Rings whose Cyclic Modules have Finitely Generated Socle*  

R. P. KURSHAN

Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey 07974

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This paper investigates various chain conditions enjoyed by a ring $R$ and the modules over it as a result of restrictions placed on the way simple modules can be embedded into cyclic modules and injective modules. Assuming all action of a ring to be on the left, a ring $R$ is said to be FGS if all cyclic $R$ modules have a finitely generated (or empty) socle (the socle of a module $M$, denoted $s(M)$, is the sum of all simple submodules); a ring with this property is half-Noetherian, as it is finite dimensional (in the sense of Goldie [7]); but the property is stronger than this and will replace Noetherian in theorems about rings with suitably restricted vertical structure (e.g., a left or right perfect ring which is FGS is artinian).

The other half of being Noetherian is the property that whenever $M$ is a submodule of a cyclic module and has a simple essential socle, then $M$ is finitely generated; a ring with this property is called TC. In Section 2 the equivalence of the following is obtained for a ring $R$: (1) $R$ is FGS and TC; (2) $R$ is Noetherian; (3) certain injective modules can be decomposed into enough summands; (4) countable direct sums of injective envelopes of simple modules are injective.

It is well-known that an arbitrary direct sum of injective modules over a Noetherian ring is injective (see [3]); by adapting a proof of Bass, the implication (4) $\Rightarrow$ (2) extends results of Bass and Chase who showed that if an arbitrary direct sum of injective modules (respectively, injective envelopes of simple modules over a semiprimary ring) is injective, then the ring is Noetherian (see [4], p. 471). Also, the implication (3) $\Rightarrow$ (2) extends a theorem of Papp (see [6], p. 208) which states that a ring $R$ is Noetherian if (*) each injective $R$ module is a direct sum of indecomposable modules; for the condition (3) [see (2.4) below] is satisfied when (**) the particular injective modules defined therein: $\prod E(S_i)$ and each $E(S_i)$ are each a

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direct sum of indecomposable modules. Furthermore, since (\(\ast\)) always holds for a Noetherian ring (see Matlis [10], Theorem 2.5), it follows that (\(\ast\ast\)) \(\Rightarrow\) (\(\ast\)).

1. Preliminaries

Throughout this paper all rings are assumed to have an identity and all modules are unital left modules (so Noetherian means left-Noetherian); a module is semisimple if it is equal to its socle; that \(\mathcal{A}\) is an essential submodule of \(\mathcal{B}\) will be denoted \(\mathcal{A} \subseteq \mathcal{B}\); the injective envelope of a module \(\mathcal{M}\) will be denoted \(E(\mathcal{M})\) (see [5]).

If \(\mathcal{I}\) is a subset of a module \(\mathcal{A}\) and \(\mathcal{B}\) is a family of submodules of \(\mathcal{A}\), then for any \(\mathcal{B} \in \mathcal{B}\) the projection of \(\mathcal{I}\) onto its set of cosets in \(\mathcal{A}/\mathcal{B}\) will be denoted by \(\pi\), the context rendering the meaning clear.

If \(\mathcal{Q}\) is an indecomposable injective module, then it can be shown that \(\text{End}_R \mathcal{Q}\) is a local ring. (A module \(\mathcal{Q}\) is indecomposable if whenever \(\mathcal{A}\) and \(\mathcal{B}\) are submodules such that \(\mathcal{A} \bigoplus \mathcal{B} = \mathcal{Q}\) then either \(\mathcal{A} = 0\) or \(\mathcal{B} = 0\).) Using this fact, Azumaya’s Theorem [1] immediately yields the following:

\begin{enumerate}
\item[1.1] \textbf{Theorem.} Let \(\mathcal{M} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{M}_\alpha\) be an arbitrary direct sum of indecomposable injective modules \(\mathcal{M}_\alpha\). Then every indecomposable direct summand of \(\mathcal{M}\) is isomorphic to one of the \(\mathcal{M}_\alpha\)’s and given a second decomposition of \(\mathcal{M}\) into a direct sum of indecomposable injective submodules, \(\mathcal{M} = \bigoplus_{\beta \in \mathcal{B}} \mathcal{N}_\beta\), there exists a bijection \(i : \mathcal{A} \rightarrow \mathcal{B}\) such that \(\mathcal{M}_\alpha \cong \mathcal{N}_{i(\alpha)}\) for all \(\alpha \in \mathcal{A}\).

A module is \textit{finite dimensional} (Goldie) if it does not contain an infinite direct sum of nonzero submodules.

\item[1.2] \textbf{Theorem.} Let \(\mathcal{M}\) be a finite dimensional module. Then there exists a positive integer \(n\) such that: Every direct sum of submodules of \(\mathcal{M}\) contains at most \(n\) components, there exists \(n\) uniform submodules of \(\mathcal{M}\) whose sum is direct, and any direct sum of uniform submodules is essential in \(\mathcal{M}\) if, and only if, it contains \(n\) nonzero components. (A module is uniform if every submodule is indecomposable.)

\textit{Proof.} This is a direct application of (1.1) to \(E(\mathcal{M})\) which must contain an essential finite direct sum of uniform submodules of \(\mathcal{M}\) since \(\mathcal{M}\) is finite dimensional.

This number \(n\) is the \textit{dimension} of \(\mathcal{M}\), denoted \(\dim \mathcal{M}\).

A module \(\mathcal{M}\) is \textit{finitely embedded} if \(s(\mathcal{M})\) is finitely generated and essential in \(\mathcal{M}\) (see Vamos [14]). The notions \textit{finitely generated} and \textit{finitely embedded} are dual in the following sense: A module is finitely generated (embedded) if, and only if, the union (intersection) of an increasing (decreasing) chain of
proper (nontrivial) submodules is proper (nontrivial)—see Osofsky [11]
(Vamos [14]).

2. FGS Rings

Since any module $M$ is an essential submodule of its injective hull, it
follows that $\dim M = \dim E(M)$. Furthermore, it is clear that isomorphic
modules have the same dimension and that the dimension of a finite direct
sum of modules is either infinite or is the sum of the dimensions of the
summands. Also, if $A$ is a submodule of a module $M$ then Zorn's lemma may
be applied to the set of submodules of $M$ having zero intersection with $A$
to obtain a module $B$ maximal with respect to this property; then $A \oplus B \subseteq M$.
Hence, we obtain:

(2.1) LEMMA. If $0 \rightarrow \cdot A \rightarrow M \rightarrow B \rightarrow 0$ is an exact sequence of modules
such that $A$ and $B$ are finite dimensional, then $M$ is also finite dimensional and
$\dim M \leq \dim A + \dim B$.

Proof. Find a submodule $C \subseteq M$ such that $C \cap A = 0$ and $A \subseteq C \subseteq M$;
then $E(M) = E(A) \oplus E(C)$. Also, $0 \rightarrow C \rightarrow M/A \cong B$ and thus surely
$\dim E(C) = \dim C \leq \dim B$. Since $A$ is finite dimensional it follows that
$E(M)$ is too, and $\dim M = \dim E(A) \oplus E(C) \leq \dim A + \dim B$.

(2.2) Proposition. The following conditions on a ring $R$ are equivalent:

(a) $R$ is FGS;

(b) All finitely generated $R$ modules are finite dimensional;

(c) Whenever $T_0 \subseteq T_1 \subseteq \cdots$ is a strictly increasing chain of submodules
of a cyclic $R$ module $C$, then there exists an integer $n$ such that $T_n \subseteq \Sigma T_i$.

When condition (c) holds, the number of modules in such a chain which are not
essential in $\Sigma T_i$ is less than $\dim C$.

Proof. (a) $\Rightarrow$ (b). Suppose $R$ is FGS, and $M$ is a module generated by
$n$ elements. The proof will proceed by induction on $n$. If $n = 1$ then $M$ is
cyclic; if $M$ is not finite dimensional, then there exists an infinite direct sum
of nonzero submodules $T = \bigoplus T_i \subseteq M$, and it may be assumed that each $T_i$
is cyclic. But then submodules $U_i$ maximal in $T_i$ may be found, and upon
setting $U = \bigoplus U_i$ it is evident that although $M/T$ is cyclic, its socle contains
the infinite direct sum $U/T$, which is contrary to the hypothesis. Hence the
assertion is true for $n = 1$.

Now assume that the assertion is true for $n \leq k$, and that
$$M = Rx_1 + \cdots + Rx_{k+1}.$$
Set \( A = R_x \) and \( B = M/A \); by hypothesis \( A \) and \( B \) are finite dimensional. Since also \( 0 \to A \to M \to B \to 0 \), it follows from (2.1) that \( M \) is finite dimensional, and the induction step is complete.

(b) \( \Rightarrow \) (c). Suppose (c) fails; let \( C \) be a cyclic module containing a strictly increasing chain of submodules \( T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots \) such that for all \( j \), \( T_j \) is not essential in \( T = \Sigma T_i \). Then there exists an infinite subchain \( T_{i_1} \subseteq T_{i_2} \subseteq \cdots \) such that \( T_{i_k} \) is not essential in \( T_{i_{k+1}} \) for \( k = 1, 2, \ldots \). Hence, nonzero modules \( U_k \), \( k = 1, 2, \ldots \) may be found such that \( U_k \subseteq T_{i_{k+1}} \) and \( U_k \cap T_{i_k} = 0 \); it follows that \( \Sigma U_k \) is direct and hence \( C \) is not finite dimensional.

(c) \( \Rightarrow \) (a). If \( T = \bigoplus S_i \) is a semisimple submodule of a cyclic module then by (c) there is an integer \( n \) such that \( S_1 \oplus \cdots \oplus S_n \subseteq T \), in which case \( T = S_1 \oplus \cdots \oplus S_n \).

(2.3) Remarks. (1) An FGS ring need not be Noetherian: For example, any non-Noetherian ring with linearly ordered ideals is FGS.

(2) A TC ring need not be Noetherian: For example, any non-Noetherian commutative regular ring (for which each simple module is injective by [13] Theorem 6).

(3) If \( R \) is FGS and either left or right perfect (see [2]), then \( R \) is artinian by [12], Lemma 11.

(2.4) Theorem. Let \( \{S_a\}_{a \in A} \) be the set of (pairwise nonisomorphic) simple \( R \) modules. Then the following are equivalent:

1. \( R \) is FGS and TC.
2. \( R \) is Noetherian.
3. (a) \( \prod_{a \in A} E(S_a) \) has as a direct summand a direct sum of indecomposable modules containing \( \bigoplus_a S_a \); and
   (b) \( E(S_a) \) has as a direct summand a direct sum of indecomposable modules containing the infinite sum \( S_a \oplus S_a \oplus \cdots \), for each \( a \in A \).
4. \( \bigoplus_{i=1}^n E_i \) is injective provided each \( E_i \simeq E(S_\alpha) \) for some \( \alpha \in A \).

Proof. (1) \( \Rightarrow \) (2). Clearly, a Noetherian ring is FGS and TC. Let \( L \) be an ideal of a ring \( R \) satisfying (1); \( R \) will be proved to be Noetherian by showing that \( L \) is finitely generated. To this end the following lemmas will be proved; they are stated in more general terms than are needed here, for later use.

(2.5) Lemma. For every submodule \( L \) of a cyclic module over an FGS ring, there exists a module \( M \subseteq L \) such that \( L/M \) is finitely embedded and such that the inverse image of \( s(L/M) \) with respect to the projection \( L \to L/M \to 0 \) is finitely generated.
(2.6) \textbf{Lemma.} For a ring \( R \), the following are equivalent:

(1) \( R \) is TC.

(2) Every finitely embedded submodule of a cyclic \( R \) module is finitely generated.

(3) Every finitely embedded cyclic \( R \) module is Noetherian.

Then, finding \( M \) as in (2.5), and letting \( T \) be the inverse image of \( s(L/M) \) in \( R \), the short exact sequence

\[
0 \to T \to L \to L/T \to 0
\]

obtains; \( T \) is finitely generated, and \( L/T \) is also finitely generated, being a projection of \( L/M \) which is finitely generated by (2.6). Hence \( L \) is finitely generated.

\textbf{Proof of (2.5).} Suppose \( L \) is a submodule of a cyclic module \( C \) over an FGS ring \( R \); since \( L \) is finite dimensional, cyclic submodules \( I_1, \ldots, I_n \) contained in \( L \) may be found such that their sum \( I \) is direct and essential in \( L \). Find submodules \( M_1, \ldots, M_m \) maximal in \( I_1, \ldots, I_n \), respectively and set \( M = \Sigma M_j \). Then in \( L/M \), \( I \) is semisimple and of finite length. If \( I \) is essential in \( L/M \), then \( L/M \) is finitely embedded. Otherwise, \( I \) is not essential in \( L/M \) and the process may be continued, finding modules \( J_1, \ldots, J_m \) contained in \( L \) such that \( J_j \subseteq M_j \) and such that \( L/M \) is direct and essential.

Proceeding by induction, \( I \) and \( M \) can be defined in this manner for all \( i \) or until for some \( k \), \( I = \cdots + I_k \) is semisimple and essential. But in the former possibility the socle of the cyclic module \( C/\Sigma_{i=1}^k M \) contains \( \bigoplus_{i=1}^k I \) and hence is not finitely generated. Hence the latter holds, and setting \( M = \cdots + M \), \( L/M \) is finitely embedded.

Furthermore, if \( \{y_i\}_{i \in I} \) are chosen so that in the appropriate factor \( M_j \) generates \( J_j \), then the finite set \( \{y_i\}_{i \in I} \) generates the inverse image of \( s(L/M) \). This completes the proof of (2.5).

(2.7) \textbf{Lemma.} For an arbitrary ring \( R \) suppose \( M \cap C \) are \( R \) modules and an \( R \) module \( B \subset C \) is chosen maximal with respect to the property \( M \cap B = 0 \). Then \( M \subset C/B \).

\textbf{Proof.} Given \( c \in C - B \), \( r \in R \) must be found such that \( 0 \neq rc \in M \); indeed, by maximality of \( B \), \( M \cap (B + Rc) \neq 0 \) so there exist \( r \in R, m \in M \) and \( b \in B \) such that \( 0 \neq m = b + rc \), whence \( 0 \neq rc = \tilde{m} \in M \).
Proof of (2.6). (1) \implies (2). Suppose $M$ is a finitely embedded submodule of
a cyclic module $C$ over a TC ring $R$, and $s(M) = S_1 \oplus \cdots \oplus S_n$ ($S_i$ simple).
Find a submodule $M_n \subseteq M$ maximal with respect to the property $S_n \subseteq M_n$ and
$(S_1 \oplus \cdots \oplus S_{n-1}) \cap M_n = 0$. Then $s(M_n) = S_n \subseteq \overline{M_n}$ and so since $R$
is TC, $M_n$ is finitely generated. Furthermore, $S_1 \oplus \cdots \oplus S_{n-1} \subseteq \overline{M/M_n}$ by
(2.7), and $\overline{M/M_n}$ is contained in the cyclic module $C/M_n$. Induction on $n$
completes the argument, showing that $M/M_n$ and hence $M$ are finitely
generated.

(2) \implies (3). Obvious.

(3) \implies (1). Suppose $S, M, C$ are $R$ modules, $S$ simple, $C$ cyclic and
$S \subseteq M \subseteq C$, and suppose further that every finitely embedded cyclic module
is Noetherian. Find a submodule $N \subseteq C$ maximal with respect to the property
that $\forall \alpha, \beta \in \mathbb{N}$, $M \cap N_\alpha \cap N_\beta = 0$. Then by (2.7) $M \subseteq C/N$ where $C/N$ is cyclic and $\overline{M} \cong \overline{M}$. Hence $C/N$ is a finitely embedded cyclic module so $M$ is finitely generated.
This proves that $R$ is TC.

The proof of the theorem now continues:

(2) \implies (3). Since $R$ is Noetherian, $\oplus_{\alpha \in \mathcal{A}} E(S_\alpha)$ is injective and as such is
a direct summand of $\prod_{\alpha \in \mathcal{A}} E(S_\alpha)$; similarly, for $E(S_\alpha)^{\mathbb{N}}$, and so (3) obtains.

(2.8) Lemma. Let $F = \{Q_\alpha\}_\alpha$ be a family of injective $R$-modules. Then
$\bigoplus Q_\alpha$ is injective if, and only if, whenever $L$ is a submodule of a cyclic module $C$
and $h \in \text{Hom}_R(L, \bigoplus Q_\lambda)$, there exists a finite subfamily of $F$, $\{Q_1, \ldots, Q_n\}$,
such that $h(L) \subseteq Q_1 \oplus \cdots \oplus Q_n$.

Proof. If $\bigoplus Q_\alpha$ is injective then every such homomorphism $h$ can be
extended to $h \in \text{Hom}_R(C, \bigoplus Q_\lambda)$, and $h(L) \subseteq h(C) \subseteq \text{Rr}(g)$ where $g$ generates
$C$; clearly $h(g)$ can have a nonzero projection onto at most a finite number of
members of $F$.

The converse is clear, as $R$ is cyclic and a finite direct sum of injective
modules is always injective.

(3) \implies (4). It is sufficient to show that a countable direct sum of copies
of $Q = \bigoplus_{\alpha \in \mathcal{A}} E(S_\alpha)$ is injective, as the sum $\bigoplus_{i=1}^\infty E_i$ will be a direct summand
of such a module. Set $Q' = \bigoplus_{i=1}^\infty Q_i$, where $Q_i \cong Q$ for $i = 1, 2, \ldots$.

It will be shown first that $\bigoplus_{\alpha \in \mathcal{A}} E(S_\alpha)$ is injective and that for each
$\alpha \in \mathcal{A}$ $\bigoplus_{i=1}^\infty E_i^\alpha$ is injective where $E_i^\alpha \cong E(S_\alpha)$ for $i = 1, 2, \ldots$. Indeed, let
$\{P_\beta\}_{\beta \in \mathcal{B}}$ be a family of indecomposable modules such that $P = \bigoplus_{\beta} P_\beta$
is a direct summand of $\prod_{\beta} E(S_\beta)$ and $\bigoplus_{\alpha} S_\alpha \subseteq P$. Each $E(S_\alpha)$ is a direct
summand of $P$ and so by (1.1) for each $\alpha \in \mathcal{A}$ there is a $\beta \in \mathcal{B}$ such that
$P_\beta \cong E(S_\alpha)$. Thus $\bigoplus_{\alpha} E(S_\alpha)$ is a direct summand of an injective module and
hence is itself injective.
Next, fixing $\alpha$, let $E = \prod_{i=1}^{\infty} E_i^\alpha$ and suppose $P = \bigoplus_{\beta} P_\beta$ is a direct summand of $E$ containing the infinite sum $S = S_a \oplus S_a \oplus \cdots$ with each $P_\beta$ indecomposable. Then $P_\beta \cap S \neq 0$ implies that $P_\beta \cong E(S_a)$ and this must be the case for an infinite number of the $\beta$'s. Hence, it follows as above that $\bigoplus_{\beta} E_\beta^\alpha$ is injective.

Now it is shown that $Q'$ is injective: let $L$ be an ideal of $R$ and $h : L \to Q'$ an $R$-map. Writing $E(i, \alpha)$ for the $E(S_i)$-component of $Q_i$, it is sufficient by (2.8) to show that there is a finite subfamily $\{E_1, \ldots, E_m\}$ of the components $\{E(i, \alpha)\}_{i, \alpha}$ such that $h(L) \subseteq E_1 \oplus \cdots \oplus E_m$. Indeed, if this were not the case then, letting $\pi(i, \alpha)$ be the projection $Q' \to E(i, \alpha) \to 0$ ($Q' \cong \bigoplus_{i, \alpha} E(i, \alpha)$), there would exist an infinite set of pairs $(i, \alpha)$ such that $\pi(i, \alpha)(h(L)) \neq 0$. If an infinite number of distinct $\alpha$'s appeared—say $\{\alpha_n\}_{n=1}^{\infty}$, then letting $\{i_n\}_{n=1}^{\infty}$ be the corresponding set of $i$'s, one obtains $\pi(i_n, \alpha_n)h(L) \neq 0$ for $n = 1, 2, \ldots$. Letting $p$ be the projection $Q' \to \bigoplus_{n=1}^{\infty} E(i_n, \alpha_n) \to 0$ and $\pi_k$ the projection $\bigoplus_{n=1}^{\infty} E(i_n, \alpha_n) \to E(i_k, \alpha_k) \to 0$ for $k = 1, 2, \ldots$, then

$$p \circ h : L \to \bigoplus_{n=1}^{\infty} E(i_n, \alpha_n)$$

and $\pi_k(p \circ h)L \neq 0$ for all $k$, which by (2.8) contradicts the assumption that $Q$ is injective. Hence, only a finite number of distinct $\alpha$'s appear, so there is a $\beta$ such that $\pi(i, \beta)h(L) \neq 0$ for an infinite number of $i$'s which similarly contradicts the fact that $\bigoplus_{i=1}^{\infty} E_i^\beta$ is injective. This completes the proof that (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (2). To show that $R$ is Noetherian, it is sufficient to show that each strictly increasing sequence of finitely generated ideals

$$0 = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$

is finite. For such a sequence, find ideals $M_1, M_2, \ldots$ such that for each $i = 1, 2, \ldots, L_{i-1} \subseteq M_i \subseteq L_i$ and $L_i/M_i$ is simple; let $L = \Sigma L_i$, let $\pi_i$ be the projection $L \to L_i/M_i \to 0$ and $k_i$ the inclusion $0 \to L_i/M_i \to E(L_i/M_i)$; let $E_i = E(L_i/M_i)$ and let $\phi_i$ be an isomorphism $\phi_i : E(L_i/M_i) \to E_i \oplus D_i$ for some $D_i$ such that

$$\begin{array}{ccc}
L_i/M_i & \longrightarrow & E(L_i/M_i) \\
\downarrow & & \downarrow \phi_i \\
E_i & \longrightarrow & E_i \oplus D_i
\end{array}$$

commutes, where the unlabelled maps are the obvious inclusions. Let $\beta_i = \phi_i \circ k_i \circ \pi_i$ and set $\beta = \Sigma \beta_i : L \to \bigoplus(E_i \oplus D_i)$ (cf. [4] p. 471; this is
well-defined since \( \beta_j(L_i) = 0 \) for \( j > i \). Finally, let \( \pi \) be the projection \( \oplus (E_i \oplus D_i) \rightarrow \oplus E_i \rightarrow 0 \) and set \( \alpha = \pi \circ \beta \). If \( x \in L_i - M_i \) then

\[
\alpha(x) = \beta_i(x) + \pi \sum_{j < i} \beta_j(x) \quad \text{and} \quad 0 \neq \beta_i(x) \in E_i.
\]

It follows by (2.8) that \( \oplus E_i \) and hence the sequence of \( L_i \)'s is finite.

(2.9) **Corollary.** Suppose \( G \) is an Abelian group such that for any subgroup \( H \), \( G/H \) is reduced and contains no infinite sum of simple groups. Then \( G \) is finitely generated.

**Proof.** Embed \( G \) into a ring with identity in the standard fashion. Clearly, \( R \) is FGS, and since \( G/H \) is reduced for all subgroups \( H \), \( R \) is TC. Hence, \( R \) is Noetherian, and so \( G \) is a finitely generated \( R \) module. But this is tantamount to \( G \) being a finitely generated \( \mathbb{Z} \) module, i.e., a finitely generated Abelian group.

(2.10) **Lemma.** If a ring \( R \) is finitely embedded (as a module over itself) and the Jacobson radical of \( R \): \( J(R) = 0 \), then \( R \) is semisimple (i.e., is equal to its socle).

**Proof.** Since \( J(R) = 0 \), \( R \) has no nilpotent ideals; hence every finitely generated semisimple ideal is generated by an idempotent, and is thus a direct summand of \( R \). In particular, \( s(R) \) is a direct summand and, being essential in \( R \), is itself equal to \( R \).

(2.11) **Corollary.** If \( R \) is a ring such that \( R/J(R) \) is finitely embedded and \( 3 - b \) of (2.4) is satisfied, then \( R \) is Noetherian.

(2.12) **Corollary.** A commutative FGS ring is von Neumann-regular if, and only if, it is semisimple.

**Proof.** A commutative semisimple ring is a direct sum of fields and hence regular. Conversely, if \( R \) is a commutative regular ring then by [13] Theorem 6 every simple \( R \) module is injective. If \( I \) is an ideal of \( R \), \( \{ S_i \} \), is a family of
simple $R$ modules and $0 \neq h \in \text{Hom}_R(L, \bigoplus S_i)$, then $L/\ker h \cong h(L)$ is semisimple; hence, if $R$ is also FGS, $h$ can have a nonzero projection onto only a finite number of the $S_i$'s. Thus by (2.8) $\bigoplus S_i$ is injective and so by (2.4) $R$ is Noetherian. But every finitely generated ideal of a regular ring is generated by an idempotent, and hence every ideal of $R$ is a direct summand, from which it follows that $R$ is semisimple.

3. FINITELY EMBEDDED MODULES

In (2.6) it was shown that the rings for which every finitely embedded cyclic module is Noetherian are exactly the TC rings. However, upon removing the word cyclic, the situation seems to be difficult to characterize in terms of internal properties of the ring.

Vamos [14] and Rosenberg and Zelinsky [13] have dealt with the respective questions: When is $P_1$ — every finitely embedded module Noetherian, and when is $P_2$ — every finitely embedded module of finite length? They showed that if $R$ is commutative, the two questions are equivalent and in that case any finitely embedded finitely generated module $M$ over a Noetherian ring has finite length.

Without some commutivity this last statement is false (see 3.5); however, it is possible to trade off commutivity for semisimplicity — a little at a time — by demanding that the annihilator (or even some power of the annihilator) of the socle of $M$ be contained in the center of the ring (so $j(R) \subseteq Z(R)$ and hence the smaller the center, the closer $R$ is to being semisimple). Then $M$ will have finite length, and hence in this case $P_1$ and $P_2$ remain equivalent.

If $M$ is an $R$ module, the annihilator of $M$ is the two-sided ideal $l(M) = \{r \in R \mid rM = 0\}$.

(3.1) PROPOSITION. Suppose $M$ is a finitely generated module over a Noetherian ring $R$ such that for some submodule $B \subseteq M$, some power of $l(B)$ is central. Then $l(B)^m M = 0$ for some integer $m$.

Proof. Let $N$ be some power of $l(B)$ which is central. Since

$$NM \supseteq N^2M \supseteq \cdots$$

gives rise to an increasing chain of ideals $l(NM) \subseteq l(N^2M) \subseteq \cdots$, there is some integer $m$ such that for $k \geq m$, $l(N^kM) = l(N^mM)$. It will suffice to show that $N^mM = 0$.

Let $\{x_1, \ldots, x_l\}$ be a set of generators for $N$, and $\{y_1, \ldots, y_m\}$ a set of generators for $M$. Then every element of $N^mM$ is a sum of elements of the form $ax_{i_1} \cdots x_{i_t} y_k$ where $a \in R$, $1 \leq i_j \leq t$ and $1 \leq k \leq u$. If $N^mM \neq 0$ order
the distinct nonzero products of the form $x_1 \cdots x_n y_k$ and denote them $X_1, \ldots, X_n$, respectively. Since $B \subseteq M$, $r_1 \in R$ may be found such that $0 \neq r_1 X_j \in B$. Suppose $i$ is the next smallest integer such that $r_1 X_i \neq 0$. Then find $r_2 \in R$ such that $0 \neq r_2 X_i \in B$. Continue in this manner, finding $r_1, \ldots, r_n$ in $R$ such that for some $w$ $0 \neq r_v \cdots r_1 X_{w}$, and $r_v \cdots r_1 X_j \in B$ for $1 \leq j \leq n$. Let $r = r_v \cdots r_1$. Since $r X_j \neq 0$, $r \notin l(N^{w-1} M)$. However, every element of $N^{w+1} M$ is a sum of elements of the form $as_1 \cdots x_n y_k$ where $a \in R$ and $1 \leq i_j \leq t$. Furthermore, $r a x_{i_1} \cdots x_{i_{w+1}} y_k = (a x_{i_1} \cdots x_{i_w}) y_k = 0$ since $ax_{i_j} \in N \subseteq l(B)$ and $r (x_{i_1} \cdots x_{i_{w+1}}) y_k \in B$. Hence $r \in l(N^{w+1} M)$ and so $l(N^{w+1} M) \subseteq l(N^{w-1} M)$, a contradiction. Thus $N^{w+1} M = 0$.

(3.3) **Lemma.** If $M_1, \ldots, M_n$ are maximal ideals of a ring $R$ and $K = \cap M_1 \cap \cdots \cap M_n$, then $R/K$ is semisimple.

**Proof.** Define $h : R/K \to R/M_1 \oplus \cdots \oplus R/M_n$ by $h(r) = (r, r, \ldots, r)$ where each coset $r$ is considered in the appropriate factor. Clearly, $h$ is an $R$ module homomorphism and $h$ is an injection since for each nonzero $r \in R/K$ there is some $M_i$ such that $r \notin M_i$. Hence, $R/K$ is semisimple, being a submodule of a semisimple module.

(3.3) **Theorem.** Suppose $M$ is a finitely generated, finitely embedded module over a Noetherian ring $R$ such that some power of $N = l(s(M))$ is central. Then $M$ has finite length.

**Proof.** By (3.1) there is some integer $m$ such that $N^{m-1} M = 0$. By (3.2) $R/N$ is a semisimple $R$ module, and in fact is a semisimple ring. But $N^{m+1} M/N^{m+1} M$ is an $R/N$ module and it follows that for $i = 0, \ldots, m - 1$ $N^i M/N^{i+1} M$ is a finite sum of simple $R/N$ and hence $R$-modules, which induces a composition series on $M$.

(3.4) **Proposition.** Suppose $R$ is a ring with maximum condition on annihilators which is finitely embedded as an $R$ module. If some power of $N = l(s(R))$ is generated by elements in the center of $R$, then $R$ is artinian.

**Proof.** The proof is analogous to that of (3.3) with the exception that in the proof of (3.1) the generators for $N$ must be chosen to be central.

Notice that since $N$ is nilpotent, $N = J(R)$.

(3.5) **Examples.** There exist nonartinian finitely embedded Noetherian rings with nil radical, and ones whose radical is not nil. An example of the former is the ring of matrices $(a_{ij})$; and an example of the latter is the ring of matrices $(a_{ij})$ where $P$ is the ring of formal power series in one indeterminate over a field, and $Q$ is its (classical) quotient ring—the radical $J$ of
this ring is \((\frac{M}{I})_0\) where \(M\) is the unique maximal ideal of \(P\), and for all \(n\)
\[ I^n = (\frac{M^n}{I^n})_0 \neq 0. \]

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