Multiplicity of Positive Solutions for Second-Order Three-Point Boundary Value Problems

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Abstract—We study the multiplicity of positive solutions for the second-order three-point boundary value problem

\[ u'' + \lambda h(t)f(u) = 0, \quad t \in (0, 1), \]

\[ u(0) = 0, \quad \alpha u(\eta) = u(1), \]

where \( 0 < \eta < 1, \) \( 0 < \alpha < 1/\eta. \) The methods employed are fixed-point index theorems and Leray-Schauder degree and upper and lower solutions. \( \odot 2000 \) Elsevier Science Ltd. All rights reserved.

Keywords—Three-point BVP, Positive solution, Cone, Fixed-point index.

1. INTRODUCTION

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Motivated by the study of Il'in and Moiseev [1,2], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multipoint boundary value problems have been studied by several authors by using the Leray-Schauder Continuation Theorem, nonlinear alternative of Leray-Schauder, or coincidence degree theory. We refer the reader to [3-8] for some existence results of nonlinear multipoint boundary value problems. Very recently, the author [9] considered the existence of positive solutions of the problem

\[ u'' + a(t)f(u) = 0, \quad t \in (0, 1), \]

\[ u(0) = 0, \quad \alpha u(\eta) = u(1), \]

where \( \eta \in (0, 1). \) By using fixed-point theorem in cone, we established the existence results for positive solutions to (1.1),(1.2), assuming that \( 0 < a\eta < 1 \) and

\[ f \in C([0, \infty), [0, \infty)), \quad \alpha \in C([0, 1], [0, \infty)), \]

and \( f \) is either superlinear or sublinear.

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In this paper, we are concerned with the existence and multiplicities of positive solutions of the problem

\[
\begin{align*}
  u'' + \lambda h(t)f(u) &= 0, & t \in (0, 1), \\
  u(0) &= 0, & \alpha u(\eta) = u(1).
\end{align*}
\]

We make the following assumptions.

(A1) \( \lambda \) is a positive parameter; \( \eta \in (0, 1) \) and \( 0 < \alpha \eta < 1 \).

(A2) \( h : [0, 1] \to [0, \infty) \) is continuous and does not vanish identically on any subset of positive measure.

(A3) \( f : [0, \infty) \to (0, \infty) \) is continuous.

(A4) \( \lim_{u \to 0^+} f(u) = \infty \).

Our main result is the following.

**Theorem 1.1.** Assume (A1)-(A4). Then there exists a positive number \( \lambda^* \) such that (1.3),(1.4) has at least two positive solutions for \( 0 < \lambda < \lambda^* \), at least one positive solution for \( \lambda = \lambda^* \), and no positive solutions for \( \lambda > \lambda^* \).

Note that we do not require any monotonicity on \( f \). Similar results were proved for a variety of two-point boundary value problems in [10].

The proof of Theorem 1.1 is based upon the method of upper and lower solutions and the degree theory and the following fixed-point index results [11].

**Lemma 1.3.** Let \( X \) be a Banach space, and let \( K \) be a cone in \( X \). For \( r > 0 \), define \( K_r = \{ x \in K \mid \|x\| < r \} \). Assume \( T : K_r \to K \) is a compact map such that \( Tx \neq x \) for \( x \in \partial K_r \).

(i) If \( \|x\| \leq \|Tx\| \) for \( x \in \partial K_r \), then

\[
i(T, K_r, K) = 0.
\]

(ii) If \( \|x\| \geq \|Tx\| \) for \( x \in \partial K_r \), then

\[
i(T, K_r, K) = 1.
\]

**2. PRELIMINARY RESULTS**

**Lemma 2.1.** For \( y \in C[0,1] \), the problem

\[
\begin{align*}
  u'' + y(t) &= 0, & t \in (0,1), \\
  u(0) &= a, & u(1) - \alpha u(\eta) = b
\end{align*}
\]

has a unique solution

\[
u(t) = \frac{b-a+\alpha a}{1-\alpha \eta} t + a - \int_0^t (t-s) y(s) \, ds - \frac{\alpha t}{1-\alpha \eta} \int_0^\eta (\eta-s) y(s) \, ds + \frac{t}{1-\alpha \eta} \int_0^1 (1-s) y(s) \, ds.
\]

**Proof.** See [4].

The following two results were essentially established in [9]. In order that this paper be self contained, we provide details here.
**Lemma 2.2.** Let $0 < \alpha < 1/\eta$, and $a \geq 0$, $b \geq 0$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution $u$ of problem (2.1), (2.2) satisfies

$$u \geq 0, \quad t \in [0,1].$$

**Proof.** We divide the proof into two steps.

**Step 1.** We deal with the special case that $a = b = 0$.

In fact, from the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$. So, if $u(1) \geq 0$, then the concavity of $u$ and the boundary condition $u(0) = 0$ imply that

$$u \geq 0, \quad t \in [0,1].$$

If $u(1) < 0$ and $0 < \alpha \leq 1$, then

$$u(\eta) < 0,$$

$$u(1) = \alpha u(\eta) \geq u(\eta).$$

This contradicts with the concavity of $u$.

If $u(1) < 0$ and $1 < \alpha < 1/\eta$, then

$$u(\eta) < 0,$$

$$1 = u(1) > \eta u(\eta).$$

This contradicts with the concavity of $u$ again.

**Step 2.** Consider the linear problem

$$u'' = 0, \quad t \in (0,1),$$

$$u(0) = a, \quad u(1) - \alpha u(\eta) = b.$$

The above problem has a solution

$$u_0(t) = \frac{b - a + a\alpha}{1 - \alpha\eta} t + a.$$

It is easy to check that $u_0(t) \geq 0$, for $t \in [0,1]$.

To sum up, the proof of Lemma 2.2 is completed.

**Remark.** If $\alpha\eta > 1$, then the following counterexample shows that $y \geq 0$ does not imply that (2.1), (2.2) has positive solutions.

Consider the linear three-point boundary value problem

$$-u'' = t, \quad t \in (0,1),$$

$$u(0) = 0, \quad 8u\left(\frac{1}{2}\right) = u(1).$$

It is easy to see that (2.5), (2.6) has a unique negative solution

$$u(t) = -\frac{1}{6}t^3.$$

**Lemma 2.3.** Let $0 < \alpha < 1/\eta$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution $u$ of the problem

$$u'' + y(t) = 0, \quad t \in (0,1),$$

$$u(0) = 0, \quad u(1) - \alpha u(\eta) = 0$$
satisfies
\[ \inf_{t \in [\eta, 1]} u(t) \geq \gamma \|u\|, \]
where
\[ \gamma = \min \left\{ \alpha \eta, \frac{\alpha (1 - \eta)}{1 - \alpha \eta}, \eta \right\}. \]

(In this paper, only the sup normal is used).

**Proof.** We divide the proof into two steps.

**Step 1.** We deal with the case $0 < \alpha < 1$. In this case, by Lemma 2, we know that

\[ u(\eta) \geq u(1). \tag{2.7} \]

Set

\[ u(\bar{t}) = \|u\|. \tag{2.8} \]

If $\bar{t} \leq \eta < 1$, then

\[ \min_{t \in [\eta, 1]} u(t) = u(1) \tag{2.9} \]

and

\[
\begin{align*}
    u(\bar{t}) &\leq u(1) + \frac{u(1) - u(\eta)}{1 - \eta} (0 - 1) = u(1) \left[ 1 - \frac{1 - (1/\alpha)}{1 - \eta} \right] \\
&= u(1) \frac{1 - \alpha \eta}{\alpha (1 - \eta)}.
\end{align*}
\]

This together with (2.9) implies that

\[ \min_{t \in [\eta, 1]} u(t) \geq \frac{\alpha (1 - \eta)}{1 - \alpha \eta} \|u\|. \tag{2.10} \]

If $\eta < \bar{t} < 1$, then

\[ \min_{t \in [\eta, 1]} u(t) = u(1). \tag{2.11} \]

From the concavity of $u$, we know that

\[ \frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}}. \tag{2.12} \]

Combining (2.12) and boundary condition $\alpha u(\eta) = u(1)$, we conclude that

\[ \frac{u(1)}{\alpha \eta} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t}) = \|u\|. \]

This is

\[ \min_{t \in [\eta, 1]} u(t) \geq \alpha \eta \|u\|. \tag{2.13} \]

**Step 2.** We deal with the case $1 \leq \alpha < 1/\eta$. In this case, we have

\[ u(\eta) \leq u(1). \tag{2.14} \]

Set

\[ u(\bar{t}) = \|u\|. \tag{2.15} \]

then we can choose $\bar{t}$ such that

\[ \eta \leq \bar{t} \leq 1. \tag{2.16} \]
We note that if \( \tilde{t} \in [0, 1] \setminus \{\eta, 1\} \), then the point \((\eta, u(\eta))\) is below the straight line determined by \((1, u(1))\) and \((\tilde{t}, u(\tilde{t}))\). This contradicts with the concavity of \(u\). From (2.14) and the concavity of \(u\), we know that
\[
\min_{t \in [\eta, 1]} u(t) = u(\eta). \tag{2.17}
\]
Using the concavity of \(u\) and Lemma 2, we have that
\[
\frac{u(\eta)}{\eta} \geq \frac{u(\tilde{t})}{\tilde{t}}. \tag{2.18}
\]
This implies
\[
\min_{t \in [\eta, 1]} u(t) \geq \eta \|u\|. \tag{2.19}
\]

### 3. EXISTENCE AND NONEXISTENCE

In this section, we prove the following.

**Theorem 3.1.** For \(\lambda\) sufficiently small, (1.3), (1.4) has at least one positive solution, whereas for \(\lambda\) sufficiently large, (1.3), (1.4) has no positive solutions.

Let \(X = C[0, 1]\) with the usual normal \(\|u\| = \max_{t \in [0, 1]} |u(t)|\). Define \(T : X \to X\) by
\[
Tu(t) = -\int_0^t (t-s)\lambda h(s)f(u(s))\,ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s)\lambda h(s)f(u(s))\,ds + \frac{t}{1 - \alpha \eta} \int_0^1 (1-s)\lambda h(s)f(u(s))\,ds.
\]
Let \(K\) be the cone defined by
\[
K = \left\{ u \in X \mid u \geq 0, \min_{t \in [\eta, 1]} u(t) \geq \eta \|u\| \right\}. \tag{3.2}
\]
Let \(C\) be the cone defined by
\[
C = \{ u \in X \mid u \geq 0 \}.
\]
Then by Lemma 2.3, we know that \(T(C) \subset K\). Clearly, \(T : X \to X\) is completely continuous.

**Proof of Theorem 3.1.** If \(q > 0\), then
\[
\beta(q) = \max_{u \in K, \|u\| = q} \left[ -\int_0^t (t-s)h(s)f(u(s))\,ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s)h(s)f(u(s))\,ds + \frac{t}{1 - \alpha \eta} \int_0^1 (1-s)h(s)f(u(s))\,ds \right] > 0. \tag{3.3}
\]
For any number \(0 < \delta_1 = r_1/\beta(r_1)\) and set
\[
K_{\delta_1} = \{ u \in X \mid \|u\| < r_1 \}.
\]
Then for \(\lambda \in (0, \delta_1)\) and \(y \in \partial K_{\delta_1}\), we have
\[
Tu(t) < \delta_1 \left[ -\int_0^t (t-s)h(s)f(u(s))\,ds - \frac{\alpha t}{1 - \alpha \eta} \int_0^\eta (\eta-s)h(s)f(u(s))\,ds + \frac{t}{1 - \alpha \eta} \int_0^1 (1-s)h(s)f(u(s))\,ds \right] \leq \delta_1 \beta(r_1) = r_1. \tag{3.4}
\]
Thus, Lemma 1.3 implies

$$i(A, K_{r_1}, K) = 1.$$  \hfill (3.5)

Since $f_\infty = \infty$, there is $H > 0$ such that $f(u) \geq \mu u$ for $u \geq H$, where $\mu$ is chosen so that

$$\frac{\lambda \mu \gamma}{1 - \alpha \eta} \int_\eta^1 (1 - s)h(s) \, ds > 1.$$  \hfill (3.6)

Let $r_2 \geq H/\gamma$, and set

$$K_{r_2} = \{ u \in X \mid \|u\| < r_2 \}.$$

If $y \in \partial K_{r_2}$, then

$$\min_{t \in [\eta, 1]} u(t) \geq \gamma \|y\| \geq H.$$

Therefore,

$$Tu(\eta) = \lambda \left[ - \int_0^\eta (\eta - s)h(s)f(u(s)) \, dt - \frac{\alpha \eta}{1 - \alpha \eta} \int_0^\eta (\eta - s)h(s)f(u(s)) \, ds \right]$$

$$+ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)f(u(s)) \, ds$$

$$= \lambda \left[ - \int_0^\eta (\eta - s)h(s)f(u(s)) \, ds + \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)f(u(s)) \, ds \right]$$

$$= \lambda \left[ - \int_0^\eta \eta h(s)f(u(s)) \, ds + \frac{1}{1 - \alpha \eta} \int_0^\eta sh(s)f(u(s)) \, ds \right]$$

$$+ \frac{\eta}{1 - \alpha \eta} \int_0^1 h(s)f(u(s)) \, ds - \frac{\eta}{1 - \alpha \eta} \int_0^1 sh(s)f(u(s)) \, ds \right], \quad \text{(3.7)}$$

$$\geq \lambda \left[ \frac{\eta}{1 - \alpha \eta} \int_0^1 h(s)f(u(s)) \, ds - \frac{\eta}{1 - \alpha \eta} \int_0^1 sh(s)f(u(s)) \, ds \right] \quad \text{(by } \eta < 1 \text{)}$$

$$= \lambda \left[ \frac{\eta}{1 - \alpha \eta} \int_\eta^1 (1 - s)h(s)f(u(s)) \, ds \right].$$

Hence,

$$\|Tu\| \geq \frac{\lambda \mu \gamma}{1 - \alpha \eta} \int_\eta^1 (1 - s)h(s) \, ds \|u\|,$$

which implies

$$\|Tu\| > \|u\|,$$

for $y \in \partial K_{r_2}$. An application of Lemma 1.3 again shows that

$$i(A, K_{r_2}, K) = 0.$$  \hfill (3.8)

Since we can adjust $r_1, r_2$ so that $r_1 < r_2$, it follows from the additivity of the fixed-point index that

$$i(A, K_{r_2} \setminus \overline{K_{r_1}}, K) = -1.$$

Thus, $T$ has a fixed point in $K_{r_2} \setminus \overline{K_{r_1}}$, which is the desired positive solution of (1.3),(1.4).

To prove the nonexistence part, we note that (A_3) and (A_4) imply the existence of a constant $c_0 > 0$ such that

$$f(u) \geq c_0 u, \quad \text{for } u \geq 0.$$
Let $u \in X$ be a positive solution of (1.3), (1.4). By Lemma 2.3, $u \in K$. Now choose $\lambda$ large enough so that
\[ \lambda c_0 \gamma \left[ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)u(s) \, ds \right] > 1. \] (3.9)

By Lemma 3.2 and the similar method used to prove (3.7), we have that
\[ u(\eta) = \lambda \left[ - \int_0^\eta (\eta - s)h(s)f(u(s)) \, dt - \frac{\alpha \eta}{1 - \alpha \eta} \int_0^\eta (\eta - s)h(s)f(u(s)) \, ds \right. \]
\[ \left. + \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)f(u(s)) \, ds \right] \]
\[ = \lambda \left[ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)f(u(s)) \, ds \right] \]
\[ \geq \lambda \left[ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s)c_0 u(s) \, ds \right] \]
\[ \geq \lambda c_0 \gamma \left[ \frac{\eta}{1 - \alpha \eta} \int_0^1 (1 - s)h(s) \, ds \right] \|u\| \]
\[ > \|u\|. \]

We have an obvious contradiction.

4. UPPER AND LOWER SOLUTIONS

In this section, we shall develop upper and lower solution methods for the boundary value problem
\[ u''(t) + \lambda h(t)f(u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1) = 0. \] (4.1)

**DEFINITION 4.1.** We say that the function $x \in C^2[0, 1]$ is an upper solution of problem (4.1), (4.2) if
\[ x''(t) + \lambda h(t)f(x(t)) \leq 0, \quad t \in (0, 1), \quad x(0) \geq 0, \quad x(1) - \alpha x(\eta) \geq 0, \] (4.3)
and $y \in C^2[0, 1]$ is a lower solution of problem (4.1), (4.2) if
\[ y''(t) + \lambda h(t)f(y(t)) \geq 0, \quad t \in (0, 1), \quad y(0) \leq 0, \quad y(1) - \alpha y(\eta) \leq 0. \] (4.5)

We now establish several lemmas that will be used throughout.

Let $x, y$ be upper and lower solutions for (4.1), (4.2) and satisfy $x(t) \geq y(t)$ on $[0, 1]$. We define $f^*$ by
\[ f^*(u(t)) = \begin{cases} f(x(t)), & u(t) \geq x(t), \\ f(u(t)), & u(t) \leq x(t) \leq y(t), \\ f(y(t)), & u(t) \leq x(t). \end{cases} \] (4.7)

Consider the following problem:
\[ u''(t) + \lambda h(t)f^*(u(t)) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1) = 0. \] (4.8)
LEMMA 4.2. If there is a solution $u$ of (4.8),(4.9), then

$$y(t) \leq u(t) \leq x(t), \quad \text{for } t \in [0,1].$$

In other words, $u$ is a solution of (4.1),(4.2).

PROOF. We first prove that $u(t) \leq x(t)$, for all $t \in (0,1]$. Suppose to the contrary that $u(t_0) > x(t_0)$ for some $t_0 \in (0,1]$.

Set

$$c = \inf\{t \in [0,1] \mid u(t) > x(t)\}, \quad (4.10)$$

then from the fact that $u(0) = 0$ and $x(0) \geq 0$, we know that $c > 0$ and

$$u(c) = x(c). \quad (4.11)$$

There are three cases as follows.

CASE 1. There exists $d \in (c,1]$, such that $u(d) = x(d)$ and $u(t) > x(t)$, for all $t \in (c,d)$.

In this case, we have

$$f^*(u(t)) = f(x(t)), \quad \text{for } t \in (c,d),$$

$$u(c) = x(c), \quad u(d) = x(d). \quad (4.12)$$

Therefore,

$$(x-u)'' \leq -\lambda h(t) [f(x(t)) - f^*(u(t))] = 0, \quad \text{for } t \in (c,d),$$

$$(x-u)(c) = (x-u)(d) = 0, \quad (4.13)$$

which, by the concavity of $x-u$, implies the contradiction $(x-u)(t) \geq 0$, for all $t \in (c,d)$.

CASE 2. $c \in (0,\eta)$ and $u(t) > x(t)$, for all $t \in (c,1]$.

In this case, we have

$$f^*(u(t)) = f(x(t)), \quad \text{for } t \in (c,1],$$

$$u(c) = x(c). \quad (4.14)$$

Therefore,

$$(x-u)'' \leq -\lambda h(t) [f(x(t)) - f^*(u(t))] = 0, \quad \text{for } t \in (c,1].$$

Using the boundary conditions $u(1) = \alpha u(\eta)$ and $x(1) \geq \alpha x(\eta)$, we know that

$$(x-u)(1) - \alpha(x-u)(\eta) \geq 0. \quad (4.15)$$

Combining (4.15) and (4.11) and using the same arguments used to prove Lemma 2.2, we can get the desired contradiction $x-u \geq 0$, for all $t \in [c,1]$.

CASE 3. $c \in [\eta,1)$ and $u(t) > x(t)$, for all $t \in (c,1]$.

In this case, we have

$$f^*(u(t)) = f(x(t)), \quad \text{for } t \in (c,1],$$

$$u(c) = x(c). \quad (4.17)$$

Therefore,

$$(x-u)'' \leq -\lambda h(t) [f(x(t)) - f^*(u(t))] = 0, \quad \text{for } t \in (c,1].$$

By the definition of $c$, we know that

$$u(t) \leq x(t), \quad \text{for all } t \in [0,c]. \quad (4.18)$$

In particular, we have that

$$u(\eta) \leq x(\eta). \quad (4.19)$$
This, together with the boundary conditions $u(1) = \alpha u(\eta)$ and $x(1) \geq \alpha x(\eta)$, implies

$$u(1) \leq x(1). \quad (4.21)$$

Combining this with (4.11) and (4.18) and using the concavity of $x - u$, we obtain the desired contradiction $(x - u)(t) \geq 0$, for all $t \in (c, 1]$.

By the same arguments, we see that $y(t) \leq u(t)$, for $x \in [0, 1]$. Since $y(t) \leq u \leq x(t)$ for $t \in [0, 1]$, it follows that $f = f^*$, and so $u$ is a solution of (4.1),(4.2).

**Lemma 4.3.** If there exist upper and lower solutions $x$ and $y$ of (4.1),(4.2) with $y(t) \leq x(t)$, for $t \in [0, 1]$, then there is a solution $u$ to (4.1),(4.2) such that

$$y(t) \leq u(t) \leq x(t), \quad \text{for } t \in [0, 1].$$

**Proof.** Consider problem (4.8),(4.9). By Lemma 2.1, we know that (4.8),(4.9) is equivalent to the integral equation

$$u(t) = -\int_0^t (t-s)\lambda h(s)f^*(y(s)) \, ds$$

$$- \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta-s)\lambda h(s)f^*(y(s)) \, ds$$

$$+ \frac{t}{1-\alpha\eta} \int_0^1 (1-s)\lambda h(s)f^*(y(s)) \, ds.$$ 

Let $T^*u(t) = -\int_0^t (t-s)\lambda h(s)f^*(y(s)) \, ds$

$$- \frac{\alpha t}{1-\alpha\eta} \int_0^\eta (\eta-s)\lambda h(s)f^*(y(s)) \, ds$$

$$+ \frac{t}{1-\alpha\eta} \int_0^1 (1-s)\lambda h(s)f^*(y(s)) \, ds.$$ 

Then $T^* : C[0,1] \to C[0,1]$ is completely continuous. Since $f^*$ is bounded, $T^*$ is bounded. By the Schauder fixed-point theorem, $T^*$ has a fixed point $u$, which is a solution of (4.8),(4.9). By Lemma 4.2, $u$ is also a solution of (4.1),(4.2).

## 5. MULTIPLICITY

In order to guarantee that all possible solutions of (1.3),(1.4) are nonnegative, we make the convention that

$$f(u) = f(0), \quad \text{if } u < 0. \quad (5.1)$$

We first need the following priori estimate.

**Lemma 5.1.** There is a constant $b_1 > 0$ such that $\|y\| \leq b_1$, for all solutions $u$ of (1.3),(1.4) where $\lambda$ belongs to a compact subset $I$ of $(0, \infty)$.

**Proof.** Now suppose there is an unbounded sequence $\{u_n\}$ of solutions of (1.3),(1.4) which corresponding $\lambda_n$ belongs to a compact subset of $(0, \infty)$. By Lemma 2.3, $u_n \in K$, which implies that

$$\min_{t \in [0,1]} u_n(t) \geq \gamma \|u_n\|.$$ 

Since $f_{\infty} = \infty$, there is a $q > 0$ such that

$$f(u) \geq \bar{\mu} u, \quad \text{for all } u \geq q.$$
where \( \bar{\mu} \) is chosen so that

\[
\inf \{ \lambda_n \} \bar{\mu} \gamma \left[ -\frac{\eta}{1 - \alpha \eta} \int_{\eta}^{1} (1 - s) a(s) \, ds \right] > 1.
\]

Choosing \( n \) large enough so that \( \gamma \|u_n\| \geq q \), then by the same arguments used to get (3.7), we have that

\[
\|u_n\| > \|u_n\|,
\]

which is a contradiction.

Now let \( F \) denote the set of \( \lambda > 0 \) such that a positive solution of (1.3),(1.4) exists. Let \( \lambda^* = \sup F \). By Theorem 3.1, \( F \) is nonempty and bounded, and thus, \( 0 < \lambda^* < \infty \). We claim that \( \lambda^* \in F \). To see this, let \( \lambda_n \to \lambda^* \), where \( \lambda_n \in \Gamma \):

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots < \lambda^*.
\]

Since the \( \{\lambda_n\} \) are bounded, Lemma 5.1 implies that the corresponding solutions \( \{u_n\} \) are bounded. By the compactness of the integral operator \( T \), it easily follows that \( \lambda^* \in F \).

Let \( u^* \) be a solution of (1.3),(1.4) corresponding to \( \lambda^* \) and define

\[
f(u(t)) = \begin{cases} 
  f(u^*(t) + \epsilon), & u(t) \geq u^*(t) + \epsilon, \\
  f(u(t)), & -\epsilon \leq u(t) \leq u^*(t) + \epsilon, \\
  f(-\epsilon), & u(t) \leq -\epsilon.
\end{cases}
\]

Let

\[
\hat{T}_\lambda u(t) = \lambda \left[ -\int_{0}^{t} (s - t) h(s) \tilde{f}(u(s)) \, ds - \frac{at}{1 - \alpha \eta} \int_{\eta}^{1} (1 - s) h(s) \tilde{f}(u(s)) \, ds \right] + \frac{t}{1 - \alpha \eta} \int_{0}^{t} (1 - s) h(s) \tilde{f}(u(s)) \, ds.
\]

Consider

\[
\Omega = \{ u \in X \mid -\epsilon \leq u(t) \leq u^*(t) + \epsilon \}.
\]

**Lemma 5.2.** There is an \( \epsilon > 0 \), sufficiently small, such that if \( u \in C[0,1] \) satisfies \( \hat{T}_\lambda u = u \) for some \( 0 < \lambda < \lambda^* \), then \( u \in \Omega \).

**Proof.** Since \( u \geq 0 \), to prove that \( u \leq u^* + \epsilon \), we first show that \( u^* + \epsilon \) is an upper solution of (1.3),(1.4). Since \( u^* \geq 0 \), there is a constant \( d_0 > 0 \) such that \( f(u^*(t)) > d_0 > 0 \), for all \( t \in [0,1] \). By uniform continuity, there is an \( \epsilon_0 > 0 \) such that

\[
|f(u^*(t) + \epsilon) - f(u^*(t))| < d_0 \frac{(\lambda^* - \lambda)}{\lambda},
\]
for all $t \in [0,1], 0 \leq \epsilon \leq \epsilon_0$. Now

$$
(u^*(t) + \epsilon)' = (u^*(t))'' = -\lambda^* h(t)f(u^*(t)) \\
= -\lambda h'(t)f(u^*(t) + \epsilon) \\
+ \lambda h(t)[f(u^*(t) + \epsilon) - f(u^*(t))] + (\lambda - \lambda^*) h(t)f(u^*(t)) \\
< -\lambda h(t)f(u^*(t) + \epsilon)
$$

and

$$
(u^* + \epsilon)(0) \geq 0, \quad (u^* + \epsilon)(1) \geq 0.
$$

Clearly, if $\epsilon > 0$, then (5.3) becomes

$$
(u^* + \epsilon)(0) > 0, \quad (u^* + \epsilon)(1) > 0.
$$

Therefore, $u^* + \epsilon$ is an upper solution of (1.3),(1.4). It follows from Lemma 4.1 that $u \leq u^* + \epsilon$.

**Proof of Theorem 1.1.** Let $\lambda \in (0, \lambda^*)$; we show that (1.3),(1.4) has at least two positive solutions. Since $u^*$ is an upper solution and 0 is a lower solution, Lemma 4.3 implies the existence of a solution $u_\lambda$ of (1.3),(1.4) such that $0 \leq u_\lambda \leq u^*$. Thus, for $0 < \lambda < \lambda^*$, a positive solution exists, whereas for $\lambda > \lambda^*$, a positive solution does not exist. Moreover, $u_\lambda \in \Omega$.

Choose $I = [0, \lambda^* + 1]$; then

$$(0, \lambda^*) \cap I \neq \emptyset$$

and

$$(\lambda^*, \infty) \cap I \neq \emptyset.$$

We next establish the existence of a second positive solution to (1.3),(1.4) for $\lambda \in (0, \lambda^*) \cap I$.

Since $T_\lambda$ is bounded for $\lambda \in I$,

$$\deg(I - T_\lambda, B(u^*_\lambda, R), 0) = 1,$$

for $R$ large enough, where $B(u^*_\lambda, R)$ is the ball centered at $u^*_\lambda$ with radius $R$ in $C[0,1]$. If there exists a $u \in \partial \Omega$ such that $u = T_\lambda(u)$, then $f = \bar{f}$, and so $u$ is a second positive solution. Now suppose $u \neq T_\lambda(u)$, for all $u \in \partial \Omega$. Then $\deg(I - T_\lambda, \Omega, 0)$ is well defined. Since Lemma 5.2 implies $T_\lambda$ has no fixed point in $B(u^*_\lambda, R) \setminus \Omega$, we have from the excision property of degree that

$$\deg(I - T_\lambda, \Omega, 0) = 1.$$

This, together with the fact that

$$T_\lambda|\Omega = T_\lambda|\Omega,$$

implies that

$$\deg(I - T_\lambda, \Omega, 0) = 1.$$

On the other hand, by Lemma 5.1, all positive solutions of (1.3),(1.4) are bounded for $\lambda \in I$, and thus,

$$\deg(I - T_\lambda, B(0, M), 0) = \text{constant}, \quad \text{for } \lambda \in I,$$

for $M$ large enough, where $B(0, M)$ is the ball centered at 0 with radius $M$ in $C[0,1]$. The late degree must equal 0, since for all $\lambda > \lambda^*$, no solutions exist. (We note that $(A_3)$ and (5.1) and Lemma 2.2 imply that all solutions of (1.3),(1.4) are positive solutions!) Finally, by the excision property

$$\deg(I - T_\lambda, B(0, M) \setminus \Omega, 0) = -1,$$

and so a second positive solution of (1.3),(1.4) exists for $\lambda \in (0, \lambda^*) \cap I$. 
REFERENCES


