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The limit of a Stanley–Wilf sequence is not always rational, and layered patterns beat monotone patterns

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Abstract

We show the first known example for a pattern q for which $L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$ is not an integer, where $S_n(q)$ denotes the number of permutations of length n avoiding the pattern q. We find the exact value of the limit and show that it is irrational, but algebraic. Then we generalize our results to an infinite sequence of patterns. We provide further generalizations that start explaining why certain patterns are easier to avoid than others. Finally, we show that if q is a layered pattern of length k, then $L(q) \geqslant (k-1)^2$ holds.

Keywords: Permutations; Patterns; Stanley–Wilf sequence; Limit

1. Introduction

Let $S_n(q)$ be the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q. For a brief introduction to the area of pattern avoidance, see [4]; for a more detailed introduction, see [5]. A recent spectacular result of Marcus and Tardos [8] shows that for any pattern q, there exists a constant c_q so that $S_n(q) < c_q^n$ holds for all n. As pointed out by Arratia [1], this is equivalent to the statement that $L(q) = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$

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exists. Let us call the sequence $\sqrt[n]{S_n(q)}$ a *Stanley–Wilf* sequence. It is a natural and intriguing question to ask what the limit $L(q)$ of a Stanley–Wilf sequence can be, for various patterns q.

The main reason this question has been so intriguing is that in all cases where $L(q)$ has been known, it has been known to be an *integer*. Indeed, the results previously known are listed below.

1. When q is of length three, then $L(q) = 4$. This follows from the well-known fact [10] that in this case, $S_n(q) = \binom{2n}{n}/(n+1)$.
2. When $q = 123 \cdots k$, or when q is such that $S_n(q) = S_n(12 \cdots k)$, then $L(q) = (k-1)^2$. This follows from an asymptotic formula of Regev [9].
3. When $q = 1342$, or when q is such that $S_n(q) = S_n(1342)$, then $L(q) = 8$. See [3] for this result and an exact formula for the numbers $S_n(1342)$.

In this paper, we show that $L(q)$ is *not* always an integer. We achieve this by proving that $14 < L(12453) < 15$. Then we compute the exact value of this limit, and see that it is not even rational; it is the number $9+4\sqrt{2}$. We compute the limit of the Stanley–Wilf sequence for an infinite sequence of patterns, and see that as the length k of these patterns grows, $L(q)$ will fall further and further below the largest known possible value, $(k-1)^2$. Finally, we show that while for certain patterns, our methods provide the exact value of the limit of the Stanley–Wilf sequence, for certain others they only provide a *lower bound* on this limit. This starts explaining why certain patterns are easier to avoid than others. Among other results, we will confirm a 7-year old conjecture by proving that in the sense of logarithmic asymptotics, a layered pattern q is always easier to avoid than the monotone pattern of the same length.

2. Proving an upper bound

Let $p = p_1 p_2 \cdots p_n$ be a permutation. Recall that p_i is called a *left-to-right minimum* of p if $p_j > p_i$ for all $j < i$. In other words, a left-to-right minimum is an entry that is smaller than everything on its left. Note that p_1 is always a left-to-right minimum, and so is the entry 1 of p. Also note that the left-to-right minima of p always form a decreasing sequence. For the rest of this paper, entries that are not left-to-right minima are called *remaining entries*.

Now we are in a position to prove our promised upper bound for the numbers $S_n(12453)$.

Lemma 2.1. *For all positive integers n, we have*

$$S_n(12453) < (9+4\sqrt{2})^n < 14.66^n.$$

Proof. Let p be a permutation counted by $S_n(12453)$, and let p have k left-to-right minima. Then we have at most $\binom{n}{k}$ choices for the set of these left-to-right minima, and we have at most $\binom{n}{k}$ choices for their positions. The string of the remaining entries has to form a 1342-avoiding permutation of length $n-k$. Indeed, if there was a copy *acdb* of 1342 among the entries that are not left-to-right minima, then we could complete it to a 12453 pattern by simply prepending it by the closest left-to-right minimum that is on the left of a. The number of 1342-avoiding permutations on $n-k$ elements is less than 8^{n-k} as we know from [3]. This shows that

$$\begin{aligned} S_n(12453) &< \sum_{k=1}^{n} \binom{n}{k}^2 \cdot 8^{n-k} \\ &< \sum_{k=1}^{n} \left(\binom{n}{k} \cdot \sqrt{8}^{n-k} \right)^2 \leqslant \left(\sum_{k=1}^{n} \binom{n}{k} \cdot \sqrt{8}^{n-k} \right)^2 \\ &< (1+\sqrt{8})^{2n} = (9+4\sqrt{2})^n \end{aligned}$$

and the proof is complete. □

Corollary 2.2. *We have*

$$L(12453) \leqslant 9 + 4\sqrt{2} < 14.66.$$

3. Proving a lower bound

We have seen in Corollary 2.2 that $L(12453) \leqslant 9 + 4\sqrt{2} < 14.66$. In order to prove that this limit is not an integer, it suffices to show that it is larger than 14. In what follows, we are going to work toward a good lower bound for the numbers $S_n(12453)$, and thus the number $L(12453)$.

Where is the waste in the proof of the upper bound in the previous section? The waste is that there are some choices for the left-to-right minima that are incompatible with some choices for the 1342-avoiding permutation of the remaining entries. This is a crucial concept of the upcoming proof, so we will make it more precise.

We have mentioned in the previous section, that determining the left-to-right minima of a permutation p means to determine the set T of positions where these minima will be, and to determine the set Z of entries that are the left-to-right minima. In other words, the ordered pair (T, Z) of equal-sized subsets of $[n] = \{1, 2, \ldots, n\}$ describes the left-to-right minima of p.

Definition 3.1. Let n be a positive integer, and let $m \leqslant n$ be a positive integer. Let T and Z be two m-element subsets of $[n]$. Finally, let S be a permutation of the elements of the set $[n] - Z$. If there exists an n-permutation p so that its left-to-right minima are precisely the elements of Z, they are located in positions belonging to T, and its string of remaining entries is S, then we say that the triple (T, Z, S) is *compatible*. Otherwise, we say that the triple (T, Z, S) is *incompatible*.

Clearly, if (T, Z, S) is *compatible*, then there is exactly one permutation p satisfying all criteria specified by (T, Z, S).

Example 3.2. If $n = 4$, and $T = \{1, 3\}$, $Z = \{1, 2\}$, and $S = 43$, then (T, Z, S) is *compatible* as shown by the permutation 2413.

Example 3.3. If $n = 4$, and $T = \{1, 3\}$, $Z = \{1, 3\}$, and $S = 24$, then (T, Z, S) is *incompatible*. Indeed, the only permutation allowed by T and S is 3214, but for this permutation $Z = \{1, 2, 3\}$, not $\{1, 3\}$.

Returning to the method by which we proved our upper bound for $L(12453)$, we will show that in a sufficient number of cases, our triples (T, Z, S) are compatible. This will show that the upper bound is quite close to the precise value of $L(12453)$.

What is a good way to check that a particular choice (T, Z) of left-to-right minima is compatible with a particular choice of S? For shortness, let us call the procedure of putting together S and a string (T, Z) of left-to-right minima *merging*. One has to check that in the permutation obtained by merging our left-to-right minima with S, the left-to-right minima are indeed the entries in Z. That is, there are no additional left-to-right minima, and the entries in Z are indeed all left-to-right minima. This is achieved exactly when any remaining entry is larger than the closest left-to-right minimum on its left.

In our efforts to find a good lower bound on $L(12453)$, we will only consider a special kind of permutations. Let N be a positive integer so that $S_n(1342) > 7.99^n$ for all $n \geqslant N$. (We know from [3] that such an N exists as $L(1342) = 8$.)

Assume first that the string S of remaining entries of our permutations has length s, where s is divisible by N. Consider permutations having the following additional property. If we cut S into s/N blocks of consecutive entries of length N each, then the entries of any given block B are all smaller than the entries of any block on the left of B, and larger than the entries of any block on the right of B. Let us call these strings S *block-structured*. See Fig. 1 for the generic diagram of a block-structured string in the (unrealistic) case of $N = 2$.

If s is not divisible by N, that is, when $s = Nt + r$ for some $r \in [1, N - 1]$, then we call S block-structured if its last r entries are its smallest entries, and they are in decreasing order, and its first $s - r$ entries have the block-structured property in the above sense. For instance, for $N = 3$ and $s = 8$, the string 687|534|21 is block structured. As the last r entries must be in decreasing order, we will not call their string a block.

Let S be a block-structured string in which each block is a 1342-avoiding substring. It is then clear that S itself is 1342-avoiding as a 1342-pattern cannot start in a block and end in another one. The definition of N implies that we have more than 7.99^N choices for the substring of each block. Therefore, we have at least 7.99^{s-r} block-structured strings S of length s that avoid 1342. (Recall that r is the remainder of s modulo N). As $r < N$, this implies that the number of block-structured strings of length s is always more than $\frac{1}{7.99^N} \cdot 7.99^s = c \cdot 7.99^s$, for an absolute constant c. (The constant c will become insignificant when we take nth roots.)

We claim that a sufficient number of these strings S will be compatible with a sufficient number of the choices (T, Z) of left-to-right minima.

First, look at the very special case when S is *decreasing*. In this case, we will write S^{dec} instead of S. Now our permutation p consists of two decreasing sequences (so it is 123-avoiding), namely the left-to-right minima and S^{dec}. The following proposition is very well-known.

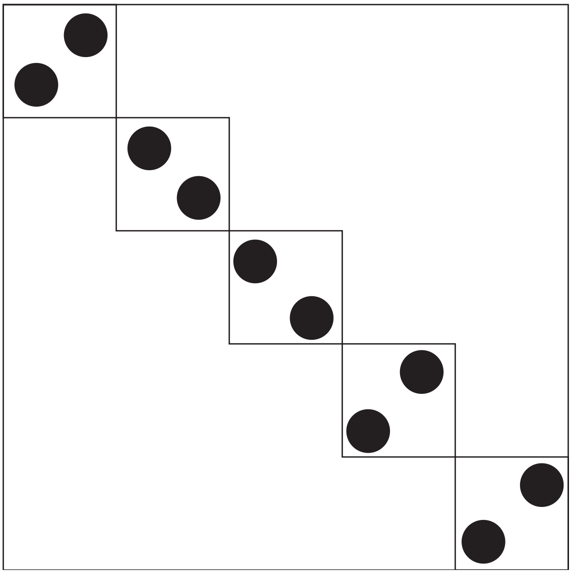

Fig. 1. A block-structured string.

Proposition 3.4. *Let* $1 \leqslant m \leqslant n$. *Then the number of* 123-*avoiding n-permutations having exactly m left-to-right minima is*

$$A(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}, \quad (1)$$

a Narayana *number*.

For a proof, see [11] or [5].

The significance of this result for us is the following. If we just wanted to merge (T, Z) and S^{dec} together, with no regard to the existing constraints, the total number of ways to do that would be of course at most $\binom{n}{m} \cdot \binom{n}{m}$. The above formula shows that roughly $\frac{1}{n}$ of these mergings will actually be good, that is, they will not violate any constraints, they will lead to compatible triples (T, Z, S). The factor $\frac{1}{n}$ is not a significant loss from our point of view, since $\lim_{n\to\infty} \sqrt[n]{1/n} = 1$.

Now let us return to the general case of block-structured strings S. In other words, take a 123-avoiding n-permutation (T, Z, S^{dec}), and replace its string S^{dec} by a block-structured string S *taken on the entries that belong to* S^{dec}. We claim that after this replacement, a sufficient number of triples (T, Z, S) will be compatible.

Here is the outline of the proof of that claim. Because of the definition of a block-structured S, it is true that every entry in S is at most N positions away from the position it was in S^{dec}. (We will take the left-to-right minima into account next.) Therefore, if we merge (T, Z) and S^{dec} together so that each left-to-right minimum y is not only smaller than all entries on its left, and smaller than all remaining entries located between y and the closest left-to-right minimum y' on the right of y, but also smaller than the N closest remaining entries on the right of y', then we will be done. Indeed, in this case replacing S^{dec} by any block-structured string S will not violate any constraints since no remaining entry moves up by more than N slots in the string of remaining entries.

For example, set $N = 3$ (which is unrealistic because in reality, N needs to be much larger). Then the permutation 592871643 has the desired property. Indeed, each left-to-right minimum y of this permutation is smaller than the three remaining entries immediately following the left-to-right minimum y' that comes after y. That is, 5 is smaller than 8, 7 and 6, and 2 is smaller than 6, 4, and 3. (The condition always vacuously holds for 1.) Therefore, if we rearrange the string 987643 so that no entry moves up by more than three slots within this string, then no constraints will be violated, that is, the obtained permutation will still have left-to-right minima 5, 2, and 1.

Therefore, we will have a lower bound for the number of compatible triples (T, Z, S) if we find a lower bound for the number of compatible triples (T, Z, S^{dec}) in which each left-to-right minimum has the mentioned stronger property.

In order to find such a lower bound, take a 123-avoiding permutation p' which is of length $n - N$. Let p' have m left-to-right minima. Denote (T', Z') the string of the left-to-right minima of p', and let $S^{\text{dec}'}$ denote the decreasing string of remaining entries of p'. Now prepend p' with the decreasing string taken on the N-element set $\{n - N + 1, n - N + 2, \ldots, n\}$, to get an n-permutation. In this n-permutation, move each of the original m left-to-right minima of p' to the left by N positions. Let us call the obtained n-permutation p''.

For example, with $n = 9$, $N = 3$, and $p' = 456123$, we first prepend p' with the string 987, to get 987456123, then move the original two left-to-right minima of p', that is, the entries 4 and 1, to the left by three positions, to get $p'' = 498715623$. Note that means that the set of positions of the left-to-right minima of p'' is actually still T'.

It is then clear that the left-to-right minima of p'' are the same as the left-to-right minima of p'. Furthermore, because of the translation we used to create our new permutation, p'' has the property that if y and y' are two left-to-right minima so that y' is the closest left-to-right minimum on the right of y, then y is smaller than the N remaining entries immediately on the right of y'. Indeed, these N remaining entries were on the right of y' in the original permutation p'.

Now we can use the argument that we outlined five paragraphs ago. For easy reference, we sketch that argument again. If S^{dec} is replaced by any block-structured permutation of the same size taken on the same set of elements, (resulting in the n-permutation $p*$) then each remaining entry x will move within its block only, that is, x will move at most N positions from its original position in the string S of remaining entries. Therefore, x will still be larger than the left-to-right minimum closest to it and preceding it.

This shows that if p' and (T', Z') lead to a compatible triple, then so too will $p*$ and (T, Z), where (T, Z) describes the left-to-right minima of $p*$. Proposition 3.4 implies that the number of compatible triples (T', Z', p') is $\frac{1}{n-N}\binom{n-N}{m}\binom{n-N}{m-1}$. As N is a constant, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n-N}\binom{n-N}{m}\binom{n-N}{m-1}} = \lim_{n\to\infty} \sqrt[n]{\binom{n}{m}\binom{n}{m}}. \tag{2}$$

Now restrict our attention to the particular case when $m = \lfloor n/3 \rfloor$. We claim that permutations of this particular type are sufficiently numerous to provide the lower bound we need. Using Stirling's formula, a routine computation yields that in this case, we have

$$\lim_{n\to\infty} \sqrt[n]{\binom{n}{m}\binom{n}{m}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3^n}{2^{2n/3}}\right)^2} \geqslant 1.88^2.$$

Besides, we have more than $c \cdot 7.99^{2n/3}$ choices for the block-structured string S by which we replace S^{dec}. Therefore, we have proved the following lower bound.

Lemma 3.5. *For n sufficiently large, the number of n-permutations of length n that avoid the pattern* 12453 *is larger than*

$$1.88^{2n} \cdot c \cdot 7.99^{2n/3} \geqslant c \cdot 14.12^n,$$

where $c = 7.99^{-N}$.

Lemma 3.5 and Corollary 2.2 together immediately yield the following.

Theorem 3.6. *We have*

$$14.12 \leqslant L(12453) \leqslant 14.66.$$

In particular, $L(12453) = \lim_{n\to\infty} \sqrt[n]{S_n(12453)}$ *is not an integer*.

4. The exact value of $L(12453)$

If we are a little bit more careful with our choice of m in the argument of the previous section, we can find the exact value of $L(12453)$. It turns out to be the upper bound proved in Corollary 2.2.

Theorem 4.1. *We have* $L(12453) = (1+\sqrt{8})^2 = 9 + 4\sqrt{2}$.

Proof. The above argument works for any $1 \leqslant m \leqslant n - N$ instead of $m = \lfloor n/3 \rfloor$, and for any positive real number $8 - \varepsilon < 8$ instead of 7.99. The inequality generalizing Lemma 3.5 we get is

$$S_n(12453) \geqslant c \cdot \frac{1}{n-N}\binom{n-N}{m}\binom{n-N}{m-1}(8-\varepsilon)^{n-m}, \tag{3}$$

where $c = (8-\varepsilon)^{-N}$.

Taking (3) for all $m \in [1, n-N]$, then summing all the obtained inequalities, we get

$$(n-N)S_n(12453) \geqslant \frac{c}{n-N}\sum_{m=1}^{n-N}\binom{n-N}{m}\binom{n-N}{m-1}(8-\varepsilon)^{n-m}. \tag{4}$$

By a routine computation, we see that

$$\binom{n-N}{m-1} \geqslant \frac{1}{n-N}\binom{n-N}{m}.$$

The last inequality and (4) together yield

$$(n-N)S_n(12453) \geqslant \frac{c}{(n-N)^2} \sum_{m=1}^{n-N} \binom{n-N}{m}^2 (8-\varepsilon)^{n-m}.$$

Finally, if $n > N$, then clearly, $S_n(12453) \geqslant S_n(1342) \geqslant (8-\varepsilon)^n$. Comparing this to the last inequality, we get

$$(n-N+1)S_n(12453) \geqslant c \cdot \frac{(8-\varepsilon)^N}{(n-N)^2} \sum_{m=0}^{n-N} \binom{n-N}{m}^2 (8-\varepsilon)^{n-N-m}. \tag{5}$$

Let us now resort to the well-known Cauchy–Schwarz inequality stating that if $a_1, a_2, \ldots, a_d$ are positive real numbers, then

$$\frac{1}{d}\,(a_1 + a_2 + \cdots + a_d)^2 \leqslant a_1^2 + a_2^2 + \cdots + a_d^2. \tag{6}$$

The right-hand side of (5) can be viewed as the sum of $(n-N+1)$ squares, namely the squares of the positive real numbers $\binom{n-N}{m}\sqrt{(8-\varepsilon)^{n-N-m}}$. Therefore, setting $d = n-N+1$, we can apply (6) to the sum on the right-hand side of (5), which then leads us to the inequality

$$\begin{aligned}\frac{1}{n-N+1}\,(1+\sqrt{8-\varepsilon})^{2(n-N)} &= \frac{1}{n-N+1}\left(\sum_{m=0}^{n-N}\binom{n-N}{m}\sqrt{8-\varepsilon}^{\,n-N-m}\right)^2 \\ &\leqslant \sum_{m=0}^{n-N}\binom{n-N}{m}^2 (8-\varepsilon)^{n-N-m}.\end{aligned}$$

Comparing this with (5), we see that

$$S_n(12453) \geqslant c \cdot \frac{(8-\varepsilon)^N}{(n-N+1)^4}\,(1+\sqrt{8-\varepsilon})^{2(n-N)}.$$

Taking nth roots, then taking limits as n goes to infinity, we see that

$$L(12453) \geqslant (1+\sqrt{8-\varepsilon})^2$$

for any positive ε, proving our claim. □

5. Some generalizations

In this Section, we will provide some interesting generalizations of our results. We will need the following simple recursive properties of pattern avoiding permutations.

Proposition 5.1. *Let q be a pattern of length k, and let q' be the pattern of length $k+1$ that is obtained from q by adding* 1 *to each entry of q and prepending it with* 1. *Let p be a permutation whose string of remaining entries is S. Then the following hold.*

1. *If S avoids q, then p avoids q'.*
2. *If q itself starts with* 1, *then p avoids q' if and only if S avoids q.*

Iteratively applying part 2 of Proposition 5.1, and the method explained in the previous sections, we get the following theorem.

Theorem 5.2. *Let $k \geqslant 4$, and let q_k be the pattern $12\cdots(k-3)(k-1)k(k-2)$. So $q_4 = 1342$, $q_5 = 12453$, and so on. Then we have*

$$L(q_k) = (k-4+\sqrt{8})^2.$$

Proof. Induction on k. For $k=4$, the result is proved in [3], and for $k=5$, we have just proved it in the previous section. Assuming that the statement is true for k, we can prove the statement for $k+1$ the very same way we proved it for $k=5$, using the result for $k=4$, and part 2 of Proposition 5.1. □

The method we used to prove Lemma 2.1 can also be used to prove the following recursive result.

Lemma 5.3. *Let q be a pattern of length k that starts with* 1, *and let q' be the pattern of length $k+1$ that is obtained from q by adding* 1 *to each entry of q and prepending it with* 1. *Let c be a constant so that $S_n(q) < c^n$ for all n. Then we have*

$$S_n(q') < (1+\sqrt{c})^{2n} = (1+c+2\sqrt{c})^n.$$

This is an improvement of the previous best result [6], that only showed $S_n(q') < (4c)^n$. The following generalization of Theorem 4.1 can be proved just as that Theorem is.

Theorem 5.4. *Let q and q' be as in Lemma 5.3. Then we have*

$$L(q') = 1 + L(q) + 2\sqrt{L(q)}.$$

In a sense, this result generalizes Regev's result [9] that showed that $L(12\cdots k) = (k-1)^2$. Our result shows that this particular growth rate, that is, that $\sqrt{L(q)}$ grows by one as the pattern grows by one, is not limited to monotone patterns.

An interesting consequence of this theorem is that if q is as above, and $L(q) < (k-1)^2$, in other words, q is harder (or easier, for that matter) to avoid than the monotonic pattern of the same length, then repeatedly prepending q with 1 will not change this. That is, the obtained new patterns will still be more difficult to avoid than the monotonic pattern of the same length.

Are the methods presented in this paper useful at all if the pattern q does not start in the entry 1? We will show that for most patterns q, the answer is in the affirmative, as far as a *lower* bound is concerned. Let us say that the pattern q is *indecomposable* if it cannot be cut into two parts so that all entries on the left of the cut are larger than all entries on the right of the cut. For instance, 1423 and 3142 are indecomposable, but 3412 is not as we could cut it after two entries. Therefore, we call 3412 *decomposable*. It is routine to verify that as k grows, the ratio of indecomposable patterns among all $k!$ patterns of length k goes to 1.

Theorem 5.5. *Let q be an* indecomposable *pattern of length k, and let $L = \lim_{n\to\infty} \sqrt[n]{S_n(q)}$. Let q' be defined as in Lemma 5.3. Then we have*

$$\lim_{n\to\infty} \sqrt[n]{S_n(q')} \geqslant 1 + L + 2\sqrt{L}.$$

Proof. This theorem can be proved as Lemma 3.5, and Theorem 4.1 are. Indeed, as q is indecomposable, any block-structured string S will avoid q if each block does. Now apply part 1 of Proposition 5.1 to see that our argument will still provide the required lower bound. □

Note that the fact that the reverse complement of an indecomposable pattern is also an indecomposable pattern makes it possible to prove an analogous version of Theorem 5.5, in which instead of prepending the indecomposable pattern q by a minimal entry, we affix a maximal entry to the end of q.

Our methods will not provide an upper bound for $\lim_{n\to\infty} \sqrt[n]{S_n(q')}$ because the string S of remaining entries of a q'-avoiding permutation does not have to be q-avoiding. (Only part 1 and not part 2 of Proposition 5.1 applies.) That condition is simply sufficient, but not necessary, in this general case. Nevertheless, Theorem 5.5 is interesting. It shows that for almost all patterns q, if we prepend q by the entry 1, the limit of the corresponding Stanley–Wilf sequence will grow at least as fast as for monotone q. If q started in 1, then this growth will be the same as for monotone q.

Now it is a little easier to understand why, in the case of length 4, the patterns that are the hardest to avoid, are along with certain equivalent ones, 1423 and 1342. Indeed, removing the starting 1 from them, we get the *decomposable* patterns 423 and 342. As these patterns are decomposable, Theorem 5.5 does not hold for them, so the limit of the Stanley–Wilf sequence for the patterns 1423 or 1342 does not have to be at least $1 + 4 + 4 = 9$, and in fact it is not.

A particularly interesting application of Theorem 5.4 is as follows. Recall that a *layered* pattern is a pattern that consists of decreasing subsequences (the layers) so that the entries increase among the layers. For instance, 3217654 is a layered pattern. In 1997, several people (including present author) have observed, using numerical evidence computed in [12], that if q is a layered pattern of length k, then for small n, the inequality $S_n(12\cdots k) \leqslant S_n(q)$ seems to hold. We will now show that this is indeed true in the sense of logarithmic asymptotics.

Theorem 5.6. *Let q be a layered pattern of length k. Then we have*

$$L(q) \geqslant (k-1)^2.$$

Equivalently, $L(q) \geqslant L(12\cdots k)$.

In order to prove Theorem 5.6, we need the following powerful Lemma, due to Backelin, West, and Xin.

Lemma 5.7 (*Backelin et al.* [2]). *Let $r < k$, and let v be any pattern of length $k - r$ taken on the set $\{r+1, r+2, \ldots, k\}$. Then for all positive integers n, we have*

$$S_n(12 \cdots rv) = S_n(r(r-1) \cdots 21v).$$

Now we are in position to prove Theorem 5.6.

Proof of Theorem 5.6. Induction on k. If q has only one layer, then q is the decreasing pattern, and the statement is obvious. Now assume q has at least two layers, and that we know the statement for all layered patterns of length $k-1$. As q is layered, it is of the form $r(r-1)\cdots 21v$ for some r, and some layered pattern v. Therefore, Lemma 5.7 applies, and we have $S_n(q) = S_n(12\cdots rv)$. If this last pattern is denoted by q^*, then we obviously also have $L(q) = L(q^*)$. We further denote by q^{*-} the pattern obtained from q^* by removing its first entry. Note that q^{*-} is still a layered pattern, just its first several layers may have length 1.

Case 1: Assume first that $r > 1$. Then note that q^{*-} starts with its smallest entry. Therefore, Theorem 5.4 applies, and by the induction hypothesis we have

$$\begin{aligned} L(q) = L(q^*) &= 1 + L(q^{*-}) + 2\sqrt{L(q^{*-})} \geqslant 1 + (k-2)^2 + 2(k-2) \\ &= (k-1)^2, \end{aligned}$$

which was to be proved.

Case 2: Now assume that $r = 1$. Then q is a layered pattern that starts with a layer of length 1. Therefore, instead of applying Theorem 5.4, we need to, and almost always can, apply Theorem 5.5 for the pattern q^{*-}. Indeed, q^{*-} is a layered pattern, and as such, is indecomposable, except when it has only one layer, that is, it is the decreasing permutation.

Subcase 2a: First look at the case when q^{*-} has more than one layers. That implies that q^{*-} is indecomposable. Therefore, we can apply Theorem 5.5 to get

$$L(q) \geqslant 1 + L(q^{*-}) + 2\sqrt{L(q^{*-})} \geqslant 1 + (k-2)^2 + 2(k-2) = (k-1)^2.$$

Subcase 2b: Finally, if q^{*-} has only one layer, then by the definition of layered patterns, q^{*-} must be the decreasing pattern. As we are in the case when $r = 1$, we simply have $q = 1k(k-1)\cdots 2$. Then we have

$$S_n(q) = S_n(k-1\cdots 21k) = S_n(12\cdots k),$$

where the first equality follows by taking reverse complements, and the second one is a special case of Lemma 5.7. Indeed, simply set $r = k-1$ in that lemma, and let v be the one-element pattern.

As we have covered all possible cases, the proof is complete. □

We point out that while Case 1 could have been treated the same way as Subcase 2a, that would have been less elucidating. Indeed, in Case 1, we prove an *equality*, while in Subcase 2a, we only prove an *inequality*. This lends some further support to the conjecture, supported by numerical evidence, that among all layered patterns q of length k, the one for which $S_n(q)$ is maximal for large n is $q = 1325476\cdots$. As this pattern has as many non-singleton layers as possible (without being equivalent to the monotone pattern), for this pattern our inductive proof will go to Subcase 2a as many times as possible.

Here is another way in which our results start explaining why certain patterns are easier to avoid than others. We formulate our observations in the following corollary.

Corollary 5.8. *Let q_1 and q_2 be patterns so that $L(q_1) \leqslant L(q_2)$. Let q_i' be the pattern obtained from q_i by prepending q_i by a* 1. *Furthermore, let q_1 start with the entry* 1, *and let q_2 be indecomposable. Then we have*

$$L(q_1') = 1 + L(q_1) + 2\sqrt{L(q_1)} \leqslant 1 + L(q_2) + 2\sqrt{L(q_2)} \leqslant L(q_2').$$

For instance, if we set $q_1 = 123$ and $q_2 = 213$, we get the well-known statement weakly comparing the limits of the Stanley–Wilf sequences of 1234 and 1324, first proved in [6].

6. Further directions

Our results raise two interesting kinds of questions. We have seen that the limit of a Stanley–Wilf sequence is not simply not always an integer, but also not always rational. Is it always an *algebraic* number? If yes, can its degree be arbitrarily high? Can it be more than two? Is it always an *algebraic integer*, that is, the root of a *monic* polynomial with integer coefficients? The results so far leave that possibility open.

The second question is related to the size of the limit of $\sqrt[n]{S_n(q)}$ if q is of length k. The largest value that this limit is known to take is $(k-1)^2$, attained by the monotonic pattern. Before present paper, the smallest known value, in terms of k, for this limit was $(k-1)^2 - 1 = 8$, attained by $q = 1342$. As Theorem 5.2 shows, the value $(k-4+\sqrt{8})^2$ is also possible. As k goes to infinity, the *difference* of the assumed maximum $(k-1)^2$ and this value also goes to infinity, while their ratio goes to 1. Is it possible to find a series of patterns q_k so that this ratio does not converge to 1? We point out that it follows from a result of P. Valtr (published in [7]) that for any pattern q of length k, we have $\lim_{n\to\infty} \sqrt[n]{S_n(q)} \geqslant e^{-3}k^2$, so the mentioned ratio cannot be more then e^3.

Finally, now that the Stanley–Wilf conjecture has been proved, and we know that the limit of a Stanley–Wilf sequence always exists, we can ask what the largest possible value of this limit is, in terms of k. In [1], Arratia conjectured that this limit is at most $(k-1)^2$, and, following the footsteps of Erdős, he offered 100 dollars for a proof or disproof of the conjecture $S_n(q) \leqslant (k-1)^{2n}$, for all n and q. Our results provide some additional support for this conjecture as they show that there is a wide array of patterns q for which $\sqrt{L(q)}$ grows by one when q is prepended by the entry 1. In fact, numerical evidence suggests that even the following stronger version of Arratia's conjecture could be true.

Conjecture 6.1. *Let q be a pattern of length k. Then $L(q) \leqslant (k-1)^2$, where equality holds if and only if q is layered, or the reverse of q is layered.*

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