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SIMPLICIAL DECOMPOSITIONS OF GRAPHS: A SURVEY OF APPLICATIONS

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We survey applications of simplicial decompositions (decompositions by separating complete subgraphs) to problems in graph theory. Among the areas of application are excluded minor theorems, extremal graph theorems, chordal and interval graphs, infinite graph theory and algorithmic aspects.

Let G be a graph, $\sigma > 0$ an ordinal (possibly finite), and let B_λ be an induced subgraph of G for every $\lambda < \sigma$. The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial decomposition* of G if the following conditions hold.

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) Each subgraph $(\bigcup_{\lambda<\mu} B_\lambda) \cap B_\mu =: S_\mu$ is complete $(0 < \mu < \sigma)$;
(S_μ is called the *simplex of attachment* of B_μ)

(S3) No S_μ contains B_μ or any other B_λ $(0 \leq \lambda < \mu < \sigma)$.

For finite graphs, these three conditions imply a fourth [19]:

(S4) Each S_μ is contained in B_λ for some $\lambda < \mu$ $(\mu < \sigma)$.

Notice that (S4) forces the factors of F into a tree structure: picking a fixed 'predecessor' $\lambda =: \tau(\mu)$ for each $\mu < \sigma$ as in (S4) (i.e. such that $S_\mu \subset B_{\tau(\mu)}$), we obtain a tree $T_F(G)$ with vertex set $\{B_\lambda \mid \lambda < \sigma\}$ and edge set $\{B_\mu B_{\tau(\mu)} \mid \mu < \sigma\}$. A family F satisfying (S1) and (S4) (but not necessarily (S2) or (S3)) is therefore called a *tree-decomposition of G*, and a family satisfying all four conditions (S1)–(S4) is a *simplicial tree-decomposition* of G (Fig. 1).

A simplicial decomposition none of whose member in turn admits a simplicial decomposition into more than one factor is called a decomposition into *primes*, or a *prime decomposition*. All finite graphs have prime decompositions, and so do all infinite graphs not containing an infinite complete subgraph [33].

The existing applications of simplicial decompositions to problems in graph theory can be roughly divided into two categories. The first kind of application typically exploits the inductive nature of their definition and the information provided by (S2): the fact that all attachment graphs S_μ are *simplices* (complete graphs) often allows one to lift assertions about the factors to similar assertions about the whole graph. For example, if each factor of G admits a k-colouring of its vertices, then so does G: since all vertices in S_μ must be coloured differently, a simple permutation of colours will adjust any k-colouring of B_μ to a given k-colouring of S_μ, and hence to any given k-colouring of $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda$; thus by induction, G can be k-coloured if every B_λ can.

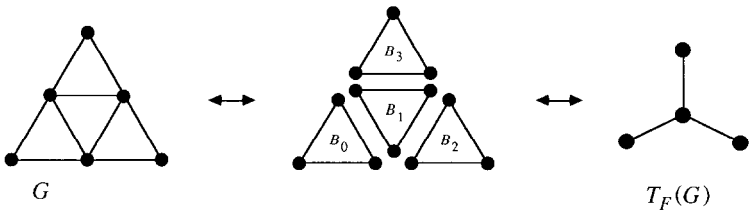


Fig. 1. A simplicial tree-decomposition.

The other line of application of simplicial decompositions places the emphasis on their tree-shape. Condition (S2) is used only to ensure (S4), and is otherwise eroded by considering not the decomposition of G itself but the decompositions induced on its subgraphs H (see e.g. [62]). As the attachment graphs $S_\mu \cap H$ will not in general be complete, such a decomposition of H may no longer be a simplicial decomposition. It will, however, still be a tree-decomposition (at least in the finite case), because it inherits (S4) from the decomposition of G. However, it is usually more convenient in such cases to work with the more general tree-decompositions rather than with simplicial decompositions in the first place.

Most of the results surveyed in this paper belong to the first of these two types of application of simplicial decompositions. Not that those of the second kind were not exciting: the results on well-quasi-ordering and embeddings of graphs recently achieved by Robertson and Seymour [61] are largely applications of tree-decompositions and would thus belong in this category. However, the object of this survey is more modest: it aims to bring to wider attention a number of interesting older results which have remained largely unknown (see particularly Section 1), to show the variety of ways in which simplicial and related decompositions can or could be used, and to present some open problems from the various fields of application.

1. Excluded minor theorems

Let H and X be graphs. In analogy to the familiar notation of TX for subdivisions of X (or 'topological' X graphs) we say that H is an HX (H for 'homomorphism') if its vertex set $V(H)$ admits a partition $\{V_x \mid x \in V(X)\}$ into *branch sets* V_x spanning connected subgraphs in H, such that H contains a $V_x - V_y$ edge if and only if x and y are adjacent in X. If H is an HX and H is a subgraph of G, then X is called a *minor* of G. For finite G this is equivalent to saying that X is obtained from a subgraph of G by contracting edges. The partition sets V_x may or may not be required to be finite; since we shall only consider finite minors (for which the sets V_x can be made finite without loss of generality), such a restriction will lead to equivalent results.

If $\mathscr{X}$ is a set of graphs, we write $\mathrm{T}\mathscr{X} := \{\mathrm{T}X \mid X \in \mathscr{X}\}$ and $\mathrm{H}\mathscr{X} := \{\mathrm{H}X \mid X \in \mathscr{X}\}$. We shall use $\mathscr{G}(\mathscr{H})$ to denote $\{G \mid H \in \mathscr{H} \Rightarrow H \not\subset G\}$; for example, $\mathscr{G}(\mathrm{H}\mathscr{X})$

contains the graphs without a minor in $\mathscr{X}$, and $\mathscr{G}(\mathrm{T}C_4)$ consists of the graphs in which every cycle is a triangle.

The classical prototype of an excluded minor theorem is Kuratowski's characterization of planar graphs: a finite graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. Thus in our notation, the finite planar graphs are precisely the finite elements of $\mathscr{G}(\mathrm{H}K_5, \mathrm{H}K_{3,3})$ (or, as in Kuratowski's original version of the theorem, the finite elements of $\mathscr{G}(\mathrm{T}K_5, \mathrm{T}K_{3,3})$).

The importance of Kuratowski's theorem has traditionally been attributed to the fact that while planarity is easy to verify (using a concrete drawing in the plane), the equivalent property of not containing a K_5 or $K_{3,3}$ minor is easy to falsify (using a concrete $\mathrm{H}K_5$ or $\mathrm{H}K_{3,3}$ subgraph). Thus, whether we want to sell a certain graph as planar or as non-planar, by Kuratowski's theorem there is always an efficient way of convincing our customers.

This feature of Kuratowski's theorem is common to all excluded minor theorems, and indeed is their raison d'être; they all assert the equivalence of some structural graph property (which is easy to verify but difficult to falsify) with the absence of certain minors (which is easy to falsify but difficult to verify). The emphasis in such equivalence theorems can lie on either side: sometimes the structural property is 'natural' and comes first (as with planarity), while in other cases the excluded minors are given and the task is to describe the structure of the graphs not containing them. (Excluded minor theorems of a somewhat different kind have recently been considered by Truemper [65].)

Since the minor relation is transitive, any class of the form $\mathscr{G}(\mathrm{H}\mathscr{X})$ is closed under taking minors. Conversely, it is easily seen that any graph property $\mathscr{G}$ which is closed under taking minors has this form: if $\bar{\mathscr{G}}$ denotes the complement of $\mathscr{G}$, then clearly $\mathscr{G} = \mathscr{G}(\mathrm{H}\bar{\mathscr{G}})$. Moreover, if we restrict ourselves to finite graphs, the recent well-quasi-ordering theorem of Robertson and Seymour [61] ('Wagner's conjecture') tells us that $\bar{\mathscr{G}}$ can in fact always be replaced with a finite set of excluded minors. In other words: a property of finite graphs is closed under taking minors if and only if it has the form $\mathscr{G}(\mathrm{H}X_1, \ldots, \mathrm{H}X_n)$.

In this paper we are interested in properties of graphs whose structure can be described in terms of simplicial decompositions. If such a property is closed under taking minors, as is the case in the following example, it gives rise to an excluded minor theorem.

The example, motivated by applications in computer science, was suggested by Chvátal and is due to Arnborg, Corneil and Proskurowski [2]. Call a graph a *k-tree* if it is recursively obtained from a K_k by the operation of joining a new vertex to all vertices in some complete subgraph of order k. Thus, a k-tree ($\neq K_k$) is simply a graph that admits a simplicial decomposition into K_{k+1}'s, all simplices of attachment having order k. The graphs considered in [2] are the *partial* k-trees, the subgraphs of k-trees. These are precisely the graphs having tree-width at most k – the *tree-width* of G is the smallest k such that G admits a tree-decomposition into factors of order at most $k+1$ – which is a minor-closed

property. The partial k-trees are therefore characterized by a unique minimal set of forbidden minors. Arnborg, Corneil and Proskurowski determined these minors for $k=3$ as K_5, the octahedron $K_{2,2,2}$, the Wagner graph W (the octagon with its four diagonals) and the 5-prism $C_5 \times K_2$ (the cartesian product of a 5-cycle with an edge). We remark that although the graphs considered in [2] are finite, the result extends to infinite graphs.

In the above example we started out from a structural graph property and had to find the corresponding excluded minors. In a sense, simplicial decompositions were merely incidental to the problem: the property considered happened to involve them, but they were not needed to solve the problem. And naturally, the genuine applications of simplicial decompositions are found in the excluded minor theorems of the opposite type: a list of forbidden minors is given, and simplicial decompositions are used to describe the structure of the graphs not containing these minors.

The first such theorem was proved by Wagner in 1937 – which is how simplicial decompositions were introduced. Wagner set out to explore how far we would be 'taken out of the plane' by graphs that were no longer forbidden to contain either K_5 or $K_{3,3}$ minors (as in Kuratowski's theorem), but only one of these two types. In particular, the question was whether the chromatic number of graphs not containing a K_5 minor (but possibly one isomorphic to $K_{3,3}$) might be higher than that of planar graphs. The fact that this is not so but rather that all such graphs can be 4-coloured (as can planar graphs) is now commonly known as the case of $k=5$ of the (later) conjecture of Hadwiger:

$$\mathbf{H}(k)\colon\ \chi(G) \geqslant k \Rightarrow G \supset \mathrm{H}K_k \quad (\forall k \in \mathbb{N}),$$

or equivalently,

$$G \in \mathcal{G}(\mathrm{H}K_k) \Rightarrow \chi(G) \leqslant k-1.$$

As indicated earlier, a graph admits a k-colouring if and only if its simplicial factors do (in any given decomposition); for a proof of $\mathbf{H}(5)$ it is therefore sufficient to show that all possible factors in prime decompositions of graphs in $\mathcal{G}(\mathrm{H}K_5)$ can be 4-coloured. Moreover, since the chromatic number of a graph cannot increase through the deleting of edges, Wagner could restrict his consideration to prime factors of graphs that are *edge-maximal* in $\mathcal{G}(\mathrm{H}K_5)$, i.e. in which any addition of a new edge creates an $\mathrm{H}K_5$. (It is easily seen that every graph in $\mathcal{G}(\mathrm{H}K_5)$ can be made edge-maximal by adding edges, and we remark that every graph in $\mathcal{G}(\mathrm{H}K_5)$ admits a simplicial decomposition into primes.) And indeed, it turned out that all possible primes of edge-maximal graphs in $\mathcal{G}(\mathrm{H}K_5)$ can be 4-coloured: they are either planar or isomorphic to the 3-chromatic graph W, the octagon with its four diagonals. (At the time of Wagner's paper, the 4-colourability of planar graphs was, of course, still the 4-Colour-*Conjecture*, and for this reason Wagner's results has become known as his 'equivalence theorem', establishing as it does the equivalence of the 4CC with Hadwiger's Conjecture for $k=5$.)

Wagner's characterization of $\mathcal{G}(\mathrm{H}K_5)$ in terms of simplicial decompositions set the trend for a number of similar excluded minor theorems, which are listed in Table 1. The general pattern is that a set $\mathcal{X}$ of finite graphs is given (the *excluded minors*), and that the theorem determines the *homomorphism base* $\mathcal{B}(\mathrm{H}\mathcal{X})$ of $\mathcal{X}$, which is the set of graphs that can occur as factors in prime decompositions of edge-maximal graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$. In addition, the theorem usually gives some structural information on the precise manner in which the base elements have to be composed in order to give the edge-maximal graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$. This information typically takes the form of prescribing the order $|S_\mu|$ of the simplices of attachment, sometimes depending on the type of the factors in which they are contained. (For a very simple example of how such a theorem is typically proved, the reader is referred to the determination of $\mathcal{B}(\mathrm{T}K_{2,3})$ $(=\mathcal{B}(\mathrm{H}K_{2,3}))$ in [10].)

Three remarks should be made at this point. Firstly, the restriction that all excluded minors be finite is essential: it ensures that every graph of $\mathcal{G}(\mathrm{H}\mathcal{X})$ is indeed contained in some edge-maximal element of $\mathcal{G}(\mathrm{H}\mathcal{X})$ (this is not so, for example, with $\mathcal{G}(\mathrm{H}P_\infty)$), and that all these graphs admit a unique simplicial decomposition into primes [33]. (For a more thorough discussion of the problem of uniqueness and existence of prime decompositions see [13, 14].)

Secondly, unless otherwise stated no restriction is imposed on the order of the graphs $G \in \mathcal{G}(\mathrm{H}\mathcal{X})$. However, by a theorem of Halin [35] elements of homomorphism bases are always countable, regardless of the cardinality of the graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$ of which they are prime factors.

The third remark concerns small elements of the homomorphism base $\mathcal{B}(\mathrm{H}\mathcal{X})$. If the smallest graph in $\mathcal{X}$ has order n, then clearly all graphs of order $<n$ are in $\mathcal{G}(\mathrm{H}\mathcal{X})$, and every complete graph of order $<n$ is in $\mathcal{B}(\mathrm{H}\mathcal{X})$ (because it is trivially edge-maximal and admits only the trivial prime decomposition, itself being the only factor). Such small complete graphs are therefore not listed among the elements of homomorphism bases in Table 1, unless they appear (non-trivially) in prime decompositions of graphs of order at least $\min\{|X| : X \in \mathcal{X}\}$.

The excluded minor theorems shown in Table 1 lend themselves to the deduction of various corollaries, which can sometimes be rather surprising. For example, any graph of chromatic number or minimal degree $\geqslant 6$ contains minors isomorphic to the graphs L and $K_{1,2,3}$; any graph of minimal degree $\geqslant 5$ contains minors isomorphic to $W_{1,5}$ and to the prism (consider the last factor in any prime decomposition of an edge-maximal graph in $\mathcal{G}(\mathrm{H}W_{1,5})$ or $\mathcal{G}(\mathrm{H}(C_3 \times K_2))$); any graph of minimal degree $\geqslant 4$ has a K_5^- minor, and so on. Or to mention just one more example, which can be read out of the excluded minor theorem for $K_{1,2,3}$: if G is a 3-connected non-planar graph of order $\geqslant 11$, then G not only has a minor isomorphic to either $K_{3,3}$ or K_5 (as by Kuratowski) – it must have a minor isomorphic to $K_{1,2,3}$, a $K_{3,3}$ plus two adjacent edges.

It is worth pointing out that all the above investigations can be carried out in a completely analogous fashion for forbidden subdivisions rather than minors. If a graph property $\mathcal{G}$ has the form $\mathcal{G} = \mathcal{G}(\mathrm{T}\mathcal{X})$, where $\mathcal{X}$ is again any set of finite

Table 1. The structure of the edge-maximal graphs G in classes of the form $\mathscr{G}(\mathrm{H}\mathscr{X})$.

Excluded minors	Homomorphism base	$\lvert S_\mu \rvert$	Ref.	Structure
 K_3	K_2	1		trees[b]
 C_4	K_2, K_3	1		no limitation[b]
 K_4^-	K_2, all cycles	1		no limitation[b]
 K_4	K_3	2	[67]	no limitation[c]
finite 3-connected graphs	K_3 ($\mathscr{G}(\mathrm{H}\mathscr{X}) = \mathscr{G}(\mathrm{H}K_4)$)	2	[67]	no limitation[c]
 C_5	K_2, K_3, K_4	1, 2	[34]	
 $C_5 + e$	K_2, K_3, K_4, all cycles C_n ($n \geqslant 5$)	1, 2		
 $K_{2,3}$	K_2, K_3, K_4	1, 2	[44] [10]	each edge is in at most 2 factors, and if so then these are both K_3's[b]
 $\mathrm{T}K_4$ of order 5	K_2, K_3, K_4	1, 2		
 $K_{1,1,3}$	K_2, K_3, $K_{3,3}$, the prism,[a] all wheels[a]	1, 2		
 $K_{1,2,2}$	K_3, K_4	2	[39]	no two K_3 factors share an edge[c,d]
 K_5^{-2}	K_2, K_3, $K_{3,3}$, the prism[a], all wheels[a]	1, 2		
 K_5^-	K_3, $K_{3,3}$, the prism[a], all wheels[a]	2	[69]	no two K_3 factors share an edge[c,d]

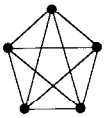
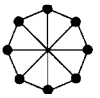
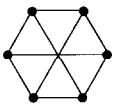
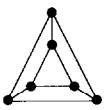
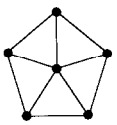
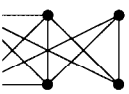
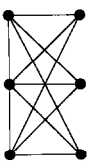
[a] the *prism* is the cartesian product $C_3 \times K_2$: the wheels are taken to include K_4 and will be denoted $W_{1,k}$.

[b] Conversely, every graph with such a decomposition is in $\mathscr{G}(\mathrm{H}\mathscr{X})$, though not necessarily edge-maximal.

[c] Conversely, every graph with such a decomposition is edge-maximal in $\mathscr{G}(\mathrm{H}\mathscr{X})$.

[d] If this rule is violated, the graph will still be in $\mathscr{G}(\mathrm{H}\mathscr{X})$ but will not longer be edge-maximal.

Table 1 (continued). The structure of the edge-maximal graphs G in classes of the form $\mathscr{G}(\mathrm{H}\mathscr{X})$.

Excluded minors	Homomorphism base	$\lvert S_\mu \rvert$	Ref.	Structure
 K_5	K_3, K_4, $\mathcal{MP}$,[a] $W =$ 	2, 3	[67], [35]	(for finite G) simplicial 2-sums[g] of W and G_3's,[b] so that no two of these G_3's share an edge[f,h]
 $K_{3,3}$	K_3, K_4, K_5, $\mathcal{MP}$[a]	2, 3	[68], [35]	(for finite G) simplicial 2-sums[g] of K_5 and planar triangulations ($\neq K_5^-$)[f], so that no two planar factors share an edge[f,h]
K_5, $K_{3,3}$	K_3, K_4, $\mathcal{MP}$[a]	2, 3	[35]	(for finite G) planar triangulations[h]
 $C_3 \times K_2$	K_3, K_4, K_5, all wheels $W_{1,k}$ ($k \geq 5$)	2, 3	[31]	simplicial 2-sums[g] of K_5, wheels and G_Δ's[c] ($\neq K_5^-$)[f] so that no K_3 factors shares an edge with the Δ of any G_Δ factor[f,h]
 $W_{1,5}$	K_3, K_4, K_5, the octahedron,[e] L,[e] the cube,[e] $Q =$ 	2, 3	[34]	simplicial 2-sums[g] of base elements of order ≥ 5 and G_Δ's[c] ($\neq K_5^-$)[f], so that no K_3 factor shares an edge with the Δ of any G_Δ factor[f,h]
 L	K_3, K_4, K_5, W $\mathcal{MP}$[a]	2, 3	[32], [35]	(for finite G) simplicial 2-sums[g] of W, K_5 and G_3's[b] ($\neq K_5$)[f], so that no two of these G_3's share an edge[f,h]
 $K_{1,2,3}$	K_3, K_4, K_5, W, L, $\mathcal{MP}$[a], 9 sporadic non-planar graphs of order ≤ 10	2, 3	[32], [35]	(for finite G) simplicial 2-sums[g] of non-planar base elements and planar triangulations ($\neq K_5^-$)[f], so that no two planar factors share an edge[f,h]
K_5, $K_{2,2,2}$, W, $C_5 \times K_2$	K_4	3	[2]	graphs of tree-width ≤ 3; no limitations[h]
finite 4-connected graphs; finite 4-edge-connected graphs; finite graphs of minimal degree 4	K_3, K_4, W, $C_5 \times K_2$ (classes $\mathscr{G}(\mathrm{H}\mathscr{X})$ coincide; [25])	2, 3	[34]	simplicial 2-sums[g] of W, $C_5 \times K_2$ and H_3's[d], so that no two of these H_3's share an edge[f,h]

[a] $\mathcal{MP}$ stands for "all countable 4-connected maximally planar graphs". (These are precisely the prime graphs among the countable maximally planar graphs; we remark that in the infinite case these graphs need not be planar triangulations [35].)

[b] A G_3 is a K_3, a K_4, or any simplicial 3-sum** of finite graphs from $\mathcal{MP}$ (i.e., of finite planar triangulations).

[c] For a fixed triangle Δ, a G_Δ is any graph equal to Δ or to a union of K_4's each containing Δ.

[d] An H_3 is a K_3 or any simplicial 3-sum** of K_4's.

[e] L is a $K_{3,3}$ plus 2 independent edges; the *cube* is the cartesian product $C_4 \times K_2$: the *octahedron* is the tripartite graph $K_{2,2,2}$.

[f] If this rule is violated, the graph will be in $\mathscr{G}(\mathrm{H}\mathscr{X})$ but will no longer be edge-maximal.

[g] A *simplicial k-sum* of graphs B_i, $i \in I$, is any graph with a simplicial decomposition into factors from $\{B_i \mid i \in I\}$ and all simplices of attachment having order k. (For a formal definition of K-sums see Section 5.)

[h] Conversely, every graph with such a decomposition is edge-maximal in $\mathscr{G}(\mathrm{H}\mathscr{X})$.

graphs, then every graph $G \in \mathcal{G}$ can be extended to an edge-maximal member of $\mathcal{G}$, and all graphs in $\mathcal{G}$ admit simplical decompositions into primes. The set of graphs that can occur as factors in prime decompositions of edge-maximal graphs in $\mathcal{G}(\mathrm{T}\mathcal{X})$ is then called the *subdivision base* of $\mathcal{X}$, and denoted by $\mathcal{B}(\mathrm{T}\mathcal{X})$.

Given the great deal of information a homomorphism or subdivision base characterization offers, it seems desirable to know for which sets $\mathcal{X}$ there is any reasonable hope to determine the corresponding base. For example, it would be a big step forward to be able to decide in which cases such a base is countable: if it is, it may be worth the effort trying to determine its elements constructively, whereas otherwise there would be little hope of doing so. However, very little is known in this direction, even if $\mathcal{X}$ consists of only one excluded minor:

Problem 1.1. For which finite graphs X is $\mathcal{B}(\mathrm{T}X)$ or $\mathcal{B}(\mathrm{H}X)$ countable? In particular, for which X does $\mathcal{B}(\mathrm{T}X)$ or $\mathcal{B}(\mathrm{H}X)$ consist entirely of finite graphs?

Let us provisionally call a homomorphism base *simple* if it is countable, or (alternatively) if all its elements are finite. (It is not known whether these two conditions coincide, that is, whether there exists a countable homomorphism base containing an infinite graph.)

Judging from what little evidence is available regarding Problem 1.1, it seems that denser excluded minors are less likely to have a simple base than sparser ones. For example, it was shown in [11] that $\mathcal{B}(\mathrm{T}X)$ and $\mathcal{B}(\mathrm{H}X)$ are uncountable if $X = K_n$ or $X = K_{n,m}$ with $n, m \geqslant 5$, of if X is such that any two vertices have at least two common neighbours. In another theorem of [11], $\mathcal{B}(\mathrm{T}\mathcal{X})$ and $\mathcal{B}(\mathrm{H}\mathcal{X})$ are shown to be uncountable for any

$$\mathcal{X} = \{X \mid \alpha(X) \geqslant n\},$$

where $n \geqslant 5$, and α denotes minimal degree, vertex-connectivity, edge-connectivity, degree of regularity or chromatic number -1. (Table 1 shows that the bases in the first three of these cases *are* countable for $n \leqslant 4$.)

To get a handle on Problem 1.1, it would be useful to know something about the relationship between different sets of graphs with simple bases. The following conjecture (whose first part is taken from [11]) is aimed in this direction.

Conjecture 1.2. (i) *If $\mathcal{B}(\mathrm{H}X)$ is simple and X' is obtained from X by deleting an edge, then $\mathcal{B}(\mathrm{H}X')$ is simple.*

(ii) *If $\mathcal{B}(\mathrm{H}X)$ is simple and X' is obtained from X by contracting an edge, then $\mathcal{B}(\mathrm{H}X')$ is simple.*

(iii) *If $\mathcal{B}(\mathrm{H}\mathcal{X})$ is simple and $\mathcal{X}' \supset \mathcal{X}$, then $\mathcal{B}(\mathrm{H}\mathcal{X}')$ is simple.*

Note that part (iii) of Conjecture 1.2 implies parts (i) and (ii), because $\mathcal{G}(\mathrm{H}X) = \mathcal{G}(\mathrm{H}\mathcal{Y})$ for $\mathcal{Y} = \{Y \mid Y \supset \mathrm{H}X\}$.

Parts (i) and (ii) of Conjecture 1.2 together say that, for single graphs X, the property of having a simple homomorphism base is itself closed under taking minors. Assuming this as true, one might find the evidence of Table 1 tempting to make the following conjecture:

Conjecture 1.3. *$\mathcal{B}(\mathrm{H}X)$ is simple if and only if X is planar.*

Conjecture 1.3, though perhaps born more of wishful thinking than insight, is made particularly attractive by the fact that the planarity or non-planarity of an excluded minor determines whether the corresponding class of graphs has bounded tree-width: by a result of Robertson and Seymour [61], the tree-widths of the graphs in $\mathcal{G}(\mathrm{H}X)$ have a uniform finite bound if and only if X is planar. (In [61] this is proved for finite graphs only; the extension to infinite graphs – but still with finite X – is due to R. Thomas.) However, it is not clear how much the order of a homomorphism base $\mathcal{B}(\mathrm{H}\mathcal{X})$ or of its elements has to do with the tree-width of the graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$:

Problem 1.4. Is there a connection between the simplicity of a homomorphism base $\mathcal{B}(\mathrm{H}\mathcal{X})$ and the tree-width of the graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$? In particular, is it true that $\mathcal{B}(\mathrm{H}\mathcal{X})$ is simple if and only if the graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$ have bounded tree-width?

It is clear that a positive answer to the second question in Problem 1.4 would imply Conjecture 1.2(iii) (and hence the whole of Conjecture 1.2), as well as Conjecture 1.3 (by the remarks above).

Finally, it may be worth recording a more immediate problem about homomorphism bases, whose solution would nevertheless clarify the situation considerably. If a graph G is edge-maximal in $\mathcal{G}(\mathrm{H}X)$ and prime, then clearly $G \in \mathcal{B}(\mathrm{H}X)$, because G has only the trivial prime decomposition with itself as the only factor. Conversely however, a homomorphism base may contain graphs which are not themselves edge-maximal but only (as by the definition of a homomorphism base) prime factors of larger edge-maximal graphs. And although it is not difficult to construct examples of such bases, in most 'natural' cases the homomorphism base of a graph X does seem to coincide with the class of prime and edge-maximal elements of $\mathcal{G}(\mathrm{H}X)$.

Problem 1.5. For which $\mathcal{X}$ are all elements of $\mathcal{B}(\mathrm{H}\mathcal{X})$ edge-maximal graphs in $\mathcal{G}(\mathrm{H}\mathcal{X})$?

2. Extremal graphs

When the edge-maximal graphs in a class of the form $\mathcal{G}(\mathcal{H})$ are known, we have a fairly good overview of all graphs in $\mathcal{G}(\mathcal{H})$, since they are precisely the

subgraphs of the edge-maximal ones. However, in cases where it is too difficult to determine all the edge-maximal members of such a class, it may still be possible to characterize an important subset of them: the so-called extremal graphs in $\mathcal{G}(\mathcal{H})$. A graph $G \in \mathcal{G}(\mathcal{H})$ of order n is called *extremal* in $\mathcal{G}(\mathcal{H})$ if it has the largest possible size (number of edges) that any n-graph in $\mathcal{G}(\mathcal{H})$ can have; this size is denoted by $\mathrm{ex}(n; \mathcal{H})$ [3]. Since this definition makes sense only for finite graphs, we shall assume for this section that all graphs considered are finite.

Table 2. The simplicial structure of the extremal graphs in $\mathcal{G}(\mathcal{H})$.

Excluded subgraphs	$\mathrm{ex}(n; \mathcal{H})$	Factors	$\lvert S_\mu \rvert$	Ref.	Comments
$\mathrm{H}K_4\,(=\mathrm{T}K_4)$	$2n-3$	K_3	2	([67])	for $n \geqslant 3$
$\mathrm{H}K_5^{-2}$, $\mathrm{H}K_5^{=}$	$2n-2$	K_4	1	[21]	for $n \geqslant 4$
$\mathrm{H}K_5^{-}$	$\frac{5}{2}n-4$	K_4	2	[21]	for $n \geqslant 4$
$\mathrm{H}K_5$	$3n-6$	$\mathcal{MP}$	3	([67])	for $n \geqslant 4$
$\mathrm{H}K_6^{-2}$, $\mathrm{H}K_6^{=}$	$3n-5$	K_5	2	[21]	for $n \geqslant 5$
$\mathrm{H}K_6^{-}$	$\frac{7}{2}n-\frac{15}{2}$	K_5	3	[21]	for $n \geqslant 5$
$\mathrm{H}K_6$	$4n-10$	K_5, $\mathcal{MP}+K_1$	4	[48]	for $n \geqslant 5$
$\mathrm{H}K_7^{-2}$, $\mathrm{H}K_7^{=}$	$4n-9$	K_6	3	[45]	for $n \geqslant 6$
$\mathrm{H}K_7^{-}$	$\frac{9}{2}n-12$	K_6, $K_4(2)$	4	[47]	for $n \geqslant 6$
$\mathrm{H}K_7$	$5n-15$	K_6, $\mathcal{MP}+K_2$	5	[48]	for $n \geqslant 6$; additional extremal graph: $G = K_{2,2,2,3}$
$\mathrm{H}K_8^{-2}$, $\mathrm{H}K_8^{=}$	$5n-14$	K_7	4	[46]	for $n \geqslant 7$
$\mathrm{H}K_8$	$6n-20$	$K_5(2)$	5	[49]	for $n \geqslant 8$
$\mathrm{T}K_5^{-2}$	$2n-2$	all wheels	1	[64]	for $n \geqslant 4$
$\mathrm{T}K_5^{-}$	$\frac{5}{2}n-4$	K_4	2	[64]	for even $n \geqslant 3$
$\mathrm{T}K_5^{-}$	$\frac{5}{2}n-\frac{9}{2}$	K_3, K_4, $W_{1,4}$	2	[64]	for odd $n \geqslant 3$; all factors except exactly one are K_4's
$\mathrm{T}L$	$3n-5$	K_5	2	[64]	for $n \geqslant 5$
$\mathrm{T}K_5$, $\mathrm{T}L$	$3n-6$	K_4, $\mathcal{MP}$	3	[48]	for $n \geqslant 4$
S_1	$2n-3$	K_3	2	[66]	for $n \geqslant 3$; cf. $\mathrm{ex}(n; \mathrm{T}K_4)$ above
S_2	$\frac{3}{2}n-\frac{3}{2}$	K_3	1	[5]	for $n \geqslant 1$
S_3	$2n-3$	K_3, $K_{3,3}$	2	[66]	for $n \geqslant 3$

K_n^- denotes the complete graph of order n minus an edge, K_n^{-2} is a K_n with two adjacent edges deleted, $K_n^=$ is a K_n with two non-adjacent edges deleted. $W_{1,4}$ is the wheel with 4 spokes. L and $\mathcal{MP}$ are defined as for Table 1; $\mathcal{MP}+K_n$ stands for the graphs obtained from the disjoint union of any $G \in \mathcal{MP}$ and a K_n by adding all edges between G and the K_n. $K_r(k)$ denotes the complete r-partite graph with k vertices in each class. The graphs S_1, S_2 and S_3 are *semitopological* subgraphs [3]: S_1 is any subdivision of a K_4 in which the three edges of a path P_3 have remained undivided; S_2 denotes any graph obtained by adding a new vertex to some cycle and joining it to exactly two vertices of that cycle; an S_3 is defined like an S_2, except that the new vertex is joined to exactly three vertices of the cycle (thus, an S_3 is a $\mathrm{T}K_4$ in which a 3-star was left undivided).

As in general a given graph $G \in \mathcal{G}(\mathcal{H})$ will not necessarily be a subgraph of an extremal member of $\mathcal{G}(\mathcal{H})$, a characterization of these extremal graphs cannot be expected to give us the same amount of information as, say, the theorems listed in Table 1. However, it will at least provide one important bit of information: it will tell us how many edges force an n-graph to contain an element of $\mathcal{H}$. In addition, a typical extremal graph theorem also determines the structure of the extremal graphs, and this structure is often quite simple.

Table 2 lists a number of extremal graph theorems where the structure of the extremal graphs can be described in terms of simplicial decompositions. In most cases the graphs property involved is again given by one or two excluded minors; in order to accommodate other forbidden structures in the same table, however, the forbidden configurations, including minors, are uniformly expressed as excluded *subgraphs*.

The extremal size $\mathrm{ex}(n; \mathcal{H})$ of an n-graph in $\mathcal{G}(\mathcal{H})$ is in all our cases roughly given by a linear polynomial in n. Its exact value however may vary a little for different values of n, depending on features like the parity of n. The polynomial shown under $\mathrm{ex}(n; \mathcal{H})$ in Table 2 always marks the top edge of this variation: for each $n \in \mathbb{N}$ it is at least as large as $\mathrm{ex}(n; \mathcal{H})$, with equality for infinitely many values of n. Similarly, the structural information provided refers only to (all) those extremal graphs in $\mathcal{G}(\mathcal{H})$ for whose size the value given under $\mathrm{ex}(n; \mathcal{H})$ is attained. Thus, a convenient translation of a row in Table 2 (with entries $\mathrm{H}X/p(n)/B_1, B_2/k/\cdots/\cdots/$say) into an extremal graph theorem would be, 'If G has n vertices and at least $p(n)$ edges, then G contains an $\mathrm{H}X$ as a subgraph (or: X as a minor), unless G has size exactly $p(n)$ and admits a simplicial decomposition into factors B_1 or B_2 in which every simplex of attachment has order k; conversely, any graph with such a decomposition has size $p(n)$ and is extremal in $\mathcal{G}(\mathrm{H}X)$'.

3. Chordal graphs

A graph is called *chordal* (or sometimes 'triangulated') if it has no induced cycles other than triangles, that is, if every cycle of length ≥ 4 has a chord. Chordal and related graphs have been studied extensively in recent years, and it would be a formidable task well beyond the scope of this paper to survey even only those results that can be proved using simplicial decompositions. Instead, we shall largely confine ourselves to pointing out the theorems that link simplicial decompositions with these graphs, thus forming the basis for the various applications.

The first of these theorems is now a classic. It is due to Dirac, and in its original form it describes the structure of all finite chordal graphs. However, the theorem extends easily to all graphs that admit a simplicial decomposition into primes. Recall that a *clique* is a maximal complete subgraph.

Theorem 3.1 [20]. *Let G be a graph with a prime decomposition $(B_\lambda)_{\lambda<\sigma}$. G is chordal if and only if every B_λ is a clique in G.*

A typical application of Dirac's theorem would be to prove that some property which holds trivially for complete graphs extends to all chordal graphs, by showing that it is preserved in the process of pasting graphs together along simplices. To consider just one example, note that perfection is such a property; a graph is *perfect* if the chromatic number of every induced subgraph equals the order of the largest clique in that subgraph. Therefore all finite chordal graphs are perfect. Since infinite graphs without infinite simplices also have prime decompositions, we even have the following result, which answers a question of Wagon [70]:

Theorem 3.2 [38]. *Every chordal graph not containing an infinite simplex is perfect.*

(We remark that chordal graphs containing infinite simplices need not be perfect; see [70] for an example of R. Laver.)

Chordality is certainly a rather restrictive graph property – a fact clearly reflected in the uniformity of structure imposed on chordal graphs by Theorem 3.1. We may therefore expect that if we slightly relax its defining condition, the graphs we obtain can still be described in terms of their simplicial decompositions.

We present two results of this kind. The first of these is due to Gallai. Call a k-cycle C in a graph G *triangulated* if it has $k-3$ pariwise non-crossing chords in G. (Two chords e_1, e_2 of C *cross* if C can be written as $C = x_1, \ldots, x_k$ such that $e_1 = x_{i_1}x_{j_1}$, $e_2 = x_{i_2}x_{j_2}$ and $i_1 < i_2 < j_1 < j_2$; it is easily seen that a k-cycle can have at most $k-3$ pairwise non-crossing chords.) A straightforward induction shows that a graph is chordal if and only if each of its cycles is triangulated. The property $\mathcal{P}_1$ that every *odd* cycle in a graph is triangulated is therefore a natural weakening of chordality. (Moreover, as is easily seen, this property is equivalent to the semmingly weaker one that every odd cycle of length at least 5 has *two* non-crossing chords.)

Gallai [28] proved that the simplicial primes among these graphs (i.e. those graphs in $\mathcal{P}_1$ that have no separating simplex) are of only two possible types: a simplex completely joined to a complete multipartite graph (in the notation of [3], a graphs of the form $K_r + K(s_1, \ldots, s_t)$),[1] or a simplex completely joined to a 2-connected bipartite graph. In both cases, either part of the sum may be empty.

Conversely, the union $G_1 \cup G_2$ of two $\mathcal{P}_1$-graphs identified along a common simplex is not necessarily again in $\mathcal{P}^1$; the simplest counterexample is a C_4 identified with a K_3 along a K_2. However, it is easily checked that any such union

[1] Several authors have misquoted this type as merely the $K(s_1, \ldots, s_t)$, without the added K_r; there is even an entire paper investigating the (misconceived) 'Gallai graphs' arising from these primes.

which avoids joining an induced even cycle to a triangle in this way will be in $\mathcal{P}_1$: an odd cycle C of $G_1 \cup G_2$ that is in neither G_i has at least two non-crossing chords. We therefore arrive at the following characterization of $\mathcal{P}_1$:

Theorem 3.3. *Let G be a finite graph with a prime decomposition $(B_\lambda)_{\lambda<\sigma}$. Then the following assertions are equivalent:*

(i) *Every odd cycle in G is triangulated.*

(ii) *Each factor B_λ is of the form $K_r + K(s_1, \ldots, s_t)$ or of the form $K_r + H$, where H is a bipartite graph. Furthermore, if $\mu < \sigma$, $e \in E(S_\mu)$, $\{G_1, G_2\} = \{G|_\mu, B_\mu\}$, and e lies on a triangle of G_1, then e does not lie on an even induced cycle in G_2.*

Let us now turn to the second result generalizing chordal graphs. The property $\mathcal{P}_2$ it describes is readily observed in planar triangulations: every induced cycle of length at least 4 separates the graph, but no proper (induced) subgraph of the cycle does. Clearly all chordal graphs have this property too, simply because they contain no such cycles. The question to what extent planar traingulations and chordal graphs are unique with this property is answered by the following theorem:

Theorem 3.4 [12]. *A finite graph has property $\mathcal{P}_2$ if and only if each of its simplicial prime factors is complete or maximality planar and every simplex of attachment contained in a non-complete factor has order* 3 *or* 4.

A subspecies of the chordal graphs that has attracted much attention are the *interval graphs,* graphs that can be represented as intersection graphs of intervals on the real line. (The intersection graph of a family of sets has these sets as its vertices, and two sets are adjacent if and only if their intersection is non-empty.) Interval graphs are clearly chordal, and their simplicial decompositions into primes can be neatly identified among those described in Theorem 3.1.

Call a finite simplicial decomposition $(B_r)_{r\leq s}$ of a graph G *consecutive* if $S_r \subset B_{r-1}$ for every $r \leq s$. The following description of finite interval graphs was first formulated by Halin [41]; however, it is related to an earlier characterization in terms of incidence matrices, due to Fulkerson and Gross [27].

Theorem 3.5. *A finite graph is an interval graph if and only if it admits a consecutive simplicial decomposition into its cliques.*

Of course, the linear arrangement of the cliques in an interval graph is not surprising: since real intervals have the Helly property (finitely many pairwise intersecting intervals have a non-empty overall intersection), every clique can be labelled by a real number contained in all its intervals, and the cliques inherit the

natural order of their labels. Conversely, an interval representation of a graph G given as a consecutive union of cliques $C_1, \ldots, C_n$ is also readily reobtained: for each vertex $v \in G$ let $I(v)$ be the real convex hull of the set $\{i \mid v \in C_i\}$ of consecutive integers.

Theorem 3.5 can be used to derive other known criteria for interval graphs without much effort; see for example [39] for a proof of Gilmore and Hoffman's characterization [30] ('G is chordal, and its complement is a comparability graph'), or [41] for a short proof of the characterization due to Lekkerkerker and Boland [55] ('G is chordal and contains no "asteroidal triple"'). Moveover, we have the following:

Corollary 3.6 [41]. *A graph G is an interval graph if and only if among any three cliques of G there is one which separates the other two.*

Note that Corollary 3.6 is still true for infinite graphs; its proof is immediate from an adaptation of Theorem 3.5 to the infinite case.

We finally mention another subspecies of the chordal graphs, which is similar to interval graphs but not quit as restricted: the tree-representable graphs. A graph is *tree-representable* if it is isomorphic to the intersection graph of a family of subtrees of some tree. Again, tree-respresentable graphs are clearly chordal. In fact, the finite tree-representable graphs coincide with the finite chordal graphs, a result independently proved by Buneman [4] and Gavril [29].

The problem of identifying the infinite tree-representable graphs however is much deeper, and it is one which leads straight into simplicial decomposition theory proper:

Theorem 3.7 [42]. *A graph is tree-representable if and only if it is chordal and admits a simplicial tree-decomposition into primes.*

Or in other words (by Theorem 3.1), a graph is tree-representable if and only if it has a simplicial tree-decomposition into cliques. Reobtaining a tree-representation of a graph G given in terms of such a decomposition $F = (B_\lambda)_{\lambda<\sigma}$ is completely analogous to the interval case: for each vertex $v \in G$ the factors B_λ containing v span a subtree T_v of the decomposition tree $T_F(G)$, and clearly two of these subtrees, T_v and T_w, intersect if and only if v and w are in a common clique of G, that is, if and only if v and w are adjacent.

The proof of Theorem 3.7 is already quite involved – it uses the notion of ends of a tree in order to adapt the Helly property of finite systems of subtrees to the infinite case – and the result certainly gives a satisfactory description of any graph *known* to be tree-representable. Yet it does not offer much help for *deciding* whether a given graph is tree-representable, at least if we have problems decomposing it into primes and are not sure whether the desired decomposition exists. This problem however, to determine the graphs that admit a simplicial

tree-decomposition (or indeed any simplicial decomposition) into primes, is still unsolved – and it is as hard for chordal graphs only as it is for arbitrary graphs.

The countable case of this problem however has recently been settled (see [13] and [15], or [17] for an overview), and we have the following corollary for tree-decompositions:

Theorem 3.8 [16]. *For a countable graph G the following assertions are equivalent.*

(i) *G is tree-representable*;

(ii) *G admits a tree-decomposition into primes*;

(iii) *G is chordal, and neither of two specified graphs is its simplicial minor.*

(For definitions of a *simplicial minor* and the two forbidden graphs see any of the given references.)

4. Infinite graphs

In this section we consider applications of simplicial decompositions to problems in purely infinite graph theory. The applications are based on two theorems: one, due to Diestel, Halin and Vogler, which relates homomorphism and subdivision bases (see Section 1) to universal graphs, and another, due to Halin, which concerns decompositions of uncountable graphs into smaller factors. The axiom of choice will be assumed throughout this section.

For our discussion of universal graphs let us assume that all graphs considered are countable. When $\mathcal{G}$ is a class of graphs and $G^* \in \mathcal{G}$, call G^* (*strongly*) *universal* in $\mathcal{G}$ if G^* contains a copy of every graph $G \in \mathcal{G}$ as a subgraph (as an induced subgraph). Universal graphs were introduced by Rado [60], who constructed a strongly universal graph R for the class of all countable graphs. (Although Rado's construction is explicit, we remark that R is isomorphic to the countably infinite random graph which occurs with probability one when the edges are chosen independently with probability $\frac{1}{2}$.)

If $\mathcal{G}$ is a given monotone decreasing graph property (i.e. if $H \subset G \in \mathcal{G}$ implies $H \in \mathcal{G}$) and G^* is universal in $\mathcal{G}$, then the subgraphs of G^* are precisely the graphs in $\mathcal{G}$. Thus, by constructing a universal graph for such a property $\mathcal{G}$ it may be possible to describe $\mathcal{G}$ 'in a nutshell'. This hope has led several authors ([59, 52, 10]) to investigate which properties have universal graphs, though often with negative results. If $\mathcal{G}$ is given by excluded minors, however, it is often possible to use the homomorphism bases of Table 1 to construct a universal graph: all we have to do is paste the graphs of the base together in a sufficiently general way, allowing for embeddings of any graph with a decomposition into base elements that conforms to the given rules. In this manner universal graphs can be

constructed for most of the classes $\mathcal{G}(\mathrm{H}X)$ where X is one of the planar excluded minors listed in Table 1 [25, 10].

To prove that a given class $\mathcal{G}$ does *not* contain a universal graph is usually not an easy matter; such negative results can be found e.g. in [59], [52] and [10]. However the following theorem, proved in [9] but essentially already contained in [25], allows us to draw on existing decomposition results for excluded minor properties, and thereby to derive easily a large number of negative universal graph theorems.

Theorem 4.1. *Let $\mathcal{X}$ be a set of finite graphs, and let $\mathcal{G} = \mathcal{G}(\mathrm{H}\mathcal{X})$ or $\mathcal{G} = \mathcal{G}(\mathrm{T}\mathcal{X})$. If the homomorphism base (or subdivision base, respectively) of $\mathcal{G}$ is uncountable, then $\mathcal{G}$ contains no universal graph.*

As a consequence of Theorem 4.1 we immediately see that none of the classes $\mathcal{G}(\mathrm{H}X)$ has a universal graph where X is any of the non-planar excluded minors listed in Table 1. Moreover, there is no universal planar graph (consider the homomorphism base for $\mathcal{X} = \{K_5, K_{3,3}\}$), a result originally due to Pach [59].

We finally mention that the converse of Theorem 4.1 does not hold: there are classes $\mathcal{G}(\mathrm{H}\mathcal{X})$ that have no universal graph but a countable, or even finite, homomorphism base [25].

We now turn to applications of simplicial decompositions to uncountable graphs. All these applications are consequences of the following fundamental decomposition theorem due to Halin. Given two vertices x, y of a graph G, let us write $\mu_G(x, y)$ for the *Menger number* of a and b in G, the supremum (in fact, the maximum) of all cardinals m such that G contains m independent $x - y$ paths.

Theorem 4.2 [37, 19]. *Let G be a graph with $|G| \geq a > \aleph_0$ for some regular cardinal a. Suppose that $G \not\supset K_a$, and that $\mu_G(x, y) < a$ whenever x and y are non-adjacent vertices of G. Then G admits a simplicial decomposition $F = (B_\lambda)_{\lambda<\sigma}$, where σ is the initial ordinal of $|G|$ and $|B_\lambda| < a$ for all $\lambda < \sigma$. F can be chosen in such a way that, for each $\mu < \sigma$, $S|_\mu$ does not separate B_μ and every vertex of $S|_\mu$ has a neighbour in $B_\mu \setminus S|_\mu$.*

Theorem 4.2 can often be used to extend results for countable graphs to uncountable ones. We give three examples of this: an infinite version of Hadwiger's conjecture, a result extending a theorem of Jung [51] on the existence of certain spanning trees, and a theorem concerning the so-called *ends* of a graph. More such applications can be found in [36, 37].

If G has chromatic number $\aleph_0$, then $G \supset \mathrm{T}K_r$ for every finite r. Indeed, if $G \not\supset \mathrm{T}K_r$ and $r \in \mathbb{N}$, then $\chi(G') \leq s$ for all finite $G' \subset G$ and some s depending on r [50]; by a well-known theorem of de Bruijn and Erdös [24] this implies that $\chi(G) \leq s$. The following theorem extends this result to arbitrary infinite graphs:

Theorem 4.3 [36]. *If G has chromatic number $\chi(G) \geq \aleph_0$, then $G \supset \mathrm{T}K_a$ for every $a < \chi(G)$.*

We remark that Theorem 4.3 is sharp: G need not contain a $\mathrm{T}K_{\chi(G)}$, even if $\chi(G)$ is a successor cardinal [39, Ch. X. 10.7].

Call a rooted spanning tree T of a graph G *normal* if every pair of adjacent vertices of G is comparable in the partial order on $V(G)$ induced by T. Jung [51] proved that every countable connected graph contains a normal rooted spanning tree. Using Theorem 4.2, this result can be extended as follows:

Theorem 4.4 [37]. *Every connected graph not containing a* $\mathrm{T}K_{\aleph_0}$ *has a normal rooted spanning tree.*

In fact, Halin conjectured that the condition of not containing a $\mathrm{T}K_{\aleph_0}$ can be weakened further:

Conjecture [37]. *A connected graph G has a normal rooted spanning tree if and only if every uncountable set $X \subset V(G)$ contains vertices x, y for which $\mu_G(x, y)$ is finite.*

A similar extension from the countable to the uncountable produces a step forward towards a solution of the following long-standing problem. Call two rays (one-way infinite paths) P, Q in a connected infinite graph G *end-equivalent* if there exists a ray $R \subset G$ which meets both P and Q infinitely often. Let $\mathscr{E}(G)$ denote the set of the corresponding equivalent classes, the *ends* of G. For example, the 2-way infinite ladder has two ends, the infinite grid $\mathbb{Z} \times \mathbb{Z}$ has one end, and the dyadic tree has $2^{\aleph_0}$ ends.

If T is a spanning tree of G and P, Q are end-equivalent rays in T, then clearly P and Q are also equivalent in G. We therefore have a natural map $\eta : \mathscr{E}(T) \to \mathscr{E}(G)$ mapping each end of T to the end of G containing it. In general, η need be neither 1–1 nor onto; if it is both, then T is called *end-faithful*. The following question was raised by Halin in 1964:

Problem [43]. Does every infinite connected graph have an end-faithful spanning tree?

For countable graphs, such a tree was already constructed as the main result in Halin [43]. Using Theorem 4.2, this construction[2] can again be extended:

Theorem 4.5 [18]. *Every connected graph not containing a* $\mathrm{T}K_{\aleph_0}$ *has an end-faithful spanning tree.*

The basic idea in the proofs of Theorems 4.4 and 4.5 is to decompose a given graph G into countable factors by Theorem 4.2, use the countable version of the

[2] Note that the existence statement of Theorem 4.5 as such follows directly from Theorem 4.4, because normal rooted spanning trees are end-faithful. The proof of Theorem 4.4 however is non-constructive.

theorem to find admissible spanning trees in each of the factors, and to combine these spanning trees into one of G. However, in order to avoid the rather restrictive condition on the Menger numbers in Theorem 4.2, Theorem 4.2 is applied not to G itself but to a slight modification of G: its $\aleph_1$-closure.

The *a-closure* $[G]_a$ of a graph G is obtained by adding all edges between non-adjacent vertices x, y with $\mu_G(x, y) \geq a$. It is not difficult to show that in the a-closure of a graph the Menger number of any two non-adjacent vertices is less than a, as required for Theorem 4.2. Moreover, the edges added in the closure operation will not jeopardize the other condition of Theorem 4.2, that $G \not\supset K_a$, since a new K_a can only be created if G already contained a $\mathrm{T}K_a$:

Theorem 4.6 [36, 19]. *For any graph G and any infinite cardinal a the following are equivalent*:

(i) $[G]_a \supset K_a$;
(ii) $[G]_a \supset \mathrm{T}K_a$;
(iii) $G \supset \mathrm{T}K_a$.

Used in conjunction with Theorem 4.6, Theorem 4.2 becomes a very powerful tool indeed for decomposing infinite graphs into smaller factors.

We conclude this section with another application of these two theorems, which generalizes a result of Dirac [22, 23].

Theorem 4.7 [37]. *Let G be an n-connected graph ($n \in \mathbb{N}$), and suppose that a is a regular cardinal with $|G| \geq a > \aleph_0$. Then $G \supset \mathrm{T}K_{n,a}$.*

Corollary 4.8. *If an uncountable graph G is n-connected, $n \in \mathbb{N}$, then $G \supset \mathrm{T}K_n$.*

5. Related decompositions

Among the motivations suggested in the introduction of this paper for decomposing graphs into simplicial factors was the prospect of being able to 'lift assertions about the factors to similar assertions about the whole graph'; k-colourability was given as an example to illustrate the idea. The value of a particular kind of decomposition for this purpose clearly depends on two features of the graph property under investigation. Firstly, the property must be wholly or at least to a controllable extent preserved in the pasting operation, and secondly, it must be easier to investigate the factors of the graph than the graph itself. However, these two objectives obviously work against each other; the more specificly we define our attachment rule, the fewer graphs will be decomposable, and the larger the primes we get – and vice versa. Finding the right kind of decomposition for a given problem is therefore a task of striking a balance.

As was illustrated (and to some degree explained) in Section 1, simplicial

decompositions seem to be just the right kind of decomposition for investigating minor-closed properties, and properties defined in terms of forbidden subdivisions. In general, however, the requirement (S2) that all attachment graphs be simplices seems to be rather on the strict side. If one focusses on monotone increasing classes of attachment graphs (and there seem to be reasons for doing so), simplicial decompositions are even an extreme case, based on the smallest possible class of attachment graphs. And indeed, while quite a few graph properties are compatible with attachment along a simplex (i.e. can be lifted from simplicial factors to the whole graph), simplicial decompositions do tend to leave rather large primes, which are often not fewer in number or simpler in structure than arbitrary graphs with that property. The k-colourable graphs are again a case in point.

Halin [40] suggested to take account of this problem by relaxing condition (S2) in the definition of simplicial decompositions if appropriate, while keeping the tree structure of the decomposition by imposing (S4). This, together with a few additional constraints, would ensure that the structural properties of the decompositions obrained would be similar to simplicial ones, enabling us to transfer some of the existing theory. However, if we place the emphasis firmly on decomposability to a high degree, keeping the tree structure seems unnecessarily restrictive: it may prevent us from decomposing a factor further, even if it has a separator that would be admissible as an attachment graph (an example will be given below).

The simplest way to ensure maximum decomposability (with respect to a fixed class $\mathcal{P}$ of admissible attachment graphs) is to use as an attachment rule the direct reversal of the process of successive separation. For properties $\mathcal{P}$ and $\mathcal{G}$ of finite graphs let us define the *$\mathcal{P}$-sum* of graphs in $\mathcal{G}$ recursively:

(1) Every $G \in \mathcal{G}$ is a $\mathcal{P}$-sum of graphs in $\mathcal{G}$;

(2) If G, G' are $\mathcal{P}$-sums of graphs in $\mathcal{G}$ and $G \cap G' \in \mathcal{P}$, then $G \cup G'$ is a $\mathcal{P}$-sum of graphs in $\mathcal{G}$.

For $\mathcal{P}_k = \{G : |G| = k\}$ and $\mathcal{P}_{\leq k} = \{G : |G| \leq k\}$, we abbreviate '$\mathcal{P}_k$-sum' to '$k$-sum', '$\mathcal{P}_{\leq k}$ sum' to '$(\leq k)$-sum', '$(\mathcal{P} \cap \mathcal{P}_k)$-sum' to '$\mathcal{P}$-$k$-sum' and so on. Moreover, we shall loosely speak of *simplicial* sums, *connected* sums etc. if $\mathcal{P}$ is the property of being complete, connected etc.

Using well-known facts about simplicial decompositions it is not difficult to show that any simplicial sum of certain graphs admits a simplicial decomposition into precisely these graphs as factors (provided only that none of them is contained in another), and with precisely those simplices as simplices of attachment that were used as attachment graphs for building the sum. For a simplicial sum we may therefore usually assume without loss of generality that it was obtained by adding only one factor at a time.

For other sums however this is not the case. As an example, consider the graphs shown in Fig. 2: it is a connected 3-sum of four K_4's, obtained by first pasting the K_4's together in pairs along triangles and then joining the two arising

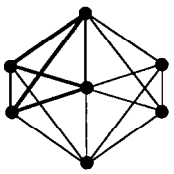


Fig. 2. A connected 3-sum of K_4's.

K_5^-'s along a common path of order 3. This graph cannot be obtained as a 3-sum of K_4's in any other way.

Connected sums are used in a recent paper of Duchet, Las Vergnas and Meyniel [26] to describe two interesting graph properties: 'well-connectedness' and 'null-homotopy'. A graph G is *well-connected* if every minimal relative separator is (non-empty and) connected, and G is *null-homotopic* if every algebraic cycle of G is the sum (mod 2) of triangles. Both these properties are compatible with connected summing: if $G = G' \cup G''$ where G' and G'' are well-connected (null-homotopic) and $G' \cap G''$ is connected, then G is well-connected (null-homotopic). Using Wagner's characterization of the finite graphs without a K_5 minor (see Table 1), Duchet, Las Vergnas and Meyniel obtain the following result:

Theorem 5.1 [26]. *For a finite graph $G \in \mathcal{G}(\mathrm{H}K_5)$ the following statements are equivalent*:

(i) *G is null-homotopic*;

(ii) *G is well-connected*;

(iii) *G is a connected ($\leqslant 3$)-sum of disc-triangulations.*

(A disc-triangulation is a plane graph in which at most one face is not a triangle.) It is interesting to note that if G is required to be planar, the connected 3-sums in Theorem 5.1 can be replaced with simplicial 3-sums [26].

Using other homomorphism bases from Table 1, Theorem 5.1 can be extended as follows:

Theorem 5.2. *If G is a finite connected graph from any of the classes $\mathcal{G}(\mathrm{H}K_{3,3})$, $\mathcal{G}(\mathrm{H}W_{1,5})$, $\mathcal{G}(\mathrm{H}L)$ or $\mathcal{G}(\mathrm{H}(C_3 \times K_2))$, the following statements are equivalent*:

(i) *G is null-homotopic*;

(ii) *G is well-connected*;

(iii) *G is a connected ($\leqslant 3$)-sum of disc-triangulations and copies of K_5.*

If $G \in \mathcal{G}(\mathrm{H}(C_3 \times K_2))$, the disc-triangulations in (iii) *can be chosen as wheels, triangles, K_5^-'s or K_2's.*

An infinite analogue to k-sums can be obtained from the definition of simplicial decompositions by replacing (S2) with a condition on the order of the attachment

graphs. Given an infinite cardinal a and a graph G, call a family $(B_\lambda)_{\lambda<\sigma}$ of induced subgraphs of G an a-decomposition of G if it satisfies (S1), (S3) and

(S_a2) Each subgraph $(\bigcup_{\lambda<\mu} B_\lambda) \cap B_\mu =: S_\mu$ has order $<a$ $(0<\mu<\sigma)$,

(S_a5) Every graph $\bigcup_{\lambda<\mu} B_\lambda$ is an induced subgraph of G $(0<\mu<\sigma)$.

As a consequence of Halin's decomposition theorem (4.2) and Theorem 4.6 we then have the following remarkable result:

Theorem 5.3. *If G is a graph, a is a regular uncountable cardinal and $G \not\supset TK_a$, then G has an a-decomposition $(B_\lambda)_{\lambda<\sigma}$ into factors of order $<a$. This decomposition can be chosen in such a way that, for every $\mu<\sigma$, S_μ separates $(\bigcup_{\lambda<\mu} B_\lambda)\setminus S_\mu$ from $B_\mu \setminus S_\mu$ in G.*

Conversely, a graph with a decomposition as in Theorem 5.3 may well contain a TK_a; for example, a TK_a in which every edge has been subdivided once has such a decomposition. No characterization of the graphs admitting an a-decomposition into factors of order $<a$ is known, for any a.

6. Algorithmic aspects of simplicial decompositions

Let us finally mention some algorithmic aspects of simplicial decompositions. Whitesides [71] and Tarjan [63] were the first to propose algorithms that decompose a given finite graph into simplicial factors. Examples of how to use these algorithms to tackle otherwise NP-complete problems in graph theory are also found in [63] and [71]. The problems considered are vertex colouring [63, 71], 'minimizing the fill-in caused by Gaussian elimination' [63], finding a clique (or a stable set of vertices) of largest weight [63, 71], and finding a maximal weight clique or stable set cover [71]. A refined version of Tarjan's algorithm which finds the unique set of simplicial primes of a finite graph is due to Leimer [54]. (Tarjan's original algorithm ends with a set of subgraphs containing the prime factors as well as some simplices of attachment.) Both algorithms run in $O(nm)$ time, where n and m are the number of vertices and of edges in the graph, respectively. A parallel algorithm for the same problem was recently proposed by Dahlhaus (private communication). This algorithm runs in $O(\log^2 n)$ time on $O(n^4)$ processors.

Algorithmic applications of simplicial decompositions to problems in statistics are considered by Lauritzen and Spiegelhalter [56]. Decompositions of chordal graphs into their cliques (as discussed in Section 3) have applications to problems in areas as diverse as measure theory and database schemes; see Lauritzen, Speed and Vijayan [57] and Beeri et al. [6, 7].

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