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Weakly hereditary regular closure operators

Maria Manuel Clementino*
Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal

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Abstract

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We present characterizations of subcategories inducing weakly hereditary regular closure operators. These characterizations are applicable, in particular, to the category $\mathcal{T}op$ of topological spaces and continuous maps and to Abelian categories. Weakly hereditary regular closure operators, in $\mathcal{T}op$ and in Abelian categories satisfying some conditions, are shown to correspond to disconnectednesses and torsion-free subcategories, respectively.

Keywords: Factorization system, closure operator, ($\mathscr{E}$-)reflective subcategory, regular closure operator, disconnectedness, torsion-free subcategory.

AMS (MOS) Subj. Class.: 18A32, 18A40, 18B30, 18E40, 54B30.

Introduction

There is a natural way of associating a closure operator $C_{\mathscr{A}}$ to a full subcategory $\mathscr{A}$ of a category $\mathscr{X}$: call a morphism *$\mathscr{A}$-regular* if it is the equalizer of a pair of morphisms with codomain in $\mathscr{A}$, and define—under mild conditions on $\mathscr{A}$ and $\mathscr{X}$—the *$\mathscr{A}$-closure* $[m]_{\mathscr{A}}:[M]_{\mathscr{A}}\to X$ of a subobject $m:M\to X$ in $\mathscr{X}$ to be the least $\mathscr{A}$-regular subobject of X containing m. A closure operator on $\mathscr{X}$ obtained this way

Correspondence to: Professor M.M. Clementino, Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal.
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is called *regular*. It was introduced first for $\mathscr{X}$ the category $\mathscr{T}op$ of topological spaces in [13], and then studied in a general setting in [7, 9].

The regular closure operator $C_{\mathscr{A}}$ is idempotent (i.e., $[[m]_{\mathscr{A}}]_{\mathscr{A}} \cong [m]_{\mathscr{A}}$) but not necessarily weakly hereditary (i.e., $[M \to [M]_{\mathscr{A}}]_{\mathscr{A}} \cong 1_{[M]_{\mathscr{A}}}$). There are various ways of describing weak hereditariness of $C_{\mathscr{A}}$. It is equivalent to the property that the factorization

$$M \to [M]_{\mathscr{A}} \to X$$

gives an orthogonal and not just a locally co-orthogonal factorization system (cf. [9, 15]), or that the composition of two $C_{\mathscr{A}}$-closed subobjects is again $C_{\mathscr{A}}$-closed. In this paper we prove a general necessary and sufficient criterion for weak hereditariness of regular closure operators (Theorem 3.3), and apply it to two important cases. We prove that the quotient-reflective subcategories of $\mathscr{T}op$ inducing a weakly hereditary closure operator are precisely the *disconnectednesses* in the sense of Arhangel'skiĭ and Wiegandt [2] (Theorem 3.4). For an Abelian category $\mathscr{X}$, we identify the epireflective subcategories inducing a weakly hereditary closure operator as the *torsion-free* subcategories in the sense of Dickson [6] (Theorem 4.4).

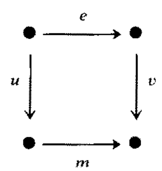
1. Basic definitions and results

In this section we recall some definitions and results from [1, 7, 9, 14].

Let $\mathscr{X}$ be a category and $\mathcal{M}$ a class of morphisms of $\mathscr{X}$ containing all isomorphisms and closed under composition. $\mathscr{X}$ is said to be $\mathcal{M}$-complete if pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$.

Then we have that (cf. [14] in the dual situation):

- $\mathcal{M}$ is a class of monomorphisms of $\mathscr{X}$.
- There exists a class $\mathscr{E}$ of morphisms such that $(\mathscr{E}, \mathcal{M})$ is a factorization system in $\mathscr{X}$, that is, every $\mathscr{X}$-morphism has an $(\mathscr{E}, \mathcal{M})$-factorization, and for each commutative diagram



with $e \in \mathscr{E}$ and $m \in \mathcal{M}$, there exists a unique morphism d satisfying the equalities $m.d = v$ and $d.e = u$.

We recall that the $\mathcal{M}$-completeness of $\mathscr{X}$ is equivalent to the existence of a factorization system of sinks, $(\mathscr{E}', \mathcal{M})$. For general information on factorization systems see [1].

• Given an $\mathscr{X}$-object X and $\mathscr{M}$-morphisms m and n with codomain X, we shall say that $m \leq n$ if there exists a morphism k (which we shall denote by m_n) such that $n.k = m$. The comma category $\mathscr{M}_X$ of $\mathscr{M}$-morphisms with codomain X equipped with this preorder is a complete class.

• For each morphism $f: X \to Y$ there exist functors $f^{-1}(\text{-}): \mathscr{M}_Y \to \mathscr{M}_X$ and $f(\text{-}): \mathscr{M}_X \to \mathscr{M}_Y$ given by pullback and $(\mathscr{E}, \mathscr{M})$-factorization, respectively, $f(\text{-})$ being left adjoint to $f^{-1}(\text{-})$.

By $\mathscr{M}$ we also denote the full subcategory of the morphisms category $\mathscr{X}^2$ whose objects are the $\mathscr{M}$-morphisms. By a closure operator C on $\mathscr{X}$ with respect to $\mathscr{M}$ we mean a functor $C: \mathscr{M} \to \mathscr{M}$ such that:

• $U.C = U$, where $U: \mathscr{M} \to \mathscr{X}$ is the "codomain functor".

• There exists a natural transformation $\gamma: \mathrm{Id}_{\mathscr{M}} \to C$ such that $U_\gamma = 1_U$.

We remark that, in order to define C, one only needs to give, for each $X \in \mathrm{Ob}\, \mathscr{X}$, a functor $c_X : \mathscr{M}_X \to \mathscr{M}_X$ such that $m \leq c_X(m)$ and $f(c_X(m)) \leq c_Y(f(m))$, for each $m, n \in \mathscr{M}_X$ and each $\mathscr{X}$-morphism $f: X \to Y$.

We shall denote $c_X(m)$ by $[m]_X$, or simply by $[m]$, when its meaning is clear from the context.

An $\mathscr{M}$-morphism m is called *C-closed* if $m \cong [m]$, and an $\mathscr{X}$-morphism $f: X \to Y$ is called *C-dense* if $[f(1_X)] \cong 1_Y$. We denote the classes of C-closed morphisms and of C-dense morphisms by $\mathscr{M}^C$ and $\mathscr{E}^C$, respectively.

The closure operator C is said to be *idempotent* (respectively *weakly hereditary*) whenever, for each $\mathscr{M}$-morphism m, $[m]$ is C-closed (respectively $m_{[m]}$ is C-dense).

Weak hereditariness of idempotent closure operators was nicely characterized in [7].

Proposition 1.1. *For an idempotent closure operator C on $\mathscr{X}$ with respect to $\mathscr{M}$, the following assertions are equivalent:*

(a) *C is weakly hereditary;*

(b) *$\mathscr{M}^C$ is closed under composition;*

(c) *$(\mathscr{E}^C, \mathscr{M}^C)$ is a factorization system in $\mathscr{X}$.*

Let $\mathscr{A}$ be a reflective subcategory of $\mathscr{X}$ (which will always be assumed to be full and isomorphism-closed). An $\mathscr{X}$-morphism is called *$\mathscr{A}$-regular* if it is the equalizer of a pair of morphisms with codomain in $\mathscr{A}$. If $\mathscr{X}$ has equalizers and $\mathscr{M}$ contains all regular monomorphisms of $\mathscr{X}$, then there is a canonical closure operator $C_{\mathscr{A}}$ with respect to $\mathscr{M}$ which assigns to each $\mathscr{M}$-morphism $m: M \to X$ the morphism defined by

$$[m]_{\mathscr{A}} = \bigwedge \{r \in \mathscr{M}_X \mid r \geq m \text{ and } r \text{ is } \mathscr{A}\text{-regular}\}.$$

If, moreover, $\mathscr{X}$ has cokernel pairs, it is easily verified that

$$[m]_{\mathscr{A}} \cong \mathrm{eq}(r_Y.i, r_Y.j)$$

with $(i, j : X \to Y)$ the cokernel pair of m, and r_Y the $\mathcal{A}$-reflection of Y. Hence, $[m]_{\mathcal{A}}$ is $\mathcal{A}$-regular.

This closure operator is called the *$\mathcal{A}$-closure* or the *regular closure operator induced by $\mathcal{A}$*. We point out that this closure operator is always idempotent.

The following proposition shows that the $\mathcal{A}$-closure of each $\mathcal{M}$-morphism is determined by the $\mathcal{A}$-closure of an $\mathcal{M}$-morphism with codomain in $\mathcal{A}$.

Proposition 1.2. *Let $\mathcal{X}$ have equalizers, cokernel pairs and a factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ contains all regular monomorphisms, and let $\mathcal{A}$ be a reflective subcategory of $\mathcal{X}$. Then, for each $\mathcal{M}$-morphism $m : M \to X$,*

$$[m]_{\mathcal{A}} \cong r_X^{-1}([r_X(m)]_{\mathcal{A}})$$

with r_X the $\mathcal{A}$-reflection of X.

Proof. Cf. [9, Proposition 3.11]. □

Throughout we consider a category $\mathcal{X}$ with equalizers, cokernel pairs, terminal object, T, and a factorization system $(\mathcal{E}, \mathcal{M})$. We shall denote by $\mathcal{A}$ an $\mathcal{E}$-reflective subcategory of $\mathcal{X}$ and by $C_{\mathcal{A}}$ the closure operator it induces.

2. Consequences of weak hereditariness

In this section we shall assume that:

(i) $\mathcal{X}$ is $\mathcal{M}$-complete;

(ii) $\mathcal{E}$ is a class of epimorphisms of $\mathcal{X}$;

(iii) pullbacks of $\mathcal{E}$-morphisms along $\mathcal{M}$-morphisms belong to $\mathcal{E}$;

(iv) $\mathcal{X}(T, X)$ is nonempty whenever the morphism $X \to T$ belongs to $\mathcal{E}$.

We already observed that $\mathcal{M}$-completeness of $\mathcal{X}$ implies that $\mathcal{M}$ is a class of monomorphisms. From (ii) it follows that $\mathcal{M}$ contains all regular monomorphisms of $\mathcal{X}$.

Let $\mathcal{S}$ be a class of morphisms of $\mathcal{X}$. The $\mathcal{A}$-reflections $\{r_X : X \to RX \mid X \in \mathrm{Ob}\, \mathcal{X}\}$ are said to be *hereditary with respect to $\mathcal{S}$* whenever the pullback, $f : Y \to S$, of r_X along an $\mathcal{S}$-morphism $s : S \to RX$, is the $\mathcal{A}$-reflection of Y, for each $X \in \mathrm{Ob}\, \mathcal{X}$.

We shall denote by $\mathcal{T}$ the class of $\mathcal{X}$-morphisms with domain T.

Proposition 2.1. *If $C_{\mathcal{A}}$ is weakly hereditary, then the $\mathcal{A}$-reflections are hereditary with respect to $\mathcal{T}$.*

Proof. First we point out that each morphism $y : T \to Y$, with $Y \in \mathrm{Ob}\, \mathcal{A}$, is a $C_{\mathcal{A}}$-closed $\mathcal{M}$-morphism because $y = \mathrm{eq}(y.g, 1_Y)$, g being the unique morphism from Y to T.

Now, let the following diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & T \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle x} \\ X & \xrightarrow[r_X]{} & RX \end{array}$$

be a pullback. Considering the morphism $t: RM \to T$ we have that $t.r_M = f$ with r_M and f in $\mathscr{E}$, by hypothesis. Hence we can conclude that t belongs to $\mathscr{E}$, using the fact that $(\mathscr{E}, \mathscr{M})$ is a factorization system (cf. [1, 14.9]). Therefore, $\mathscr{X}(T, RM)$ is not empty, by (iv). Consider $y: T \to RM$, and let $m': M' \to M$ be its pullback along r_M. Then $m.m'$ is $C_{\mathscr{A}}$-closed because m and m' are $C_{\mathscr{A}}$-closed and $C_{\mathscr{A}}$ is weakly hereditary (cf. Proposition 1.1). Moreover, since $[m.m'] \cong r_X^{-1}([r_X(m.m')])$ (by Proposition 1.2), and $r_X^{-1}([r_X(m.m')]) \cong r_X^{-1}(x) \cong m$, m' is an isomorphism. Using again properties of factorization systems (cf. [1, 14.6, 14.9]), one derives that y belongs to $\mathscr{E}$, hence, y is an isomorphism too. Therefore, up to isomorphism, the pullback of r_X along x, f, coincides with the $\mathscr{A}$-reflection of M. □

Let $m \cong r_X^{-1}(n)$, $m: M \to X$, for some $\mathscr{M}$-morphism $n: N \to RX$. Note that N is an $\mathscr{A}$-object since $\mathscr{A}$ is $\mathscr{E}$-reflective. We denote by f^m the unique morphism such that $f^m.r_M = f$, with $f: M \to N$ the pullback of r_X along n. We point out that this morphism is always in $\mathscr{E}$.

According to [8], an $\mathscr{X}$-morphism $f: X \to Y$ is called *$C_{\mathscr{A}}$-preserving* whenever $f([m]) \cong [f(m)]$, for every $m \in \mathscr{M}_X$.

Proposition 2.2. *Let $C_{\mathscr{A}}$ be weakly hereditary. Then, for each $\mathscr{M}^{C_{\mathscr{A}}}$-morphism $m: M \to X$, f^m is $C_{\mathscr{A}}$-preserving.*

Proof. Let $m: M \to X$ be a $C_{\mathscr{A}}$-closed morphism. By Proposition 1.2 we may write $m \cong r_X^{-1}(n)$, with n $C_{\mathscr{A}}$-closed. If $n': N' \to RM$ is $C_{\mathscr{A}}$-closed, $m' \cong r_M^{-1}(n')$ and $n'' \cong f^m(n')$, then $m.m'$ is $C_{\mathscr{A}}$-closed and $r_X(m.m') \cong n.n''$. Hence,

$$r_X^{-1}(n.n'') \geq m.m' \cong [m.m'] \cong r_X^{-1}([n.n'']).$$

Since, by (iii), $r_X(r_X^{-1}(w)) \cong w$ for each $w \in \mathscr{M}_{RX}$, it follows that $n.n'' \geq [n.n'']$. That is, $n.n''$ is $C_{\mathscr{A}}$-closed, hence, n'' is $C_{\mathscr{A}}$-closed too. Thus, f^m is $C_{\mathscr{A}}$-preserving. □

3. A characterization which leads to disconnectednesses

In this section we replace conditions (iii) and (iv) of the previous section by the stronger condition

(iii′) $\mathscr{E} = \{f \in \operatorname{Mor} \mathscr{X} \mid T \text{ is projective with respect to } f\}$

that is, we assume that $(\mathscr{E}, \mathscr{M})$ satisfies conditions (i′) = (i), (ii′) = (ii) and (iii′).

Under these assumptions, $\mathscr{X}(T, \text{-})$ reflects epimorphisms. This implies that $\mathscr{X}(T, \text{-})$ is faithful, hence, T is a generator. Conversely, if T is a generator and $\mathscr{E} = \text{Epi}\,\mathscr{X}$, then (iii′) holds.

Examples of this situation are given by the topological categories (in the sense of [11]) with the usual factorization system.

Proposition 3.1. *If $\mathscr{A}$ is $\mathscr{E}$-reflective such that $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$ then, for each $\mathscr{M}^{C_{\mathscr{A}}}$-morphism $m: M \to X$, f^m is a monomorphism.*

Proof. Let $m: M \to X$ be $C_{\mathscr{A}}$-closed, with $m \cong r_X^{-1}(n)$, and let $f^m.y = f^m.z$. Since T is a generator we can assume that the domain of y and z is T. Consider $x = n.f^m.y = n.f^m.z$, $w \cong r_X^{-1}(x)$, $w' \cong r_M^{-1}(y)$ and $w'' \cong r_M^{-1}(z)$. Then $w \leq m$ and, since $r_X(m.w') \cong x$ and $r_X(m.w'') \cong x$, it is obvious that $m.w' \leq w$ and $m.w'' \leq w$.

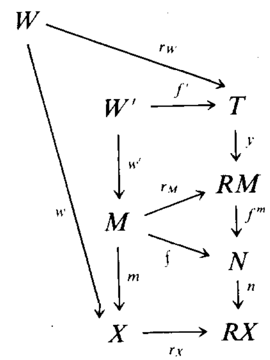
Since the $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$, the $\mathscr{A}$-reflection of the domain W of w is $r_W: W \to T$. Hence, there exists a unique $a: T \to RM$ such that $a.r_W = r_M.w_m$, and so, $r_M(w') \leq r_M(w_m) \cong a$ and $r_M(w'') \leq r_M(w_m) \cong a$. Therefore $y = a$ and $z = a$, that is, $y = z$. □

The following lemma will be important in the sequel.

Lemma 3.2. *Under the hypotheses of Proposition* 3.1, *if $m: M \to X$ is $C_{\mathscr{A}}$-closed (with $m \cong r_X^{-1}(n)$) and $y: T \to RM$, one has*

$$r_X^{-1}(n.f^m.y) \cong m.r_M^{-1}(y).$$

Proof. Let $x = n.f^m.y$, $w \cong r_X^{-1}(x)$ and $w' \cong r_M^{-1}(y)$.



Then $w \leq m$, and, as

$$n.f^m.r_M.w_m = r_X.w = n.f^m.y.r_W$$

and $n.f^m$ is a monomorphism by Proposition 3.1, $r_M.w_m = y.r_W$. Hence, by definition of pullback, there exists a morphism $h: W \to W'$ such that $w'.h = w_m$, that is, $w_m \leq w'$. On the other hand, also by definition of pullback, from

$$n.f^m.y.f' = r_X.m.w'$$

it follows that there exists a morphism $h': W' \to W$ such that $w_m.h' = w'$, hence, $w' \leq w_m$. Therefore, $m.w' \cong w$. □

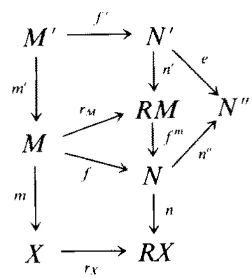
Theorem 3.3. *Let* $(\mathscr{E}, \mathscr{M})$ *be a factorization system in* $\mathscr{X}$ *satisfying* (i), (ii) *and* (iii'). *For an* $\mathscr{E}$*-reflective subcategory* $\mathscr{A}$, *the following assertions are equivalent*:

(a) $C_{\mathscr{A}}$ *is weakly hereditary.*

(b) $C_{\mathscr{A}}$ *is weakly hereditary when restricted to* $\mathscr{A}$, $\mathscr{A}$*-reflections are hereditary with respect to* $\mathscr{T}$ *and, for each* $\mathscr{M}^{C_{\mathscr{A}}}$*-morphism* $m: M \to X, f^m$ *is* $C_{\mathscr{A}}$*-preserving.*

Proof. The nontrivial part of (a)⇒(b) follows from Propositions 2.1 and 2.2.

In order to prove the converse, we shall show that $\mathscr{M}^{C_{\mathscr{A}}}$ is closed under composition. Let $m: M \to X$ and $m': M' \to M$ be $C_{\mathscr{A}}$-closed morphisms, with $m \cong r_X^{-1}(n)$ and $m' \cong r_M^{-1}(n')$. For $n'' \cong f^m(n')$, the morphism $n.n''$ is $C_{\mathscr{A}}$-closed because n and n' are $C_{\mathscr{A}}$-closed, f^m is $C_{\mathscr{A}}$-preserving and $C_{\mathscr{A}}$ is weakly hereditary in $\mathscr{A}$.



Let P be the domain of $[m.m']$. We have $[m.m'] \cong r_X^{-1}(n.n'')$ since $r_X(m.m') \cong n.n''$ and $n.n''$ is $C_{\mathscr{A}}$-closed. We shall show that T is projective with respect to $(m.m')_{[m.m']}$. This morphism is then an isomorphism, hence $m.m'$ is $C_{\mathscr{A}}$-closed. Indeed, if $y: T \to P$ and $x = r_X.[m.m'].y$, then $x \leq n.n''$ and, as T is projective with respect to $e.f'$, there exists a morphism $x': T \to M'$ such that $e.f'.x' = x_{n.n''}$. Hence,

$$x = n.n''.e.f'.x' = n.f^m.n'.f'.x'.$$

Therefore, by the above lemma, we conclude that $r_X^{-1}(x) \cong m.r_M^{-1}(n'.f'.x')$. Now, since $m' \geq r_M^{-1}(n'.f'.x')$, we have that

$$m.m' \geq m.r_M^{-1}(n'.f'.x') \cong r_X^{-1}(x) \geq [m.m'].y.$$

Therefore, $C_{\mathscr{A}}$ is weakly hereditary as claimed. □

We shall analyse the consequences of Theorem 3.3 when $\mathscr{X}$ is the category $\mathscr{T}op$ of topological spaces, $\mathscr{E}$ is the class of surjective continuous maps, $\mathscr{M}$ is the class of embeddings, and $\mathscr{A}$ is an extremal-epireflective subcategory of $\mathscr{T}op$.

A subcategory $\mathscr{A}$ of $\mathscr{T}op$ is called a *disconnectedness* if there exists a class of topological spaces $\mathscr{D}$ such that a space X is in $\mathscr{A}$ if and only if any morphism with domain in $\mathscr{D}$ and codomain X is constant (see [2]).

Theorem 3.4. *For an extremal-epireflective subcategory $\mathscr{A}$ of $\mathscr{T}op$, the following conditions are equivalent:*

(a) *$\mathscr{A}$ is a disconnectedness.*

(b) *$\mathscr{A}$ induces a weakly hereditary regular closure operator.*

Proof. By Proposition 2.1, if $\mathscr{A}$ induces a weakly hereditary closure operator then $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$, and this implies that $\mathscr{A}$ is a disconnectedness. Indeed, if $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$, it is easily checked that a topological space X belongs to $\mathscr{A}$ if and only if any morphism with domain in $\mathscr{D} = \{Y \in \mathrm{Ob}(\mathscr{T}op) \mid RY \cong T\}$ and with codomain X is constant, that is, if $\mathscr{A}$ is a disconnectedness.

To prove the converse, we first recall that disconnectednesses have reflections which are hereditary with respect to $\mathscr{T}$ (cf. [2, Theorem 3.7]).

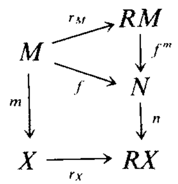
Now, let $\mathscr{A}$ be a disconnectedness. If $\mathscr{A} = \mathscr{T}op$, it is easily seen that $C_{\mathscr{A}}$ is discrete, hence, $C_{\mathscr{A}}$ is weakly hereditary. If $\mathscr{A} = \mathscr{T}op_0$, $C_{\mathscr{A}}$ is the well-known *b-closure operator*, and it is easy to verify that it is weakly hereditary. If $\mathscr{A} \neq \mathscr{T}op$ and $\mathscr{A} \neq \mathscr{T}op_0$, then $\mathscr{A} \subseteq \mathscr{T}op_1$ (cf. [10, Proposition 1.1]). Hence, $C_{\mathscr{A}}$ is discrete in $\mathscr{A}$ (cf. [3, 4]). Therefore $C_{\mathscr{A}}$ is weakly hereditary in $\mathscr{A}$, and, for each $C_{\mathscr{A}}$-closed morphism $m: M \to X$, the $\mathscr{A}$-morphism f^m is $C_{\mathscr{A}}$-preserving. Then, from Theorem 3.3, it follows that $C_{\mathscr{A}}$ is weakly hereditary. □

4. Torsion-free subcategories

Next, we relate weak hereditariness of regular closure operators in Abelian categories with *torsion-free subcategories* in the sense of [6]. Consequently, we shall be working in an Abelian category $\mathscr{X}$. Furthermore, we assume the Mono $\mathscr{X}$-completeness of $\mathscr{X}$ and that $(\mathscr{E}, \mathscr{M}) = (\mathrm{Epi}\,\mathscr{X}, \mathrm{Mono}\,\mathscr{X})$. It is easy to check that in this situation conditions (i)–(iv) of Section 2 are satisfied.

Lemma 4.1. *Let $\mathscr{X}$ be a* Mono *$\mathscr{X}$-complete Abelian category and $\mathscr{A}$ an epireflective subcategory of $\mathscr{X}$ such that the $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$. Then, for each $\mathscr{M}^{C_{\mathscr{A}}}$-morphism $m: M \to X$, f^m is an isomorphism.*

Proof. Let $m: M \to X$ be a $C_{\mathscr{A}}$-closed morphism, with $m \cong r_X^{-1}(n)$.



Since $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$, $\ker(r_X.m) \cong \ker(r_M)$. Hence,

$$\ker(f^m.r_M) \cong \ker(n.f^m.r_M) \cong \ker(r_X.m) \cong \ker(r_M).$$

Then it turns out that f^m is an isomorphism, since r_M and f^m are epimorphisms. □

From this lemma it follows that if $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$ then they are hereditary with respect to Mono $\mathscr{X}$.

Proposition 4.2. *For a regular closure operator $C_{\mathscr{A}}$ induced by an epireflective subcategory $\mathscr{A}$ of an Abelian category $\mathscr{X}$ which is* Mono *$\mathscr{X}$-complete, the following conditions are equivalent*:

(a) *$C_{\mathscr{A}}$ is weakly hereditary.*

(b) *$C_{\mathscr{A}}$ is weakly hereditary when restricted to $\mathscr{A}$ and $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$.*

Proof. By Proposition 2.1, if $C_{\mathscr{A}}$ is weakly hereditary, then $\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$. Hence, (a) implies (b).

In order to prove the converse, let $m: M \to X$ and $m': M' \to M$ be $C_{\mathscr{A}}$-closed morphisms. That is, $m \cong r_X^{-1}(n)$ and $m' \cong r_M^{-1}(n')$, with n and n' $C_{\mathscr{A}}$-closed. Then $m.m' \cong r_X^{-1}(n.f^m.n')$ with $n.f^m.n'$ $C_{\mathscr{A}}$-closed because f^m is an isomorphism and $C_{\mathscr{A}}$ is weakly hereditary in $\mathscr{A}$. Thus, $m.m'$ is $C_{\mathscr{A}}$-closed too. □

Using this result we are able to relate epireflective subcategories of $\mathscr{X}$ which induce weakly hereditary regular closure operators with torsion-free subcategories.

The following proposition is essentially known (cf. [6]).

Proposition 4.3. *For an epireflective subcategory $\mathscr{A}$ of an Abelian category, the following assertions are equivalent*:

(a) *$\mathscr{A}$-reflections are hereditary with respect to $\mathscr{T}$.*

(b) *$\mathscr{A}$ is closed under extensions, that is, $X \in \mathrm{Ob}\,\mathscr{A}$ whenever M, $Y \in \mathrm{Ob}\,\mathscr{A}$ in the exact sequence*

$$0 \to M \to X \to Y \to 0.$$

(c) *$\mathscr{A}$ is a torsion-free subcategory.*

Theorem 4.4. *For an epireflective subcategory $\mathscr{A}$ of an Abelian category $\mathscr{X}$ which is* Mono *$\mathscr{X}$-complete, the following assertions are equivalent*:

(a) *$\mathscr{A}$ is a torsion-free subcategory.*

(b) *$\mathscr{A}$ induces a weakly hereditary closure operator.*

Proof. By Propositions 2.1 and 4.3 it follows that $\mathscr{A}$ is a torsion-free subcategory whenever it induces a weakly hereditary closure operator.

Before proving that (a) implies (b), let us point out that, for $m: M \to X$ in $\mathscr{M}$, m is $C_{\mathscr{A}}$-closed if and only if $\text{coker}(m) \in \text{Mor}\, \mathscr{A}$ (cf. [5]). Indeed, if $\text{coker}(m): X \to X/M$ is an $\mathscr{A}$-morphism, we obviously have $m = \text{eq}(\text{coker}(m), 0)$. Conversely, if $m = \text{eq}(f, g)$ with the codomain of f and g in $\mathscr{A}$, then $m = \ker(f-g)$, hence

$$f - g = \ker(\text{coker}(f-g)).\text{coker}(m).$$

Consequently, as $\ker(\text{coker}(f-g))$ belongs to $\mathscr{M}$ and its codomain belongs to $\mathscr{A}$, we can conclude that its domain, X/M, belongs to $\mathscr{A}$. Therefore, $\text{coker}(m) \in \text{Mor}\, \mathscr{A}$.

Now, let $m: M \to X$ and $m': M' \to M$ be $C_{\mathscr{A}}$-closed, X being an $\mathscr{A}$-object. This means that $X/M \in \text{Ob}\, \mathscr{A}$ and $M/M' \in \text{Ob}\, \mathscr{A}$. Consider the exact sequence

$$0 \to M/M' \to X/M' \to (X/M')/(M/M') \to 0.$$

As $(X/M')/(M/M') \cong X/M$ and M/M' and X/M are $\mathscr{A}$-objects, it turns out that also X/M' belongs to $\mathscr{A}$, hence, $m.m'$ is $C_{\mathscr{A}}$-closed. Therefore, $C_{\mathscr{A}}$ is weakly hereditary in $\mathscr{A}$, and the conclusion follows from Proposition 4.2. □

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