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On the 2-sum in rigidity matroids

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Dedicated to Tom Brylawski, who got great pleasure from small counterexamples.

ABSTRACT

We show that the graph 2-sum of two frameworks is the underlying framework for the 2-sum of the infinitesimal and generic rigidity matroids of the frameworks. However, we show that, unlike the cycle matroid of a graph, these rigidity matroids are not closed under 2-sum decomposition.

Given a graph $G = (V, E)$, we consider an embedding $\mathbf{p} : V \to \mathbb{R}^d$ of the vertices into d-dimensional Euclidean space. The graph G and the embedding $\mathbf{p}$ may be used to model a framework, $G(\mathbf{p})$ of rigid rods connected to one another via ball joints. The vertices correspond to the joints, and each edge is interpreted as a length constraint. If $G(\mathbf{p})$ is a framework, then for each edge $e = (v, w)$ there is a linear constraint

$$(\mathbf{p}(v) - \mathbf{p}(w)) \cdot \mathbf{p}'(v) = (\mathbf{p}(v) - \mathbf{p}(w)) \cdot \mathbf{p}'(w) \tag{1}$$

on the vertex velocities $\mathbf{p}'(v)$ which implies that the distance between v and w is infinitesimally unaltered. This linear system induces a matroid, $\mathfrak{R}(\mathbf{p})$, on E, in which a set of edges is independent if and only if the corresponding set of linear constraints is independent. We call $\mathfrak{R}(\mathbf{p})$ the *infinitesimal rigidity matroid* corresponding to $\mathbf{p}$. Since all constraints of the form (1) are satisfied by the elements of the $d(d+1)/2$-dimensional space of infinitesimal isometries, the rank of E is at most $d|V| - d(d+1)/2$. In the case $\text{rank}(E) = d|V| - d(d+1)/2$, we say that E is *infinitesimally rigid in dimension d*.

In the framework of Fig. 1, the infinitesimal vertex motion indicated by the arrows on the left extends to one of the *trivial* motions of the plane, in this case, an infinitesimal isometry induced by a rotation about vertex 0. The motion indicated on the right is non-trivial, so this framework is not infinitesimally rigid in dimension 2.

The linear constraints (1) may be encoded in $|E|$ by $d|V|$ matrix $R(\mathbf{p})$ whose kernel corresponds to the infinitesimal motions $\mathbf{p}'(v)$ which do not violate (1) and whose cokernel is the space of *prestresses*; see [5,2,1].

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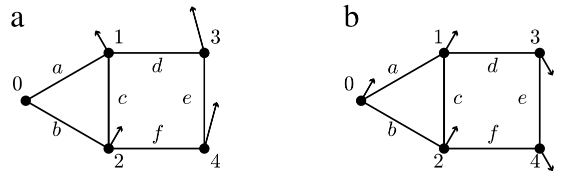

Fig. 1. A framework with a trivial infinitesimal motion, (a), and a non-trivial infinitesimal motion, (b).

For example, the rigidity matrix of the 2-dimensional framework pictured in Fig. 1 is given by

$$\begin{bmatrix} (\mathbf{p}(1)-\mathbf{p}(0)) & (\mathbf{p}(0)-\mathbf{p}(1)) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{p}(2)-\mathbf{p}(0)) & \mathbf{0} & (\mathbf{p}(0)-\mathbf{p}(2)) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{p}(2)-\mathbf{p}(1)) & (\mathbf{p}(1)-\mathbf{p}(2)) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{p}(3)-\mathbf{p}(1)) & \mathbf{0} & (\mathbf{p}(1)-\mathbf{p}(3)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{p}(4)-\mathbf{p}(3)) & (\mathbf{p}(3)-\mathbf{p}(4)) \\ \mathbf{0} & \mathbf{0} & (\mathbf{p}(4)-\mathbf{p}(2)) & \mathbf{0} & (\mathbf{p}(2)-\mathbf{p}(4)) \end{bmatrix}$$

whose rows correspond to the edges $a, \ldots, f$ and entries are 2-dimensional row vectors. Note that, using this vector notation, the rigidity matrix for any framework on G in any dimension will have this form.

The independence of the constraints depends on the embedding $\mathbf{p}$, so a given graph may have several different infinitesimal rigidity matroids $\mathfrak{R}(\mathbf{p})$ in dimension d. If, however, the embedding is *generic*, for example if the coordinates of the vertices are algebraically independent, then the matroid is determined by the dimension and the graph G alone, and we say that it is the *generic d-dimensional rigidity matroid of* G, $\mathfrak{R}_d(G)$.

Note that every embedding of V into $\mathbb{R}^1$ is generic and G is rigid in dimension 1 if and only if G is connected, so $\mathfrak{C}(G) = \mathfrak{R}_1(G)$, where $\mathfrak{C}(G)$ is well known as the *cycle matroid of* G, [6].

A partition $\{E_1, E_2\}$ of E is called a *k-separator* of a matroid $\mathfrak{M}$ on E if $|E_i| \geq k$ and

$$\text{rank}(E_1) + \text{rank}(E_2) \leq \text{rank}(E) + k - 1.$$

Tutte [10] calls $\mathfrak{M}$ *n-connected* if there is no k-separator for $k < n$. With this definition every matroid is 1-connected.

A matroid is 2-connected if there is no partition of E into two sets E_1 and E_2 such that $|E_i| \geq 1$ and $\text{rank}(E_1) + \text{rank}(E_2) \leq \text{rank}(E)$, i.e., if it is not the direct sum of its restrictions to the E_i's. It is clear that every matroid can be uniquely decomposed into a direct sum such that each of the summands is 2-connected. Note that many authors call a matroid *connected* if it is 2-connected in the Tutte sense. We choose to use Tutte's 2-connectivity, so that 2-connectivity of the graph G is equivalent to 2-connectivity of its cycle matroid $\mathfrak{C}(G)$.

It is well known, see for example [6] or [7], that a matroid is 2-connected if and only if for any partition of the ground set into two sets, there is a cycle C intersecting both of them. In fact an even stronger conclusion holds, namely a matroid is 2-connected if and only if any pair of its edges is contained in a cycle.

The 2-*sum*, $\mathfrak{M}_1 \bigoplus_{2/e} \mathfrak{M}_2$, of two matroids $\mathfrak{M}_1$ and $\mathfrak{M}_2$, both containing at least 3 elements and having exactly one element e in common, where e is neither dependent (a loop) or a bridge (a coloop) in either of the $\mathfrak{M}_i$, is a matroid on the union of the ground sets of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ excluding e and the cycles of $\mathfrak{M}_1 \bigoplus_{2/e} \mathfrak{M}_2$ consist of cycles of $\mathfrak{M}_i$ not containing e and of sets of the form $(C_1 \bigcup C_2) \setminus e$ where C_i is a cycle of $\mathfrak{M}_i$ containing e. A matroid is 3-connected if and only if it cannot be written as a 2-sum.

The 2-sum is also defined for 2-connected graphs, but here one cannot identify two edges without specifying which pairs of endpoints are to be identified, in other words, without specifying an orientation on the edges to be amalgamated; see Fig. 2. For 2-connected graphs, the cycle matroid of the 2-sum of two graphs is identical with the 2-sum of their cycle matroids. We would like to show that the same is true for the infinitesimal rigidity matroids.

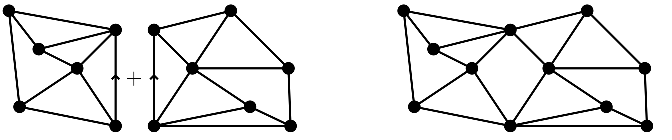

Fig. 2. The 2-sum of two r-cycles.

Theorem 1. *Suppose we have two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and edges $e_i = (x_i, y_i) \in E_i$, and a framework $\mathbf{p}$ in d-dimensional Euclidean space on the 2-sum $G = G_1 \bigoplus_{e_1,e_2} G_2$, so $\mathbf{p} : V_1 \cup V_2 \to \mathbb{R}^d$. Suppose also that $\mathbf{p}(V_1)$ and $\mathbf{p}(V_2)$ each contain at least d points in general position. If E_1 and E_2 are cycles in the infinitesimal rigidity matroid induced by $\mathbf{p}$, then so are the edges E of G, $E = (E_1 - \{e_1\}) \cup (E_2 - \{e_2\})$.*

Proof. If E_i is a cycle, then the cokernel of the rigidity matrix $R(\mathbf{p}_i)$ is generated by a vector ω_i, a prestress, all of whose entries are non-zero. So, for each vertex $v \in V_i$, we have

$$\sum_{(v,w)\in E_i} \omega_i(v, w)(\mathbf{p}(w) - \mathbf{p}(v)) = 0. \tag{2}$$

We may regard both ω_1 and ω_2 as prestresses on all of $G + e$, that is, G plus the edge of attachment $e = e_1 = e_2$, by giving the value zero to the unassigned edges. So both ω_1 and ω_2 are prestresses on $G + e$, as well as $\omega = \omega_1/\omega_1(e_1) - \omega_2/\omega_2(e_2)$. Since $\omega(e) = 0$, ω is actually a prestress on G which is non-zero on all edges of G. Therefore, the edges E of G will be a cycle if ω generates all the prestresses of G, that is, the space of prestresses is of rank 1.

Since $\mathbf{p}(V_i)$ contains at least d points in general position, its space of trivial motions is of dimension $d(d + 1)/2$, and

$$d|V_i| = \mathrm{rank}(R(E_i)) + m_i + d(d + 1)/2.$$

The number m_i defined by this equation is often referred to as the number of internal degrees of freedom. Since E_i is a cycle, $\mathrm{rank}(R(E_i)) = |E_i| - 1$, so

$$d|V_i| = |E_i| - 1 + m_i + d(d + 1)/2.$$

We need to show that $\mathrm{rank}(R(E)) = |E| - 1$.

We have $|V| = |V_1| + |V_2| - 2$, and $|E| = |E_1| + |E_2| - 2$, and we now want to compute the number m of internal degrees of freedom of G. Since e is dependent on both $E_1 - e$ and $E_2 - e$, the distance between its endpoints must be fixed infinitesimally in G. So, without loss of generality, we may work in the space of motions which fix both x and y, the endpoints of e. The trivial motions in this space have dimension $(d - 1)(d - 2)/2$. The space of infinitesimal motions which fix all vertices in V_2 has dimension $m_1 + (d - 1)(d - 2)/2$, and the space of infinitesimal motions which fix all vertices in V_1 has dimension $m_2 + (d - 1)(d - 2)/2$. Every infinitesimal motion of G which fixes x and y is a combination of these, so the dimension of infinitesimal motions of G is $m_1 + m_2 + (d - 1)(d - 2)$. Subtracting the $(d - 1)(d - 2)/2$ trivial motions fixing x and y we have that G has internal degree of freedom $m = m_1 + m_2 + (d - 1)(d - 2)/2$, hence $\mathrm{rank}(E) = |E| - 1$. □

Theorem 1 can be used to produce interesting examples of cycles in various rigidity matroids. A famous example is the so-called *double banana*, the 2-sum of two K_5's. K_5 is a cycle in the 3-dimensional generic rigidity matroid, so the double banana is a cycle as well. The double banana has just enough edges to be rigid in 3-space and none of its edges is wasted by over-bracing a proper subset of vertices. The edge removed in taking the 2-sum is in the closure of either banana, so in dimension 3 and higher dimensions, there are novel ways to waste edges.

Theorem 1 implies the following.

Corollary 1. *If the 2-sum of $\mathfrak{R}(G_1) \bigoplus_e \mathfrak{R}(G_2)$ is defined, so e is neither a loop nor a coloop in either of the matroids, then*

$$\mathfrak{R}(G_1) \bigoplus_e \mathfrak{R}(G_2) = \mathfrak{R}(G_1 \oplus_e G_2).$$

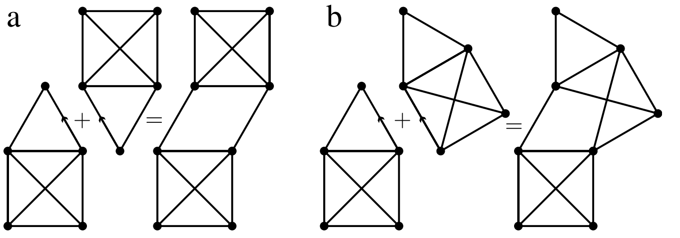

Fig. 3. Graph 2-sums only.

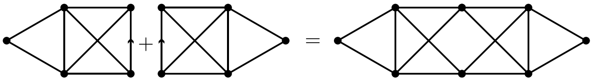

Fig. 4. Graph- and rigidity matroid 2-sum.

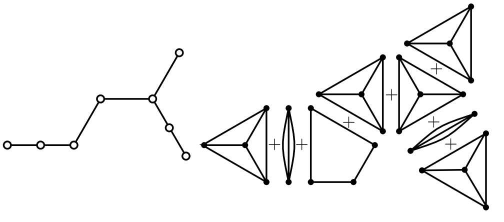

Fig. 5. A 2-sum tree.

Note that the graphic 2-sum may be defined when the rigidity matroid 2-sum is not. This is shown in Fig. 3 where (a) the 2-sum of the graphs is taken along coloops of the rigidity matroid, so the rank of the rigidity matroid of the 2-sum graph is too small to be the 2-sum of the corresponding matroids, or in (b) the 2-sum is taken along a coloop of the rigidity matroid of one of the summands. Here the rank of the rigidity matroid of the resulting framework is 13, as it should be, but the separator induced by the summands is actually a 1-separator. The rigidity matroid of the 2-sum graph in Fig. 4 is indeed the 2-sum of the rigidity matroids of the summands.

Theorem 1 allows us to build up frameworks with 2-connected rigidity matroids using merely the 2-sum of the underlying frameworks. A natural question to ask is whether this process can be reversed, as has been classically done for the cycle matroid.

Clearly the 2-sum of graphs is associative provided that the edges to be amalgamated are distinct, and so it is convenient to represent the result of a succession of 2-sums as a tree in which the nodes of the tree encode the graphs to be joined, and the edges of the tree encode the (oriented) edges of the graphs to be amalgamated; see Fig. 5. If all the graphs corresponding to the nodes in the amalgamation tree are 2-connected, then the graph which is the result of the joins encoded by the tree is also 2-connected. We consider the case when each of the graphs corresponding to the nodes in the tree is a 3-*block*, that is, either 3-connected, a simple cycle with at least 3 edges, or a k-*link* which is graph consisting solely of two vertices and $k \geq 3$ parallel edges. If all the graphs corresponding to the nodes are 3-blocks with the restriction that no adjacent nodes correspond to cycles, and no adjacent nodes correspond to k-links, then the resulting 2-sum tree is called a 3-*block tree*; see Fig. 5. Tutte proved the following deep theorem characterizing finite 2-connected graphs; see [9,4] (Fig. 6).

Theorem 2 ([9]). *A 2-connected graph G is uniquely encoded by its 3-block tree.*

This result has been generalized to matroids. Every 2-connected matroid has a unique encoding as a 3-block tree in which the 3-blocks are 3-connected matroids, bonds (matroids in which every 2-element subset is a cycle) and polygons (matroids consisting of a single cycle), such that no two bonds are adjacent, nor two polygons; see [3], Theorem 18.

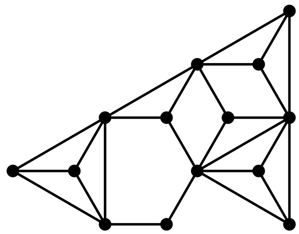

Fig. 6. The 3-block tree in Fig. 5 encodes this graph.

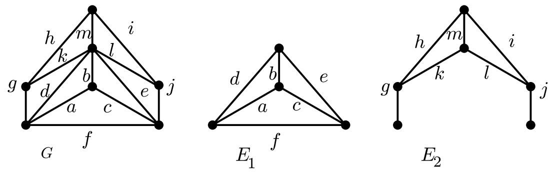

Fig. 7. A 2-separator of $\mathfrak{R}_2(G)$.

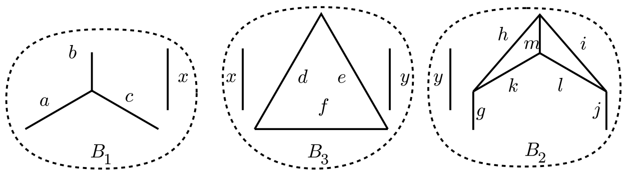

Fig. 8. 3-block decomposition of $\mathfrak{R}(G)$, with 2-sums along x and y.

Note that in forming the 3-block decomposition of a matroid, each cycle must be considered indecomposable since any non-trivial partition of the edge set forms a matroid 2-separation.

Given a graph G, the 3-block decomposition of $\mathfrak{C}(G)$ only involves graphic matroids and the class of graphic matroids is closed under 2-sum decomposition. In fact, given a graphic matroid $\mathfrak{C}$, one can construct all the graphs G with $\mathfrak{C}(G) = \mathfrak{C}$ from the 3-block tree of $\mathfrak{C}$ by assigning orientations to the amalgamated edges, and ordering the edges around each cycle [11,8].

By contrast, and despite Corollary 1, the class of rigidity matroids does not behave as well as the class of graphs under 2-sum decomposition. The nodes of the tree in the Cunningham and Edmonds generalization of Theorem 2 need not correspond to rigidity matroids, nor even graphic matroids. The following example gives a rigidity matroid $\mathfrak{R}(G)$ illustrating this phenomenon. Here the 3-block decomposition of $\mathfrak{R}(G)$ leads to a 2-separator $\{E_1, E_2\}$, which is not a 2-separator of the underlying graph, since the edge sets intersect in more than two vertices; see Fig. 7.

More importantly, the 3-block decomposition of $\mathfrak{R}_2(G)$ may involve 3-blocks which are not the infinitesimal rigidity matroid of any framework. See Fig. 8, in which the matroid $\mathfrak{R}(G)$ from Fig. 7 has been decomposed into its 3-blocks, B_i. The matroids B_1 and B_2 are both cycles and the matroid B_3 is the 3-connected matroid in which every 3-element subset is a basis. The matroid B_3 is not the infinitesimal rigidity matroid of any framework in any dimension, since the graph would have to have at least 4 vertices, so there would have to be at least one vertex of valence 2, and the edges at that vertex could not be in the closure of the other three.

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