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Spaces of Kudrjavcev Type
II. Spaces of Distributions: Duality, Interpolation

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1. INTRODUCTION

This note is the continuation of [3]. So we shall use all the notations and results of [3] without explaining them again here. In [3] we considered the spaces $h^s_{p,\mu}(R_n)$ and $b^s_{p,q,\mu}(R_n)$ (and their restrictions on $R_n{}^+$), where $0 \leqslant s < \infty$; $1 < p < \infty$; $1 \leqslant q \leqslant \infty$ and $-\infty < \mu < \infty$. The index s indicates the order of differentiation, while μ represents the weight function. The restriction $s \geqslant 0$ was essentially for the definition of these spaces. The same restriction appears in all papers, known to the author, dealing with the interpolation theory for spaces with weights (in particular in Chapter 3 of [2]). (Of course, one can define spaces with negative order of differentiation by duality. But what is meant here is the constructive definition, where the duality is a consequence of such a definition.) On the other hand it is well-known that for the Lebesgue spaces $H_p{}^s(R_n)$ and the Besov spaces $B^s_{p,q}(R_n)$ without weights there exist constructive descriptions for all values of s; $-\infty < s < \infty$ [2, Chap. 2; or 1], and the references given there). So it will be desirable to extend the definition of spaces with weights on values $s < 0$. This will be done systematically in the forthcoming paper [4]. The main tool will be the extension of the Michlin–Hörmander multiplier theorem (respectively its vector-valued generalization obtained in 2.2.4 in [2] or in [1, Theorem 3.5]) on L_p-spaces with weights. But for special spaces with weights, there exists a simple other method, which will be described here for spaces of Kudrjavcev type. But this method is not restricted on these spaces. Section 2 contains the definitions and first results. In Section 3 we shall prove a duality theorem, and Section 4 deals with the interpolation of these spaces.

2. THE SPACES $b^s_{p,q,\mu}(R_n)$ AND $f^s_{p,q,\mu}(R_n)$

R_n denotes the Euclidean n-space. $B^s_{p,q}(R_n)$, where $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$; are the usual Besov-spaces. (Definitions may be found, for instance, in [1; or 2, Chap. 2].) $F^s_{p,q}(R_n)$, where $-\infty < s < \infty$; $1 < p < \infty$; and $1 < q < \infty$; are the spaces introduced in [1; 2, Chap. 2]. It holds that

$$F^s_{p,2}(R_n) = H_p{}^s(R_n), \qquad F^s_{p,p}(R_n) = B^s_{p,p}(R_n),$$

where $H_p{}^s(R_n)$ are the Lebesgue spaces. In particular, the spaces

$$F^s_{p,q}(R_n), \qquad \min(2, p) \leqslant q \leqslant \max(2, p), \tag{1}$$

connect the spaces $H_p{}^s(R_n)$ and $B^s_{p,p}(R_n)$. So, we shall be concerned in the first line with the spaces $B^s_{p,q}(R_n)$ and the spaces in (1).

LEMMA. *Let $\zeta(x)$ be a complex-valued infinitely differentiable function defined in R_n, such that all derivatives $D^\alpha\zeta(x)$ (inclusively the function $\zeta(x)$ itself) are bounded,*

$$| D^\alpha\zeta(x)| \leqslant c_\alpha \qquad \textit{for all} \quad x \in R_n .$$

(a) *Let $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$. Then there exists a number $C > 0$ depending only on a finite number of c_α (and of n, s, p, q) such that*

$$\| \zeta f \|_{B^s_{p,q}(R_n)} \leqslant C \| f \|_{B^s_{p,q}(R_n)} \tag{2}$$

for all $f \in B^s_{p,q}(R_n)$.

(b) *Let $-\infty < s < \infty$; $1 < p < \infty$, and $\min(2, p) \leqslant q \leqslant \max(2, p)$. Then there exists a number $C' > 0$ depending only on a finite number of c_α (and of n, s, p, q) such that*

$$\| \zeta f \|_{F^s_{p,q}(R_n)} \leqslant C' \| f \|_{F^s_{p,q}(R_n)} \tag{3}$$

for all $f \in F^s_{p,q}(R_n)$.

Proof. If $s = m$ is a natural number, then

$$F^m_{p,2}(R_n) = H_p{}^m(R_n) = W_p{}^m(R_n); \qquad 1 < p < \infty;$$

where $W_p{}^m(R_n)$ are the usual Sobolev spaces, normed by

$$\| f \|_{W_p{}^m(R_n)} = \sum_{|\alpha| \leqslant m} \| D^\alpha f \|_{L_p} .$$

For these spaces the statements of the lemma are obvious. Using $(H_p{}^s)' = H^{-s}_{p'}$, where $(1/p) + (1/p') = 1$, and the fact that the multiplication with $\zeta(x)$ is a self-adjoint operation, it follows by duality that the lemma is true for $H^{-m}_p(R_n)$, too. (m is again a natural number). Now we use interpolation methods. The spaces $B^s_{p,q}(R_n)$; where $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$; can be obtained by real interpolation, and the spaces $H_p{}^s(R_n)$; where $-\infty < s < \infty$; $1 < p < \infty$; by complex interpolation of $H_p{}^m(R_n)$ and $H_p^{-m}(R_n)$, provided that m is a sufficiently large natural number. Further, it holds that[1]

$$[H_p{}^s(R_n), B^s_{p,p}(R_n)]_\Theta = F^s_{p,q}(R_n); \qquad \frac{1}{q} = \frac{1-\Theta}{2} + \frac{\Theta}{p}. \tag{4}$$

Hence, the spaces described in (1) are interpolation spaces of $H_p{}^s(R_n)$ and $B^s_{p,p}(R_n)$. But now the lemma is a consequence of the interpolation property.

Problem. The proof of part (b) is based on the interpolation formula (4). It would be of interest to give a direct proof using only the definition of $F^s_{p,q}(R_n)$. Further it should be possible to extend the result on all spaces $F^s_{p,q}(R_n)$; where $-\infty < s < \infty$; $1 < p < \infty$; $1 < q < \infty$.[2]

DEFINITION 1. *Let Z be the system of functions described in Definition 1 in* [3]. *Let $-\infty < s < \infty$; $-\infty < \mu < \infty$; and $1 < p < \infty$.*

(a) *If $1 \leqslant q \leqslant \infty$ and if $\{\zeta_j\}_{j=0}^\infty \in Z$, then*

$$b^s_{p,q,\mu}(R_n) = \Big\{ g \mid g \in S'(R_n), \| g \|_{b^s_{p,q,\mu}(R_n)} = \Big(\sum_{j=0}^\infty 2^{j\mu - jsp + jn} \| (g\zeta_j)(2^j x) \|^p_{B^s_{p,q}(R_n)} \Big)^{1/p} < \infty \Big\}. \tag{5}$$

(b) *If $\min(2, p) \leqslant q \leqslant \max(2, p)$, and if $\{\zeta_j\}_{j=0}^\infty \in Z$, then*

$$f^s_{p,q,\mu}(R_n) = \Big\{ g \mid g \in S'(R_n), \| g \|_{f^s_{p,q,\mu}(R_n)} = \Big(\sum_{j=0}^\infty 2^{j\mu - jsp + jn} \| (g\zeta_j)(2^j x) \|^p_{F^s_{p,q}(R_n)} \Big)^{1/p} < \infty \Big\}. \tag{6}$$

Remark 1. If $g \in S'(R_n)$, then holds $g\zeta_j \in S'(R_n)$. Further we recall the meaning of the distribution $(g\zeta_j)(2^j x)$: If $\varphi \in S(R_n)$ then by definition,

$$[(g\zeta_j)(2^j x)](\varphi) = 2^{-jn}(g\zeta_j)(\varphi(2^{-j}x)). \tag{7}$$

We shall show that the spaces $b^s_{p,q,\mu}(R_n)$ and $f^s_{p,q,\mu}(R_n)$ are independent of the choice of the system $\{\zeta_j\}_{j=0}^\infty \in Z$ (equivalent norms !). This justifies the

[1] All assertions for the spaces $B^s_{p,q}$, $F^s_{p,q}$, $H_p{}^s$, and $W_p{}^s$ used in this paper may be found in [2, Chap. 2] (or in [1]).

[2] It is not very hard to give an affirmative answer to this problem. But this will be done in a later paper in a more general context.

notation. If we assume that the above mentioned problem is solved, then one can extend the definition of the spaces $f^s_{p,q,\mu}(R_n)$ on all values $1 < q < \infty$. (This is also true for all the following results of the paper.)

Remark 2. We set

$$h^s_{p,\mu}(R_n) = f^s_{p,2,\mu}(R_n). \tag{8}$$

Comparison with formula (29) of [3] shows that the spaces $h^s_{p,\mu}(R_n)$ coincide with the spaces $h^s_{p,\mu}(R_n)$ defined there, provided that $s \geqslant 0$. The same holds for the spaces $b^s_{p,q,\mu}(R_n)$, provided that $s > 0$. Hence, the above defined spaces are extensions of the spaces introduced in [3] on negative values of derivations.

THEOREM 1. (a) *$b^s_{p,q,\mu}(R_n)$, where $-\infty < \mu < \infty$; $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqslant q \leqslant \infty$; and $f^s_{p,q,\mu}(R_n)$; where $-\infty < \mu < \infty$; $-\infty < s < \infty$; $1 < p < \infty$; $\min(2, p) \leqslant q \leqslant \max(2, p)$; are Banach spaces. All these spaces are independent of the choice of $\{\zeta_j\}_{j=0}^\infty \in Z$ (equivalent norms).*

(b) *$C_0^\infty(R_n)$ and $S(R_n)$ are dense subsets in $b^s_{p,q,\mu}(R_n)$, where $-\infty < \mu < \infty$; $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqslant q < \infty$; and in $f^s_{p,q,\mu}(R_n)$; where $-\infty < \mu < \infty$; $-\infty < s < \infty$; $1 < p < \infty$; $\min(2, p) \leqslant q \leqslant \max(2, p)$.*

Proof. Step 1. Let $\{\zeta_j\}_{j=0}^\infty \in Z$ and $\{\tilde{\zeta}_j\}_{j=0}^\infty \in Z$ be two systems of functions The definition of these systems shows

$$\zeta_j(x)\, \tilde{\zeta}_k(x) \equiv 0 \qquad \text{for} \quad | k - j | > 2.$$

For fixed j, where $j = 0, 1, 2,\ldots$, and $g \in b^s_{p,q,\mu}(R_n)$ we obtain

$$\|(g\zeta_j)\,(2^j x)\|_{B^s_{p,q}(R_n)} = \Big\| \sum_{k=j-2}^{j+2} \Big(g \frac{\zeta_j \tilde{\zeta}^k}{\sum_{l=0}^\infty \tilde{\zeta}_l}\Big)\,(2^j x)\Big\|_{B^s_{p,q}(R_n)}$$

$$\leqslant \sum_{k=j-2}^{j+2} \Big\| \Big(g\tilde{\zeta}_k \frac{\zeta_j}{\sum_{l=0}^\infty \tilde{\zeta}_l}\Big)\,(2^j x)\Big\|_{B^s_{p,q}(R_n)},$$

(here we set $\tilde{\zeta}_k(x) \equiv 0$ for $k < 0$). Now we apply the above lemma, whe[n] $\zeta(x)$ must be replaced by $(\zeta_j/\sum_{l=0}^\infty \tilde{\zeta}_l)(2^j x)$. The properties of the syste[m] $\{\zeta_j\}_{j=0}^\infty$ (see [3]) yield that one can estimate all the derivations of these functio[n] by constants which are independent of j. Hence, it follows that

$$\|(g\zeta_j)\,(2^j x)\|_{B^s_{p,q}(R_n)} \leqslant c \sum_{k=j-2}^{j+2} \|(g\tilde{\zeta}_k)\,(2^j x)\|_{B^s_{p,q}(R_n)},$$

where c is independent of j. Using the properties of the spaces $B^s_{p,q}(R_n)$ (for instance their explicit definition given in [1; or 2, Chap. 2] one obtains

$$\|(g\zeta_j)(2^j x)\|_{B^s_{p,q}(R_n)} \leqslant c' \sum_{k=j-2}^{j+2} \|(g\tilde{\zeta}_k)(2^k x)\|_{B^s_{p,q}(R_n)},$$

where again c' is independent of j. This proves the equivalence of the norms of $b^s_{p,q,\mu}(R_n)$ corresponding to $\{\zeta_j\}_{j=0}^\infty$, respectively $\{\tilde{\zeta}_j\}_{j=0}^\infty$. A similar conclusion may be made for the spaces $f^s_{p,q,\mu}(R_n)$.

Step 2. The proof that $b^s_{p,q,\mu}(R_n)$ and $f^s_{p,q,\mu}(R_n)$ are Banach spaces can be obtained in a standard way.

Step 3. Finally we must prove that $C_0^\infty(R_n)$ and $S(R_n)$ are dense subsets in the considered spaces, provided that $q < \infty$. First we remark that

$$b^s_{p,q,\mu}(R_n) \supset h^m_{p,\mu}(R_n), \qquad \text{and} \qquad f^s_{p,q,\mu}(R_n) \supset h^m_{p,\mu}(R_n),$$

provided that m is a sufficiently large natural number. But $h^m_{p,\mu}(R_n) = w^m_{p,\mu}(R_n)$ are the Kudrjavcev spaces which can be normed by Theorem 2 of [3]. This yields $h^m_{p,\mu}(R_n) \supset S(R_n)$. Now it follows that

$$b^s_{p,q,\mu}(R_n) \supset S(R_n), \qquad \text{and} \qquad f^s_{p,q,\mu}(R_n) \supset S(R_n), \tag{9}$$

where "$\supset$" is to be understood in the topological sense. ((9) is valid also for $q = \infty$ in the case of the b-spaces). Let $\zeta(x) \in C_0^\infty(R_n)$ and $g \in b^s_{p,q,\mu}(R_n)$, respectively $g \in f^s_{p,q,\mu}(R_n)$. Using again the above lemma it follows that by an appropriate choice of ζ the distribution $g\zeta$ approximates the distribution g in the corresponding spaces. But $g\zeta \in B^s_{p,q}(R_n)$, respectively $g\zeta \in F^s_{p,q}(R_n)$, can be approximated by functions belonging to $C_0^\infty(R_n)$ (and so also to $S(R_n)$), provided that $q < \infty$. This is also an approximation in the sense of part (b) of the theorem.

Remark 3. It is well known that $C_0^\infty(R_n)$ is not dense in $B^s_{p,\infty}(R_n)$.[1] Hence, $C_0^\infty(R_n)$ is also not dense in $b^s_{p,\infty,\mu}(R_n)$. This justifies the following definition.

DEFINITION 2. *Let* $-\infty < s < \infty$; $1 < p < \infty$; *and* $-\infty < \mu < \infty$. *Then* $\mathring{b}^s_{p,\infty,\mu}(R_n)$ *denotes the completion of* $C_0^\infty(R_n)$ *in* $b^s_{p,\infty,\mu}(R_n)$.

3. DUALITY

The space of tempered distributions is denoted by $S'(R_n)$. It is the dual space to $S(R_n)$. Part (b) of Theorem 1 and (9) give the possibility to interpret the dual spaces of $b^s_{p,q,\mu}(R_n)$ and $f^s_{p,q,\mu}(R_n)$ (and hence in particular of the spaces $h^s_{p,\mu}(R_n)$ and $w^s_{p,\mu}(R_n)$) as subspaces of $S'(R_n)$. This means, that $g \in S'(R_n)$ belongs to $(f^s_{p,q,\mu}(R_n))'$ if and only if there exists a positive number C such that

$$|g(\varphi)| \leqslant C \|\varphi\|_{f^s_{p,q,\mu}} \qquad \text{for all } \varphi \in S(R_n).$$

THEOREM 2. (a) *Let* $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqslant q < \infty$; *and* $-\infty < \mu < \infty$. *Then*

$$(b^s_{p,q,\mu p}(R_n))' = b^{-s}_{p',q',-\mu p'}(R_n), \tag{10}$$

where

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{11}$$

(b) *Let* $-\infty < s < \infty$; $1 < p < \infty$; *and* $-\infty < \mu < \infty$. *Then*

$$(\mathring{b}^s_{p,\infty,\mu p}(R_n))' = b^{-s}_{p',1,-\mu p'}(R_n). \tag{12}$$

(c) *Let* $-\infty < s < \infty$; $1 < p < \infty$; $\min(2, p) \leqslant q \leqslant \max(2, p)$; *and* $-\infty < \mu < \infty$. *Then*

$$(f^s_{p,q,\mu p}(R_n))' = f^{-s}_{p',q',-\mu p'}(R_n). \tag{13}$$

In particular

$$(h^s_{p,\mu p}(R_n))' = h^{-s}_{p',-\mu p'}(R_n). \tag{14}$$

Proof. Step 1. Let the hypotheses of part (a) be satisfied. Further let $f \in b^{-s}_{p',q',-\mu p'}(R_n)$ and let $\varphi \in S(R_n)$. We choose two systems

$$\{\zeta_j\}_{j=0}^{\infty} \in \mathring{Z}, \qquad \{\sigma_j\}_{j=0}^{\infty} \in \mathring{Z},$$

where "$\circ$" indicates that

$$\sum_{j=0}^{\infty} \zeta_j(x) \equiv \sum_{j=0}^{\infty} \sigma_j(x) \equiv 1,$$

see [3, Definition 1]. Then

$$f(\varphi) = \sum_{j=0}^{\infty} (f\zeta_j)(\varphi) = \sum_{j=0}^{\infty} \sum_{r=-2}^{2} (f\zeta_j)(\sigma_{j+r}\varphi).$$

(Here $\sigma_k \equiv 0$ for $k < 0$). Using $(B^s_{p,q})' = B^{-s}_{p',q'}$ [1] it follows that

$$|f(\varphi)| \leqslant \sum_{r=-2}^{2} \sum_{j=0}^{\infty} 2^{jn} \,|(f\zeta_j)\,(2^j x)\,[(\sigma_{j+r}\varphi)\,(2^j x)]|$$

$$\leqslant \sum_{r=-2}^{2} \sum_{j=0}^{\infty} 2^{jn} \,\|(f\zeta_j)\,(2^j x)\|_{B^{-s}_{p',q'}} \,\|(\sigma_{j+r}\varphi)\,(2^j x)\|_{B^s_{p,q}}$$

$$\leqslant C \sum_{r=-2}^{2} \sum_{j=0}^{\infty} 2^{jn} \,\|(f\zeta_j)\,(2^j x)\|_{B^{-s}_{p',q'}} \,\|(\sigma_{j+r}\varphi)\,(2^{j+r} x)\|_{B^s_{p,q}}$$

$$\leqslant C' \sum_{r=-2}^{2} \left(\sum_{j=0}^{\infty} 2^{jn-j\mu p'+jsp'} \,\|(f\zeta_j)\,(2^j x)\|^{p'}_{B^{-s}_{p',q'}}\right)^{1/p'}$$

$$\times \left(\sum_{j=0}^{\infty} 2^{jn+j\mu p-jsp} \,\|(\sigma_{j+r}\varphi)\,(2^{j+r} x)\|^{p}_{B^s_{p,q}}\right)^{1/p}$$

$$\leqslant C'' \,\|f\|_{b^{-s}_{p',q',-\mu p'}} \,\|\varphi\|_{b^s_{p,q,\mu p}}\,.$$

This proves $f \in (b^s_{p,q,\mu p}(R_n))'$ and

$$b^{-s}_{p',q',-\mu p'}(R_n) \subset (b^s_{p,q,\mu p}(R_n))'. \tag{15}$$

Step 2. To prove the inverse embedding of (15) we start with some preliminaries. Let $f \in (b^s_{p,q,\mu p}(R_n))'$ ($\subset S'(R_n)$), and let $\{\zeta_j\}_{j=0}^{\infty}$ be the system of the first step. By a good choice of this system it will be possible to construct a second system $\{\eta_j\}_{j=0}^{\infty} \in Z$ such that

$$\eta_j(x) = 1 \qquad \text{for} \quad x \in \operatorname{supp} \zeta_j\,. \tag{16}$$

First we want to show that $\sum_{j=0}^{\infty} \zeta_{3j+k}(x) f$ belongs also to $(b^s_{p,q,\mu p}(R_n))'$. Here $k = 0, 1, 2$. Let $\varphi \in S(R_n)$. Using the above lemma it follows that

$$\left|\left(\sum_{j=0}^{\infty} \zeta_{3j+k}(x)\, f\right)(\varphi)\right| = \left| f\left(\sum_{j=0}^{\infty} \zeta_{3j+k}(x)\, \varphi(x)\right)\right|$$

$$\leqslant \|f\|_{(b^s_{p,q,\mu p})'} \left\| \sum_{j=0}^{\infty} \zeta_{3j+k}(x)\, \varphi(x) \right\|_{b^s_{p,q,\mu p}} \tag{17}$$

$$\leqslant C\, \|f\|_{(b^s_{p,q,\mu p})'} \,\|\varphi\|_{b^s_{p,q,\mu p}}\,.$$

This proves

$$\sum_{j=0}^{\infty} \zeta_{3j+k}(x)\, f \in (b^s_{p,q,\mu p}(R_n))'.$$

Further one obtains $\zeta_{3j+k}f \in (B^s_{p,q})' = B^s_{p',q'}$. It follows that

$$\Big\| \sum_{j=0}^{\infty} \zeta_{3j+k} f \Big\|_{(b^s_{p,q,\mu p})'}$$

$$= \sup_{\|\varphi\|_{b^s_{p,q,\mu p}} \leq 1} \Big| \Big(\sum_{j=0}^{\infty} \zeta_{3j+k} f \Big)(\varphi) \Big|$$

$$= \sup_{\|\varphi\|_{b^s_{p,q,\mu p}} \leq 1} \sum_{j=0}^{\infty} | \zeta_{3j+k} f(\eta_{3j+k}\varphi)|$$

$$= \sup_{\|\varphi\|_{b^s_{p,q,\mu p}} \leq 1} \sum_{j=0}^{\infty} 2^{n(3j+k)} |(\zeta_{3j+k} f)(2^{3j+k}x)\,[(\eta_{3j+k}\varphi)(2^{3j+k}x)]| .$$

But now it is not hard to see that the last supremum can be taken separately in each summand. (Here one must use again the above lemma). It follows

$$\Big\| \sum_{j=0}^{\infty} \zeta_{3j+k} f \Big\|_{(b^s_{p,q,\mu p})'} \geq c \sup_{\|\varphi\|_{b^s_{p,q,\mu p}} \leq 1} \sum_{j=0}^{\infty} 2^{n(3j+k)} \| \zeta_{3j+k} f(2^{3j+k}x)\|_{B^{-s}_{p',q'}} \| \eta_{3j+k}\varphi(2^{3j+k}x)\|_{B^s_{p,q}} , \tag{18}$$

where $c > 0$. All the numbers

$$a_{3j+k} = \| \eta_{3j+k}\varphi(2^{3j+k}x)\|_{B^s_{p,q}}$$

can be chosen in an arbitrary way, provided that

$$\sum_{j=0}^{\infty} a^p_{3j+k} 2^{(3j+k)(\mu p - sp + n)} \leq 1.$$

Hence, one obtains

$$\Big(\sum_{j=0}^{\infty} 2^{(3j+k)(n-\mu p'+sp')} \| \zeta_{3j+k} f(2^{3j+k}x)\|^{p'}_{B^{-s}_{p',q'}} \Big)^{1/p'}$$

$$\leq C \Big\| \sum_{j=0}^{\infty} \zeta_{3j+k} f \Big\|_{(b^s_{p,q,\mu p})'} \leq C' \|f\|_{(b^s_{p,q,\mu p})'} .$$

Since the last estimate holds for $k = 0, 1, 2$ it follows $f \in b^{-s}_{p',q',-\mu p'}(R_n)$, and one obtains the opposite relation to (15). This proves part (a).

Step 3. In the same way one proves (b) and (c). One has to use

$$(\mathring{B}^{s}_{p,\infty}(R_n))' = B^{-s}_{p',1}(R_n), \qquad (F^{s}_{p,q}(R_n))' = F^{-s}_{p',q'}(R_n),$$

(see [1; 2, Chap. 2]).

Remark 4. The last theorem shows that (9) can be reinforced by

$$S(R_n) \subset b^{s}_{p,q,\mu}(R_n) \subset S'(R_n); \qquad S(R_n) \subset f^{s}_{p,q,\mu}(R_n) \subset S'(R_n)$$

for all admissible values of the parameters.

3. Interpolation

The interpolation theory for these spaces is rather simple: One can carry over the method of the proof to Theorem 3 in [3].

Theorem 3. *Let* $-\infty < \mu_0 < \infty$; $-\infty < \mu_1 < \infty$; $-\infty < s_0 < \infty$; $-\infty < s_1 < \infty$; $1 < p_0 < \infty$; *and* $1 < p_1 < \infty$;

$$1 \leqslant q_0 \leqslant \infty \quad \text{and} \quad 1 \leqslant q_1 \leqslant \infty \quad \text{for the } b\text{-spaces}$$

$$\left.\begin{matrix} \min(2, p_0) \leqslant q_0 \leqslant \max(2, p_0) \\ \min(2, p_1) \leqslant q_1 \leqslant \max(2, p_1) \end{matrix}\right\} \quad \text{for the } f\text{-spaces.}$$

Let $0 < \Theta < 1$ *and*

$$s = (1-\Theta)\, s_0 + \Theta s_1; \qquad \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}; \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1};$$

$$\frac{\mu}{p} = (1-\Theta)\frac{\mu_0}{p_0} + \Theta\frac{\mu_1}{p_1}.$$

(a) *If additionally* $p_0 = p_1 = p$ *and* $s_0 \neq s_1$, *then*

$$b^{s}_{p,p,\mu}(R_n) = (f^{s_0}_{p,q_0,\mu_0}(R_n), f^{s_1}_{p,q_1,\mu_1}(R_n))_{\Theta,p}$$

$$= (f^{s_0}_{p,q_0,\mu_0}(R_n), b^{s_1}_{p,q_1,\mu_1}(R_n))_{\Theta,p}$$

$$= (b^{s_0}_{p,q_0,\mu_0}(R_n), b^{s_1}_{p,q_1,\mu_1}(R_n))_{\Theta,p}.$$

(b) *It holds that*

$$(b^{s_0}_{p_0,p_0,\mu_0}(R_n), b^{s_1}_{p_1,p_1,\mu_1}(R_n))_{\Theta,p} = b^{s}_{p,p,\mu}(R_n).$$

(c) *If additionally* $s_0 \neq s_1$, *then*

$$(f^{s_0}_{p_0,q_0,\mu_0}(R_n), f^{s_1}_{p_1,q_1,\mu_1}(R_n))_{\Theta,p} = b^s_{p,p,\mu}(R_n).$$

(d) *If additionally* $s_0 = s_1 = s$; $q_0 = q_1 = q$ *and* $p_0 \neq p_1$, *then*

$$(f^s_{p_0,q,\mu_0}(R_n), f^s_{p_1,q,\mu_1}(R_n))_{\Theta,p} = f^s_{p,q,\mu}(R_n).$$

(e) *It holds*

$$[b^{s_0}_{p_0,q_0,\mu_0}(R_n), b^{s_1}_{p_1,q_1,\mu_1}(R_n)]_\Theta = b^s_{p,q,\mu}(R_n),$$

$$[f^{s_0}_{p_0,q_0,\mu_0}(R_n), f^{s_1}_{p_1,q_1,\mu_1}(R_n)]_\Theta = f^s_{p,q,\mu}(R_n).$$

Proof. The proof is the same as the proof of Theorem 3 of [3]. The needed interpolation results for the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ may be found in [2, Sect. 2.4; or 1].

Remark 5. We recall that

$$f^s_{p,2,\mu}(R_n) = h^s_{p,\mu}(R_n).$$

So one can formulate many special cases of the above theorem. In particular, the last theorem is an extension of Theorem 3 in [3].

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