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Multiplicity of Positive Solutions for Second-Order Three-Point Boundary Value Problems

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Abstract—We study the multiplicity of positive solutions for the second-order three-point boundary value problem

$$u'' + \lambda h(t) f(u) = 0, \qquad t \in (0,1),$$
$$u(0) = 0, \qquad \alpha u(\eta) = u(1),$$

where $\eta : 0 < \eta < 1$, $0 < \alpha < 1/\eta$. The methods employed are fixed-point index theorems and Leray-Schauder degree and upper and lower solutions.

Keywords—Three-point BVP, Positive solution, Cone, Fixed-point index.

1. INTRODUCTION

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Motivated by the study of Il'in and Moiseev [1,2], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint boundary value problems have been studied by several authors by using the Leray-Schauder Continuation Theorem, nonlinear alternative of Leray-Schauder, or coincidence degree theory. We refer the reader to [3–8] for some existence results of nonlinear multipoint boundary value problems. Very recently, the author [9] considered the existence of positive solutions of the problem

$$u'' + a(t) f(u) = 0, \qquad t \in (0,1), \tag{1.1}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1), \tag{1.2}$$

where $\eta \in (0,1)$. By using fixed-point theorem in cone, we established the existence results for positive solutions to (1.1),(1.2), assuming that $0 < \alpha\eta < 1$ and

$$f \in C([0,\infty),[0,\infty)), \qquad a \in C([0,1],[0,\infty)),$$

and f is either superlinear or sublinear.

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In this paper, we are concerned with the existence and multiplicities of positive solutions of the problem

$$u'' + \lambda h(t) f(u) = 0, \qquad t \in (0,1), \tag{1.3}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1). \tag{1.4}$$

We make the following assumptions.

(A_1) λ is a positive parameter; $\eta \in (0,1)$ and $0 < \alpha\eta < 1$.

(A_2) $h : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subset of positive measure.

(A_3) $f : [0,\infty) \to (0,\infty)$ is continuous.

(A_4)

$$f_\infty := \lim_{u\to\infty} \frac{f(u)}{u} = \infty.$$

Our main result is the following.

THEOREM 1.1. *Assume (A_1)–(A_4). Then there exists a positive number λ^* such that (1.3),(1.4) has at least two positive solutions for $0 < \lambda < \lambda^*$, at least one positive solution for $\lambda = \lambda^*$, and no positive solutions for $\lambda > \lambda^*$.*

Note that we do not require any monotonicity on f. Similar results were proved for a variety of two-point boundary value problems in [10].

The proof of Theorem 1.1 is based upon the method of upper and lower solutions and the degree theory and the following fixed-point index results [11].

LEMMA 1.3. *Let X be a Banach space, and let K be a cone in X. For $r > 0$, define $K_r = \{x \in K \mid \|x\| < r\}$. Assume $T : \overline{K_r} \to K$ is a compact map such that $Tx \neq x$ for $x \in \partial K_r$.*

(i) *If $\|x\| \le \|Tx\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|x\| \ge \|Tx\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

2. PRELIMINARY RESULTS

LEMMA 2.1. *For $y \in C[0,1]$, the problem*

$$u'' + y(t) = 0, \qquad t \in (0,1), \tag{2.1}$$

$$u(0) = a, \qquad u(1) - \alpha u(\eta) = b \tag{2.2}$$

has a unique solution

$$u(t) = \frac{b - a + a\alpha}{1 - \alpha\eta} t + a - \int_0^t (t-s) y(s)\, ds - \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta - s) y(s)\, ds + \frac{t}{1-\alpha\eta} \int_0^1 (1-s) y(s)\, ds.$$

PROOF. See [4].

The following two results were essentially established in [9]. In order that this paper be self contained, we provide details here.

LEMMA 2.2. *Let $0 < \alpha < 1/\eta$, and $a \geq 0$, $b \geq 0$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution u of problem (2.1),(2.2) satisfies*

$$u \geq 0, \qquad t \in [0,1].$$

PROOF. We divide the proof into two steps.

STEP 1. We deal with the special case that $a = b = 0$.

In fact, from the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on (0,1). So, if $u(1) \geq 0$, then the concavity of u and the boundary condition $u(0) = 0$ imply that

$$u \geq 0, \qquad \text{for } t \in [0,1].$$

If $u(1) < 0$ and $0 < \alpha \leq 1$, then

$$\begin{aligned} u(\eta) &< 0, \\ u(1) = \alpha u(\eta) &\geq u(\eta). \end{aligned} \tag{2.3}$$

This contradicts with the concavity of u.

If $u(1) < 0$ and $1 < \alpha < 1/\eta$, then

$$\begin{aligned} u(\eta) &< 0, \\ u(1) = \alpha u(\eta) &> \frac{1}{\eta} u(\eta). \end{aligned} \tag{2.4}$$

This contradicts with the concavity of u again.

STEP 2. Consider the linear problem

$$\begin{aligned} u'' &= 0, & t &\in (0,1), \\ u(0) &= a, & u(1) - \alpha u(\eta) &= b. \end{aligned}$$

The above problem has a solution

$$u_0(t) = \frac{b - a + a\alpha}{1 - \alpha\eta} t + a.$$

It is easy to check that $u_0(t) \geq 0$, for $t \in [0,1]$.

To sum up, the proof of Lemma 2.2 is completed.

REMARK. If $\alpha\eta > 1$, then the following counterexample shows that $y \geq 0$ does not imply that (2.1),(2.2) has positive solutions.

Consider the linear three-point boundary value problem

$$-u'' = t, \qquad t \in (0,1), \tag{2.5}$$

$$u(0) = 0, \qquad 8u\left(\frac{1}{2}\right) = u(1). \tag{2.6}$$

It is easy to see that (2.5),(2.6) has a unique negative solution

$$u(t) = -\frac{1}{6} t^3.$$

LEMMA 2.3. *Let $0 < \alpha < 1/\eta$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution u of the problem*

$$\begin{aligned} u'' + y(t) &= 0, & t &\in (0,1), \\ u(0) &= 0, & u(1) - \alpha u(\eta) &= 0 \end{aligned}$$

satisfies

$$\inf_{t\in[\eta,1]} u(t) \geq \gamma\|u\|,$$

where

$$\gamma = \min\left\{\alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \eta\right\}.$$

(In this paper, only the sup normal is used).

PROOF. We divide the proof into two steps.

STEP 1. We deal with the case $0 < \alpha < 1$. In this case, by Lemma 2, we know that

$$u(\eta) \geq u(1). \tag{2.7}$$

Set

$$u(\bar{t}) = \|u\|. \tag{2.8}$$

If $\bar{t} \leq \eta < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1) \tag{2.9}$$

and

$$u(\bar{t}) \leq u(1) + \frac{u(1)-u(\eta)}{1-\eta}(0-1) = u(1)\left[1 - \frac{1-(1/\alpha)}{1-\eta}\right] = u(1)\frac{1-\alpha\eta}{\alpha(1-\eta)}.$$

This together with (2.9) implies that

$$\min_{t\in[\eta,1]} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\|. \tag{2.10}$$

If $\eta < \bar{t} < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1). \tag{2.11}$$

From the concavity of u, we know that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}}. \tag{2.12}$$

Combining (2.12) and boundary condition $\alpha u(\eta) = u(1)$, we conclude that

$$\frac{u(1)}{\alpha\eta} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t}) = \|u\|.$$

This is

$$\min_{t\in[\eta,1]} u(t) \geq \alpha\eta\|u\|. \tag{2.13}$$

STEP 2. We deal with the case $1 \leq \alpha < 1/\eta$. In this case, we have

$$u(\eta) \leq u(1). \tag{2.14}$$

Set

$$u(\bar{t}) = \|u\|, \tag{2.15}$$

then we can choose $\bar{t}$ such that

$$\eta \leq \bar{t} \leq 1. \tag{2.16}$$

(We note that if $\bar{t} \in [0,1] \setminus [\eta, 1]$, then the point $(\eta, u(\eta))$ is below the straight line determined by $(1, u(1))$ and $(\bar{t}, u(\bar{t}))$. This contradicts with the concavity of u.) From (2.14) and the concavity of u, we know that

$$\min_{t\in[\eta,1]} u(t) = u(\eta). \tag{2.17}$$

Using the concavity of u and Lemma 2, we have that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}}. \tag{2.18}$$

This implies

$$\min_{t\in[\eta,1]} u(t) \geq \eta \|u\|. \tag{2.19}$$

3. EXISTENCE AND NONEXISTENCE

In this section, we prove the following.

THEOREM 3.1. *For λ sufficiently small, (1.3),(1.4) has at least one positive solution, whereas for λ sufficiently large, (1.3),(1.4) has no positive solutions.*

Let $X = C[0,1]$ with the usual normal $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define $T : X \to X$ by

$$\begin{aligned} Tu(t) = &-\int_0^t (t-s)\lambda h(s) f(u(s))\,ds \\ &- \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)\lambda h(s) f(u(s))\,ds \\ &+ \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\lambda h(s) f(u(s))\,ds. \end{aligned} \tag{3.1}$$

Let K be the cone defined by

$$K = \left\{ u \in X \mid u \geq 0,\ \min_{t\in[\eta,1]} u(t) \geq \gamma\|u\| \right\}. \tag{3.2}$$

Let C be the cone defined by

$$C = \{u \in X \mid u \geq 0\}.$$

Then by Lemma 2.3, we know that $T(C) \subset K$. Clearly, $T : X \to X$ is completely continuous.

PROOF OF THEOREM 3.1. If $q > 0$, then

$$\begin{aligned} \beta(q) = \max_{u\in K,\|u\|=q} \Bigg[&-\int_0^t (t-s) h(s) f(u(s))\,ds \\ &- \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s) h(s) f(u(s))\,ds \\ &+ \frac{t}{1-\alpha\eta}\int_0^1 (1-s) h(s) f(u(s))\,ds \Bigg] > 0. \end{aligned} \tag{3.3}$$

For any number $0 < r_1$, let $\delta_1 = r_1/\beta(r_1)$ and set

$$K_{r_1} = \{u \in X \mid \|u\| < r_1\}.$$

Then for $\lambda \in (0, \delta_1)$ and $y \in \partial K_{r_1}$, we have

$$\begin{aligned} Tu(t) &< \delta_1 \left[-\int_0^t (t-s) h(s) f(u(s))\,ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s) h(s) f(u(s))\,ds \right. \\ &\quad \left. + \frac{t}{1-\alpha\eta}\int_0^1 (1-s) h(s) f(u(s))\,ds \right] \\ &\leq \delta_1 \beta(r_1) = r_1. \end{aligned} \tag{3.4}$$

Thus, Lemma 1.3 implies

$$i(A, K_{r_1}, K) = 1. \tag{3.5}$$

Since $f_\infty = \infty$, there is $H > 0$ such that $f(u) \geq \mu u$ for $u \geq H$, where μ is chosen so that

$$\frac{\lambda \mu \eta \gamma}{1 - \alpha\eta} \int_\eta^1 (1-s)h(s)\,ds > 1. \tag{3.6}$$

Let $r_2 \geq H/\gamma$, and set

$$K_{r_2} = \{u \in X \mid \|u\| < r_2\}.$$

If $y \in \partial K_{r_2}$, then

$$\min_{t \in [\eta, 1]} u(t) \geq \gamma\|y\| \geq H.$$

Therefore,

$$\begin{aligned}
Tu(\eta) &= \lambda \left[-\int_0^\eta (\eta - s)h(s)f(u(s))\,dt - \frac{\alpha\eta}{1-\alpha\eta} \int_0^\eta (\eta - s)h(s)f(u(s))\,ds \right. \\
&\quad \left. + \frac{\eta}{1-\alpha\eta} \int_0^1 (1-s)h(s)f(u(s))\,ds \right] \\
&= \lambda \left[-\frac{1}{1-\alpha\eta} \int_0^\eta (\eta - s)h(s)f(u(s))\,ds + \frac{\eta}{1-\alpha\eta} \int_0^1 (1-s)h(s)f(u(s))\,ds \right] \\
&= \lambda \left[-\frac{1}{1-\alpha\eta} \int_0^\eta \eta h(s)f(u(s))\,ds + \frac{1}{1-\alpha\eta} \int_0^\eta sh(s)f(u(s))\,ds \right. \\
&\quad \left. + \frac{\eta}{1-\alpha\eta} \int_0^1 h(s)f(u(s))\,ds - \frac{\eta}{1-\alpha\eta} \int_0^1 sh(s)f(u(s))\,ds \right] \qquad (3.7) \\
&= \lambda \left[\frac{\eta}{1-\alpha\eta} \int_\eta^1 h(s)f(u(s))\,ds + \frac{1}{1-\alpha\eta} \int_0^\eta sh(s)f(u(s))\,ds \right. \\
&\quad \left. - \frac{\eta}{1-\alpha\eta} \int_0^1 sh(s)f(u(s))\,ds \right] \\
&\geq \lambda \left[\frac{\eta}{1-\alpha\eta} \int_\eta^1 h(s)f(u(s))\,ds - \frac{\eta}{1-\alpha\eta} \int_\eta^1 sh(s)f(u(s))\,ds \right] \quad (\text{by } \eta < 1) \\
&= \lambda \left[\frac{\eta}{1-\alpha\eta} \int_\eta^1 (1-s)h(s)f(u(s))\,ds \right].
\end{aligned}$$

Hence,

$$\|Tu\| \geq \frac{\lambda \mu \eta \gamma}{1-\alpha\eta} \int_\eta^1 (1-s)h(s)\,ds \|u\|,$$

which implies

$$\|Tu\| > \|u\|,$$

for $y \in \partial K_{r_2}$. An application of Lemma 1.3 again shows that

$$i(A, K_{r_2}, K) = 0. \tag{3.8}$$

Since we can adjust r_1, r_2 so that $r_1 < r_2$, it follows from the additivity of the fixed-point index that

$$i\left(A, K_{r_2} \setminus \overline{K_{r_1}}, K\right) = -1.$$

Thus, T has a fixed point in $K_{r_2} \setminus \overline{K_{r_1}}$ which is the desired positive solution of (1.3),(1.4).

To prove the nonexistence part, we note that (A_3) and (A_4) imply the existence of a constant $c_0 > 0$ such that

$$f(u) \geq c_0 u, \qquad \text{for } u \geq 0.$$

Let $u \in X$ be a positive solution of (1.3),(1.4). By Lemma 2.3, $u \in K$. Now choose λ large enough so that

$$\lambda c_0 \gamma \left[\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)u(s)\,ds\right] > 1. \tag{3.9}$$

By Lemma 3.2 and the similar method used to prove (3.7), we have that

$$\begin{aligned} u(\eta) &= \lambda\left[-\int_0^\eta (\eta-s)h(s)f(u(s))\,dt - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta-s)h(s)f(u(s))\,ds \right.\\ &\quad \left. +\frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)h(s)f(u(s))\,ds\right]\\ &= \lambda\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)f(u(s))\,ds\right]\\ &\geq \lambda\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)c_0u(s)\,ds\right]\\ &\geq \lambda c_0\gamma\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)h(s)\,ds\right]\|u\|\\ &> \|u\|. \end{aligned}$$

We have an obvious contradiction.

4. UPPER AND LOWER SOLUTIONS

In this section, we shall develop upper and lower solution methods for the boundary value problem

$$u''(t) + \lambda h(t)f(u(t)) = 0, \qquad t \in (0,1), \tag{4.1}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1) = 0. \tag{4.2}$$

DEFINITION 4.1. *We say that the function $x \in C^2[0,1]$ is an upper solution of problem (4.1),(4.2) if*

$$x''(t) + \lambda h(t)f(x(t)) \leq 0, \qquad t \in (0,1), \tag{4.3}$$

$$x(0) \geq 0, \qquad x(1) - \alpha x(\eta) \geq 0, \tag{4.4}$$

and $y \in C^2[0,1]$ is a lower solution of problem (4.1),(4.2) if

$$y''(t) + \lambda h(t)f(y(t)) \geq 0, \qquad t \in (0,1), \tag{4.5}$$

$$y(0) \leq 0, \qquad y(1) - \alpha y(\eta) \leq 0. \tag{4.6}$$

We now establish several lemmas that will be used throughout.

Let x, y be upper and lower solutions for (4.1),(4.2) and satisfy $x(t) \geq y(t)$ on $[0,1]$. We define f^* by

$$f^*(u(t)) = \begin{cases} f(x(t)), & u(t) \geq x(t),\\ f(u(t)), & y(t) \leq u(t) \leq x(t),\\ f(y(t)), & u(t) \leq x(t). \end{cases} \tag{4.7}$$

Consider the following problem:

$$u''(t) + \lambda h(t)f^*(u(t)) = 0, \qquad t \in (0,1), \tag{4.8}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1) = 0. \tag{4.9}$$

LEMMA 4.2. *If there is a solution u of (4.8),(4.9), then*

$$y(t) \leq u(t) \leq x(t), \qquad \text{for } t \in [0,1].$$

In other words, u is a solution of (4.1),(4.2).

PROOF. We first prove that $u(t) \leq x(t)$, for all $t \in (0,1]$. Suppose to the contrary that $u(t_0) > x(t_0)$ for some $t_0 \in (0,1]$.

Set

$$c = \inf\{t \in [0,1] \mid u(t) > x(t)\}, \tag{4.10}$$

then from the fact that $u(0) = 0$ and $x(0) \geq 0$, we know that $c > 0$ and

$$u(c) = x(c). \tag{4.11}$$

There are three cases as follows.

CASE 1. There exists $d \in (c,1]$, such that $u(d) = x(d)$ and $u(t) > x(t)$, for all $x \in (c,d)$.

In this case, we have

$$\begin{aligned} f^*(u(t)) &= f(x(t)), \qquad &&\text{for } t \in (c,d), \\ u(c) &= x(c), &&u(d) = x(d). \end{aligned} \tag{4.12}$$

Therefore,

$$\begin{aligned} (x-u)'' \leq -\lambda h(t)\left[f(x(t)) - f^*(u(t))\right] = 0, \qquad &\text{for } t \in (c,d), \\ (x-u)(c) = (x-u)(d) = 0, \end{aligned} \tag{4.13}$$

which, by the concavity of $x - u$, implies the contradiction $(x-u)(t) \geq 0$, for all $t \in (c,d)$.

CASE 2. $c \in (0,\eta)$ and $u(t) > x(t)$, for all $t \in (c,1]$.

In this case, we have

$$\begin{aligned} f^*(u(t)) &= f(x(t)), \qquad \text{for } t \in (c,1], \\ u(c) &= x(c). \end{aligned} \tag{4.14}$$

Therefore,

$$(x-u)'' \leq -\lambda h(t)\left[f(x(t)) - f^*(u(t))\right] = 0, \qquad \text{for } t \in (c,1]. \tag{4.15}$$

Using the boundary conditions $u(1) = \alpha u(\eta)$ and $x(1) \geq \alpha x(\eta)$, we know that

$$(x-u)(1) - \alpha(x-u)(\eta) \geq 0. \tag{4.16}$$

Combining (4.16) and (4.11) and using the same arguments used to prove Lemma 2.2, we can get the desired contradiction $x - u \geq 0$, for all $t \in [c,1]$.

CASE 3. $c \in [\eta,1)$ and $u(t) > x(t)$, for all $t \in (c,1]$.

In this case, we have

$$\begin{aligned} f^*(u(t)) &= f(x(t)), \qquad \text{for } t \in (c,1], \\ u(c) &= x(c). \end{aligned} \tag{4.17}$$

Therefore,

$$(x-u)'' \leq -\lambda h(t)\left[f(x(t)) - f^*(u(t))\right] = 0, \qquad \text{for } t \in (c,1]. \tag{4.18}$$

By the definition of c, we know that

$$u(t) \leq x(t), \qquad \text{for all } t \in [0,c]. \tag{4.19}$$

In particular, we have that

$$u(\eta) \leq x(\eta). \tag{4.20}$$

This, together with the boundary conditions $u(1) = \alpha u(\eta)$ and $x(1) \geq \alpha x(\eta)$, implies

$$u(1) \leq x(1). \tag{4.21}$$

Combining this with (4.11) and (4.18) and using the concavity of $x - u$, we obtain the desired contradiction $(x - u)(t) \geq 0$, for all $t \in (c, 1]$.

By the same arguments, we see that $y(t) \leq u(t)$, for $x \in [0, 1]$. Since $y(t) \leq u \leq x(t)$ for $t \in [0, 1]$, it follows that $f = f^*$, and so u is a solution of (4.1),(4.2).

LEMMA 4.3. *If there exist upper and lower solutions x and y of (4.1),(4.2) with $y(t) \leq x(t)$, for $t \in [0, 1]$, then there is a solution u to (4.1),(4.2) such that*

$$y(t) \leq u(t) \leq x(t), \qquad \text{for } t \in [0, 1].$$

PROOF. Consider problem (4.8),(4.9). By Lemma 2.1, we know that (4.8),(4.9) is equivalent to the integral equation

$$\begin{aligned} u(t) = &-\int_0^t (t - s)\lambda h(s) f^*(y(s))\, ds \\ &- \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta - s)\lambda h(s) f^*(y(s))\, ds \\ &+ \frac{t}{1 - \alpha\eta} \int_0^1 (1 - s)\lambda h(s) f^*(y(s))\, ds. \end{aligned}$$

Let

$$\begin{aligned} T^* u(t) = &-\int_0^t (t - s)\lambda h(s) f^*(y(s))\, ds \\ &- \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta - s)\lambda h(s) f^*(y(s))\, ds \\ &+ \frac{t}{1 - \alpha\eta} \int_0^1 (1 - s)\lambda h(s) f^*(y(s))\, ds. \end{aligned}$$

Then $T^* : C[0, 1] \to C[0, 1]$ is completely continuous. Since f^* is bounded, T^* is bounded. By the Schauder fixed-point theorem, T^* has a fixed point u, which is a solution of (4.8),(4.9). By Lemma 4.2, u is also a solution of (4.1),(4.2).

5. MULTIPLICITY

In order to guarantee that all possible solutions of (1.3),(1.4) are nonnegative, we make the convention that

$$f(u) = f(0), \qquad \text{if } u < 0. \tag{5.1}$$

We first need the following *priori* estimate.

LEMMA 5.1. *There is a constant $b_I > 0$ such that $\|y\| \leq b_I$, for all solutions u of (1.3),(1.4) where λ belongs to a compact subset I of $(0, \infty)$.*

PROOF. Now suppose there is an unbounded sequence $\{u_n\}$ of solutions of (1.3),(1.4) which corresponding λ_n belongs to a compact subset of $(0, \infty)$. By Lemma 2.3, $u_n \in K$, which implies that

$$\min_{t \in [\eta, 1]} u_n(x) \geq \gamma \|u_n\|.$$

Since $f_\infty = \infty$, there is a $q > 0$ such that

$$f(u) \geq \bar{\mu} u, \qquad \text{for all } u \geq q,$$

where $\bar{\mu}$ is chosen so that

$$\inf\{\lambda_n\}\bar{\mu}\gamma\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)\,ds\right]>1.$$

Choosing n large enough so that $\gamma\|u_n\|\geq q$, then by the same arguments used to get (3.7), we have that

$$\begin{aligned}
u_n(\eta)&=\lambda_n\left[-\int_0^\eta(\eta-s)a(s)f(u_n(s))\,dt-\frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta(\eta-s)a(s)f(u_n(s))\,ds\right.\\
&\quad\left.+\frac{\eta}{1-\alpha\eta}\int_0^1(1-s)a(s)f(u_n(s))\,ds\right]\\
&=\lambda_n\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)f(u_n(s))\,ds\right]\\
&\geq\lambda_n\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)\bar{\mu}u_n(s)\,ds\right]\\
&\geq\lambda_n\bar{\mu}\gamma\left[\frac{\eta}{1-\alpha\eta}\int_\eta^1(1-s)a(s)\,ds\right]\|u_n\|>\|u_n\|,
\end{aligned}$$

which is a contradiction.

Now let Γ denote the set of $\lambda>0$ such that a positive solution of (1.3),(1.4) exists. Let $\lambda^*=\sup\Gamma$. By Theorem 3.1, Γ is nonempty and bounded, and thus, $0<\lambda^*<\infty$. We claim that $\lambda^*\in\Gamma$. To see this, let $\lambda_n\to\lambda^*$, where $\lambda_n\in\Gamma$:

$$\lambda_1<\lambda_2<\cdots<\lambda_{n-1}<\lambda_n<\cdots<\lambda^*.$$

Since the $\{\lambda_n\}$ are bounded, Lemma 5.1 implies that the corresponding solutions $\{u_n\}$ are bounded. By the compactness of the integral operator T, it easily follows that $\lambda^*\in\Gamma$.

Let u^* be a solution of (1.3),(1.4) corresponding to λ^* and define

$$\tilde{f}(u(t))=\begin{cases}f(u^*(t)+\epsilon), & u(t)\geq u^*(t)+\epsilon,\\ f(u(t)), & -\epsilon\leq u(t)\leq u^*(t)+\epsilon,\\ f(-\epsilon), & u(t)\leq-\epsilon.\end{cases}$$

Let

$$\begin{aligned}
\tilde{T}_\lambda u(t)&=\lambda\left[-\int_0^t(t-s)h(s)\tilde{f}(u(s))\,ds-\frac{\alpha t}{1-\alpha\eta}\int_0^\eta(\eta-s)h(s)\tilde{f}(u(s))\,ds\right.\\
&\quad\left.+\frac{t}{1-\alpha\eta}\int_0^1(1-s)h(s)\tilde{f}(u(s))\,ds\right].
\end{aligned}$$

Consider

$$\Omega=\{u\in X\mid-\epsilon\leq u(t)\leq u^*(t)+\epsilon\}.$$

LEMMA 5.2. *There is an $\epsilon>0$, sufficiently small, such that if $u\in C[0,1]$ satisfies $\tilde{T}_\lambda u=u$ for some $0<\lambda<\lambda^*$, then $u\in\bar{\Omega}$.*

PROOF. Since $u\geq0$, to prove that $u\leq u^*+\epsilon$, we first show that $u^*+\epsilon$ is an upper solution of (1.3),(1.4). Since $u^*\geq0$, there is a constant $d_0>0$ such that $f(u^*(t))>d_0>0$, for all $t\in[0,1]$. By uniform continuity, there is an $\epsilon_0>0$ such that

$$|f(u^*(t)+\epsilon)-f(u^*(t))|<d_0\frac{(\lambda^*-\lambda)}{\lambda},$$

for all $t \in [0,1], 0 \le \epsilon \le \epsilon_0$. Now

$$\begin{aligned}(u^*(t)+\epsilon)'' &= (u^*(t))'' \\ &= -\lambda^* h(t) f(u^*(t)) \\ &= -\lambda h(t) f(u^*(t)+\epsilon) \\ &\quad + \lambda h(t)\left[f(u^*(t)+\epsilon) - f(u^*(t))\right] + (\lambda - \lambda^*) h(t) f(u^*(t)) \\ &< -\lambda h(t) f(u^*(t)+\epsilon)\end{aligned} \tag{5.2}$$

and

$$(u^*+\epsilon)(0) \ge 0, \qquad (u^*+\epsilon)(1) \ge 0. \tag{5.3}$$

Clearly, if $\epsilon > 0$, then (5.3) becomes

$$(u^*+\epsilon)(0) > 0, \qquad (u^*+\epsilon)(1) > 0. \tag{5.4}$$

Therefore, $u^*+\epsilon$ is an upper solution of (1.3),(1.4). It follows from Lemma 4.1 that $u \le u^* + \epsilon$.

PROOF OF THEOREM 1.1. Let $\lambda \in (0, \lambda^*)$; we show that (1.3),(1.4) has at least two positive solutions. Since u^* is an upper solution and 0 is a lower solution, Lemma 4.3 implies the existence of a solution u_λ of (1.3),(1.4) such that $0 \le u_\lambda \le u^*$. Thus, for $0 < \lambda < \lambda^*$, a positive solution exists, whereas for $\lambda > \lambda^*$, a positive solution does not exist. Moreover, $u_\lambda \in \Omega$.

Choose $I = [0, \lambda^* + 1]$; then

$$(0, \lambda^*) \cap I \ne \emptyset$$

and

$$(\lambda^*, \infty) \cap I \ne \emptyset.$$

We next establish the existence of a second positive solution to (1.3),(1.4) for $\lambda \in (0,\lambda^*) \cap I$.

Since $\tilde{T}_\lambda$ is bounded for $\lambda \in I$,

$$\deg\left(I - \tilde{T}_\lambda, B(u_\lambda, R), 0\right) = 1, \tag{5.5}$$

for R large enough, where $B(u_\lambda, R)$ is the ball centered at u_λ with radius R in $C[0,1]$. If there exists a $u \in \partial\Omega$ such that $u = \tilde{T}_\lambda(u)$, then $f = \tilde{f}$, and so u is a second positive solution. Now suppose $u \ne \tilde{T}_\lambda(u)$, for all $u \in \partial\Omega$. Then $\deg(I - \tilde{T}_\lambda, \Omega, 0)$ is well defined. Since Lemma 5.2 implies $\tilde{T}_\lambda$ has no fixed point in $B(u_\lambda, R) \setminus \Omega$, we have from the excision property of degree that

$$\deg\left(I - \tilde{T}_\lambda, \Omega, 0\right) = 1. \tag{5.6}$$

This, together with the fact that

$$\tilde{T}_\lambda\,|_\Omega = T_\lambda|_\Omega\,,$$

implies that

$$\deg(I - T_\lambda, \Omega, 0) = 1. \tag{5.7}$$

On the other hand, by Lemma 5.1, all positive solutions of (1.3),(1.4) are bounded for $\lambda \in I$, and thus,

$$\deg(I - T_\lambda, B(0,M), 0) = \text{constant}, \qquad \text{for } \lambda \in I, \tag{5.8}$$

for M large enough, where $B(0,M)$ is the ball centered at 0 with radius M in $C[0,1]$. The late degree must equal 0, since for all $\lambda > \lambda^*$, no solutions exist. (We note that (A_3) and (5.1) and Lemma 2.2 imply that all solutions of (1.3),(1.4) are positive solutions!) Finally, by the excision property

$$\deg(I - T_\lambda, B(0,M) \setminus \Omega, 0) = -1, \tag{5.9}$$

and so a second positive solution of (1.3),(1.4) exists for $\lambda \in (0, \lambda^*) \cap I$.

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