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Duality in Nondifferentiable Vector Programming

R. Osuna-Gómez, A. Rufián-Lizana, and P. Ruíz-Canales

Departamento de Estadística e Investigación Operativa, Fac. de Matematicas

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In this paper we study the saddle point optimality conditions and Lagrange duality in multiobjective optimization for generalized subconvex-like functions. We obtain results which will allow us to characterize the solutions for multiobjective programming problems from the saddle point conditions and allow us to relate them to the dual problem solutions which will be adequately defined. We also define a new dual problem for the multiobjective programming problem with the special property of being a scalar programming problem. © 2001 Academic Press

1. INTRODUCTION

Lagrange duality is an attractive topic in optimization theory. In the past few years, several studies have been dedicated to this subject, discussing it within the multiobjective optimization theory framework [1, 2, 5, 9, 11, 12]. One of the basic questions is how to weaken the assumptions of the known results, as well as defining adequate dual problems that might facilitate the search for solutions of multiobjective optimization problems.

The vector optimization problem considered in this paper can be formulated as

$$\begin{array}{lll} \textbf{(VOP)} & \text{Min} & f(x) \\ & \text{s.t.} & g(x) \leqq 0, \\ & & x \in S \subseteq \mathbb{R}^n, \end{array}$$

where $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Let us denote by X the set of feasible points for this problem, that is, $X = \{x \in S \subseteq \mathbb{R}^n \text{ such that } g(x) \leqq 0\}$.

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There does not exist a unique solution concept for vectorial programming problems such as occurs for scalar programming problems. Amongst the numerous definitions of solutions for multiobjective optimization problems which exist in the literature, we will emphasize those we consider the most important, and those will be the ones used in this work.

DEFINITION 1.1. $\bar{x} \in X$ is said to be an efficient solution (a weakly efficient solution) of Problem (VOP) if there exists no other feasible x such that $f(x) \leq f(\bar{x})$ $(f(x) < f(\bar{x}))$.

Kuhn and Tucker noted that some efficient solutions presented an undesirable property with respect to the ratio between the marginal profit of an objective function and the loss of some other. To these solutions, they introduced the concept of the noninferior proper solution. Subsequently, Geoffrion [6] modified the concept slightly and defined the properly efficient solutions for a multiobjective problem as follows.

DEFINITION 1.2. $\bar{x} \in X$ is said to be a properly efficient solution of Problem (VOP) if it is efficient and if there exists a scalar $M > 0$ such that, for each i, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} < M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in X$ and $f_i(x) < f_i(\bar{x})$.

This paper consists of six parts. In Section 2, some basic definitions and theorems are first introduced. Sections 3 and 4 discuss saddle point theorems for multiobjective programming problems. Section 5 addresses the Lagrange duality, and Section 6 introduces a new dual problem for the multiobjective problem with the special property of being a scalar programming problem. Some conclusions are given in Section 7.

2. BASIC RESULTS AND PRELIMINARIES

First, we introduce a few notations and definitions.

Let $x = (x_1, \dots, x_p)^T$, $y = (y_1, \dots, y_p)^T \in \mathbb{R}^p$, then

$$x = y \quad \text{iff } x_i = y_i,\ i = 1, \dots, p;$$

$$x \leqq y \quad \text{iff } x_i \leq y_i,\ i = 1, \dots, p;$$

$$x \leq y \quad \text{iff } x_i \leq y_i,\ i = 1, \dots, p,$$

with strict inequality holding for at least one i;

$$x < y \quad \text{iff } x_i < y_i,\ i = 1, \dots, p.$$

If $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}^p$, we denote by f_i the ith component of f; i.e., $f(x) = (f_1(x), \ldots, f_p(x))^T$.

Yang [15] defined the concept of generalized subconvex-like functions and provided an alternative theorem for these functions.

DEFINITION 2.1. Let $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}^p$. f is said to be generalized subconvex-like on S if $\exists u \in \mathbb{R}^p$, $u > 0$, such that $\forall \alpha \in (0,1)$, $\forall x_1, x_2 \in S$, and $\forall \epsilon > 0$, $\exists x_3 \in S$, $\exists \rho > 0$ such that

$$\epsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \geq \rho f(x_3).$$

For these functions was proved the following generalized alternative theorem [15].

THEOREM 2.1 (Generalized Alternative Theorem). *Let S be a nonempty set in $\mathbb{R}^n$ and let $f\colon S \to \mathbb{R}^p$ be a generalized subconvex-like function on S. Then either*

(i) $f(x) < 0$ *has a solution* $x \in S$, *or*

(ii) $w^T f(x) \geq 0$ *for all* $x \in S$, *for some* $w \in \mathbb{R}^p$, $w \geq 0$,

but both alternatives are never true.

Note 2.1. In the previous theorem, we can suppose that $w^T e = 1$ since if not, defining $v = w/(\sum_{j=1}^p w_j)$, we have that $v^T e = 1$, and this v verifies that $v^T f(x) \geq 0$ $\forall x \in S$.

Now we show some useful properties of the generalized subconvex-like functions that will be used subsequently.

LEMMA 2.1. *If f is a generalized subconvex-like function and $M > 0$, then Mf is a generalized subconvex-like function with respect to the same point.*

Proof. If there exists $u \in \mathbb{R}^p$, $u > 0$, such that $\forall \alpha \in (0,1)$, $\forall x_1, x_2 \in S$, and $\forall \epsilon > 0$, $\exists x_3 \in S$ and $\exists \rho > 0$ such that

$$\epsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \geq \rho f(x_3),$$

then for $u' = Mu > 0$ and $\rho' = M\rho > 0$, Mf is generalized subconvex-like on S with respect to the same point x_3. ∎

LEMMA 2.2. *Let $f\colon S \to \mathbb{R}^p$ be a generalized subconvex-like function on S. Then for all $i, j = 1, \ldots, p$, $f_i + f_j$ is a generalized subconvex-like function with respect to the same point.*

Proof. If f is generalized subconvex-like, then for any $i, j = 1, \ldots, p$ there exist u_i, $u_j > 0$ such that $\forall \alpha \in (0,1)$, $\forall x_1, x_2 \in S$, and $\forall \epsilon > 0$ $\exists x_3 \in S$, $\exists \rho > 0$ such that

$$\epsilon u_i + \alpha f_i(x_1) + (1-\alpha) f_i(x_2) \geq \rho f_i(x_3)$$

and

$$\epsilon u_j + \alpha f_j(x_1) + (1-\alpha) f_j(x_2) \geq \rho f_j(x_3).$$

Then, taking $u = u_i + u_j > 0$ we have that $f_i + f_j$ is generalized subconvex-like. ∎

Generalized subconvex-like functions present a special type of irregularity, which no other generalized convex function presents. If f is generalized subconvex-like and $a \in \mathbb{R}^p$, then the function $a + f$ does not have to be a generalized subconvex-like function, as is shown in the following example.

EXAMPLE 2.1. Let $f(x, y) = (x, y)$, and f be generalized subconvex-like on $S = \mathbb{R}^2_+ - \{0 \leq x \leq 1, 0 \leq y \leq 1\}$. But $f(x, y) - (1,1) = (x-1, y-1)$ is not a generalized subconvex-like function on S because for $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 1)$, and $\alpha = \frac{1}{2}$ there would have to exist a $u > 0$ and a $\rho > 0$ such that $\forall \epsilon > 0$,

$$\epsilon(u_1, u_2) + \left(-\frac{1}{2}, -\frac{1}{2}\right) \geq \rho(x_3 - 1, y_3 - 1) \text{ with } (x_3, y_3) \in S.$$

But this is impossible.

3. EFFICIENCY CONDITIONS

In order to operationalize the concept of solutions for a multiobjective programming problem we should relate them to familiar concepts. The most common strategy is to characterize them in terms of optimal solutions of appropriate scalar optimization problems [3, 10]. Among the many possible ways of obtaining a scalar problem associated with (VOP), the following is known as a scalar weighting problem.

$$
\begin{array}{lll}
(\mathbf{VP}_\lambda) & \text{Min} & \lambda^T f(x) \\
 & \text{s.t.} & g(x) \leqq 0, \\
 & & x \in S \subseteq \mathbb{R}^n,
\end{array}
$$

where $\lambda \in \mathscr{L} = \{\lambda \in \mathbb{R}^p / \lambda_j \geq 0 \text{ and } \sum_{j=1}^p \lambda_j = 1\}$.

Geoffrion [6] established the following fundamental result.

THEOREM 3.1. *Let $\lambda > 0$ ($\lambda \geq 0$) be fixed. If $\bar{x}$ is optimal in* (VP$_\lambda$), *then $\bar{x}$ is properly efficient* (*weakly efficient*) *in* (VOP).

Assuming that f and g are convex functions and that S is a convex set, Geoffrion also established the converse of the above theorem. This result is based on Gordan's alternative theorem. Hence by replacing Gordan's alternative theorem with the Generalized Alternative Theorem (Theorem 2.1) we obtain the following result.

THEOREM 3.2. *Let $\bar{x}$ be a properly efficient solution in* (VOP) *and let $f - f(\bar{x})$ be generalized subconvex-like on X. Then there exists $\overline{\lambda} \in \mathbb{R}^p$, $\overline{\lambda} > 0$, such that $\bar{x}$ is optimal in* ($VP_{\overline{\lambda}}$).

Proof. If $\bar{x}$ is properly efficient, then there exists a scalar $M > 0$ such that, for each $i = 1, \dots, p$, the system

$$f_i(x) < f_i(\bar{x}),$$
$$f_i(x) + Mf_j(x) < f_i(\bar{x}) + Mf_j(\bar{x}) \qquad \text{for all } j \neq i$$

admits no solution in X. By Lemma 2.1 and Lemma 2.2 and the Generalized Alternative Theorem, for each $i = 1, \dots, p$ there exist $w^i \in \mathbb{R}$, $w^i \geq 0$, with $\sum_{j=1}^{p} w_j^i = 1$, such that

$$w_i^i f_i(x) + \sum_{j \neq i} w_j^i \big(f_i(x) + Mf_j(x) \big) \geq w_i^i f_i(\bar{x}) + \sum_{j \neq i} w_j^i \big(f_i(\bar{x}) + Mf_j(\bar{x}) \big),$$

or equivalently

$$f_i(x) + M \sum_{j \neq i} w_j^i f_j(x) \geq f_i(\bar{x}) + M \sum_{j \neq i} w_j^i f_j(\bar{x}), \tag{2}$$

for each $i = 1, \dots, p$ and for all $x \in X$.

Summing (2) over i yields, after some rearrangement,

$$\sum_{j=1}^{p} \left(1 + M \sum_{i \neq j} w_j^i \right) f_j(x) \geq \sum_{j=1}^{p} \left(1 + M \sum_{i \neq j} w_j^i \right) f_j(\bar{x}),$$

for all $x \in X$.

Then, taking $\overline{\lambda}_j = (1 + M \sum_{i \neq j} w_j^i)$, $\bar{x}$ is optimal in ($VP_{\overline{\lambda}}$). ∎

The next theorem proves an analogous result for weakly efficient solutions.

THEOREM 3.3. *Let $\bar{x}$ be a weakly efficient solution in* (VOP), *and let $f - f(\bar{x})$ be generalized subconvex-like on X. Then there exists $\overline{\lambda} \in \mathbb{R}^p$, $\overline{\lambda} \geq 0$, such that $\bar{x}$ is optimal in* ($VP_{\overline{\lambda}}$).

Proof. If $\bar{x}$ is a weakly efficient solution, then the system

$$f_i(x) - f_i(\bar{x}) < 0, \qquad i = 1, \dots, p,$$

has no solution at $x \in X$. By the Generalized Alternative Theorem, there exists $\overline{\lambda} \geq 0$ such that

$$\overline{\lambda}^T (f(x) - f(\bar{x})) \geq 0 \qquad \forall x \in X,$$

which implies that

$$\overline{\lambda}^T f(x) \geq \overline{\lambda}^T f(\bar{x}) \qquad \forall x \in X.$$

Thus $\bar{x}$ is the optimal solution for $(\mathrm{VP}_{\overline{\lambda}})$. ∎

We remark that no assumption on the convexity of the set X is made in the above theorems.

4. SADDLE POINTS CONDITIONS

For scalar mathematical programming the relationships between the solutions of a constrained scalar programming problem and the points which fulfill certain conditions known as the saddle point optimality criteria are well known [8]. In this section we extend these results to multiobjective programming problems. To do this we begin by giving new definitions of saddle points for the vector case.

DEFINITION 4.1. $(\bar{x}, \bar{r}, \bar{v}) \in \mathbb{R}^n * \mathbb{R}^p * \mathbb{R}^m$ is said to be a *vector Fritz–John saddle point* for Problem (VOP) if $(\bar{r}, \bar{v}) \geq 0$, and the following inequalities hold $\forall v \geqq 0$ and $\forall x \in S$:

$$\bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \leq \bar{r}^T f(x) + \bar{v}^T g(x). \tag{3}$$

DEFINITION 4.2. $(\bar{x}, \bar{r}, \bar{v}) \in \mathbb{R}^n * \mathbb{R}^p * \mathbb{R}^m$ is said to be a *vector Kuhn–Tucker saddle point* for Problem (VOP) if $(\bar{r}, \bar{v}) \geq 0$, $\bar{r} \neq 0$, and the following inequalities hold $\forall v \geqq 0$ and $\forall x \in S$:

$$\bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \leq \bar{r}^T f(x) + \bar{v}^T g(x). \tag{4}$$

Let us note that Definition 4.1 and Definition 4.2 coincide with the Fritz–John and Kuhn–Tucker saddle-point definitions if f is a numerical function.

The above definitions have several advantages over those already existing in the literature [2, 4, 7, 13, 14, 16]. First, the multiplier for the restrictions is a vector and not a function or a matrix. Second and more important, the vector saddle point conditions are scalar conditions, not vector conditions. Thus, it is not necessary to solve any vector problem in order to find the vector saddle points, which simplifies the task.

Problem (VOP) is said to satisfy the generalized Slater constraint qualification if there exists a $\hat{x} \in X$ such that $g(\hat{x}) < 0$. We use this constraint qualification to prove the following result that relates vector Kuhn–Tucker saddle points with vector Fritz–John saddle points.

LEMMA 4.1. *If $(\bar{x}, \bar{r}, \bar{v})$ is a vector Fritz–John saddle point and the generalized Slater constraint qualification is satisfied, then $(\bar{x}, \bar{r}, \bar{v})$ is a vector Kuhn–Tucker saddle point.*

Proof. Let us suppose that $\bar{r} = 0$, then the vector Fritz–John saddle point conditions are

$$v^T g(\bar{x}) \leq \bar{v}^T g(\bar{x}) \leq \bar{v}^T g(x), \tag{5}$$

$\forall v \geqq 0$ and $\forall x \in S$. For $v = 0$, the inequalities (5) become

$$0 \leq \bar{v}^T g(\bar{x}) \leq \bar{v}^T g(x), \tag{6}$$

$\forall x \in S$. Therefore, $0 \leq \bar{v}^T g(x)$ for all $x \in S$.

Since the generalized Slater constraint qualification is satisfied, there exists a $\hat{x} \in S$ such that $g(\hat{x}) < 0$. Then for this $\hat{x}$, $\bar{v}^T g(\hat{x}) < 0$. But, by (6), $\bar{v}^T g(\hat{x}) \geq 0$, and this is a contradiction. ∎

The following result proves that vector Kuhn–Tucker saddle points are weakly efficient points for (VOP) without requiring additional conditions, as in the scalar case.

THEOREM 4.1. *If $(\bar{x}, \bar{r}, \bar{v})$ is a vector Kuhn–Tucker saddle point, then $\bar{x}$ is weakly efficient for* (VOP).

Proof. If $\bar{r} \neq 0$, by (4), $(\bar{x}, \bar{r})$ solves a Kuhn–Tucker saddle point problem for the scalar programming problem ($VP_{\bar{r}}$), and thus $\bar{x}$ is optimal for ($VP_{\bar{r}}$). As $\bar{r} \geq 0$, from Theorem 3.1, $\bar{x}$ is a weakly efficient point for (VOP). ∎

Under a certain convexity condition the following result shows the reverse of the above theorem.

THEOREM 4.2. *Let $(f - f(\bar{x}), g)$ be a generalized subconvex-like function on S, and let $\bar{x}$ be a weakly efficient solution to* (VOP). *Then there exists $(\bar{r}, \bar{v}) \geq 0$ such that $(\bar{x}, \bar{r}, \bar{v})$ is a vector Fritz–John saddle point for* (VOP).

Proof. If $\bar{x}$ is a weakly efficient solution for (VOP) then the system

$$\begin{aligned} f(x) - f(\bar{x}) &< 0 \\ g(x) &\leqq 0 \end{aligned}$$

has no solution in S, therefore the system

$$\begin{aligned} f(x) - f(\bar{x}) &< 0 \\ g(x) &< 0 \end{aligned}$$

has no solution in S.

By Theorem 2.1, there exist $(\bar{r}, \bar{v}) \in \mathbb{R}^{p+m}$ with $(\bar{r}, \bar{v}) \geq 0$ such that

$$\bar{r}^T f(x) + \bar{v}^T g(x) \geq \bar{r}^T f(\bar{x}) \qquad \forall x \in S. \tag{7}$$

In particular, we have that

$$\bar{v}^T g(\bar{x}) \geq 0. \tag{8}$$

Because $\bar{x}$ is feasible we also have

$$\bar{v}^T g(\bar{x}) \leq 0. \tag{9}$$

By (8) and (9) we have that $\bar{v}^T g(\bar{x}) = 0$. Hence, by (7)

$$\bar{r}^T f(x) + \bar{v}^T g(x) \geq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) = \bar{r}^T f(\bar{x}) \geq \bar{r}^T f(\bar{x}) + v^T g(\bar{x})$$

$\forall x \in S$ and $\forall v \geqq 0$, and thus $\bar{x}$ is a vector Fritz–John saddle point. ∎

From Lemma 4.1 and Theorem 4.2 we have the following.

THEOREM 4.3. *Let $(f - f(\bar{x}), g)$ be a generalized subconvex-like function and let $\bar{x}$ be a weakly efficient solution. Suppose that the Problem* (VOP) *satisfies the generalized Slater constraint qualification. Then there exists $(\bar{r}, \bar{v}) \geq 0$ such that $(\bar{x}, \bar{r}, \bar{v})$ is a vector Kuhn–Tucker saddle point for* (VOP).

From Theorem 3.1 it is easy to show the following result for properly efficient solutions.

THEOREM 4.4. *Let $(\bar{x}, \bar{r}, \bar{v})$ be a vector Kuhn–Tucker saddle point with $\bar{r} > 0$, then $\bar{x}$ is a properly efficient solution for* (VOP).

As before, under generalized convexity conditions, we prove the reverse.

THEOREM 4.5. *Suppose that $\bar{x}$ is a properly efficient solution of Problem* (VOP). *If $(f - f(\bar{x}), g)$ is generalized subconvex-like on S and the generalized Slater qualification constraint is satisfied, then there exist $\bar{r} > 0$ and $\bar{v} \geqq 0$ such that $(\bar{x}, \bar{r}, \bar{v})$ is a vector Kuhn–Tucker saddle point for* (VOP).

Proof. If $\bar{x}$ is a properly efficient solution for (VOP), the system

$$f_i(x) - f_i(\bar{x}) < 0$$

$$f_i(x) + Mf_j(x) - f_i(\bar{x}) - Mf_j(\bar{x}) < 0 \quad \text{for all } j \neq i$$

$$g(x) < 0$$

admits no solution in S for each $i = 1, \ldots, p$. Thus there exist $r^i \in \mathbb{R}^p$ and $v^i \in \mathbb{R}^m$, with $(r^i, v^i) \geq 0$, and $\sum_{j=1}^{p} r_j^i = 1$, for each $i = 1, \ldots, p$, such that

$$f_i(x) + M \sum_{j \neq i} r_j^i f_j(x) + v^i g(x) \geq f_i(\bar{x}) + M \sum_{j \neq i} r_j^i f_j(\bar{x}) \qquad \forall x \in S. \tag{10}$$

From (10), $x = \bar{x} \Rightarrow v^i g(\bar{x}) \geq 0$ for all $i = 1, \ldots, p$. On the other hand, $v^i g(\bar{x}) \leq 0$ for all $i = 1, \ldots, p$. Therefore

$$v^i g(\bar{x}) = 0. \tag{11}$$

Summing over i yields (10), and by (11) we get

$$\sum_{j=1}^{p} \left(1 + M \sum_{i \neq j} r_j^i \right) f_j(x) + \sum_{i=1}^{p} v^i g(x) \geq \sum_{j=1}^{p} \left(1 + M \sum_{i \neq j} r_j^i \right) f_j(\bar{x}) + \left(\sum_{i=1}^{p} v^i \right) g(\bar{x}).$$

Assuming that $\bar{r}_j = 1 + M \sum_{i \neq j} r_j^i > 0$ for each $j = 1, \ldots, p$ and $\bar{v} = \sum_{i=1}^{p} v^i$, we have

$$\bar{r}^T f(x) + \bar{v}^T g(x) \geq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) = \bar{r}^T f(\bar{x}) \geq \bar{r}^T f(\bar{x}) + \bar{v}^T g(x),$$

for all $x \in S$ and for all $v \in \mathbb{R}^n$ with $v \geqq 0$. ▮

5. LAGRANGE DUALITY FOR A MULTIOBJECTIVE PROBLEM

We define the vector-valued Lagrange function with respect to Problem (VOP) as

$$L(x, \lambda) = f(x) + \lambda^T g(x) e, \qquad (x, \lambda) \in S \times \mathscr{L},$$

where $e = (1, \ldots, 1) \in \mathbb{R}^p$ and $\mathscr{L} = \{\lambda \in \mathbb{R}^m / \lambda_i \geq 0,\ \lambda^T e = 1\}$.

Let us denote by $\mathscr{W}(\lambda)$ the set of weakly efficient solutions for the following vectorial programming problem:

$$\begin{array}{ll} \text{Min} & L(x, \lambda) \\ \text{s.t.} & x \in S. \end{array}$$

Let $\Omega(\lambda) = \{f(x) + \lambda^T g(x) e \text{ with } x \in \mathscr{W}(\lambda)\}$.

For (VOP), the corresponding Lagrange dual problem is the following:

$$\begin{array}{lll} (\textbf{DVP}) & \text{Max} & \Omega(\lambda) \\ & \text{s.t.} & \lambda \in \mathscr{L}. \end{array}$$

Now we prove the classical duality theorems between (VOP) and (DVP).

THEOREM 5.1 (Weak Duality Theorem). *For any* $x \in X$ *and* $y \in \bigcup_{\lambda \in \mathscr{L}} \Omega(\lambda)$, *we have that* $y \ngtr f(x)$.

Proof. Let $y \in \bigcup_{\lambda \in \mathscr{L}} \Omega(\lambda)$. There exists a $\lambda \in \mathscr{L}$ such that $y \in \Omega(\lambda)$. Hence, there does not exist an $x \in S$ such that

$$y > f(x) + \lambda^T g(x) e. \tag{12}$$

Since $g(x) \leqq 0$ for any $x \in X$ and $\lambda_i \geq 0$ for each $i = 1, \dots, p$, we have

$$\lambda^T g(x) \leq 0.$$

If $y > f(x')$ for some $x' \in X$, then $y > f(x') + \lambda^T g(x')e$, which contradicts (12). Therefore $y \ngtr f(x)$ for all $x \in X$. ∎

THEOREM 5.2 (Strong Duality Theorem). *Let* $\bar{x}$ *be a weakly efficient solution of Problem* (VOP). *Suppose that* $(f - f(\bar{x}), g)$ *is generalized subconvex-like on* S *and that the generalized Slater constraint qualification is satisfied. Then* $f(\bar{x})$ *is a weakly efficient solution of Problem* (DVP).

Proof. As we are under the hypotheses of Theorem 4.3, there exists $(\bar{r}, \bar{v}) \geq 0$, $\bar{r} \neq 0$, such that $\bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \leq \bar{r}^T f(x) + \bar{v}^T g(x)$, $\forall x \in S$. Since $\bar{x} \in S$ and $g(\bar{x}) \leqq 0$ we have that $\bar{v}^T g(\bar{x}) = 0$, and thus

$$\bar{r}^T (f(x) - f(\bar{x})) + \bar{v}^T g(x) \geq 0 \qquad \forall x \in S. \tag{13}$$

Now we will show that $f(\bar{x})$ is a weakly efficient solution for the vector optimization problem

$$\begin{aligned} &\text{Min } f(x) + \bar{v}^T g(x) = L(x, \bar{v}) \\ &\text{s.t. } x \in S. \end{aligned}$$

If this was not the case, there would exist $x_0 \in S$ such that

$$f(x_0) - f(\bar{x}) + \bar{v}^T g(x_0) e < 0.$$

We can suppose that $\bar{r}^T e = 1$ and thus that

$$\bar{r}^T (f(x_0) - f(\bar{x})) + \bar{v}^T g(x_0) < 0,$$

which contradicts expression (13). Therefore $f(x) \in \Omega(\bar{v})$.

Now, if $f(\bar{x})$ is not a weakly efficient point of Problem (DVP), there exists a $y' \in \bigcup_{\lambda \in \mathscr{L}} \Omega(\lambda)$ such that $f(\bar{x}) < y'$. Let λ' be such that $y' \in \Omega(\lambda')$. Since $\lambda' g(\bar{x}) \leq 0$, we have $f(\bar{x}) + \lambda' g(\bar{x}) < y'$, which contradicts $y' \in \Omega(\lambda')$. So $f(\bar{x})$ is a weakly efficient point of Problem (DVP). ∎

6. SCALAR LAGRANGE DUALITY FOR A MULTIOBJECTIVE PROBLEM

We define the dual problem for (VOP) as

$$(\mathbf{D}_1) \quad \textbf{Max} \quad l(r,v) \quad \text{s.t. } v \geqq 0, \quad v \in \mathbb{R}^m,$$

where $l(r,v) = \text{INF}\{r^T f(x) + v^T g(x): x \in S\}$ and $r \in \mathbb{R}^p$, $r \geq 0$.

This formulation of the Lagrangian dual problem for the vector case simplifies remarkably those studied up to now [4, 7, 14], since $l(r,v)$ is defined as the infimum of a scalar function.

We have defined the dual problem excluding the case $r = 0$, since if $r = 0$ the objective function f does not appear in the dual problem.

As in the scalar programming problems, the next weak duality theorem is established amongst the (VOP) and (D_l) feasible solutions without additional hypotheses.

THEOREM 6.1 (Weak Duality Theorem). *If $x_0 \in X$, $r_0 \in \mathbb{R}^p$, $r_0 \geq 0$, and $v_0 \in \mathbb{R}^m$, $v_0 \geqq 0$, then $r_0^T f(x_0) \geq l(r_0, v_0)$.*

Proof. We have that

$$l(r_0, v_0) = \text{INF}\left\{r_0^T f(x) + v_0^T g(x): x \in S\right\} \leq r_0^T f(x_0) + v_0^T g(x_0).$$

On the other hand, as $g(x_0) \leqq 0$ and $v_0 \geqq 0$, $v_0^T g(x_0) \leq 0$. Therefore

$$l(r_0, v_0) \leq r_0^T(x_0).$$

∎

From Theorem 6.1, we have the following corollaries.

COROLLARY 6.1.1. $\text{INF}\{r_0^T f(x): x \in X\} \geq \text{SUP}\{l(r_0, v): v \geqq 0\}\ \forall r_0 \geq 0$.

COROLLARY 6.1.2. *If $\bar{r}^T f(\bar{x}) \leq l(\bar{r}, \bar{v})$, where $\bar{r} \in \mathbb{R}^p$ with $\bar{r} \geq 0$, $\bar{v} \in \mathbb{R}^m$ with $\bar{v} \geqq 0$, and $\bar{x} \in X$, then $\bar{v}$ is optimal in* (D_l) *and $\bar{x}$ is weakly efficient in* (VOP).

Proof. From Theorem 6.1, we have that for all (VOP) feasible points $\bar{r}^T f(x) \geq l(\bar{r}, \bar{v})$.

If in addition $l(\bar{r}, \bar{v}) \geq \bar{r}^T f(\bar{x})$, then $\bar{r}^T f(x) \geq \bar{r}^T f(\bar{x})$, $\forall x \in X$ with $\bar{r} \geq 0$. Therefore $\bar{x}$ is an optimal solution for a scalar weighting problem with $\bar{r} \geq 0$, and this guarantees that $\bar{x}$ is weakly efficient in (VOP).

On the other hand, $l(\bar{r}, v) \leq \bar{r}^T f(\bar{x}) \leq l(\bar{r}, \bar{v})$, $\forall v \geq 0$. Then $\bar{v}$ is optimal in (D_l). ∎

From Corollary 6.1.2 the following result is immediate.

COROLLARY 6.1.3. *If* $\bar{r}^T f(\bar{x}) \leq l(\bar{r}, \bar{v})$ *with* $r > 0$, $\bar{v} \geqq 0$, *and* $\bar{x} \in X$, *then* $\bar{v}$ *is optimal in* (D_l) *and* $\bar{x}$ *is properly efficient in* (VOP).

COROLLARY 6.1.4. *If* $\mathrm{SUP}\{l(\bar{r}, v)\colon v \geqq 0\} = +\infty$ *then* (VOP) *is infeasible*.

Proof. For all $x \in X$, $v \geqq 0$, and $\bar{r} \geq 0$, it is verified that $\bar{r}^T f(x) \geq l(\bar{r}, v)$, and then

$$\bar{r}^T f(x) \geq \mathrm{SUP}\{L(\bar{r}, v)\colon v \geqq 0\} = +\infty.$$

This implies that $\bar{r}^T f(x) = +\infty$, for all $x \in X$; therefore the scalar weighting problem is infeasible. Since the feasible sets of both problems coincide, (VOP) is an infeasible problem. ∎

Up to now no hypotheses have been imposed on the functions of the problem. For generalized subconvex-like functions we obtain the following result.

THEOREM 6.2 (Strong Duality Theorem). *If* $\bar{x}$ *is a weakly efficient point for* (VOP) *and* $(f - f(\bar{x}), g)$ *is a generalized subconvex-like function, then there exist* $(\bar{r}, \bar{v}) \geq 0$ *such that* $\bar{v}$ *is an optimal solution for* (D_l) *and* $\bar{r}^T f(\bar{x}) = l(\bar{r}, \bar{v})$.

Proof. If $\bar{x}$ is a weakly efficient solution for (VOP) and $(f - f(\bar{x}), g)$ is a generalized subconvex-like function, from Theorem 4.2 there exist $\bar{r}$ and $\bar{v}$ with $(\bar{r}, \bar{v}) \geq 0$, verifying that $(\bar{x}, \bar{r}, \bar{v})$ is a vector Fritz–John saddle point for (VOP); that is,

$$\bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \leq \bar{r}^T f(x) + \bar{v}^T g(x) \qquad \forall x \in S, \forall v \geqq 0. \quad (14)$$

By definition,

$$\begin{aligned} l(\bar{r}, v) &= \mathrm{INF}\{\bar{r}^T f(x) + v^T g(x)\colon x \in S\} \\ &\leq \bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \\ &= \mathrm{INF}\{\bar{r}^T f(x) + \bar{v}^T g(x)\} = L(\bar{r}, \bar{v}). \end{aligned}$$

Then $l(\bar{r}, v) \leq l(\bar{r}, \bar{v})$, $\forall v \geqq 0$, and so $\bar{v}$ is an optimal solution for the dual problem.

On the other hand, $\bar{v}^T g(\bar{x}) = 0$. By the second inequality in (14),

$$l(\bar{r}, \bar{v}) = \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) = \bar{r}^T f(\bar{x}).$$

Finally, up to now we have considered $\bar{r} \geq 0$; if $\bar{r} = 0$, (14) becomes

$$v^T g(\bar{x}) \leq \bar{v}^T g(\bar{x}) \leq \bar{v}^T g(x) \qquad \forall x \in S, \forall v \geqq 0.$$

From the second of these inequalities, $\bar{v}$ is an optimal solution for (D_l), and from the first inequality we have that $\bar{v}^T g(\bar{x}) = 0$, which coincides with $\bar{r}^T f(\bar{x})$. ∎

Without convexity hypotheses we prove the next theorem.

THEOREM 6.3 (Inverse Duality). *Let* $r \geq 0$, $\bar{v} \geqq 0$, *such that* $l(\bar{r}, \bar{v}) = \bar{r}^T f(\bar{x})$ *for a feasible point* $\bar{x}$. *Then* $\bar{x}$ *is weakly efficient for* (VOP).

Proof. By definition,

$$l(\bar{r}, \bar{v}) = \mathrm{INF}\{\bar{r}^T f(x) + \bar{v}^T g(x) \colon x \in S\} \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}).$$

We are assuming that $l(\bar{r}, \bar{v}) = \bar{r}^T f(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x})$, and thus $0 \leq \bar{v}^T g(\bar{x})$. Because $\bar{x}$ is feasible, we have that $\bar{v}^T g(\bar{x}) \leq 0$. Hence $\bar{v}^T g(\bar{x}) = 0$. Then $\forall v \geqq 0$ we have that

$$\bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}). \tag{15}$$

On the other hand,

$$\mathrm{INF}\{\bar{r}^T f(x) + \bar{v}^T g(x) \colon x \in S\} = \bar{r}^T f(\bar{x}) = \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}).$$

Then for all $x \in S$,

$$\bar{r}^T f(x) + \bar{v}^T g(x) \geq \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}). \tag{16}$$

From (15) and (16) we have that $(\bar{x}, \bar{r}, \bar{v})$ is a vector Kuhn–Tucker saddle point and therefore $\bar{x}$ is weakly efficient in (VOP) (Theorem 4.1). ∎

From Theorem 4.4 we have the following result for properly efficient solutions.

THEOREM 6.4 (Inverse Duality). *If* $\bar{r} > 0$, $\bar{v} \geqq 0$ *such that* $l(\bar{r}, \bar{v}) = \bar{r}^T f(\bar{x})$ *for* $\bar{x} \in X$, *then* $\bar{x}$ *is properly efficient in* (VOP).

7. CONCLUSIONS

In this paper we presented new definitions of saddle point conditions for vector optimization problems. Using these definitions, which are scalar conditions, we characterize the solutions of multiobjective programming problems satisfying a weakened convexity condition. We also defined a scalar dual problem for multiobjective programming problems not assumed to be differentiable. Based on these definitions we proved associated duality theorems which weaken the convexity hypotheses; the functions in the multiobjective programming are assumed to be generalized subconvex-like functions.

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