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$\mathbf{T}_s$-Tribes and $\mathbf{T}_s$-Measures

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We show that any fundamental triangular norm-based $\mathbf{T}_s$-tribe $\mathscr{T}$, $s \in (0, \infty)$, is a weakly generated tribe. Consequently, $\mathscr{T}$ is a $\mathbf{T}$-tribe for any measurable t-norm $\mathbf{T}$ if and only if it is a $\mathbf{T}_s$-tribe for some $s \in (0, \infty)$. Further we prove that each $\mathbf{T}_s$-measure $\mathbf{m}$, $s \in (0, \infty]$, defined on a $\mathbf{T}_s$-tribe $\mathscr{T}$, is a generated measure; i.e., the irreducible part in the Butnariu–Klement decomposition of $\mathbf{T}_s$-measures is always trivial.

1. INTRODUCTION

Triangular norm-based measures ($\mathbf{T}$-measures) appear under various names, and in specific analytical form, in such fields as mathematical statistics, capacity theory, probability and measure theory, pattern recognition, and game theory. For more details see, e.g., [2, 3]. Throughout this paper, we will deal with fundamental triangular norm-based measures, as far as the class $(\mathbf{T}_s;\ s \in [0, \infty])$ of Frank's fundamental t-norms [4] is of interest in most of the applications mentioned above. For more details about t-norms see Section 2. Recall that the family $(\mathbf{T}_s)$ appeared first in

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Frank's [4] investigation of the functional equation

$$\mathbf{T}(a,b) + \mathbf{S}(a,b) = a + b, \qquad a,b \in [0,1], \tag{1}$$

where $\mathbf{T}$ is a triangular norm and $\mathbf{S}$ is an associative function on the unit square. Note that the only strict solutions of (1) are just t-norms $\mathbf{T}_s$ for $s \in (0,\infty)$ (and $\mathbf{T}_0$ with $\mathbf{T}_\infty$ are the limits of these $\mathbf{T}_s$) and the corresponding $\mathbf{S}_s$ are just the dual t-conorms, i.e., $\mathbf{S}_s(a,b) = 1 - \mathbf{T}_s(1-a, 1-b)$. Frank's family of fundamental t-norms is given by

$$\begin{aligned} \mathbf{T}_s(a,b) &= \min(a,b) && \text{if } s = 0 \\ &= a \cdot b && \text{if } s = 1 \\ &= \max(a+b-1, 0) && \text{if } s = \infty \\ &= \log_s\big(1+(s^a-1)\cdot(s^b-1)/(s-1)\big) && \text{if } s \in (0,1) \cup (1,\infty). \end{aligned}$$

$\mathbf{T}$-measures are defined on subsets of the unit cube $[0,1]^{\mathbf{X}} = \mathscr{F}(\mathbf{X})$ which form triangular norm based tribes ($\mathbf{T}$-tribes); see [6]. Note that a generalization to subsets of $[0,C]^{\mathbf{X}}$, $C \in (0,\infty)$, is immediate. Let $(\mathbf{X}, \mathscr{S})$ be a measurable space, i.e., $\mathscr{S}$ is a σ-algebra of (crisp) subsets of $\mathbf{X}$. If one takes a system $\mathscr{T}(\mathscr{S})$ of all $\mathscr{S}$-measurable functions with the range in the unit interval, then $\mathscr{F}(\mathscr{S})$ is a $\mathbf{T}$-tribe for any measurable t-norm $\mathbf{T}$. $\mathscr{T}(\mathscr{S})$ is called a *generated tribe*. In Section 2 we give a characterization of generated tribes which is due to Klement [6] (these are exactly those $\mathbf{T}_s$-tribes, for some $s \in (0,\infty)$, that contain all constant functions from $\mathscr{F}(\mathbf{X})$). For a denumerable space $\mathbf{X}$, Mesiar [9] showed the structure of a $\mathbf{T}_s$-tribe $\mathscr{T}$ based on a strict fundamental t-norm $\mathscr{T}_s$ which ensures that $\mathscr{T}$ is also a $\mathbf{T}$-tribe with respect to any measurable t-norm $\mathbf{T}$. We extend this result to the general case showing that $\mathscr{T}$ is a $\mathbf{T}$-tribe for each measurable t-norm $\mathbf{T}$ if and only if it is a $\mathbf{T}_s$-tribe for some strict $\mathbf{T}_s$ (i.e., $\mathbf{s} \in (0,\infty)$). For some details on this topic see also [10, 11].

Let $\mathscr{T} \subset \mathscr{T}(\mathscr{S})$ be a $\mathbf{T}_s$-tribe. Let $\mathbf{M}$ be a finite σ-additive measure on $\mathscr{S}$ and let g, h be two $\mathbf{M}$-integrable $\mathscr{S}$-measurable non-negative functions on $\mathbf{X}$. Then any real-valued mapping $\mathbf{m}$ defined on $\mathscr{T}$ via

$$\mathbf{m}(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\mathbf{M}, \qquad A \in \mathscr{T}, \tag{2}$$

where the right-hand-side integral is a Lebesgue−Stieltjes integral, is a well defined finite monotone $\mathbf{T}_s$-measure, $s < \infty$. The measure $\mathbf{m}$ defined by (2) is called a *generated measure*. Note that a generated measure $\mathbf{m}$ is a $\mathbf{T}_\infty$-measure if and only if $g = 0$ ($\mathbf{M}$-a.e.). The question is whether any finite monotone $\mathbf{T}_s$-measure is necessarily a generated measure. The answer for $s = 0$ is negative; see Klement [5]. Recall only that any

$\mathbf{T}_0$-measure on a generated tribe can be expressed in an integral form by means of Markov kernels. For $s = \infty$, the answer is affirmative; see Butnariu [1]. Any $\mathbf{T}_\infty$-measure $\mathbf{m}$ defined on a $\mathbf{T}_\infty$-tribe $\mathscr{T}$ is a Zadeh measure [13], i.e.,

$$\mathbf{m}(A) = \int A\, d\mathbf{M}, \qquad A \in \mathscr{T}. \tag{3}$$

In the case of strict fundamental triangular norm-based measures, an affirmative answer was given by Klement [7] under the additional assumption that $\mathscr{T}$ is a generated tribe.

The main result of Butnariu and Klement [2] shows that, in general, any finite monotone $\mathbf{T}_s$-measure, $s \in (0, \infty)$, defined on a $\mathbf{T}_s$-tribe $\mathscr{T}$, can be uniquely decomposed into a sum of a generated measure and a so-called monotonically irreducible $\mathbf{T}_s$-measure. A natural problem arose: is there any non-trivial monotonically irreducible $\mathbf{T}_s$-measure? This open problem from [3] is equivalent to the above problem whether any finite monotone $\mathbf{T}_s$-measure is a generated one. A partial solution (for denumerable $\mathbf{X}$) was found by Mesiar [9]. The main result of this paper is a complete solution of the above-mentioned problems. We show that any finite monotone $\mathbf{T}_s$-measure, $s \in (0, \infty)$, defined on some $\mathbf{T}_s$-tribe $\mathscr{T}$, is necessarily a generated measure.

Our representation theorem for $\mathbf{T}_s$-measures is valid for monotone (finite) $\mathbf{T}_s$-measures. The question whether it is true also for non-monotone $\mathbf{T}_s$-measures remains open (see also [3]) and it is equivalent to the existence of Jordan decomposition of $\mathbf{T}_s$-measures.

2. $\mathbf{T}_s$-TRIBES

A triangular norm (t-norm for short) is a two place function $\mathbf{T}$: $[0,1] \times [0,1] \to [0,1]$ which is associative, commutative, and non-decreasing in each component and satisfies the boundary condition $\mathbf{T}(a, 1) = a$. In what follows, we deal with Borel-measurable t-norms only. Note that, e.g., any continuous t-norm is Borel-measurable. Due to the associativity, we are able to extend $\mathbf{T}$ to be an n-ary operation, $n \in \mathbf{N}$. The monotonicity and boundary conditions allow one to extend $\mathbf{T}$ to work on countable sequences.

A t-norm $\mathbf{T}$ is called strict if it is continuous and strictly monotone on $(0,1) \times (0,1)$. A continuous t-norm $\mathbf{T}$ is called Archimedean if $\mathbf{T}(a, a) < a$ for all $a \in (0,1)$. Recall that any strict t-norm is Archimedean. Any fundamental triangular norm is continuous, all $\mathbf{T}_s$ with $s > 0$ are Archimedean, and $\mathbf{T}_s$, $s \in (0, \infty)$, are strict t-norms.

By Ling [8], any continuous Archimedean t-norm $\mathbf{T}$ is generated by an additive generator f, $f\colon [0,1] \to [0,\infty]$, f is continuous strictly decreasing, and $f(1) = 0$, so that

$$\mathbf{T}(a,b) = f^{-1}(\min(f(0), f(a) + f(b))), \qquad a,b \in [0,1].$$

The generator f is unique up to a positive multiplicative constant. The case $f(0) = \infty$ corresponds to the strict t-norm case and then

$$\mathbf{T}(a,b) = f^{-1}(f(a) + f(b)).$$

Let $\mathbf{X}$ be a non-void set. Recall that a function $A\colon \mathbf{X} \to [0,1]$ was introduced by Zadeh [12] to generalize the concept of a Cantorian subset A of $\mathbf{X}$ which can be identified with its characteristic function $A\colon \mathbf{X} \to \{0,1\}$ (and then A is called a crisp subset of $\mathbf{X}$). Recall that the collection of all functions on $\mathbf{X}$ with the range in the unit interval $[0,1]$ is denoted by $\mathscr{F}(\mathbf{X})$. For a fixed number $d \in [0,1]$, we denote by $\mathbf{d}_{\mathbf{X}}$ the corresponding constant function on $\mathbf{X}$. Let $\mathbf{T}$ be a given t-norm. We extend $\mathbf{T}$ to $\mathscr{F}(\mathbf{X})$ pointwise as usual, i.e., $(A\mathbf{T}B)(x) = \mathbf{T}(A(x), B(x))$, $x \in \mathbf{X}$. This operation can be considered as an intersection. The complement is defined by

$$A'(x) = 1 - A(x), \qquad x \in \mathbf{X} \qquad (\text{Zadeh } [12]).$$

A dual operation to $\mathbf{T}$, $\mathbf{S}(a,b) = 1 - \mathbf{T}(1-a, 1-b)$, $a,b \in [0,1]$, is called a dual t-conorm. Its pointwise extension to $\mathscr{F}(\mathbf{X})$ can be considered as a union. If f is an additive generator of a t-norm $\mathbf{T}$, then g, $g(a) = f(1-a)$, $a \in [0,1]$, is an additive generator of the dual t-conorm $\mathbf{S}$.

DEFINITION 2.1 (Klement [6]). Let $\mathbf{T}$ be a t-norm. A subfamily $\mathscr{T} \subset \mathscr{F}(\mathbf{X})$ containing $\mathbf{0}_{\mathbf{X}}$, being closed under complementation and under countable intersections induced by $\mathbf{T}$, is called a $\mathbf{T}$-*tribe*.

Obviously, by the duality of $\mathbf{T}$ and $\mathbf{S}$, the closedness with respect to $\mathbf{T}$ is equivalent to the closedness with respect to $\mathbf{S}$ in the definition above.

EXAMPLE 2.2. (i) Let $\mathscr{T} \subset \mathscr{F}(\mathbf{X})$ consist of crisp subsets of $\mathbf{X}$ only. Then $\mathscr{T}$ is a $\mathbf{T}$-tribe for some t-norm $\mathbf{T}$ if and only if $\mathscr{T}$ is a σ-algebra (we identify an ordinary subset of $\mathbf{X}$ and its characteristic function). Consequently, a σ-algebra $\mathscr{T}$ is a $\mathbf{T}$-tribe for any t-norm $\mathbf{T}$.

(ii) Let $\mathscr{S}$ be a σ-algebra of crisp subsets of $\mathbf{X}$ and let $\mathscr{T}(\mathscr{S})$ be a generated tribe (i.e., the system of all $\mathscr{S}$-measurable functions from $\mathscr{F}(\mathbf{X})$). Then $\mathscr{T}(\mathscr{S})$ is a $\mathbf{T}$-tribe for any (measurable) t-norm $\mathbf{T}$.

(iii) (Mesiar [9]) Let $\mathscr{S}$ be a σ-algebra of crisp subsets of $\mathbf{X}$ and let $\mathbf{Y} \in \mathscr{S}$. Then $\mathscr{T}(\mathscr{S}, \mathbf{Y}) = \{A \in \mathscr{T}(\mathscr{S});\ A\mathbf{T}\mathbf{Y}' \in \mathscr{S}\}$ (where $\mathbf{T}$ is any t-norm; note that $A\mathbf{T}\mathbf{Y}'$ equals zero, resp. A, on $\mathbf{Y}$, resp. $\mathbf{Y}^{c} = \mathbf{X} \setminus \mathbf{Y}$, independently of the choice of $\mathbf{T}$) is called a *semigenerated tribe*. This tribe consists of all $\mathcal{S}$-measurable functions from $\mathcal{F}(\mathbf{X})$ which possess trivial values out of $\mathbf{Y}$. Each semigenerated tribe is a $\mathbf{T}$-tribe for any (measurable) t-norm $\mathbf{T}$ and it is a generated tribe only for trivial $\mathbf{Y}$ (i.e., when $\mathbf{Y} = \mathbf{X}$). If $\mathbf{Y} = \emptyset$, then $\mathcal{T}(\mathcal{S}, \mathbf{Y}) = \mathcal{S}$; i.e., the corresponding semigenerated tribe is a σ-algebra of crisp subsets of $\mathbf{X}$.

(iv) (Navara [11]) Let $\Delta \subset \mathcal{S}$ be a σ-ideal (i.e., Δ contains all measurable subsets of each of its members and it is closed under countable unions). Then $\mathcal{T}(\mathcal{S}, \Delta) = \{A \in \mathcal{T}(\mathcal{S}); \{x \in \mathbf{X};\ A(x) \in (0,1)\} \in \Delta\}$ is called a *weakly generated tribe*. A weakly generated tribe is a $\mathbf{T}$-tribe for any (measurable) t-norm $\mathbf{T}$. A weakly generated tribe is a semigenerated tribe if and only if Δ is a principal σ-ideal, $\Delta = \Delta(\mathbf{Y}) = \{Z \in \mathcal{S};\ Z \subseteq \mathbf{Y}\}$, $\mathbf{Y} \in \mathcal{S}$.

(v) (Klement [6]) The family $\mathcal{T}$ consisting of all functions on $\mathbf{X} = [0,1]$ with the range $[0,1]$ which either are constant or have all their values in the interval $[\frac{1}{3}, \frac{2}{3}]$ is a $\mathbf{T}_0$-tribe, but it is not a $\mathbf{T}$-tribe for any continuous Archimedean t-norm $\mathbf{T}$.

We recall some results of Butnariu and Klement for $\mathbf{T}_s$-tribes; see [2, 3, 6, 7].

THEOREM 2.3. *Let $\mathcal{T}$ be a $\mathbf{T}_s$-tribe on $\mathbf{X}$, $s > 0$, and let $\mathcal{T}^{\vee}$ be the system of all crisp subsets of $\mathbf{X}$ contained in $\mathcal{T}$. Then $\mathcal{T}^{\vee}$ is a σ-algebra and $\mathcal{T}$ is contained in the corresponding generated tribe, $\mathcal{T} \subset \mathcal{T}(\mathcal{T}^{\vee})$.*

THEOREM 2.4. *Let $\mathcal{T}$ be a $\mathbf{T}_s$-tribe on $\mathbf{X}$. Then $\mathcal{T}$ is also a $\mathbf{T}_0$-tribe; i.e., $\mathcal{T}$ is closed under countable infima and suprema.*

THEOREM 2.5. *For any fundamental t-norm $\mathbf{T}_s$ with $s > 0$, a $\mathbf{T}_s$-tribe $\mathcal{T}$ on $\mathbf{X}$ is a generated tribe if and only if it contains all constant functions from $\mathcal{F}(\mathbf{X})$.*

One of the authors (Mesiar [9]) has already shown that if $\mathbf{T}$ is a strict t-norm, then a $\mathbf{T}$-tribe $\mathcal{T}$ on $\mathbf{X}$ contains all constant functions $\mathbf{d}_{\mathbf{X}}$, $d \in [0,1]$, if and only if $\mathcal{T}$ contains some non-trivial constant function $\mathbf{d}_{\mathbf{X}}$, i.e., there is a $d \in (0,1)$ so that $\mathbf{d}_{\mathbf{X}} \in \mathcal{T}$.

COROLLARY 2.6. *A $\mathbf{T}_s$-tribe $\mathcal{T}$, $s \in (0,\infty)$, is a generated tribe if and only if there is some $d \in (0,1)$ so that $\mathbf{d}_{\mathbf{X}} \in \mathcal{T}$.*

The main result of Mesiar [9] characterizes the structure of $\mathbf{T}_s$-tribes, $s \in (0,\infty)$, on a denumerable set $\mathbf{X}$.

THEOREM 2.7. *Let $\mathbf{X}$ be denumerable. Then a subsystem $\mathcal{T} \subset \mathcal{F}(\mathbf{X})$ is a $\mathbf{T}_s$-tribe, $s \in (0,\infty)$, if and only if $\mathcal{T}$ is a semigenerated tribe (and hence it is a $\mathbf{T}$-tribe for any t-norm $\mathbf{T}$).* ∎

For a general $\mathbf{X}$, the first steps in characterizing a $\mathbf{T}_s$-tribe, $s \in (0, \infty)$, were done by Navara [11] and Mesiar [10]. In the following theorem, we summarize the main results of [10, 11] and we give a shorter proof.

THEOREM 2.8. *Let $\mathscr{T}$ be a system of functions from $\mathscr{F}(\mathbf{X})$. The following are equivalent*:

(α) *$\mathscr{T}$ is a $\mathbf{T}$-tribe for each (measurable) t-norm $\mathbf{T}$;*

(β) *$\mathscr{T}$ is a $\mathbf{T}_s$-tribe for some $s \in (0, \infty)$;*

(γ) *$\mathscr{T}$ is a weakly generated tribe.*

The implications $(\alpha) \Rightarrow (\beta)$ and $(\gamma) \Rightarrow (\alpha)$ are evident. We need only prove $(\beta) \Rightarrow (\gamma)$. The proof is divided into several steps.

LEMMA 2.9. *Let $\mathscr{T}$ be a $\mathbf{T}_s$-tribe on $\mathbf{X}$ for some $s \in (0, \infty)$ and let A be a function contained in $\mathscr{T}$. Put $F(A) = \{x \in \mathbf{X};\ A(x) \in (0,1)\}$. Then the restriction $\mathscr{T}|F(A) = \{B|F(A);\ B \in \mathscr{T}\}$ is a generated tribe on $F(A)$.*

Proof. It is evident that $\mathscr{T}|F(A)$ is a $\mathbf{T}_s$-tribe on $F(A)$. Now, due to Corollary 2.6, it is enough to show that some non-trivial constant function from $\mathscr{S}(F(A))$, say $(0.5)_{F(A)}$, is contained in $\mathscr{T}|F(A)$. For a given t-norm $\mathbf{T}_s$, $s \in (0, \infty)$, put

$$P_{n,m}(a) = \mathbf{T}_s^n\big(\mathbf{S}_s^m(a)\big)$$

for any $n, m \in \mathbf{N}$, $a \in (0,1)$. Here $\mathbf{T}_s^1(a) = \mathbf{S}_s^1(a) = a$ and for $n = 1, 2, \dots,$ we put $\mathbf{T}_s^{n+1}(a) = \mathbf{T}_s(\mathbf{T}_s^n(a), a)$ and $\mathbf{S}_s^{n+1}(a) = \mathbf{S}_s(\mathbf{S}_s^n(a), a)$. It is evident that $p_{n,m}$, being defined on $[0,1]$, can be extended pointwise to $\mathscr{F}(\mathbf{X})$ and that for any $B \in \mathscr{T}$, $p_{n,m}(B)$ is also contained in $\mathscr{T}$.

Claim 1. For each $a, b, t \in (0,1)$, $a < b$, there are integers n, $m \in \mathbf{N}$ such that

$$p_{n,m}(t) \in [a, b].$$

Indeed, let f be an additive generator of $\mathbf{T}_s$. The strictness of $\mathbf{T}_s$, $s \in (0, \infty)$, ensures $f(0) = \infty$. Further,

$$p_{n,m}(t) = f^{-1}\Big(n \cdot f\Big(g^{-1}\big(m \cdot g(t)\big)\Big)\Big),$$

where $g(t) = f(1 - t)$ is an additive generator of the dual t-conorm $\mathbf{S}_s$. Put $q = f(a) - f(b) \in (0, \infty)$. Then there is an integer $m \in \mathbf{N}$ such that

$$m \geq f\big(1 - f^{-1}(q)\big)/f(1 - t),$$

and consequently

$$\infty > q = f(a) - f(b) \geq f\Big(1 - f^{-1}\big(m \cdot f(1 - t)\big)\Big) = f\Big(g^{-1}\big(m \cdot g(t)\big)\Big) > 0.$$

Now, it is easy to see that there is some integer $n \in \mathbf{N}$ such that

$$f(a) \geq n \cdot f\big(g^{-1}(m \cdot g(t))\big) \geq f(b),$$

i.e., $a \leq p_{n,m}(t) \leq b$.

Claim 2. There is a function $C \in \mathscr{T}$ such that $C|F(A) = (0.5)_{F(A)}$. Put

$$C = \sup\big\{(p_{n,m}(A))\mathbf{T}_0(p_{n,m}(A))'; n, m \in \mathbf{N}\big\}.$$

Due to Theorem 2.4, C is contained in $\mathscr{T}$. Further, Claim 1 ensures that $C(x) = 0.5$ whenever $A(x) \in (0,1)$ and $C(x) = 0$ whenever $A(x) \in \{0,1\}$, which proves Claim 2.

By Claim 2, there is a non-trivial constant function contained in the $\mathbf{T}_s$-tribe $\mathscr{T}|F(A)$ and consequently $\mathscr{T}|F(A)$ is a generated tribe. ∎

LEMMA 2.10. *Let $\mathscr{T}$ be a $\mathbf{T}_s$-tribe on X for some $s \in (0,\infty)$. Put*

$$\Delta = \{F(A); A \in \mathscr{T}\}.$$

Then Δ is a σ-ideal of $\mathscr{T}^{\vee}$ (i.e., of crisp subsets of X contained in $\mathscr{T}$).

Proof. Theorem 2.3 ensures $\Delta \subset \mathscr{T}^{\vee}$. Let $F(A)$, $A \in \mathscr{T}$, be a given element of Δ and let D be any subset of $F(A)$ contained in $\mathscr{T}^{\vee}$. Due to Lemma 2.9, $\mathscr{T}|F(A)$ is a generated tribe and hence it contains a function E such that $E(x) = 0.5$ whenever $x \in D$ and $E(x) = 0$ otherwise. Consequently, there is a function $B \in \mathscr{T}$ such that $B|F(A) = E$. $F(A)$ can be understood as a crisp subset of X contained in $\mathscr{T}$ and consequently $C = B\mathbf{T}_s F(A) \in \mathscr{T}$. It is easy to see that $F(C) = D$, which implies $D \in \Delta$.

Further, let $\{D_n\}_{n \in \mathbf{N}} \subset \Delta$ be a countable sequence of elements of Δ. Then there are some functions $A_n \in \mathscr{T}$ such that $D_n = F(A_n)$, $n \in \mathbf{N}$. Similarly, the functions C_n, such that $C_n(x) = 0.5$ whenever $x \in D_n$ and $C(x) = 0$ otherwise, $n \in \mathbf{N}$, are contained in $\mathscr{T}$. By Theorem 2.4, $C = \sup_n C_n$ is contained in $\mathscr{T}$ and therefore $D = \bigcup_n D_n = F(C)$ is contained in Δ. Hence Δ is a σ-ideal. ∎

Proof of Theorem 2.8. Let $\mathscr{T}$ be a $\mathbf{T}_s$-tribe on X for some $s \in (0,\infty)$ and let Δ be a corresponding σ-ideal from Lemma 2.10 inducing a weakly generated tribe $\mathscr{T}(\mathscr{T}^{\vee}, \Delta)$. Lemma 2.9 ensures $\mathscr{T}(\mathscr{T}^{\vee}, \Delta) \subset \mathscr{T}$ and Lemma 2.10 ensures $\mathscr{T} \subset \mathscr{T}(\mathscr{T}^{\vee}, \Delta)$, i.e., $\mathscr{T}$ is the weakly generated tribe, $\mathscr{T} = \mathscr{T}(\mathscr{T}^{\vee}, \Delta)$. ∎

Note that any σ-ideal Δ on a denumerable space X is a principal σ-ideal. Consequently any $\mathbf{T}_s$-tribe, $s \in (0,\infty)$, on a denumerable X is a semigenerated tribe; i.e., Theorem 2.7 is a corollary of Theorem 2.8.

EXAMPLE 2.11. Let $\mathbf{X}$ be an uncountable set and let $\mathcal{T}$ be a system of all functions from $\mathcal{F}(\mathbf{X})$ differing from 0 or 1 in at most countably many points. Then $\mathcal{T}$ is a weakly generated tribe which is not a semigenerated tribe. Indeed, $\mathcal{T} = \mathcal{T}(2^{\mathbf{X}}, \Delta)$, where Δ is a σ-ideal of all denumerable subsets of $\mathbf{X}$ and hence Δ is not a principal σ-ideal.

We have just shown that a $\mathbf{T}_s$-tribe $\mathcal{T}$ (for some strict fundamental t-norm $\mathbf{T}_s$) is also a $\mathbf{T}$-tribe for any (measurable) t-norm $\mathbf{T}$. Is this assertion true for each/(some other) strict t-norm $\mathbf{T}$? It is evident that non-strict t-norms do not possess this property.

Open Problem 2.12. Let $\mathcal{T}$ be a $\mathbf{T}^*$-tribe for some strict t-norm $\mathbf{T}^*$. Is there some other measurable t-norm $\mathbf{T}$ so that $\mathcal{T}$ is not a $\mathbf{T}$-tribe?

3. $\mathbf{T}_s$-MEASURES

Let $\mathcal{T}$ be a $\mathbf{T}$-tribe for some t-norm $\mathbf{T}$. A mapping $\mathbf{m}: \mathcal{T} \to \overline{\mathbf{R}}$ is called a $\mathbf{T}$-measure (Klement [7]) if the following conditions are fulfilled:

$$\mathbf{m}(\mathbf{0}_{\mathbf{X}}) = 0 \tag{4}$$

$$A, B \in \mathcal{T} \Rightarrow \mathbf{m}(A\mathbf{T}B) + \mathbf{m}(A\mathbf{S}B) = \mathbf{m}(A) + \mathbf{m}(B) \tag{5}$$

$$\{A_n\}_{n\in\mathbf{N}} \subset \mathcal{T},\ A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \Rightarrow \lim_n \mathbf{m}(A_n) = \mathbf{m}\Big(\lim_n A_n\Big). \tag{6}$$

If, moreover, $\mathbf{m}(\mathbf{1}_{\mathbf{X}}) < \infty$ and $A \le B$ implies $\mathbf{m}(A) \le \mathbf{m}(B)$, then $\mathbf{m}$ is called a finite monotone $\mathbf{T}$-measure. Note that if $\mathcal{T}$ consists of crisp elements only then a $\mathbf{T}$-measure $\mathbf{m}$ may be considered as an ordinary σ-additive measure.

Zadeh [13] in 1968 introduced a probability measure on a generated tribe $\mathcal{T}(\mathcal{S})$ via

$$\mathbf{p}(A) = \int A\, d\mathbf{P}, \tag{7}$$

where $\mathbf{P}$ is a probability measure on σ-algebra $\mathcal{S}$ and the right-hand-side integral is a Lebesgue–Stieltjes integral. The additivity of the Lebesgue–Stieltjes integral ensures that $\mathbf{p}$ is a finite monotone $\mathbf{T}$-measure on $\mathcal{T}(\mathcal{S})$ if and only if the t-norm $\mathbf{T}$ fulfills the functional equation (1). The same is true for any Zadeh measure (3) (note that (3) is a generalization of (7), replacing a probability $\mathbf{P}$ by a finite σ-additive monotone measure $\mathbf{M}$). Hence any Zadeh measure is a finite monotone $\mathbf{T}_s$-measure for each $s \in [0, \infty]$. Butnariu [1] (see also [2, 3]) showed a converse in the case $s = \infty$.

THEOREM 3.1. *Let* $\mathbf{m}$ *be a* $\mathbf{T}_\infty$*-measure on a* $\mathbf{T}_\infty$*-tribe* $\mathcal{T} \subset \mathcal{F}(\mathbf{X})$. *Then for each element* $A \in \mathcal{T}$ *we have*

$$\mathbf{m}(A) = \int A \, d\mathbf{M},$$

where $\mathbf{M}$ *is the restriction of* $\mathbf{m}$ *to* $\mathcal{T}^\vee$, $\mathbf{M} = \mathbf{m}|\mathcal{T}^\vee$.

Let $\mathbf{M}$ be a finite σ-additive monotone measure on a σ-algebra $\mathcal{S}$ of subsets of a given set $\mathbf{X}$ and let $\mathcal{T}(\mathcal{S})$ be the corresponding generated tribe. For $A \in \mathcal{T}(\mathcal{S})$ put

$$\mathbf{m}(A) = \mathbf{M}(\operatorname{supp} A), \tag{8}$$

where $\operatorname{supp} A = \{x \in \mathbf{X};\ A(x) > 0\}$. It is easy to see that (8) defines a finite monotone $\mathbf{T}$-measure $\mathbf{m}$ if and only if the t-norm $\mathbf{T}$ fulfills $\mathbf{T}(a, b) = 0 \Leftrightarrow a = 0$ or $b = 0$. Hence $\mathbf{m}$ defined by (8) is a $\mathbf{T}_s$-measure for each finite s but not for $s = \infty$.

Combining the $\mathbf{T}$-measures of type (3) (i.e., Zadeh measures) and $\mathbf{T}$-measures of type (8) we get just the generated measures; see (2). Recall that a generated measure $\mathbf{m}$ defined on a generated tribe $\mathcal{T}(\mathcal{S})$ can be uniquely decomposed to a sum of a Zadeh measure and a measure of type (8),

$$\begin{aligned}\mathbf{m}(A) &= \int_{\{A>0\}} (g + h \cdot A) \, d\mathbf{M} \\ &= \int_{\{A>0\}} g \, d\mathbf{M} + \int_{\{A>0\}} A \cdot h \, d\mathbf{M} \\ &= \int A \, d\mathbf{L} + \mathbf{K}(\operatorname{supp} A),\end{aligned}$$

where $\mathbf{K}$ and $\mathbf{L}$ are finite monotone σ-additive measures whose Radon–Nikodym derivatives with respect to $\mathbf{M}$ are g and h, respectively. Any generated measure, even restricted to a subtribe of a generated tribe, is a $\mathbf{T}_s$-measure for each finite s (and it is a $\mathbf{T}_\infty$-measure if and only if it is continuous from above in $\mathbf{0}_\mathbf{X}$, i.e., when $g = 0$ ($\mathbf{M}$-a.e.)). The converse assertion for strict fundamental t-norms was shown by Klement [7] in the case when the domain of $\mathbf{m}$ is a generated tribe.

THEOREM 3.2. *Let* $\mathbf{m}$ *be a finite monotone* $\mathbf{T}_s$*-measure,* $s \in (0, \infty)$, *on a generated tribe* $\mathcal{T}(\mathcal{S})$. *Then* $\mathbf{m}$ *is a generated measure and for each element*

$A \in \mathcal{T}(\mathcal{S})$ we have

$$\mathbf{m}(A) = \int_{\{A>0\}} (f + (\mathbf{1}_{\mathbf{X}} - f) \cdot A)\, d\mathbf{M}, \tag{9}$$

where $\mathbf{M} = \mathbf{m}|\mathcal{S}$ is a finite monotone σ-additive measure and f is an $\mathbf{M}$-a.e. uniquely determined $\mathcal{S}$-measurable function with range in the unit interval (i.e., $f \in \mathcal{T}(\mathcal{S})$).

Recall again that the case of $\mathbf{T}_0$-measures (on a generated tribe) was completely solved by Klement [5] by means of Markov kernels and therefore this case will be omitted in this paper.

If $\mathcal{T}$ is a semigenerated tribe, any $\mathbf{T}_s$-measure $\mathbf{m}$ on $\mathcal{T}$ can be decomposed into a $\mathbf{T}_s$-measure $\mathbf{k}$ on a generated tribe $\mathcal{T}|\mathbf{Y}$ and a σ-additive measure $\mathbf{n}$ on a σ-algebra $\mathcal{T}|\mathbf{Y}^c$. Consequently, we have the following corollary (see also Mesiar [9]).

COROLLARY 3.3. *Let $\mathbf{X}$ be a denumerable set and let $\mathbf{m}$ be a finite monotone $\mathbf{T}_s$-measure on some $\mathbf{T}_s$-tribe $\mathcal{T}$ on $\mathbf{X}$ for some $s \in (0, \infty)$. Then $\mathbf{m}$ is a generated measure and it can be expressed by* (9). ∎

Let $\mathbf{m}$ be a $\mathbf{T}_s$-measure, $s \in (0, \infty)$, on a (non-generated) $\mathbf{T}_s$-tribe $\mathcal{T}$. Butnariu and Klement [2] have defined a $\mathbf{T}_s$-measure $\mathbf{m}^*$ on $\mathcal{T}$ to be *monotonically irreducible*, if it is monotone and if there is no nonzero generated measure $\mathbf{q}$ on $\mathcal{T}$ such that $(\mathbf{m}^* - \mathbf{q})$ is monotone on $\mathcal{T}$. The main result of [2] (see also [3]) is summarized in the following theorem.

THEOREM 3.4. *Let $\mathbf{m}$ be a finite monotone $\mathbf{T}_s$-measure, $s \in (0, \infty)$, on a $\mathbf{T}_s$-tribe $\mathcal{T} \subset \mathcal{F}(\mathbf{X})$. Then $\mathbf{m}$ can be uniquely decomposed to a sum of a monotonically irreducible $\mathbf{T}_s$-measure $\mathbf{m}^*$ and a generated measure $\mathbf{m}_1$,*

$$\mathbf{m}(A) - \mathbf{m}^*(A) = \mathbf{m}_1(A) = \int_{\{A>0\}} (g + h \cdot A)\, d\mathbf{M} \qquad \text{for each } A \in \mathcal{T}, \tag{10}$$

where $\mathbf{M} = \mathbf{m}|\mathcal{T}^{\vee}$ is the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$ and $g, h \in \mathcal{T}(\mathcal{T}^{\vee})$ are two $\mathbf{M}$-a.e. uniquely determined $\mathcal{T}^{\vee}$-measurable functions with the range in the unit interval.

As a main result of this paper, we give a solution of an open problem from [3] concerning the structure of a finite monotone $\mathbf{T}_s$-measure $\mathbf{m}$ on a general $\mathbf{T}_s$-tribe $\mathcal{T}$, where $s \in (0, \infty)$. Showing that there is no non-trivial monotonically irreducible $\mathbf{T}_s$-measure $\mathbf{m}^*$, by Theorem 3.4 we obtain immediately that the only $\mathbf{T}_s$-measures are the generated measures.

THEOREM 3.5. *Let $\mathcal{T} = \mathcal{T}(\Delta)$ be a weakly generated tribe and let $\mathbf{m}^*$ be a monotonically irreducible $\mathbf{T}_s$-measure on $\mathcal{T}$, $s \in (0, \infty)$. Then $\mathbf{m}^*$ is identically zero.*

Proof. Let $D \in \Delta$. Then $\mathscr{T}|D$ is a generated tribe on D and hence $\mathbf{m}_D = \mathbf{m}^*|(\mathscr{T}|D)$ is a generated $\mathbf{T}_s$-measure. It is evident that $\mathbf{m}_D^*$, $\mathbf{m}_D^*(A) = \mathbf{m}_D(A|D)$, $A \in \mathscr{T}$, forms a generated $\mathbf{T}_s$-measure on $\mathscr{T}$ and that $(\mathbf{m}^* - \mathbf{m}_D^*)$ is a monotone $\mathbf{T}_s$-measure on $\mathscr{T}$. But then $\mathbf{m}_D^*$ must be a zero measure and thus $\mathbf{m}^*(D) = 0$ for each $D \in \Delta$.

Now, take any element $A \in \mathscr{T}$. Put

$$Z(A) = \{x \in \mathbf{X};\ A(x) = 0\}$$

$$U(A) = \{x \in \mathbf{X};\ A(x) = 1\}$$

and recall that $F(A) = (Z(A) \cup U(A))^{c}$. The valuation property of $\mathbf{m}^*$, the boundary condition $\mathbf{m}^*(\mathbf{0_X}) = 0$, and the fact that $F(A) \in \Delta$ (i.e., $\mathbf{m}^*(F(A)) = 0$) implies that

$$\begin{aligned}\mathbf{m}^*(A) &= \mathbf{m}^*(U(A)) = \mathbf{M}^*(U(A)) \\ &= \mathbf{M}^*(U(A) \cup F(A)) = \int_{\{A>0\}} \mathbf{1_X}\, d\mathbf{M}^*,\end{aligned}$$

where $\mathbf{M}^* = \mathbf{m}^*|\mathscr{T}^{\vee}$, which means that $\mathbf{m}^*$ is a generated measure. The monotonicity of $(\mathbf{m}^* - \mathbf{m}^*)$ ensures the result. ▮

COROLLARY 3.6. *Let* $\mathbf{m}$ *be a finite monotone* $\mathbf{T}_s$*-measure on a* $\mathbf{T}_s$*-tribe* $\mathscr{T}$ *of functions from* $\mathscr{I}(\mathbf{X})$, $s \in (0, \infty)$. *Then* $\mathbf{m}$ *is a generated measure,*

$$\forall A \in \mathscr{T}: \qquad \mathbf{m}(A) = \int_{\{A>0\}} \big(f + (\mathbf{1_X} - f) \cdot A\big)\, d\mathbf{M},$$

where $\mathbf{M} = \mathbf{m}|\mathscr{T}^{\vee}$ *is a restriction of* $\mathbf{m}$ *to the* σ*-algebra* $\mathscr{T}^{\vee}$ *of all crisp subsets of* $\mathbf{X}$ *contained in* $\mathscr{T}$ *and* f *is an* $\mathbf{M}$-a.e. *uniquely determined* $\mathscr{T}^{\vee}$*-measurable function with the range in the unit interval* $[0, 1]$, $f \in \mathscr{I}(\mathscr{T}^{\vee})$.

The proof follows from Theorems 2.8, 3.4, and 3.5. Note that we have just shown that if $\mathbf{m}$ is a finite monotone $\mathbf{T}_s$-measure for *some* $s \in (0, \infty)$, then it is a finite monotone $\mathbf{T}_s$-measure for *each* $s \in (0, \infty)$. Further, it is easy to see that a finite monotone $\mathbf{T}_s$-measure $\mathbf{m}$, $s \in (0, \infty)$, is a $\mathbf{T}_\infty$-measure (i.e., f is identically zero and $\mathbf{m}(A)$ is an integral of A with respect to $\mathbf{M}$, which means $\mathbf{m}$ is a Zadeh measure) if and only if $\mathbf{m}$ is continuous from above in $\mathbf{0_X}$ (i.e., $\{A_n\} \subset \mathscr{T}$, $A_1 \geq A_2 \geq \dots$, $\lim A_n = \mathbf{0_X}$ implies $\lim_n \mathbf{m}(A_n) = 0\}$. Finally, note that for any finite monotone $\mathbf{T}_s$-measure $\mathbf{m}$ defined on a $\mathbf{T}_s$-tribe $\mathscr{T}$, $s \in (0, \infty]$, we have the "additivity" property

$$\mathbf{m}(A) + \mathbf{m}(B) = \mathbf{m}(C) + \mathbf{m}(D)$$

whenever $A, B, C, D \in \mathscr{T}$, the algebraic sums $(A + B)$ and $(C + D)$ are equal, and $\{x \in \mathbf{X};\ A(x) \cdot B(x) = 0\} = \{x \in \mathbf{X};\ C(x) \cdot D(x) = 0\}$.

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