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Invariants, alignment and the pattern of fermion masses and mixing

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ABSTRACT

We show that the main features of the pattern of fermion masses and mixing can be expressed in terms of simple relations among weak-basis invariants. In the quark sector, we identify the weak-basis invariants which signal the observed alignment of the up and down quark mass matrices in flavour space. In the lepton sector, we indicate how a set of conditions on weak-basis invariants can lead to the observed pattern of leptonic mixing, including the recent measurement of U_{e3} by the Daya Bay Collaboration. We also show the usefulness of these invariants in the study of specific ansätze for the flavour structure of fermion mass matrices.

1. Introduction

In the past few years there has been a remarkable progress in the determination of fermion masses and mixing [1], involving advances both in theory and experiment. In the quark sector, input from experiment includes the knowledge of $|V_{us}|$, $|V_{cb}|$, $|V_{ub}/V_{cb}|$, $|V_{td}|$, $|V_{ts}|$, together with the measurement of the rephasing invariant angles β and γ. In the framework of the Standard Model (SM), these results constrain the location of the vertex of the unitarity triangle to a small region. The measurement of the angle γ is especially important since it provides clear evidence that the Cabibbo–Kobayashi–Maskawa (CKM) matrix [2] is complex, even if one allows for the presence of New Physics beyond the SM [3]. In the leptonic sector, non-vanishing neutrino masses and mixing have been established [4]. However, there are still important open questions like the nature of the neutrino mass spectrum (normal hierarchy, inverted hierarchy or quasi degenerate), the determination of whether neutrinos are Majorana or Dirac particles, as well as the search for leptonic CP violation. In this respect, the recent evidence in favour of a non-vanishing value of U_{e3} provides the hope of discovering leptonic CP violation in neutrino oscillations. It is well known that having a non-vanishing U_{e3} is a necessary requirement in order to have leptonic Dirac-type CP violation, which is detectable in neutrino oscillations.

In spite of these developments, one does not have yet a standard theory of flavour. One may adopt a bottom-up approach and try to discover a symmetry principle from the observed pattern of fermion masses and mixing. One of the difficulties that one encounters in following this approach, stems from the fact that there is a large redundancy in the Yukawa couplings Y_u, Y_d which generate the quark masses M_u, M_d. One can make weak-basis (WB) transformations which change M_u, M_d, but do not alter their physical content. The above considerations also apply to flavour models which postulate the existence of so-called "texture-zeros". It is clear that these zeros only exist in a particular WB.

The above redundancy in Yukawa couplings and fermion mass matrices motivates the use of WB invariants, i.e. functions of quark masses which do not change when one preforms a WB transformation. These WB invariants are very useful in the analysis of CP violation [5], where they have been derived from first principles [6] and have been applied to both the quark [7] and lepton [8] sectors, including leptogenesis, as well as to the Higgs sector [9].

In this Letter we show that the main features of the pattern of fermion masses and mixing can be expressed in terms of simple relations involving only WB invariants. We introduce the concept of "alignment", which can be understood in the following way. For definiteness, let us consider the quark sector, where small flavour mixing is indicated by experiment. Small mixing implies that there is a WB where both M_u and M_d are close to the diagonal form. However, experiment shows more than that, it tells us that the quark mass matrices are aligned in flavour space, meaning that there is a basis where $M_d = diag(m_d, m_s, m_b)$ and M_u is close to $diag(m_u, m_c, m_t)$ and not to $diag(m_t, m_c, m_u)$, for example. Obviously, only the relative ordering of the eigenvalues of M_u, M_d is physically meaningful, since by making a WB transformation, one can change simultaneously the eigenvalue ordering in M_u, M_d.

At this point, it is worth recalling that in the context of the SM, the Yukawa couplings leading to M_u and M_d are entirely independent, there is no "dialogue" between Y_u and Y_d. Therefore, in the

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context of the SM, or its minimal supersymmetric extension, alignment is in no way more "natural" than misalignment. Quite on the contrary, if one considers the manifold of matrices M_u, M_d leading to "small mixing" as previously defined, the probability of having alignment is only 1/6.

In this Letter we will show how WB invariants can distinguish not only between small mixing and large mixing, but also between alignment and misalignment.

The Letter is organized as follows. In the next section, we consider the quark sector, where we illustrate the usefulness of WB invariants for two and three generations. In Section 3, we apply WB invariants to the study of some ansätze for quark mass matrices, while in Section 4 we briefly study the lepton sector, showing in particular how the observed pattern of leptonic mixing can be obtained through a set of conditions on WB invariants. Our conclusions are contained in Section 5.

2. The quark sector

2.1. Two quark generations

For simplicity and in order to explain the concept of alignment and its connection to weak-basis invariants, we consider first the case of two generations. The three important features of quark masses and mixing are:

(i) Hierarchical quark masses,
(ii) Small mixing,
(iii) Alignment.

We consider separately each one of these features, emphasizing that they are logically independent, at least in the context of the SM. We shall identify how each one of these features can be expressed in terms of weak-basis invariants.

(i) Hierarchical masses.

For two generations, the fact that quark masses are hierarchical can be expressed in terms of invariants as,

$$r_1 \equiv \frac{Det[H]}{(\frac{1}{2}\,Tr[H])^2} \ll 1 \tag{2.1}$$

where $H \equiv MM^\dagger$ denotes either H_d or H_u. The case of exact degeneracy corresponds to $r = 1$.

(ii) Small mixing.

It can be readily verified that the following relation holds

$$Tr\big([H_u, H_d]^2\big) = -\frac{1}{2}\big(\Delta_{12}^d\big)^2\big(\Delta_{12}^u\big)^2 \sin^2(2\theta) \tag{2.2}$$

where $\Delta_{12}^d = m_d^2 - m_s^2$, $\Delta_{12}^u = m_c^2 - m_u^2$ and θ denotes the Cabibbo angle. Let us consider the following invariant ratio

$$r_2 \equiv \frac{|\,Tr([H_u, H_d]^2)|}{\frac{1}{2}(Tr[H_d])^2(Tr[H_u])^2} \tag{2.3}$$

Assuming that quark masses are hierarchical, which can be guaranteed through the invariant condition of Eq. (2.1), it is clear from Eq. (2.2), that

$$r_2 \approx \sin^2(2\theta) \tag{2.4}$$

Therefore small mixing can be achieved through the invariant condition

$$r_2 \ll 1 \tag{2.5}$$

Maximal mixing corresponds to $\theta = 45^\circ$, i.e.

$$r_2 = 1 \tag{2.6}$$

(iii) Alignment.

Small mixing means that there is a weak-basis where both H_d and H_u are close to the diagonal form. As mentioned before, this is not sufficient to have "alignment", since it does not guarantee the same "ordering" in both H_d and H_u. Alignment means, of course, that in a WB where H_d is close to $diag(m_d^2, m_s^2)$, H_u is close to $diag(m_u^2, m_c^2)$ and not $diag(m_c^2, m_u^2)$. As previously emphasized, one can change the ordering simultaneously in the up and down sectors through a WB transformation. Only the relative ordering in the up and down quark sectors is physically meaningful. In the case of two generations, assuming hierarchical quark masses and small mixing, one has alignment if for the following invariant

$$\begin{aligned} I_1 &\equiv \frac{Tr[H_u]\,Tr[H_d] - Tr[H_u H_d]}{Tr[H_u]\,Tr[H_d]} \\ &= \sin^2(\theta) + \left(\frac{(m_d/m_s)^2}{(1+(m_d/m_s)^2)} + \frac{(m_u/m_c)^2}{(1+(m_u/m_c)^2)}\right. \\ &\quad \left. - \frac{2(m_d/m_s)^2(m_u/m_c)^2}{(1+(m_d/m_s)^2)(1+(m_u/m_c)^2)}\right)\cos(2\theta) \end{aligned} \tag{2.7}$$

the following condition is satisfied:

$$I_1 \ll 1 \tag{2.8}$$

On the contrary, assuming again hierarchical quark masses and small mixing, misalignment implies:

$$I_1 \approx 1 \tag{2.9}$$

2.2. Three quark generations

2.2.1. Hierarchy of quark masses

The hierarchy of quark masses in both the up and down quark sectors, namely

$$m_1^2 \ll m_2^2 \ll m_3^2 \tag{2.10}$$

can be translated into invariant conditions. We introduce the Hermitian quark mass matrices $H \equiv MM^\dagger$ and the corresponding invariants $\det(H)$, $Tr[H]$ together with the third invariant $\chi[H]$ which stands for $\chi[H] \equiv m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2$. Note that for a Hermitian 3×3 matrix H, one has

$$\chi[H] = \frac{1}{2}\Big(\big(Tr[H]\big)^2 - Tr\big[H^2\big]\Big) \tag{2.11}$$

The following invariant condition

$$R_1 \equiv \frac{\chi[H]}{Tr[H]^2} \ll 1 \tag{2.12}$$

implies that one of the eigenvalues of H is much larger that the other two. Finally, it can be readily verified that the condition

$$R_2 \equiv \frac{Tr[H]\,Det[H]}{(\chi[H])^2} \ll 1 \tag{2.13}$$

together with Eq. (2.12) implies that of the two smaller eigenvalues, one of them is much smaller than the other one, i.e. $m_1^2 \ll m_2^2$.

2.2.2. Invariants and the pattern of mixing

Previously [10], invariants were used to study specific ansätze where the quark mass matrices were written in a Hermitian basis. Here, we consider WB invariants which can be applied in an arbitrary basis, not necessarily Hermitian.

It is convenient to introduce the following dimensionless matrices with unit trace, $Tr[h_{u,d}] = 1$:

$$h_u = \frac{H_u}{Tr[H_u]}; \qquad h_d = \frac{H_d}{Tr[H_d]} \tag{2.14}$$

and their difference:

$$A \equiv h_d - h_u \tag{2.15}$$

By construction, one has $Tr[A] = 0$, which in turn implies

$$\chi[A] = -\frac{1}{2} Tr\big[A^2\big] \tag{2.16}$$

There is a relation between $\chi(A)$ and I_1, defined in Eq. (2.7). Indeed, from Eqs. (2.7), (2.14), one obtains

$$I_1 = 1 - Tr[h_u h_d] \tag{2.17}$$

while Eqs. (2.15), (2.16) lead to

$$\chi[A] = Tr[h_u h_d] - \frac{1}{2} Tr\big[h_u^2\big] - \frac{1}{2} Tr\big[h_d^2\big] \tag{2.18}$$

From Eqs. (2.17), (2.18), one therefore gets

$$\chi[A] = 1 - I_1 - \frac{1}{2} Tr\big[h_u^2\big] - \frac{1}{2} Tr\big[h_d^2\big] \tag{2.19}$$

Assuming hierarchy of the quark masses, which can be implemented through the invariants of Eqs. (2.12), (2.13), one obtains

$$\begin{aligned} Tr\big[h_d^2\big] &= 1 - 2\left(\frac{m_s}{m_b}\right)^2 + O\left(\left(\frac{m_s}{m_b}\right)^4\right) \\ Tr\big[h_u^2\big] &= 1 - 2\left(\frac{m_c}{m_t}\right)^2 + O\left(\left(\frac{m_c}{m_t}\right)^4\right) \end{aligned} \tag{2.20}$$

On the other hand, an explicit evaluation of I_1 for three generations in terms of $|V_{ij}|$ and quark mass ratios gives

$$\begin{aligned} I_1 = {} & |V_{23}|^2 + |V_{13}|^2 + \left(\frac{m_s}{m_b}\right)^2 + \left(\frac{m_c}{m_t}\right)^2 \\ & + O\left(\left(\frac{m_s}{m_b}\right)^4\right) \end{aligned} \tag{2.21}$$

Using Eqs. (2.19), (2.20), (2.21), one finally gets:

$$\big|\chi[A]\big| = |V_{23}|^2 + |V_{13}|^2 + O\left(\left(\frac{m_s}{m_b}\right)^4\right) \tag{2.22}$$

The usefulness of $\chi[A]$ is clear from Eq. (2.22): it gives to an excellent approximation the value of $|V_{23}|^2 + |V_{13}|^2$.

At this stage, it is worth recalling that one of the main features of quark mixing is the fact that the 3rd generation almost decouples from the other two. The deviation of exact decoupling is given by the size of $|V_{23}|^2 + |V_{13}|^2$. The experimental measurement of $|V_{23}|$ and $|V_{13}|$ shows that:

$$\big\{|V_{23}|^2 + |V_{13}|^2\big\}^{\exp} = O\left(\left(\frac{m_s}{m_b}\right)^2\right) \tag{2.23}$$

This input from experiment, can be written in terms of a simple relation among WB invariants

$$\chi[A] = O\left(\frac{\chi[H_d]}{(Tr[H_d])^2}\right) = O\big(\chi[h_d]\big) \tag{2.24}$$

It is worth emphasizing that, for three generations, $\chi[A]$ is also a measure of the alignment of the down and up quark mass matrices. Working in a WB where the up quarks are diagonal, one can, without loss of generality, order the up quarks in such a way that $H_u = diag(m_u^2, m_c^2, m_t^2)$. In the context of small mixing, alignment means that, in the above basis, H_d is close to $diag(m_d^2, m_s^2, m_b^2)$. In this case, $\chi[A]$ is small. In fact, if we take the limit $m_t \to \infty$, $m_b \to \infty$, one has $h_u = diag(0,0,1)$, $h_d = diag(0,0,1)$ and A vanishes, so $\chi[A] = 0$. On the other hand, if there is small mixing, but no alignment, in the WB where $H_u = diag(m_u^2, m_c^2, m_t^2)$, one may have that H_d is close to $diag(m_b^2, m_s^2, m_d^2)$. In this case, $\chi[A]$ is large. Indeed in the limit $m_t \to \infty$, $m_b \to \infty$, one has, for this case, $h_u = diag(0,0,1)$ but $h_d = diag(1,0,0)$, which leads to $|\chi[A]| = 1$, signalling total misalignment.

Next, we address the question of how to use invariants to constrain separately $|V_{23}|^2$ and $|V_{13}|^2$. This is a more difficult task, involving more complicated invariants, as it was to be expected. In order to constrain $|V_{13}|$, let us consider the following WB invariant:

$$I_2 \equiv 1 - \frac{Tr[H_u]Tr[H_u H_d] - Tr[H_u^2 H_d]}{\chi[H_u]Tr[H_d]} \tag{2.25}$$

This invariant can be readily calculated and one obtains in the chiral limit, i.e. when $m_u, m_d = 0$:

$$I_2 = \frac{|V_{13}|^2 + \frac{m_s^2}{m_b^2}|V_{12}|^2}{1 + \frac{m_s^2}{m_b^2}} \tag{2.26}$$

It is clear from Eq. (2.26) that if we constrain I_2 to be of order λ^6, λ denoting the Cabibbo angle, then $|V_{13}|$ is at most of order λ^3. It can be shown that this conclusion holds when one does not assume the chiral limit. Indeed, an exact calculation gives:

$$\begin{aligned} I_2 = {} & \frac{1}{[1 + (\frac{m_s}{m_b})^2 + (\frac{m_d}{m_b})^2][1 + (\frac{m_u}{m_c})^2 + (\frac{m_u}{m_t})^2]} \\ & \cdot \left(|V_{13}|^2 + \left(\frac{m_u}{m_c}\right)^2 |V_{23}|^2 + \left(\frac{m_u}{m_t}\right)^2 |V_{33}|^2 \right. \\ & + \left(\frac{m_s}{m_b}\right)^2 \left[|V_{12}|^2 + \left(\frac{m_u}{m_c}\right)^2 |V_{22}|^2 + \left(\frac{m_u}{m_t}\right)^2 |V_{32}|^2\right] \\ & \left. + \left(\frac{m_d}{m_b}\right)^2 \left[|V_{11}|^2 + \left(\frac{m_u}{m_c}\right)^2 |V_{21}|^2 + \left(\frac{m_u}{m_t}\right)^2 |V_{31}|^2\right]\right) \end{aligned} \tag{2.27}$$

From Eq. (2.27), and given the quark mass hierarchy, one concludes that putting $I_2 \approx \lambda^6$ constrains $|V_{13}|$ to be at most of order λ^3. Then, from Eq. (2.22), it follows that setting $\chi[A] \approx \lambda^4$ constrains $|V_{23}|$ to be of order λ^2, as indicated by experiment. We have thus shown how to fix separately $|V_{23}|$ and $|V_{13}|$ through WB invariants.

In order to constrain $|V_{12}|$, it is convenient to use WB invariants involving $H_{u,d}^{-1}$. Let us define

$$\widehat{A} = \widehat{h}_d - \widehat{h}_u \tag{2.28}$$

where

$$\widehat{h}_u = \frac{H_u^{-1}}{Tr[H_u^{-1}]}; \qquad \widehat{h}_d = \frac{H_d^{-1}}{Tr[H_d^{-1}]} \tag{2.29}$$

We have assumed that none of the quark masses vanish, as indicated by experiment and theory. In the weak-basis where the up quark mass matrix is diagonal, we have

$$\begin{aligned} \widehat{h}_u &= \frac{1}{(1 + \frac{m_u^2}{m_c^2} + \frac{m_u^2}{m_t^2})} diag\left(1, \frac{m_u^2}{m_c^2}, \frac{m_u^2}{m_t^2}\right) \\ \widehat{h}_d &= \frac{1}{(1 + \frac{m_d^2}{m_s^2} + \frac{m_d^2}{m_b^2})} V \cdot diag\left(1, \frac{m_d^2}{m_s^2}, \frac{m_d^2}{m_b^2}\right) \cdot V^\dagger \end{aligned} \tag{2.30}$$

Note the eigenvalues of $\widehat{h}_{u,d}$, denoted by $\widehat{\mu}_i$, satisfy an inverted hierarchy: $\widehat{\mu}_1 \gg \widehat{\mu}_2 \gg \widehat{\mu}_3$. We evaluate now $\chi[\widehat{A}]$, obtaining:

$$\left|\chi(\widehat{A})\right| = |V_{12}|^2 + |V_{13}|^2 - 2\left(\frac{m_d}{m_s}\right)^2 |V_{12}|^2 + O\left(\lambda^8\right) \tag{2.31}$$

If one constrains $\chi[\widehat{A}]$ to be of order λ^2, one necessarily has $|V_{12}| \cong \lambda$, taking into account that $|V_{13}|^2$ was already constrained to be of order λ^6.

In order to complete the determination of V^{CKM} through WB invariants, we have to address the question of CP violation. It has been shown [6] from first principles that the vanishing of the following WB invariant is a necessary condition for CP invariance in the SM, for an arbitrary number of generations:

$$I^{CP} \equiv Tr\left([H_u, H_d]^3\right) \tag{2.32}$$

For three generations $I^{CP} = 0$ is a necessary and sufficient condition for CP invariance. In terms of physical quantities, one has

$$I^{CP} = G\,\mathrm{Im}\left[V_{12}V_{23}V_{13}^*V_{22}^*\right] \tag{2.33}$$

where $G = 6i(m_b^2 - m_s^2)(m_b^2 - m_d^2)(m_s^2 - m_d^2)(m_t^2 - m_c^2)(m_t^2 - m_u^2)(m_c^2 - m_u^2)$. If we now set $\frac{1}{G}I^{CP}$ to be of order λ^6, and take into account that $\mathrm{Im}[V_{12}V_{23}V_{13}^*V_{22}^*] = |V_{12}||V_{23}||V_{13}||V_{22}|\sin(\phi)$, (with $\phi = \arg[V_{12}V_{23}V_{13}^*V_{22}^*]$), then we conclude that this constrains $\sin(\phi)$ to be of order one.

We have thus shown, without going through any process of diagonalization of the quark mass matrices, how a set of WB invariants can completely fix the pattern of mixing and strength of CP violation present in V^{CKM}. Indeed, by constraining $\chi[A]$ in Eq. (2.22) to be of order λ^4, I_2 in Eq. (2.27) to be of order λ^6 and $\frac{1}{G}I^{CP}$ in Eq. (2.33) of order λ^6, we guarantee that $|V_{23}|$ is of order λ^2, $|V_{13}|$ of order λ^3 and that the CP violation angle $\sin(\phi)$ is of order one.

3. Applying invariants to various ansätze

3.1. General remark

Next, we show the usefulness of the WB invariants introduced in the previous section and apply these invariants to some specific ansätze.

First, we derive some general results which apply to any flavour model where both the up and down Hermitian squared quark mass matrices with trace normalized to unity are equal to some fixed matrix Δ_o of order one plus some small perturbation denoted by $(\varepsilon A)_{u,d}$:

$$h_d = \Delta_o + \varepsilon_d A_d; \qquad h_u = \Delta_o + \varepsilon_u A_u \tag{3.34}$$

An example could be the case where Δ_o stands for the so-called democratic matrix, where all elements are equal, but our results apply to a broader class of flavour matrices. It is clear that Eq. (3.34) is a sufficient condition to obtain alignment, since it follows from Eq. (3.34), that $A = h_d - h_u = \varepsilon_d A_d - \varepsilon_u A_u$ is small. Let us consider now those ansätze where the following further conditions are satisfied

$$\begin{aligned}&|\varepsilon_u| \ll |\varepsilon_d|; \qquad Tr[A]_{u,d} = 0; \qquad \left(Tr[\Delta_o]\right)^2 = Tr\left[\Delta_o^2\right]\\ &Tr[A_{u,d}\Delta_o] \leqslant O(\varepsilon)_{u,d}\end{aligned} \tag{3.35}$$

The motivation for these conditions is clear: $|\varepsilon_u| \ll |\varepsilon_d|$ follows from the up and down quark hierarchies, $Tr[A]_{u,d} = 0$ from the normalization condition $Tr[h_{u,d}] = 1$, and since we are dealing with hierarchical mass matrices in whatever weak-basis, we impose just as in the democratic case $(Tr[\Delta_o])^2 = Tr[\Delta_o^2]$. Finally, $Tr[A_{u,d}\Delta_o] \leqslant O(\varepsilon)_{u,d}$ is satisfied in many "Fritzsch-like" models, in the Raimond–Roberts–Ross class of models, in USY models [11–13] and in many other "realistic" models.

It follows from Eq. (3.35), that to a good approximation $A \approx \varepsilon_d A_d$ and one obtains

$$\left|\chi(A)\right| = \varepsilon_d^2\left|\chi\left[A_d^2\right]\right| = \frac{1}{2}\varepsilon_d^2\, Tr\left[A_d^2\right] = O\left(\varepsilon_d^2\right) \tag{3.36}$$

Furthermore, using the conditions of Eq. (3.35) and computing $\chi(h_d)$, one gets

$$\begin{aligned}\left|\chi(h_d)\right| &= \frac{1}{2}\Big|\left(Tr[\Delta_o]\right)^2 - Tr\left[\Delta_o^2\right]\\ &\quad - \varepsilon_d\, Tr[A_d\Delta_o + \Delta_o A_d] - \varepsilon_d^2\, Tr\left[A_d^2\right]\Big|\\ &= O\left(\varepsilon_d^2\right)\end{aligned} \tag{3.37}$$

Therefore, one finds

$$\left|\chi(A)\right| = O\left(\left|\chi(h_d)\right|\right) = O\left(\left(\frac{m_s}{m_b}\right)^2\right) \tag{3.38}$$

Note that Eq. (3.38) coincides with Eq. (2.24) and therefore, for the whole class of ansätze satisfying the generic conditions of Eqs. (3.34), (3.35), one has the correct prediction

$$|V_{23}|^2 + |V_{13}|^2 = O\left(\left(\frac{m_s}{m_b}\right)^2\right) \tag{3.39}$$

This is a remarkable result. Using WB invariants, one can show that a whole class of ansätze for M_u, M_d satisfying the generic of Eq. (3.35), satisfies Eq. (3.39), which is one of the experimentally observed salient features of V^{CKM}.

3.2. The USY ansatz

We now apply our invariants to the hypothesis of Universality of Strength of Yukawa (USY) couplings [11–13], where all Yukawa couplings have equal moduli, the flavour dependence being all contained in their phases. For definiteness, let us consider the case where $M_{u,d}$ have the symmetric form:

$$M_{u,d} = c_{u,d}\begin{pmatrix} 1 & 1 & e^{i(\alpha-\beta)}\\ 1 & 1 & e^{i(\alpha)}\\ e^{i(\alpha-\beta)} & e^{i(\alpha)} & e^{i(\alpha)}\end{pmatrix}_{u,d} \tag{3.40}$$

Computing the invariants of the associated $h_{u,d}$,

$$\begin{aligned}Det[h] &= \frac{4^2}{9^3}\sin^4\left(\frac{\beta}{2}\right)\\ \chi[h] &= \frac{4}{9^2}\left[\sin^2\left(\frac{\alpha}{2}\right) + 4\sin^2\left(\frac{\beta}{2}\right) + \sin^2\left(\frac{\alpha-2\beta}{2}\right)\right.\\ &\quad \left. + 2\sin^2\left(\frac{\alpha-\beta}{2}\right)\right]\end{aligned} \tag{3.41}$$

we find that in leading order the parameters $\alpha_{u,d}$ and $\beta_{u,d}$ are small,

$$\begin{aligned}|\alpha_d| &= \frac{9}{2}\frac{m_s}{m_b}; \qquad |\beta_d| = 3\sqrt{3}\frac{\sqrt{m_d m_s}}{m_b}\\ |\alpha_u| &= \frac{9}{2}\frac{m_c}{m_t}; \qquad |\beta_u| = 3\sqrt{3}\frac{\sqrt{m_u m_c}}{m_t}\end{aligned} \tag{3.42}$$

and that the $h_{u,d}$ computed from Eq. (3.40) have the form:

$$h_{u,d} = \frac{\Delta}{3} + (\varepsilon A)_{u,d}; \quad \Delta = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}_{u,d} \tag{3.43}$$

where Δ is the democratic mass matrix and $(\varepsilon A)_{u,d}$ are matrices of order $\alpha_{u,d}$ and $\beta_{u,d}$. Up to second order in the largest parameter α, we find

$$(\varepsilon A)_{u,d} = \frac{1}{9}\begin{pmatrix} 0 & -i\beta & -2i\alpha-\alpha^2 \\ i\beta & 0 & -2i\alpha-\alpha^2+i\beta \\ 2i\alpha-\alpha^2 & 2i\alpha-\alpha^2-i\beta & 0 \end{pmatrix}_{u,d} \tag{3.44}$$

The form of $h_{u,d}$ corresponds to our general conditions in Eqs. (3.34), (3.35) and we find that $|\chi(A)|$ is indeed small. Thus, the USY scenario implies alignment. Furthermore, we find that in leading order

$$\left|\chi(A)\right| = \left|\chi(\varepsilon_d A_d)\right| \tag{3.45}$$

and that

$$\left|\chi(\varepsilon_d A_d)\right| = 2\left|\chi(h_d)\right| \tag{3.46}$$

Therefore, combining with Eq. (2.22), we obtain in leading order

$$|V_{23}|^2 + |V_{13}|^2 = 2\left(\frac{m_s}{m_b}\right)^2 \tag{3.47}$$

In addition, one obtains for the invariant I_2 associated with $|V_{13}|$ of Eq. (2.25) and in the limit $m_u = 0$ the exact result

$$I_2 = \frac{2}{9}\sin^2\left(\frac{\beta_d}{2}\right) \tag{3.48}$$

which combined with Eqs. (2.26), (3.42) leads in leading order to the following expression:

$$\frac{m_s^2}{m_b^2}|V_{12}|^2 + |V_{13}|^2 = \frac{3}{2}\frac{m_d m_s}{m_b^2} \tag{3.49}$$

The results, expressed in Eqs. (3.47), (3.49) are in agreement with the results which were obtained for this ansatz [13], where in leading order, it was found that $|V_{23}| = \sqrt{2}\frac{m_s}{m_b}$.

With respect to $|V_{12}|$ and the second invariant in Eq. (2.31), we compute $\widehat{h}_u = \frac{H_u^{-1}}{Tr[H_u^{-1}]}$, $\widehat{h}_d = \frac{H_d^{-1}}{Tr[H_d^{-1}]}$ and find in leading order

$$\begin{aligned}\widehat{h}_{u,d} &= \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)_{u,d}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\ &= \Delta_o + \widehat{\varepsilon}_{u,d}\widehat{A}_{u,d}\end{aligned} \tag{3.50}$$

which then leads to

$$\left|\chi(\widehat{A})\right| = \left|\chi(\widehat{\varepsilon}_d \widehat{A}_d)\right| = \frac{3}{4}\left(\frac{\beta_d}{\alpha_d}\right)^2 \tag{3.51}$$

Combining with Eqs. (3.42), (2.31) and with the results already obtained for Eqs. (3.47), (3.49), we find in leading order

$$|V_{12}|^2 = \frac{m_d}{m_s} \tag{3.52}$$

which corresponds exactly to what was known for this USY ansatz.

Finally, putting together Eqs. (3.47), (3.49), (3.52) one obtains the correct USY approximate expression for $|V_{13}| = \frac{1}{\sqrt{2}}\frac{\sqrt{m_d m_s}}{m_b}$.

3.3. Asymmetry in the NNI weak-basis

It has been shown [14], that starting with arbitrary quark mass matrices M_u°, M_d°, in the framework of the SM, it is possible to make a WB transformation such that M_u, M_d acquire the Nearest Neighbour Interaction (NNI) form:

$$M_u = c_u\begin{pmatrix} 0 & a_u & 0 \\ \widehat{a}_u & 0 & b_u \\ 0 & \widehat{b}_u & 1 \end{pmatrix}; \qquad M_d = c_d K\cdot\begin{pmatrix} 0 & a_d & 0 \\ \widehat{a}_d & 0 & b_d \\ 0 & \widehat{b}_d & 1 \end{pmatrix} \tag{3.53}$$

where all $a_{u,d}$, $\widehat{a}_{u,d}$, $b_{u,d}$, $\widehat{b}_{u,d}$, $c_{u,d}$ are real and the matrix $K = diag(1, e^{i\phi_1}, e^{i\phi_2})$. In the limit that $\widehat{a}_{u,d} = a_{u,d}$ and $\widehat{b}_{u,d} = b_{u,d}$, one obtains the Fritzsch ansatz [15], which has been eliminated by experiment, namely by the large value of the top quark and the observed value of $|V_{cb}|$.

In the following, we use our invariants to find out the minimal asymmetry which is required in M_u, M_d, when written in the NNI basis, in order to conform with experiment. Let us define the asymmetries

$$\varepsilon_u \equiv \frac{\widehat{b}_u - b_u}{\widehat{b}_u + b_u}; \qquad \varepsilon_d \equiv \frac{\widehat{b}_d - b_d}{\widehat{b}_d + b_d} \tag{3.54}$$

and the total asymmetry

$$\varepsilon = \sqrt{\varepsilon_u^2 + \varepsilon_d^2} \tag{3.55}$$

Note that alignment and hierarchy of the quark mass matrices are guaranteed in Eq. (3.53) by taking $(a,b)_{u,d}$, $(\widehat{a},\widehat{b})_{u,d}$ much smaller than 1. Computing the invariants associated to h_u and h_d as in Eq. (2.14), and taking into account the hierarchy of the quark mass matrices, one obtains in good approximation

$$|a|^2|\widehat{a}|^2 = \frac{|m_1 m_2|}{m_3^2}; \qquad |b|^2|\widehat{b}|^2 = \left(\frac{m_2}{m_3}\right)^2 \tag{3.56}$$

Then, combining Eqs. (3.54), (3.56) we obtain:

$$b_u^2 = \frac{m_c}{m_t}\left(\frac{1-\varepsilon_u}{1+\varepsilon_u}\right); \qquad b_d^2 = \frac{m_s}{m_b}\left(\frac{1-\varepsilon_d}{1+\varepsilon_d}\right) \tag{3.57}$$

Now, computing $\chi(A)$ as in Eq. (2.15) with h_u and h_d obtained from the NNI form in Eq. (3.53), and using Eq. (3.56), we get an expression which relates the experimental value for $|V_{cb}|$ and the $|\chi(A)|$ as in Eq. (2.22) in terms of the parameters of the NNI form. We find in leading order

$$b_d^2 - 2b_u b_d\cos(\phi) + b_u^2 - b_d^4 = |V_{cb}|^2 + 2\left(\frac{m_s}{m_b}\right)^2 \tag{3.58}$$

where $\phi = \phi_1 - \phi_2$ is a complex phase resulting from the diagonal matrix K in Eq. (3.53). This expression is obtained taking into account that $(a,\widehat{a}) = O(\frac{\sqrt{|m_1 m_2|}}{m_3})$ and that $(b,\widehat{b}) = O(\sqrt{|\frac{m_2}{m_3}|})$ as implied by Eqs. (3.56), (3.57).

From Eqs. (3.57), (3.58) we find that there is a connection between the required asymmetries ε_u, ε_d of the up and down quark sectors in order to conform to experiment. This connection can be understood as follows. Take the case when $\phi = 0$ and $\varepsilon_d = 0$, then from the second relation in Eq. (3.57) it follows that $b_d = \sqrt{\frac{m_s}{m_b}}$, but then the expression of Eq. (3.58) forces also $b_u = \sqrt{\frac{m_s}{m_b}}$ in leading order, and therefore, from the first relation in Eq. (3.57), one gets $\varepsilon_u \approx -1 + 2(\frac{m_c}{m_t})/(\frac{m_s}{m_b})$. Therefore, when the asymmetry in the down sector is small, the required asymmetry in the up sector is large, and vice versa. It can be readily verified that

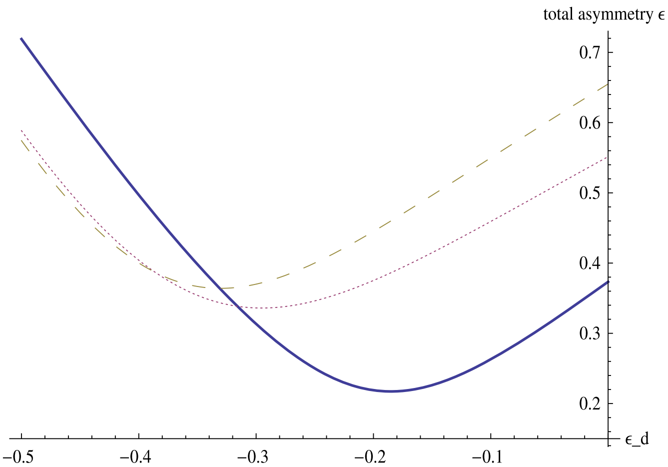

Fig. 1. Total required asymmetry ε as a function of the down quark mass matrix asymmetry ε_d. The full line is for the values of $m_s = 60$ MeV, $\phi = 0$, the tiny dashed line for values of $m_s = 100$ MeV, $\phi = 0$ and large dashed line for values of $m_s = 80$ MeV, $\phi = 0.35$. For all curves, we took the values for $m_b = 3.0$ GeV, $m_c = 680$ MeV, $m_t = 181$ GeV and $|V_{cb}| = 0.037$ at M_Z.

when $\phi \neq 0$, this result also holds. Indeed, by eliminating from Eqs. (3.57), (3.58) both b_u and b_d one finds ε_u as a function of ε_d and ϕ

$$\varepsilon_u = \varepsilon_u(\varepsilon_d, \phi) \tag{3.59}$$

and one then computes the total asymmetry ε in Eq. (3.55). This total asymmetry can thus be written as a function of ε_d and ϕ. One finds that it increases for all values of $\phi \neq 0$, and it has a minimum for a certain value of ε_d (and $\phi = 0$).

We have plotted the total asymmetry ε in Fig. 1 as a function of ε_d for typical values of $\frac{m_c}{m_t}$, $\frac{m_s}{m_b}$ and $|V_{cb}|$ at M_Z. As can be seen from the plot, the minimal required total asymmetry is about $\varepsilon = 0.2$, which indicates clearly that ansätze, written in the NNI basis, require quark mass matrices with a considerable amount of asymmetry in order to conform to experiment. This finding agrees with the result previously obtained [16] by explicitly diagonalizing the quark mass matrices written in the NNI basis.

4. Leptons

4.1. Lepton masses

The hierarchy of lepton masses may also be expressed in terms of invariants of H_l, H_ν. For the charged leptons one may use essentially the same invariants as the quarks. For the neutrinos, the invariant

$$R_1 \equiv \frac{4\chi[H_\nu]}{Tr[H_\nu]^2} \tag{4.60}$$

may distinguish normal hierarchy corresponding to $R_1 \ll 1$ from inverted hierarchy ($R_1 = 1$) and degeneracy ($R_1 = \frac{4}{3}$). The invariant

$$R_2 \equiv \frac{3\,Tr[H_\nu]\,Det[H_\nu]}{(\chi[H_\nu])^2} \tag{4.61}$$

may also be used to distinguish inverted hierarchy when R_2 is small from degeneracy when $R_2 = 1$. Furthermore, for normal hierarchy, it can distinguish the cases when one of the two smaller masses is much smaller then the other one, $R_2 \ll 1$, or the case when these two small masses are of the same order. In this case R_2 is of order one. Thus, we have

$$\begin{array}{llll} \text{Normal}_1 & \text{Normal}_2 & \text{Inverted} & \text{Degenerate} \\ R_1 \ll 1 & R_1 \ll 1 & R_1 = 1 & R_1 = \frac{4}{3} \\ R_2 \ll 1 & R_2 = O(1) & R_2 \ll 1 & R_2 = 1 \end{array} \tag{4.62}$$

4.2. Leptonic mixing and CP violation

Next, we show how the leptonic mixing and CP violation can be fixed by a set of WB invariants. For definiteness, let us consider the case of strong normal hierarchy of neutrino masses:

$$m_1^2 \ll m_2^2, m_3^2 \tag{4.63}$$

which in turn implies:

$$m_2^2 \approx \Delta m_{solar}^2; \qquad m_3^2 \approx \Delta m_{atm}^2 \tag{4.64}$$

Let us now consider the following invariant:

$$I_\nu \equiv 1 - \frac{Tr[H_l]\,Tr[H_l H_\nu^2] - Tr[H_l^2 H_\nu^2]}{\chi[H_l]\,Tr[H_\nu^2]} \tag{4.65}$$

In the limit $m_e/m_\tau = 0$, one obtains the exact result:

$$I_\nu = \frac{|V_{13}|^2 + (\frac{m_2}{m_3})^4 |V_{12}|^2 + (\frac{m_1}{m_3})^4 |V_{11}|^2}{1 + (\frac{m_2}{m_3})^4 + (\frac{m_1}{m_3})^4} \tag{4.66}$$

Recently, the Daya Bay Collaboration [17] has made a measurement of $|V_{13}| \equiv |U_{e3}|$, leading to the result:

$$\sin^2(2\theta_{13}) = 0.092 \pm 0.016\ (\text{stat}) \pm 0.005\ (\text{syst}) \tag{4.67}$$

where, the standard parametrization has been used, with $|V_{13}| = \sin(\theta_{13})$. Given the measured value of $|V_{13}|$ and taking into account Eqs. (4.64), (4.66), it is clear that one fixes $|V_{13}|^2$ to an excellent approximation by constraining I_ν to satisfy the constraint:

$$I_\nu = |V_{13}|^2_{\text{exp}} \tag{4.68}$$

Let us now consider the invariant:

$$I'_\nu \equiv 1 - \frac{Tr[H_\nu]\,Tr[H_\nu H_l] - Tr[H_\nu^2 H_l]}{\chi[H_\nu]\,Tr[H_l]} \tag{4.69}$$

In the limit $m_1/m_3 = 0$, one obtains:

$$I'_\nu = \frac{|V_{31}|^2 + \frac{m_\mu^2}{m_\tau^2}|V_{21}|^2 + \frac{m_e^2}{m_\tau^2}|V_{11}|^2}{1 + \frac{m_\mu^2}{m_\tau^2} + \frac{m_e^2}{m_\tau^2}} \tag{4.70}$$

Given the experimental value of $|V_{31}|^2$, together with Eq. (4.70), it is clear that one fixes the values of $|V_{31}|^2$ to an excellent approximation by constraining I'_ν to satisfy:

$$I'_\nu = |V_{31}|^2_{\text{exp}} \tag{4.71}$$

So far, we have shown how one can obtain the correct value of $|V_{13}|^2$, $|V_{31}|^2$ by constraining WB invariants to satisfy the relations of Eqs. (4.68), (4.71). Note that this procedure can be applied to any flavour model and does not involve any diagonalization of lepton mass matrices. In order to constrain $|V_{23}|^2 + |V_{13}|^2$, one can use the leptonic equivalent of Eqs. (2.21), (2.22), which lead to:

$$\left|\chi[A_\nu]\right| = |V_{23}|^2 + |V_{13}|^2 + O\left(\left(\frac{m_2}{m_3}\right)^4\right) \tag{4.72}$$

It is clear that using Eq. (4.72) and the fact that $|V_{13}|^2$ was already fixed, one can then fix the value of $|V_{23}|^2$ by constraining $\chi[A_\nu]$. From the knowledge of $|V_{13}|$, $|V_{23}|$, $|V_{31}|$ one can fix the mixing angles $\sin(\theta_{13})$, $\sin(\theta_{23})$, $\sin(\theta_{12})$. In order to fix the strength of Dirac CP violation, one can use the equivalent of Eqs. (2.32), (2.33) for the leptonic sector.

We have thus shown how to obtain the observed pattern of leptonic mixing through a set of invariant conditions.

5. Conclusions

We pointed out that the use of weak-basis invariants can avoid the well known redundancy of free parameters in the flavour structure of mass matrices. These invariants are especially useful when one opts for a bottom-up approach to the study of the flavour structure of Yukawa couplings and fermion mass matrices. In particular, we have shown that the pattern of fermion mixing both in the quark and lepton sectors can be expressed in terms of relations only involving weak-basis invariants. We have also pointed out that the observed alignment of the up and down quark mass matrices in flavour space can also be guaranteed through a weak-basis invariant condition. It was emphasized that in the context of the SM, the above alignment in no way follows automatically from the Yukawa couplings structure, since Y_u and Y_d are independent. On the other hand, this alignment may arise naturally, e.g. in left–right symmetric theories or in $SO(10)$, where Y_u and Y_d may be approximately proportional to each other. Finally, we have included in our analysis the recent Daya Bay result which provided a measurement of U_{e3} [17]. This important result has had an impact on the construction of viable lepton flavour models [18].

In summary, WB invariants may play an important rôle in a systematic search for patterns of fermion mass matrices consistent with experiment and may thus help to uncover a possible flavour symmetry chosen by nature.

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