \phi(ST) = \phi(S)T + S\phi(T) - S\phi(I)T \)) for any \( S, T \in \mathcal{A} \) with \( ST = Z \). We say that an operator \( Z \in \mathcal{A} \) is an all-derivable point of \( \mathcal{A} \) for the norm-topology (strongly operator topology, etc.) if, every norm-topology (strongly operator topology, etc.) continuous derivable mapping \( \phi \) at \( Z \) is a derivation.

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In recent years there has been considerable interest in studying which linear mapping on operator algebras are derivations. We describe some of the results related to ours. Jin et al. [7] showed that every derivable mapping $\phi$ at 0 with $\phi(I) = 0$ on nest algebras is an inner derivation. Zhu and Xiong in [15,17] proved that every norm-continuous generalized derivable mapping at 0 on finite CSL algebras is a generalized derivation, and every strongly operator topology continuous derivable mapping at unit operator $I$ in nest algebras is a derivation (i.e. the unit operator is an all-derivable point of the nest algebra $\text{alg} \mathcal{N}$ for the strongly operator topology). Šemrl [10] presented the notion of 2-local derivation and showed that every 2-local derivation on $B(H)$ is a derivation (no linearity is assumed), where $\dim H = \infty$. For other results, see [1–4,8–18].

It is the aim of this paper to prove the following two statements:

In Section 2, we show that every invertible operator in a nest algebra is an all-derivable point of the algebra for the strongly operator topology.

In Section 3, we show that $A \in \mathcal{A}$ is an all-derivable point if and only if $A \neq 0$, where $\mathcal{A}$ is the algebra of all $2 \times 2$ upper triangular matrixes.

The following notations will be used in this paper:

The symbols $B(H)$ and $F(H)$ stand for the set of all bounded linear operators on $H$ and the set of all finite rank operators on $H$, respectively. We use the symbols $I$ to denote the unit operator on $H$. If we denotes by $\mathcal{N}$ a complete nest on $H$, then the nest algebra $\text{alg} \mathcal{N}$ is the set of all operators which leave every member of $\mathcal{N}$ invariant. The algebra $\text{alg} \mathcal{N}$ is a Banach algebra.

2. All-derivable points on nest algebras

Lemma 2.1. Let $\mathcal{A}$ be an operator algebra with an invertible all-derivable point $Z$, and let $A, B \in \mathcal{A}$ with $A + B = Z$. If $\phi$ is a derivable mapping at $A$ and $B$ on the algebra $\mathcal{A}$, then $\phi$ is a derivation.

Proof. Since $Z$ is an all-derivable point of $\mathcal{A}$, we only need to prove that $\phi$ is a derivable mapping at $Z$ on $\mathcal{A}$. In fact, for arbitrary $S, T \in \mathcal{A}$ with $ST = Z$, then $STZ^{-1}A = A$ and $STZ^{-1}B = B$. Thus we have

$$\phi(A) = \phi(S) TZ^{-1}A + S\phi(TZ^{-1}A),$$

and

$$\phi(B) = \phi(S) TZ^{-1}B + S\phi(TZ^{-1}B).$$

Combining the two above equations, we get that

$$\phi(Z) = \phi(A + B) = \phi(S) TZ^{-1}(A + B) + S\phi(TZ^{-1}(A + B)) = \phi(S)T + S\phi(T),$$

i.e. $\phi$ is a derivable mapping at $Z$. This completes the proof. □

Lemma 2.2. Let $\mathcal{A}$ be an operator algebra. If there exists an invertible operator $Z \in \mathcal{A}$ such that $\phi$ is a derivable mapping at $Z$, then $\phi(I) = 0$.

Proof. Since $IZ = Z$, $\phi(Z) = \phi(I)Z + I\phi(Z)$. Thus we have $\phi(I)Z = 0$. Notice that $Z$ is invertible. Hence $\phi(I) = 0$. This completes the proof. □

Theorem 2.3. Let $\mathcal{N}$ be a complete nest. Then every invertible operator $Z$ in $\text{alg} \mathcal{N}$ is an all-derivable point of the algebra for the strongly operator topology.
Proof. For arbitrary idempotent operator $p \in \text{alg } \mathcal{N}$, i.e. $p^2 = p$, we have $I - p + p^2 = I$. Furthermore $Z = (I - p + p^2)Z = (I - \alpha p)(I - \beta p)Z$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha + \beta = 1$ and $\alpha \beta = 1$. It follows that

$$\varphi(Z) = \varphi(I - \alpha p)(I - \beta p)Z + (I - \alpha p)\varphi((I - \beta p)Z)$$

$$= -\alpha \varphi(p)Z + \alpha \beta \varphi(p)pZ + \varphi(Z) - \beta \varphi(pZ) - \alpha p \varphi(Z) + \alpha \beta p \varphi(pZ).$$

Furthermore we have

$$-\alpha \varphi(p)Z + \varphi(p)pZ - \beta \varphi(pZ) - \alpha p \varphi(Z) + \alpha \beta p \varphi(pZ) = 0. \quad (1)$$

On the other hand, $Z = (I - p + p^2)Z = (I - \beta p)(I - \alpha p)Z$, so we have

$$\varphi(Z) = \varphi(I - \beta p)(I - \alpha p)Z + (I - \beta p)\varphi((I - \alpha p)Z)$$

$$= -\beta \varphi(p)Z + \alpha \beta \varphi(p)pZ + \varphi(Z) - \alpha \varphi(pZ) - \beta p \varphi(Z) + \alpha \beta p \varphi(pZ).$$

Furthermore we have

$$-\beta \varphi(p)Z + \varphi(p)pZ - \alpha \varphi(pZ) - \beta p \varphi(Z) + \alpha \beta p \varphi(pZ) = 0. \quad (2)$$

Combining Eqs. (1) and (2), we get that

$$\varphi(pZ) = \varphi(I - \alpha p)(I - \beta p)Z + \varphi((I - \beta p)(I - \alpha p)Z)$$

$$= -\alpha \varphi(p)Z + \alpha \beta \varphi(p)pZ + \varphi(Z) - \beta \varphi(pZ) - \alpha p \varphi(Z) + \alpha \beta p \varphi(pZ).$$

Notice that every rank one operator in $\text{alg } \mathcal{N}$ may be denoted as a linear combination of at most four idempotents in $\text{alg } \mathcal{N}$ (see [6]), and every finite rank operator in $\text{alg } \mathcal{N}$ may be represented as a sum of rank one operators in $\text{alg } \mathcal{N}$ (see [5]). Hence we have

$$\varphi(FZ) = \varphi(F)Z + F \varphi(Z), \quad \forall F \in F(H) \cap \text{alg } \mathcal{N}.$$

Since $\varphi$ is a strongly operator topology continuous mapping and $\text{alg } \mathcal{N} \cap F(H)^{\text{SOT}} = \text{alg } \mathcal{N}$ (see [5]), we get that

$$\varphi(TZ) = \varphi(T)Z + T \varphi(Z), \quad \forall T \in \text{alg } \mathcal{N}. \quad (3)$$

For arbitrary $S, T \in \text{alg } \mathcal{N}$ with $ST = I$, then $STZ = Z$. Notice that $\varphi$ is a derivable mapping at $Z$. It follows from Eq. (3) that

$$\varphi(Z) = \varphi(S)TZ + S \varphi(TZ) = \varphi(S)TZ + S \varphi(T)Z + T \varphi(Z))$$

$$= \varphi(S)TZ + S \varphi(T)Z + \varphi(Z).$$

Thus $0 = \varphi(S)TZ + S \varphi(T)Z$. Furthermore $0 = \varphi(S)T + S \varphi(T)$. By Lemma 2.2, we know that $\varphi(I) = 0$. Hence $\varphi(I) = \varphi(S)T + S \varphi(T)$, i.e. $\varphi$ is a derivable mapping at $I$. It follows from Theorem 1.1 in [17] that $\varphi$ is a derivation. This completes the proof. \(\square\)

3. Two examples

In this section, we always write $\mathcal{A}$ for the algebra of all $2 \times 2$ upper triangular matrices, and

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lemma 3.1. Let $\mathcal{A}$ be the algebra of all $2 \times 2$ upper triangular matrices, and let $\varphi : \mathcal{A} \to \mathcal{A}$ be a linear mapping as defined the following:

$$\begin{align*}
\varphi(E_{11}) &= a_{11}E_{11} + a_{12}E_{12} + a_{22}E_{22}, \\
\varphi(E_{12}) &= b_{11}E_{11} + b_{12}E_{12} + b_{22}E_{22}, \\
\varphi(E_{22}) &= c_{11}E_{11} + c_{12}E_{12} + c_{22}E_{22}.
\end{align*}$$
Then the following statements are equivalent.

1. $\varphi$ is a derivable mapping at $E_{11}$.
2. $\varphi$ is a derivable mapping at $E_{22}$.
3. $a_{11} = b_{11} = c_{11} = a_{22} = b_{22} = c_{22} = 0$ and $a_{12} + c_{12} = 0$.
4. $\varphi$ is a derivation.

**Proof.** (3) $\Rightarrow$ (4). Suppose that $a_{11} = b_{11} = c_{11} = a_{22} = b_{22} = c_{22} = 0$ and $a_{12} + c_{12} = 0$. For arbitrary $S = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ and $T = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}$ in $\mathcal{A}$, then $ST = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} = \begin{bmatrix} xu & xv + yw \\ 0 & zw \end{bmatrix}$.

Using straightforward matrix computations, we have

$$
\varphi(ST) = \varphi \begin{bmatrix} xu & xv + yw \\ 0 & zw \end{bmatrix} = \begin{bmatrix} a_{12} xu & a_{12} xy + b_{12} xv + b_{12} yw + c_{12} zw \\ 0 & 0 \end{bmatrix}
$$

Thus we have

$$
\varphi(ST) = \varphi(ST) = \varphi(S)T + S\varphi(T).
$$

Hence $\varphi$ is a derivation.

(4) $\Rightarrow$ (1) and (2) are obvious.

(1) $\Rightarrow$ (3). Suppose that $\varphi$ is a derivable mapping at $E_{11}$. For arbitrary $S = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ and $T = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}$ in $\mathcal{A}$ with $ST = E_{11}$, then $SU = 1, XV + YW = 0$ and $ZW = 0$. Thus we have

$$
a_{11}E_{11} + a_{12}E_{12} + a_{22}E_{22}
= \varphi(E_{11}) = \varphi(ST) = \varphi(ST) = \varphi(S)T + S\varphi(T)
= (a_{11}xU + b_{11}yU + c_{11}zU + a_{11}xU + b_{11}xV + c_{11}xW)E_{11}
+ (a_{11}xV + b_{11}yV + c_{11}zV + a_{12}xU + b_{12}xV + c_{12}xW)
+ a_{12}xW + b_{12}yW + c_{12}zW + a_{22}yU + b_{22}yV + c_{22}yW)E_{12}
+ (a_{22}xW + b_{22}yW + c_{22}zW + a_{22}zU + b_{22}zV + c_{22}zW)E_{22}.
$$

Taking $X = U = 1$ and $Y = Z = V = W = 0$ in the above equation, then $a_{11} = 0$. Using the same methods as the above, we may prove that $b_{11} = c_{11} = a_{22} = b_{22} = c_{22} = 0$ and $a_{12} + c_{12} = 0$. By imitating the proof of (1) $\Rightarrow$ (3), we may prove that (2) implies (3). □

Using the same methods as the proof of Lemma 3.1, we may prove the following the lemma.

**Lemma 3.2.** Let $\mathcal{A}$ be the algebra of all $2 \times 2$ upper triangular matrices, and let $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ be a linear mapping as defined the following:

$$
\left\{ \begin{array}{l}
\varphi(E_{11}) = a_{11}E_{11} + a_{12}E_{12} + a_{22}E_{22}, \\
\varphi(E_{12}) = b_{11}E_{11} + b_{12}E_{12} + b_{22}E_{22}, \\
\varphi(E_{22}) = c_{11}E_{11} + c_{12}E_{12} + c_{22}E_{22}.
\end{array} \right.
$$

If $\lambda \in \mathcal{C}$, then the following statements hold.

1. If $\varphi$ is a derivable mapping at $A = \lambda E_{11} + E_{12}$, then $\varphi$ is a derivation.
2. If $\varphi$ is a derivable mapping at $A = \lambda E_{22} + E_{12}$, then $\varphi$ is a derivation.
3. $\varphi$ be a derivable mapping at $0$ if and only if $b_{11} = c_{11} = a_{22} = b_{22} = 0, a_{12} + c_{12} = 0$ and $a_{11} = c_{22}$. In particular, $0$ is not an all-derivable point of $\mathcal{A}$. 

By Theorem 2.3 and Lemmas 3.1 and 3.2, we may obtain the following two examples.

**Example 3.3.** Let $\mathcal{A}$ be the algebra of all $2 \times 2$ upper triangular matrices, and let $A \in \mathcal{A}$. Then $A$ is a all-derivation point of $\mathcal{A}$ if and only if $A \neq 0$.

It is a nature problem whether every nonzero point in an operator algebra is all-derivable. We show that the question has negative answer.

**Example 3.4.** Let $\mathcal{B}$ be the algebra of all $2 \times 2$ upper triangular matrices and $\mathcal{A} = \{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \mathcal{B} \}$. Then there exists $0 \neq Z \in \mathcal{A}$ and $Z$ is not all-derivable of $\mathcal{A}$. We show that $Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A}$ is not an all-derivable point of $\mathcal{A}$. In fact, we only need to find a derivable mapping $\varphi$ at $Z$ from $\mathcal{A}$ into itself and $\varphi$ is not a derivation. By Example 3.3, we may find a linear mapping $\varphi : \mathcal{A} \to \mathcal{A}$ as the following:

$$
\varphi \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & \varphi_2(B) \end{bmatrix},
$$

where $\varphi_1$ is a nonzero derivation from $\mathcal{B}$ into itself, and $\varphi_2$ is a derivable mapping at 0 from $\mathcal{B}$ into itself, but $\varphi_2$ is not a derivation. It is easy to verify that $\varphi$ is a derivable mapping at $G$ on $\mathcal{A}$, but it is not a derivation. Hence $G$ is not an all-derivable point of $\mathcal{A}$.

**References**