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Journal of Approximation Theory 134 (2005) 257–266

JOURNAL OF  
Approximation  
Theory[www.elsevier.com/locate/jat](http://www.elsevier.com/locate/jat)

# A strong convergence theorem for relatively nonexpansive mappings in a Banach space

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Received 10 July 2003; received in revised form 5 January 2005; accepted 16 February 2005

Communicated by Frank Deutsch

Available online 2 April 2005

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## Abstract

In this paper, we prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.

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*MSC:* primary 47H09; secondary 47H05

*Keywords:* Relatively nonexpansive mapping; Nonexpansive mapping; Asymptotic fixed point; Generalized projection; Maximal monotone operator

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## 1. Introduction

Let  $E$  be a smooth Banach space and let  $E^*$  be the dual of  $E$ . The function  $\phi : E \times E \rightarrow \mathbf{R}$  is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all  $x, y \in E$ , where  $J$  is the normalized duality mapping from  $E$  to  $E^*$ . Let  $C$  be a closed convex subset of  $E$ , and let  $T$  be a mapping from  $C$  into itself. We denote by  $F(T)$

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the set of fixed points of  $T$ . A point  $p$  in  $C$  is said to be an asymptotic fixed point of  $T$  [13] if  $C$  contains a sequence  $\{x_n\}$  which converges weakly to  $p$  such that the strong  $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$ . The set of asymptotic fixed points of  $T$  will be denoted by  $\hat{F}(T)$ . A mapping  $T$  from  $C$  into itself is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$  and relatively nonexpansive [3–5] if  $\hat{F}(T) = F(T)$  and  $\phi(p, Tx) \leq \phi(p, x)$  for all  $x \in C$  and  $p \in F(T)$ . The asymptotic behavior of a relatively nonexpansive mapping was studied in [3–5]. On the other hand, Nakajo and Takahashi [9] obtained strong convergence theorems for nonexpansive mappings in a Hilbert space. In particular, they studied the strong convergence of the sequence  $\{x_n\}$  generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) Sx_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, n = 0, 1, 2, \dots, \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$ ,  $S$  is a nonexpansive mapping from  $C$  into itself and  $P_{C_n \cap Q_n}$  is the metric projection from  $C$  onto  $C_n \cap Q_n$ .

Motivated by Nakajo and Takahashi [9], our purpose in this paper is to prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.

**2. Preliminaries**

Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and let  $E^*$  be the dual of  $E$ . Denote by  $\langle \cdot, \cdot \rangle$  the duality product. The normalized duality mapping  $J$  from  $E$  to  $E^*$  is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for  $x \in E$ . When  $\{x_n\}$  is a sequence in  $E$ , we denote strong convergence of  $\{x_n\}$  to  $x \in E$  by  $x_n \rightarrow x$  and weak convergence by  $x_n \rightharpoonup x$ .

A Banach space  $E$  is said to be strictly convex if  $\|\frac{x+y}{2}\| < 1$  for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $x \neq y$ . It is also said to be uniformly convex if  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$  for any two sequences  $\{x_n\}, \{y_n\}$  in  $E$  such that  $\|x_n\| = \|y_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|\frac{x_n + y_n}{2}\| = 1$ . Let  $U = \{x \in E : \|x\| = 1\}$  be the unit sphere of  $E$ . Then the Banach space  $E$  is said to be smooth provided

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each  $x, y \in U$ . It is also said to be uniformly smooth if the limit is attained uniformly for  $x, y \in U$ . It is well known that if  $E$  is smooth, then the duality mapping  $J$  is single valued. It is also known that if  $E$  is uniformly smooth, then  $J$  is uniformly norm-to-norm continuous on each bounded subset of  $E$ . Some properties of the duality mapping have been given in [6,12,16,17]. A Banach space  $E$  is said to have the Kadec–Klee property if a sequence  $\{x_n\}$  of  $E$  satisfying that  $x_n \rightharpoonup x \in E$  and  $\|x_n\| \rightarrow \|x\|$ , then  $x_n \rightarrow x$ . It is

known that if  $E$  is uniformly convex, then  $E$  has the Kadec–Klee property; see [6,16,17] for more details. Let  $E$  be a smooth Banach space. The function  $\phi : E \times E \rightarrow \mathbf{R}$  is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for  $x, y \in E$ . It is obvious from the definition of the function  $\phi$  that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad (2.1)$$

for all  $x, y \in E$ .

**Remark 2.1.** If  $E$  is a strictly convex and smooth Banach space, then for  $x, y \in E$ ,  $\phi(y, x) = 0$  if and only if  $x = y$ . It is sufficient to show that if  $\phi(y, x) = 0$  then  $x = y$ . From (2.1), we have  $\|x\| = \|y\|$ . This implies  $\langle y, Jx \rangle = \|y\|^2 = \|Jx\|^2$ . From the definition of  $J$ , we have  $Jx = Jy$ . Since  $J$  is one-to-one, we have  $x = y$ ; see [6,16,17] for more details.

Recently, Kamimura and Takahashi [7] proved the following result. This plays an important role in the proof of the main theorem.

**Proposition 2.1** (Kamimura and Takahashi [7]). *Let  $E$  be a uniformly convex and smooth Banach space and let  $\{y_n\}, \{z_n\}$  be two sequences of  $E$ . If  $\phi(y_n, z_n) \rightarrow 0$  and either  $\{y_n\}$  or  $\{z_n\}$  is bounded, then  $y_n - z_n \rightarrow 0$ .*

Let  $C$  be a nonempty closed convex subset of  $E$ . Suppose that  $E$  is reflexive, strictly convex and smooth. Then, for any  $x \in E$ , there exists a point  $x_0 \in C$  such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping  $P_C : E \rightarrow C$  defined by  $P_C x = x_0$  is called the generalized projection [1,2,7]. The following are well-known results. For example, see [1,2,7]

**Proposition 2.2** (Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]). *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$  and  $x \in E$ . Then,  $x_0 = P_C x$  if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$$

for  $y \in C$ .

**Proposition 2.3** (Alber [1], Alber and Reich [2], Kamimura and Takahashi [7]). *Let  $E$  be a reflexive, strictly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $x \in E$ . Then*

$$\phi(y, P_C x) + \phi(P_C x, x) \leq \phi(y, x)$$

for all  $y \in C$ .

Concerning the set of fixed points of a relatively nonexpansive mapping, we can prove the following result.

**Proposition 2.4.** *Let  $E$  be a strictly convex and smooth Banach space, let  $C$  be a closed convex subset of  $E$ , and let  $T$  be a relatively nonexpansive mapping from  $C$  into itself. Then  $F(T)$  is closed and convex.*

**Proof.** We first show that  $F(T)$  is closed. Let  $\{x_n\}$  be a sequence of  $F(T)$  such that  $x_n \rightarrow \hat{x} \in C$ . From the definition of  $T$ ,

$$\phi(x_n, T\hat{x}) \leq \phi(x_n, \hat{x})$$

for each  $n \in \mathbb{N}$ . This implies,

$$\phi(\hat{x}, T\hat{x}) = \lim_{n \rightarrow \infty} \phi(x_n, T\hat{x}) \leq \lim_{n \rightarrow \infty} \phi(x_n, \hat{x}) = \phi(\hat{x}, \hat{x}) = 0.$$

Therefore we obtain  $\hat{x} = T\hat{x}$ . So, we have  $\hat{x} \in F(T)$ . Next, we show that  $F(T)$  is convex. For  $x, y \in F(T)$  and  $t \in (0, 1)$ , put  $z = tx + (1 - t)y$ . It is sufficient to show  $Tz = z$ . In fact, we have

$$\begin{aligned} \phi(z, Tz) &= \|z\|^2 - 2\langle z, JTz \rangle + \|Tz\|^2 \\ &= \|z\|^2 - 2\langle tx + (1 - t)y, JTz \rangle + \|Tz\|^2 \\ &= \|z\|^2 - 2t\langle x, JTz \rangle - 2(1 - t)\langle y, JTz \rangle + \|Tz\|^2 \\ &= \|z\|^2 + t\phi(x, Tz) + (1 - t)\phi(y, Tz) - t\|x\|^2 - (1 - t)\|y\|^2 \\ &\leq \|z\|^2 + t\phi(x, z) + (1 - t)\phi(y, z) - t\|x\|^2 - (1 - t)\|y\|^2 \\ &= \|z\|^2 - 2\langle tx + (1 - t)y, Jz \rangle + \|z\|^2 \\ &= \|z\|^2 - 2\langle z, Jz \rangle + \|z\|^2 = 0. \end{aligned}$$

This implies  $z = Tz$ .  $\square$

### 3. Main result

Now, we can prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming.

**Theorem 3.1.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$ , let  $T$  be a relatively nonexpansive mapping from  $C$  into itself, and let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ . Suppose that  $\{x_n\}$  is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, n = 0, 1, 2, \dots, \end{cases}$$

where  $J$  is the duality mapping on  $E$ . If  $F(T)$  is nonempty, then  $\{x_n\}$  converges strongly to  $P_{F(T)}x$ , where  $P_{F(T)}$  is the generalized projection from  $C$  onto  $F(T)$ .

**Proof.** We first show that  $H_n$  and  $W_n$  are closed and convex for each  $n \in \mathbf{N} \cup \{0\}$ . From the definition of  $H_n$  and  $W_n$ , it is obvious that  $H_n$  is closed and  $W_n$  is closed and convex for each  $n \in \mathbf{N} \cup \{0\}$ . We show that  $H_n$  is convex. Since  $\phi(z, y_n) \leq \phi(z, x_n)$  is equivalent to

$$2\langle z, Jx_n - Jy_n \rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

it follows that  $H_n$  is convex.

Next, we show that  $F(T) \subset H_n \cap W_n$  for each  $n \in \mathbf{N} \cup \{0\}$ . Let  $u \in F(T)$  and let  $n \in \mathbf{N} \cup \{0\}$ . Then from

$$\begin{aligned} \phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n)JT x_n \rangle + \|\alpha_n Jx_n + (1 - \alpha_n)JT x_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n) \langle u, JT x_n \rangle + \alpha_n \|x_n\|^2 \\ &\quad + (1 - \alpha_n) \|T x_n\|^2 \\ &= \alpha_n (\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2) + (1 - \alpha_n) (\|u\|^2 - 2\langle u, JT x_n \rangle \\ &\quad + \|T x_n\|^2) \\ &= \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, T x_n) \\ &\leq \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, x_n) \\ &= \phi(u, x_n), \end{aligned}$$

we have  $u \in H_n$ . Therefore we obtain

$$F(T) \subset H_n$$

for each  $n \in \mathbf{N} \cup \{0\}$ . On the other hand, it is clear that  $F(T) \subset H_0 \cap W_0$ . Suppose that  $F(T) \subset H_k \cap W_k$  for some  $k \in \mathbf{N}$ . There exists an element  $x_{k+1} \in H_k \cap W_k$  such that  $x_{k+1} = P_{H_k \cap W_k} x$ . From Proposition 2.2, there holds

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0$$

for each  $z \in H_k \cap W_k$ . Since  $F(T) \subset H_k \cap W_k$ , we have  $\langle x_{k+1} - u, Jx - Jx_{k+1} \rangle \geq 0$  for every  $u \in F(T)$  and hence  $F(T) \subset W_{k+1}$ . Therefore we have  $F(T) \subset H_{k+1} \cap W_{k+1}$ . This implies that  $\{x_n\}$  is well defined. It follows from the definition of  $W_n$  and Proposition 2.2 that  $x_n = P_{W_n} x$ . Using  $x_n = P_{W_n} x$  and Proposition 2.3, we have

$$\phi(x_n, x) = \phi(P_{W_n} x, x) \leq \phi(u, x) - \phi(u, x_n) \leq \phi(u, x)$$

for each  $u \in F(T) \subset W_n$  for each  $n \in \mathbf{N} \cup \{0\}$ . Therefore,  $\phi(x_n, x)$  is bounded. Moreover, from (2.1), we have that  $\{x_n\}$  is bounded.

Since  $x_{n+1} = P_{H_n \cap W_n} x \in W_n$  and Proposition 2.3, we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x)$$

for each  $n \in \mathbf{N} \cup \{0\}$ . Therefore  $\{\phi(x_n, x)\}$  is nondecreasing. So there exists the limit of  $\phi(x_n, x)$ . From Proposition 2.3, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, P_{W_n} x) \leq \phi(x_{n+1}, x) - \phi(P_{W_n} x, x) \\ &= \phi(x_{n+1}, x) - \phi(x_n, x) \end{aligned}$$

for each  $n \in \mathbf{N} \cup \{0\}$ . This implies that  $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$ . Since  $x_{n+1} = P_{H_n \cap W_n} x \in H_n$ , from the definition of  $H_n$ , we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$$

for each  $n \in \mathbf{N} \cup \{0\}$ . Tending  $n \rightarrow \infty$ , we have  $\lim_{n \rightarrow \infty} \phi(x_{n+1}, y_n) = 0$ . Using Proposition 2.1, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Since  $J$  is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - Jy_n\| = \lim_{n \rightarrow \infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.1}$$

On the other hand, we have, for each  $n \in \mathbf{N} \cup \{0\}$ ,

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1 - \alpha_n)JT x_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1 - \alpha_n)(Jx_{n+1} - JT x_n)\| \\ &= \|(1 - \alpha_n)(Jx_{n+1} - JT x_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1 - \alpha_n)\|Jx_{n+1} - JT x_n\| - \alpha_n\|Jx_n - Jx_{n+1}\| \end{aligned}$$

and hence

$$\begin{aligned} \|Jx_{n+1} - JT x_n\| &\leq \frac{1}{1 - \alpha_n} (\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|) \\ &\leq \frac{1}{1 - \alpha_n} (\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|). \end{aligned}$$

From (3.1) and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ , we obtain

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - JT x_n\| = 0.$$

Since  $J^{-1}$  is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - Tx_n\| = \lim_{n \rightarrow \infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(JT x_n)\| = 0.$$

From

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - x_{n+1} + x_{n+1} - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|, \end{aligned}$$

we have  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . Therefore, if  $\{x_{n_k}\}$  is a subsequence of  $\{x_n\}$  such that  $x_{n_k} \rightharpoonup \hat{x} \in C$ , then  $\hat{x} \in \hat{F}(T) = F(T)$ .

Finally, we show that  $x_n \rightarrow P_{F(T)}x$ . Let  $\{x_{n_k}\}$  be a subsequence of  $\{x_n\}$  such that  $x_{n_k} \rightharpoonup \hat{x} \in F(T)$  and  $w = P_{F(T)}x$ . For any  $n \in \mathbf{N}$ , from  $x_{n+1} = P_{H_n \cap W_n}x$  and  $w \in F(T) \subset H_n \cap W_n$ , we have  $\phi(x_{n+1}, x) \leq \phi(w, x)$ . On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned} \phi(\hat{x}, x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx \rangle + \|x\|^2 \\ &\leq \liminf_{k \rightarrow \infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx \rangle + \|x\|^2) \end{aligned}$$

$$\begin{aligned} &= \liminf_{k \rightarrow \infty} \phi(x_{n_k}, x) \\ &\leq \limsup_{k \rightarrow \infty} \phi(x_{n_k}, x) \\ &\leq \phi(w, x). \end{aligned}$$

From the definition of  $P_{F(T)}x$ , we obtain  $\hat{x} = w$  and hence  $\lim_{k \rightarrow \infty} \phi(x_{n_k}, x) = \phi(w, x)$ . So, we have

$$\lim_{k \rightarrow \infty} \|x_{n_k}\| = \|w\|.$$

Using the Kadec–Klee property of  $E$ , we obtain that  $\{x_{n_k}\}$  converges strongly to  $P_{F(T)}x$ . Since  $\{x_{n_k}\}$  is an arbitrary weakly convergent sequence of  $\{x_n\}$ , we can conclude that  $\{x_n\}$  converges strongly to  $P_{F(T)}x$ .  $\square$

### 4. Applications

In this section, we discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space. Using Theorem 3.1, we first obtain the result of [9].

**Theorem 4.1** (Nakajo and Takahashi[9]). *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$ , and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T)$  is nonempty. Suppose that  $\{x_n\}$  is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, n = 0, 1, 2, \dots, \end{cases}$$

where  $\{\alpha_n\} \subset [0, a)$  for some  $a \in [0, 1)$  and  $P_{C_n \cap Q_n}$  is the metric projection from  $C$  onto  $C_n \cap Q_n$ . Then  $\{x_n\}$  converges strongly to  $P_{F(T)}x$ , where  $P_{F(T)}$  is the metric projection from  $C$  onto  $F(T)$ .

**Proof.** It is sufficient to prove that if  $T$  is nonexpansive, then  $T$  is relatively nonexpansive. It is obvious that  $F(T) \subset \hat{F}(T)$ . If  $u \in \hat{F}(T)$ , then there exists  $\{x_n\} \subset C$  such that  $x_n \rightarrow u$  and  $x_n - Tx_n \rightarrow 0$ . Since  $T$  is nonexpansive,  $T$  is demiclosed. So, we have  $u = Tu$ . This implies  $F(T) = \hat{F}(T)$ . Further, in a Hilbert space  $H$ , we know that

$$\phi(x, y) = \|x - y\|^2$$

for every  $x, y \in H$ . So,  $\|Tx - Ty\| \leq \|x - y\|$  is equivalent to  $\phi(Tx, Ty) \leq \phi(x, y)$ . Therefore,  $T$  is relatively nonexpansive. Using Theorem 3.1, we obtain the desired result.  $\square$

Let  $A$  be a multivalued operator from  $E$  to  $E^*$  with domain  $D(A) = \{z \in E : Az \neq \emptyset\}$  and range  $R(A) = \cup\{Az : z \in D(A)\}$ . An operator  $A$  is said to be monotone if

$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$  for each  $x_i \in D(A)$  and  $y_i \in Tx_i, i = 1, 2$ . A monotone operator  $A$  is said to be maximal if its graph  $G(A) = \{(x, y) : y \in Ax\}$  is not properly contained in the graph of any other monotone operator. We know that if  $A$  is a maximal monotone operator, then  $A^{-1}0$  is closed and convex. The following result is also well-known.

**Theorem 4.2** (Rockafellar [14]). *Let  $E$  be a reflexive, strictly convex and smooth Banach space and let  $A$  be a monotone operator from  $E$  to  $E^*$ . Then  $A$  is maximal if and only if  $R(J + rA) = E^*$  for all  $r > 0$ .*

Let  $E$  be a reflexive, strictly convex and smooth Banach space, and let  $A$  be a maximal monotone operator from  $E$  to  $E^*$ . Using Theorem 4.2 and strict convexity of  $E$ , we obtain that for every  $r > 0$  and  $x \in E$ , there exists a unique  $x_r \in D(A)$  such that

$$Jx \in Jx_r + rAx_r.$$

If  $J_r x = x_r$ , then we can define a single valued mapping  $J_r : E \rightarrow D(A)$  by  $J_r = (J + rA)^{-1}J$  and such a  $J_r$  is called the resolvent of  $A$ . We know that  $A^{-1}0 = F(J_r)$  for all  $r > 0$ ; see[16,17] for more details. Using Theorem 3.1, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [7–11,13,15].

**Theorem 4.3.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space, let  $A$  be a maximal monotone operator from  $E$  to  $E^*$ , let  $J_r$  be a resolvent of  $A$ , where  $r > 0$  and let  $\{\alpha_n\}$  be a sequence of real numbers such that  $0 \leq \alpha_n < 1$  and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ . Suppose that  $\{x_n\}$  is given by*

$$\begin{cases} x_0 = x \in E, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_r x_n), \\ H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, n = 0, 1, 2, \dots, \end{cases}$$

where  $J$  is the duality mapping on  $E$ . If  $A^{-1}0$  is nonempty, then  $\{x_n\}$  converges strongly to  $P_{A^{-1}0}x$  where  $P_{A^{-1}0}$  is the generalized projection from  $E$  onto  $A^{-1}0$ .

**Proof.** We first show that  $\hat{F}(J_r) \subset A^{-1}0$ . Let  $p \in \hat{F}(J_r)$ . Then, there exists  $\{z_n\} \subset E$  such that  $z_n \rightarrow p$  and  $\lim_{n \rightarrow \infty} (z_n - J_r z_n) = 0$ . Since  $J$  is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r}(Jz_n - JJ_r z_n) \rightarrow 0.$$

It follows from  $\frac{1}{r}(Jz_n - JJ_r z_n) \in AJ_r z_n$  and the monotonicity of  $A$  that

$$\langle w - J_r z_n, w^* - \frac{1}{r}(Jz_n - JJ_r z_n) \rangle \geq 0$$

for all  $w \in D(A)$  and  $w^* \in Aw$ . Letting  $n \rightarrow \infty$ , we have

$$\langle w - p, w^* \rangle \geq 0$$



for all  $w \in D(A)$  and  $w^* \in Aw$ . Therefore from the maximality of  $A$ , we obtain  $p \in A^{-1}0$ . On the other hand, we know that  $F(J_r) = A^{-1}0$  and  $F(J_r) \subset \hat{F}(J_r)$ , therefore  $A^{-1}0 = F(J_r) = \hat{F}(J_r)$ . Next we show that  $J_r$  is a relatively nonexpansive mapping with respect to  $A^{-1}0$ . Let  $w \in E$  and  $p \in A^{-1}0$ . From the monotonicity of  $A$ , we have

$$\begin{aligned}
 \phi(p, J_r w) &= \|p\|^2 - 2\langle p, J J_r w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 + 2\langle p, J w - J J_r w - J w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 + 2\langle p, J w - J J_r w \rangle - 2\langle p, J w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 - 2\langle J_r w - p - J_r w, J w - J J_r w \rangle - 2\langle p, J w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 - 2\langle J_r w - p, J w - J J_r w \rangle \\
 &\quad + 2\langle J_r w, J w - J J_r w \rangle - 2\langle p, J w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 - 2r\langle J_r w - p, \frac{1}{r}(J w - J J_r w) \rangle \\
 &\quad + 2\langle J_r w, J w - J J_r w \rangle - 2\langle p, J w \rangle + \|J_r w\|^2 \\
 &\leq \|p\|^2 + 2\langle J_r w, J w - J J_r w \rangle - 2\langle p, J w \rangle + \|J_r w\|^2 \\
 &= \|p\|^2 - 2\langle p, J w \rangle + \|w\|^2 - \|J_r w\|^2 + 2\langle J_r w, J w \rangle - \|w\|^2 \\
 &= \phi(p, w) - \phi(J_r w, w) \\
 &\leq \phi(p, w).
 \end{aligned}$$

This implies that  $J_r$  is a relatively nonexpansive mapping. Using Theorem 3.1, we can conclude that  $\{x_n\}$  converges strongly to  $P_{A^{-1}0}x$ .  $\square$

## Acknowledgments

The authors would like to thank the referee for valuable suggestions which helped to improve this manuscript.

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