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Willmore spheres in compact Riemannian manifolds

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Abstract

The paper is devoted to the variational analysis of the Willmore and other L^2 curvature functionals, among immersions of 2-dimensional surfaces into a compact Riemannian m-manifold (M^m, h) with $m > 2$. The goal of the paper is two-fold, on one hand, we give the right setting for doing the calculus of variations (including minmax methods) of such functionals for immersions into manifolds and, on the other hand, we prove the existence results for possibly branched Willmore spheres under various constraints (prescribed homotopy class, prescribed area) or under curvature assumptions for M^m. To this aim, using the integrability by compensation theory, we first establish the regularity for the critical points of such functionals. We then prove a rigidity theorem concerning the relation between CMC and Willmore spheres. Then we prove that, for every non null 2-homotopy class, there exists a representative given by a Lipschitz map from the 2-sphere into M^m realizing a connected family of conformal smooth (possibly branched) area constrained Willmore spheres (as explained in the introduction, this comes as a natural extension of the minimal immersed spheres in homotopy class constructed by Sacks and Uhlenbeck (1981) in, [38], in situations when they do not exist). Moreover, for every $A > 0$ we minimize the Willmore functional among connected families of weak, possibly branched, immersions of the 2-sphere having prescribed total area equal to A and we prove full regularity for the minimizer. Finally, under a mild curvature condition on (M^m, h), we minimize the sum of the area with the square of the L^2 norm of the second fundamental form, among weak possibly branched immersions of the two spheres and we prove the regularity of the minimizer.

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1. Introduction

Throughout the paper (M^m, h) will be a compact connected m-dimensional Riemannian manifold. For a smooth immersion $\vec{\Phi}$ of a compact 2-dimensional surface Σ into (M^m, h) recall the definition of the *Willmore functional*

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2 d\text{vol}_g, \tag{1.1}$$

where the mean curvature $\vec{H}$ is half the trace of the second fundamental form $\mathbb{I}$ and vol_g is the volume form associated to the pullback metric $g := \vec{\Phi}^* h$, of the *energy functional F*

$$F(\vec{\Phi}) := \frac{1}{2} \int_\Sigma |\mathbb{I}|^2 d\text{vol}_g, \tag{1.2}$$

and of the *conformal Willmore functional* W_{conf}

$$W_{\text{conf}}(\vec{\Phi}) := \int_\Sigma \left(|\vec{H}|^2 + \bar{K}(T\vec{\Phi}) \right) d\text{vol}_g, \tag{1.3}$$

where $\bar{K}(T\vec{\Phi})$ is the sectional curvature of the ambient manifold (M^m, h) computed on the tangent space of $\vec{\Phi}(\Sigma)$; recall moreover that W_{conf} is conformally invariant (i.e. is invariant under conformal changes of the ambient metric h); see [43].

Remark 1.1. Observe that, by the Gauss–Bonnet Theorem, for immersions in the Euclidean space $\mathbb{R}^m$ the three functionals W, W_{conf} and F differ just by a topological constant, so they are equivalent from the variational point of view. This is not the case for immersions in Riemannian manifolds, moreover in literature there is not a universal agreement about which L^2-curvature functional has to be called Willmore functional. Indeed, since in Riemannian manifolds W_{conf} is conformal invariant while W is not, the conformal geometry community usually considers W_{conf} and calls it "Willmore functional"; in addition, the pullback of the standard Willmore functional of $\mathbb{R}^3$ into $\mathbb{S}^3$ via the stereographic projection is exactly $W_{\text{conf}} = \int(|\vec{H}|^2 + 1)$, so people interested in the properties of surfaces immersed in $\mathbb{R}^3$ (for instance people working on the Willmore conjecture, see for example [21,22,37,42]) called W_{conf} "Willmore functional". On the other hand, classically, the Willmore surfaces were introduced by Blaschke in the XX's century as generalized minimal surfaces (clearly every minimal surface is stationary for W under compactly supported variations, moreover in $\mathbb{R}^3$ there are no closed minimal surfaces by the maximum principle, so the closed Willmore surfaces in $\mathbb{R}^3$ are in a natural sense "generalized minimal surfaces"), therefore from this point of view it is more natural to call $\int |\vec{H}|^2$ the Willmore functional. Moreover, for immersions in a Riemannian manifold, the functional W appears in the expression of the Hawking mass in general relativity hence, from the physical point of view, W is the interesting quatity to look at (see for instance the introduction of [20] and the references therein for more details). These are the reasons why we call $\int |\vec{H}|^2$ the "Willmore functional" and $\int |\vec{H}|^2 + \bar{K}$ the "conformal Willmore functional". □

The first goal of the present paper is to develop the *analysis* of the Willmore and the others L^2 curvature functionals in Riemannian manifolds of any dimension. Indeed, as for immersions in the euclidean space, there is the following functional analysis paradox: though the Willmore functional W defined in (1.1) has perfect meaning for $W^{2,2} \cap W^{1,\infty}$ weak immersions, the classical form of its Euler–Lagrange equation (derived in [43]) does not make sense for such weak objects (which are the natural ones for doing the analysis of the Willmore functional, as it will be explained below; exactly as the Sobolev spaces are the natural framework to studying PDEs). Indeed it requires L^3 integrability of second derivatives being

$$\Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} - \tilde{R}(\vec{H}) = 0, \tag{1.4}$$

where $\tilde{R} : T_{\vec{\Phi}(x)}M \to T_{\vec{\Phi}(x)}M$ is the curvature endomorphism defined by

$$\forall \vec{X} \in T_{\vec{\Phi}(x)}M \quad \tilde{R}(\vec{X}) := -\pi_{\vec{n}} \left[\sum_{i=1}^{2} \mathrm{Riem}^h(\vec{X}, \vec{e}_i)\vec{e}_i \right], \tag{1.5}$$

and $\tilde{A} : T_{\vec{\Phi}(x)}M \to T_{\vec{\Phi}(x)}M$ is defined as

$$\forall \vec{X} \in T_{\vec{\Phi}(x)}M \quad \tilde{A}(\vec{X}) := \sum_{i,j=1}^{2} \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j)\langle \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j), \vec{X}\rangle, \tag{1.6}$$

where $\pi_{\vec{n}}$ is the projection onto the normal space of $\vec{\Phi}$ and $(\vec{e}_1, \vec{e}_2)$ is an orthonormal basis of $T\Sigma$ for the induced metric $g := \vec{\Phi}^*h$. The same problem appears in the other L^2 curvature functionals since the difference in the Euler–Lagrange equations is given just by lower order terms.

A first achievement of the present work is to rewrite the Euler–Lagrange equation in a conservative form which makes sense for such weak immersions. In order to be more accessible, before we perform the computations and we present the equations in the codimension one case in Section 2, then we pass to the more delicate higher codimensional case.

For this purpose, up to a reparametrization and working on a parametrizing disc D^2, we can assume that the immersion $\vec{\Phi}$ is conformal so that it makes sense to consider the standard complex structure of the disc D^2; exploiting the complex notation is very convenient and simplifies also the initial presentation given by the second author in [32] for immersions in the euclidean space. The main result of Section 3 is the following theorem. Before stating it let us define $R^\perp_{\vec{\Phi}}(T\vec{\Phi})$ as

$$R^\perp_{\vec{\Phi}}(T\vec{\Phi}) := \left(\pi_T \left[\mathrm{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H}\right]\right)^\perp \tag{1.7}$$

where $(\vec{e}_1, \vec{e}_2)$ is a positive orthonormal basis of TD^2 for the induced metric $g := \vec{\Phi}^*h$, $\pi_T : T_{\vec{\Phi}}M \to \vec{\Phi}_*(TD^2)$ is the tangential projection and $.^\perp$ denotes the rotation of an angle $\frac{\pi}{2}$ in $\vec{\Phi}_*(TD^2)$ in the direction from $\vec{e}_1$ towards $\vec{e}_2$, intrinsically it can be written as $\vec{X}^\perp = (\vec{\Phi}_*) \circ *_g \circ (\vec{\Phi}_*)^{-1}(\vec{X})$ for any $\vec{X} \in \vec{\Phi}_*(TD^2)$ where $*_g$ is the Hodge duality operator on (TD^2, g). We shall also denote by $\pi_{\vec{n}}$ the orthogonal projection $\pi_{\vec{n}} : T_{\vec{\Phi}}M \to (\vec{\Phi}_*(TD^2))^\perp$ from the tangent space to M^m onto the normal space to $\vec{\Phi}(D^2)$.

We shall denote by D the Levi-Civita connection of (M^m, h) and by an abuse of notations we also denote by D the associated covariant exterior derivative. We also denote by D_g the pull back by $\vec{\Phi}$ of D which is a connection – respectively a covariant exterior derivative – of the pull-back bundle $\vec{\Phi}^{-1}TM^m$. $D_g^{*_g}$ is the adjoint of the covariant derivative D^g for the induced metric $g = \vec{\Phi}^*h$. $\star_h$ is the Hodge operator associated to h on multi-vectors of M^m from $\wedge^p M^m$ into $\wedge^{m-p} M^m$. All these objects are defined in Section 3 in (3.1)–(3.5) and (3.7).

Theorem 1.1. *Let $\vec{\Phi}$ be a smooth immersion of the two dimensional disc D^2 into an m-dimensional Riemannian manifold (M^m, h), then the following identity holds*

$$\begin{aligned}&\frac{1}{2}D_g^{*_g}\left[D_g\vec{H} - 3\pi_{\vec{n}}(D_g\vec{H}) + \star_h\left((*_g\,D_g\vec{n})\wedge_M\vec{H}\right)\right]\\&\quad= \Delta_\perp\vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\,\vec{H} - R^\perp_{\vec{\Phi}}(T\vec{\Phi})\end{aligned}\tag{1.8}$$

where $\Delta_\perp$ is the negative covariant Laplacian on the normal bundle to the immersion, $\tilde{A}$ is the linear map given in (1.6), $R^\perp$ is defined in (1.7). □

Notice that though the right hand side does not make sense for $W^{1,\infty}\cap W^{2,2}$ weak immersions, the left hand side does. Therefore a straightforward but important consequence of Theorem 1.1 is the following conservative form of Willmore surfaces equation making sense for $W^{1,\infty}\cap W^{2,2}$ weak immersions.

Corollary 1.1. *A smooth immersion $\vec{\Phi}$ of a 2-dimensional disc D^2 in (M^m, h) is Willmore if and only if*

$$\frac{1}{2}D_g^{*_g}\left[D_g\vec{H} - 3\pi_{\vec{n}}(D_g\vec{H}) + \star_h\left((*_g\,D_g\vec{n})\wedge_M\vec{H}\right)\right] = \tilde{R}(\vec{H}) - R^\perp_{\vec{\Phi}}(T\vec{\Phi}),\tag{1.9}$$

where $\tilde{R}$ and $R^\perp$ are the curvature endomorphisms defined respectively in (1.5) and (1.7). □

Remark 1.2. The Euler–Lagrange equations of the other L^2 curvature functionals are computed in Section 3 and differ just by terms completely analogous to the right hand side terms of (1.9). □

Another important corollary is the conservative form of the *constrained-conformal* Willmore equation. Let Σ be a smooth closed surface and $\vec{\Phi}:\Sigma\hookrightarrow M$ be a smooth immersion; the pullback metric $g := \vec{\Phi}^*h$ induces a complex structure J on Σ, and in the associated conformal class there exists a unique constant curvature metric c_0 with total area 1 (see [14]); notice that by construction $\vec{\Phi}:(\Sigma, c_0)\hookrightarrow M$ is a conformal immersion. Recall that the smooth immersion $\vec{\Phi}$ of (Σ, c_0) is said to be *constrained-conformal Willmore* if and only if it is a critical point of the Willmore functional under the constraint that the conformal class is fixed. Before writing the conservative form of the Willmore functional under constraint on the conformal class let us introduce some notation. Call $Q(J)$ the space of holomorphic quadratic differentials on (Σ, J) and let $q\in Q(J)$ written in local complex coordinates as $q = f(z)dz\otimes dz$; let $\vec{H}_0$ be the Weingarten map given in local coordinates, for a conformal immersion with conformal factor $\lambda = \log(|\partial_x\vec{\Phi}|)$, by

$$\begin{aligned}\vec{H}_0 &:= \frac{1}{2}e^{-2\lambda}\pi_{\vec{n}}\left(\partial^2_{x^2}\vec{\Phi} - \partial^2_{y^2}\vec{\Phi} - 2i\,\partial^2_{xy}\vec{\Phi}\right)\\&= \frac{1}{2}\left[\vec{\mathbb{I}}(\vec{e}_1,\vec{e}_1) - \vec{\mathbb{I}}(\vec{e}_2,\vec{e}_2) - 2\,i\,\vec{\mathbb{I}}(\vec{e}_1,\vec{e}_2)\right],\end{aligned}\tag{1.10}$$

where $\vec{n}_{\vec{\Phi}}$ is the normal space to $\vec{\Phi}$ and $(\vec{e}_1,\vec{e}_2) = e^{-\lambda}(\partial_x\vec{\Phi}, \partial_y\vec{\Phi})$ is a positively oriented orthonormal frame of $T\vec{\Phi}$; recall also the definition of the Weingarten operator $\vec{h}_0$ given locally by

$$\vec{h}_0 := 2\,\pi_{\vec{n}}(\partial^2_{z^2}\vec{\Phi})\,dz\otimes dz = e^{2\lambda}\,\vec{H}_0\,dz\otimes dz.\tag{1.11}$$

We introduce on the space $\wedge^{1-0} D^2 \otimes \wedge^{1-0} D^2$ of $1-0 \otimes 1-0$ form on D^2 the following hermitian product[1] depending on the conformal immersion $\vec{\Phi}$:

$$(\psi_1\, dz \otimes dz, \psi_2\, dz \otimes dz)_{WP} := e^{-4\lambda}\ \psi_1(z)\ \overline{\psi_2(z)} \tag{1.12}$$

where $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. We observe that for a conformal change of coordinate $w(z)$ (i.e. w is holomorphic in z) and for ψ_i' satisfying

$$\psi_i' \circ w\, dw \otimes dw = \psi_i\, dz \otimes dz$$

one has, using the conformal immersion $\vec{\Phi} \circ w$ in the l.h.s.

$$(\psi_1'\, dw \otimes dw, \psi_2'\, dw \otimes dw)_{WP} = (\psi_1\, dz \otimes dz, \psi_2\, dz \otimes dz)_{WP}$$

for more information about the Weil–Peterson product see [14,35,36]. Now we can write the constrained-conformal Willmore equation in conservative form.

Corollary 1.2. *Let $\vec{\Phi} : \Sigma \hookrightarrow M$ be a smooth immersion into the $m \geq 3$-dimensional Riemannian manifold (M^m, h) and call c the conformal structure associated to $g = \vec{\Phi}^* h$. Then $\vec{\Phi}$ is a constrained-conformal Willmore immersion if and only if there exists an holomorphic quadratic differential $q \in Q(c)$ such that*

$$\begin{aligned}&\frac{1}{2} D_g^{*_g}\left[D_g\vec{H} - 3\pi_{\vec{n}}(D_g\vec{H}) + \star_h\left((*_g\, D_g\vec{n}) \wedge_M \vec{H}\right)\right]\\ &\quad = \Im(q, \vec{h}_0)_{WP} + \tilde{R}(\vec{H}) - R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}). \quad \square\end{aligned} \tag{1.13}$$

Observe that, in local complex coordinates, $\Im[(q, \vec{h}_0)_{WP}] = e^{-2\lambda}\Im[f(z)\overline{\vec{H}_0}]$.

Notice that also the constrained-conformal equations of the other L^2 curvature functional differ just by terms completely analogous to the right hand side terms of (1.9).

Exploiting the conservative form just showed, in Section 5 we prove that the constrained-conformal Willmore equation is equivalent to a system of conservation laws (see Theorem 5.1) and in Section 6 we prove that weak solutions to this system of conservation laws are smooth. For proving the regularity it is crucial to construct from the system of conservation laws some potentials $\vec{R}$ and S which satisfy a critical Wente type elliptic system (see the system (6.17)). Using integrability by compensation we gain some regularity on $\vec{R}$ and S which bootstrapped, after some work, gives the smoothness of weak solutions to the constrained-conformal Willmore equation. Therefore we are able to prove the following full regularity theorem for weak solutions to the constrained-conformal Willmore equation.

Theorem 1.2 (*Regularity of Weak Constrained-Conformal Willmore Immersions*)**.** *Let $\vec{\Phi}$ be a $W^{1,\infty}$ conformal immersion of the disc D^2 taking values into a sufficiently small open subset of the Riemannian manifold (M, h), with second fundamental form in $L^2(D^2)$ and conformal factor $\lambda := \log|\partial_{x^1}\vec{\Phi}| \in L^\infty(D^2)$. If $\vec{\Phi}$ is a constrained-conformal Willmore immersion then $\vec{\Phi}$ is C^∞.* $\square$

Remark 1.3. As the reader will see, the proof of the regularity is not just a straightforward adaptation of the Euclidean one. Indeed in the euclidean case $\vec{R}$ and S were real valued and

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[1] This hermitian product integrated on D^2 is the *Weil–Peterson product*.

their existence was ensured by a direct application of the Poincaré Lemma. Here the curvature terms make the situation more delicate. Indeed $\vec{R}$ and S, which now are complex valued, are constructed using the D_z and ∂_z operators (see Lemma 6.2), and their construction makes use of singular integrals and Fourier analysis (see the Appendix). Notice that, in case of null curvature, the imaginary parts of $\vec{R}$ and S vanish and the two, a priori different, constructions coincide. Therefore our construction is canonical and has a geometric, beside analytic, meaning. □

Remark 1.4. The regularity issues regarding minimizers of L^2 curvature functionals in 3-dimensional Riemannian manifolds have been studied also in [15] using techniques from [39]. Beside the fact that here we deal with higher codimensions, the real advantage of this new approach is that it permits to infer that *any weak solution* to the equation is smooth, while in the former the regularity crucially used the minimality property. Therefore our new approach is more flexible and it is suitable for studying existence of more general critical points of *saddle* type. To this purpose, we recall that in [2] the analysis of Palais–Smale sequences for the Willmore functional in the Euclidean space is performed. Since the Riemannian curvature terms are subcritical, the same results should hold for immersions in Riemannian manifolds. □

Remark 1.5. Since the difference between the Willmore equation and the Euler–Lagrange equations of the other L^2 curvature functionals F and W_{conf} (also under area or conformal type constraint) is made of subcritical terms, the Regularity Theorem 1.2 applies to them as well. □

Another application of the conservative form of the equation is the following. Recall that an immersion is called *conformal Willmore* if it is a critical point of the conformal Willmore functional W_{conf} defined in (1.3), and is called *constrained-conformal conformal Willmore* if it is a critical point of W_{conf} under the constraint of fixed conformal class. Notice that, since by the Uniformization Theorem there is just one smooth conformal class on $\mathbb{S}^2$, the two notions coincide for smooth immersions of $\mathbb{S}^2$. Recall also that a smooth immersion $\vec{\Phi} : \Sigma \hookrightarrow M^m$ of the surface Σ has *parallel mean curvature* if the normal projection of the covariant derivative of the mean curvature $\vec{H}$ with respect to tangent vectors to $\vec{\Phi}$ is null:

$$\pi_{\vec{n}}(D\vec{H}) = 0. \tag{1.14}$$

Observe that in codimension one a surface has parallel mean curvature if and only if it has a constant mean curvature, i.e. it is a CMC surface.

In Section 4, we prove the following proposition (the analogous proposition for immersions in the Euclidean space appears in [36]) which ensures abundance of constrained-conformal conformal Willmore surfaces in space forms. Since, as explained above, the conformal constraint for smooth immersions of a 2-sphere is trivial, the proposition ensures also abundance of conformal Willmore spheres in space forms.

Proposition 1.1. *Let (M^m, h) be an m-dimensional Riemannian manifold of constant sectional curvature $\bar{K}$ and let $\vec{\Phi} : \Sigma \hookrightarrow M^m$ be a smooth immersion of the smooth surface Σ.*

If $\vec{\Phi}$ has parallel mean curvature then $\vec{\Phi}$ is constrained-conformal conformal Willmore. □

The assumption on the sectional curvature is not trivial, indeed combining results of [30,25] we get the following rigidity theorem.

Theorem 1.3 (*Rigidity for Willmore*)**.** *Let (M^3, h) be a compact 3-dimensional Riemannian manifold with constant scalar curvature. Then M has constant sectional curvature if and only if every smooth constant mean curvature sphere is conformal Willmore.* □

After having studied the analysis of the Euler–Lagrange equation of the mentioned L^2 curvature functionals we move to establish existence of minimizers of such functionals. We will study both *curvature* and *topological* conditions which ensure the existence of a minimizer.

Before passing to the existence theorems observe that minimizing the Willmore and the other L^2 curvature functionals among smooth immersion is of course a-priori an ill posed variational problem. In [35] (see also [33]), the second author introduced the suitable setting for dealing with minimization problems whose highest order term is given by the Willmore energy. We now recall the notion of *weak branched immersions with finite total curvature*.

By virtue of the Nash theorem we can always assume that M^m is isometrically embedded in some euclidean space $\mathbb{R}^n$. We first define the Sobolev spaces from $\mathbb{S}^2$ into M^m as follows: for any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$

$$W^{k,p}(\mathbb{S}^2, M^m) := \left\{ u \in W^{k,p}(\mathbb{S}^2, \mathbb{R}^n) \text{ s.t. } u(x) \in M^m \text{ for a.e. } x \in \mathbb{S}^2 \right\}.$$

Now we introduce the space of *possibly branched Lipschitz immersions*: a map $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is a *possibly branched Lipschitz immersion* if

(i) there exists $C > 1$ such that

$$\forall x \in \mathbb{S}^2 \quad C^{-1} |d\vec{\Phi}|^2(x) \leq |d\vec{\Phi} \wedge d\vec{\Phi}|(x) \leq |d\vec{\Phi}|^2(x) \tag{1.15}$$

where the norms of the different tensors have been taken with respect to the standard metric on $\mathbb{S}^2$ and with respect to the metric h on M^m and where $d\vec{\Phi} \wedge d\vec{\Phi}$ is the tensor given in local coordinates on $\mathbb{S}^2$ by

$$d\vec{\Phi} \wedge d\vec{\Phi} := 2\, \partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}\, dx_1 \wedge dx_2 \in \wedge^2 T^*\mathbb{S}^2 \otimes \wedge^2 T_{\vec{\Phi}(x)} M^m.$$

(ii) There exists at most finitely many points $\{a_1 \cdots a_N\}$ such that for any compact $K \subset \mathbb{S}^2 \setminus \{a_1 \cdots a_N\}$

$$\operatorname*{ess\,inf}_{x \in K} |d\vec{\Phi}|(x) > 0. \tag{1.16}$$

For any *possibly branched Lipschitz immersion* we can define almost everywhere the *Gauss map*

$$\vec{n}_{\vec{\Phi}} := \star_h \frac{\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}}{|\partial_{x_1} \vec{\Phi} \wedge \partial_{x_2} \vec{\Phi}|} \in \wedge^{m-2} T_{\vec{\Phi}(x)} M^m$$

where (x_1, x_2) is a local arbitrary choice of coordinates on $\mathbb{S}^2$ and $\star_h$ is the standard Hodge operator associated to the metric h on multi-vectors in TM.

With these notations we define the following.

Definition 1.1. A Lipschitz map $\vec{\Phi} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is called "weak, possibly branched, immersion" if $\vec{\Phi}$ satisfies (1.15) for some $C \geq 1$, if it satisfies (1.16) and if the Gauss map satisfies

$$\int_{\mathbb{S}^2} |D\vec{n}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_g < +\infty \tag{1.17}$$

where $d\mathrm{vol}_g$ is the volume form associated to $g := \vec{\Phi}^* h$ the pull-back metric of h by $\vec{\Phi}$ on $\mathbb{S}^2$, D denotes the covariant derivative with respect to h and the norm $|D\vec{n}_{\vec{\Phi}}|$ of the tensor $D\vec{n}_{\vec{\Phi}}$ is taken with respect to g on $T^*\mathbb{S}^2$ and h on $\wedge^{m-2} TM$. The space of "weak, possibly branched, immersion" of $\mathbb{S}^2$ into M^m is denoted $\mathcal{F}_{\mathbb{S}^2}$. $\square$

Using Müller–Sverak theory of weak isothermic charts (see [28]) and the Hélein moving frame technique (see [13]) one can prove the following proposition (see [33]).

Proposition 1.2. *Let $\vec{\Phi}$ be a weak, possibly branched, immersion of $\mathbb{S}^2$ into M^m in $\mathcal{F}_{\mathbb{S}^2}$ then there exists a bi-Lipschitz homeomorphism Ψ of $\mathbb{S}^2$ such that $\vec{\Phi} \circ \Psi$ is weakly conformal: it satisfies almost everywhere on $\mathbb{S}^2$*

$$\begin{cases} |\partial_{x_1}(\vec{\Phi} \circ \Psi)|_h^2 = |\partial_{x_2}(\vec{\Phi} \circ \Psi)|_h^2 \\ h(\partial_{x_1}(\vec{\Phi} \circ \Psi), \partial_{x_2}(\vec{\Phi} \circ \Psi)) = 0 \end{cases}$$

where (x_1, x_2) are local arbitrary conformal coordinates in $\mathbb{S}^2$ for the standard metric. Moreover $\vec{\Phi} \circ \Psi$ is in $W^{2,2} \cap W^{1,\infty}(\mathbb{S}^2, M^m)$. □

Remark 1.6. In view of Proposition 1.2 a careful reader could wonder why we do not work with *conformal* $W^{2,2}$ weak, possibly branched, immersion only and why we do not impose for the membership in $\mathcal{F}_{\mathbb{S}^2}$, $\vec{\Phi}$ to be conformal from the beginning. The reason why this would be a wrong strategy and why we have to keep the flexibility for weak immersions not to be necessarily conformal is clear in the proof of the existence theorems, Section 8 and in the Appendix where we will study the variations of the functionals under general perturbations which do not have to respect infinitesimally the conformal condition. □

Now that we have introduced the right framework we pass to discuss the existence theorems.

Fix a point $\bar{p} \in M^m$ and a 3-dimensional subspace $\mathfrak{S} < T_{\bar{p}}M$ of the tangent space to M at $\bar{p}$. We denote

$$R_{\bar{p}}(\mathfrak{S}) := \sum_{i \neq j,\ i,j=1,2,3} \bar{K}_{\bar{p}}\left(\vec{E}_i, \vec{E}_j\right) \tag{1.18}$$

where $\{\vec{E}_1, \vec{E}_2, \vec{E}_3\}$ is an orthonormal basis of $\mathfrak{S}$ and $\bar{K}_{\bar{p}}(\vec{E}_i, \vec{E}_j)$ denotes the sectional curvature of (M, h) computed on the plane spanned by $(\vec{E}_i, \vec{E}_j)$ contained in $T_{\bar{p}}M$. Notice that $R_{\bar{p}}(\mathfrak{S})$ coincides with the scalar curvature at $\bar{p}$ of the 3-dimensional submanifold of M obtained exponentiating $\mathfrak{S}$. Under a condition on $R_{\bar{p}}(\mathfrak{S})$, in the following theorem we minimize the functional F_1 defined on $\mathcal{F}$ as

$$F_1(\vec{\Phi}) := \int_{\mathbb{S}^2} \left(\frac{1}{2}|\mathbb{I}|^2 + 1\right) d\mathrm{vol}_g = F(\vec{\Phi}) + A(\vec{\Phi}). \tag{1.19}$$

Theorem 1.4. *Let (M^m, h) be a compact Riemannian manifold and assume there is a point $\bar{p}$ and a 3-dimensional subspace $\mathfrak{S} < T_{\bar{p}}M$ such that $R_{\bar{p}}(\mathfrak{S}) > 6$, where $R_{\bar{p}}(\mathfrak{S})$ is the curvature quantity defined in (1.18). Then there exists a branched conformal immersion $\vec{\Phi}$ of $\mathbb{S}^2$ into (M^m, h) with finitely many branched points $b^1, \dots, b^N$, smooth on $\mathbb{S}^2 \setminus \{b^1, \dots, b^N\}$, minimizing the functional F_1 in $\mathcal{F}_{\mathbb{S}^2}$, i.e. among weak branched immersions with finite total curvature.* □

Observe that the unit round m-dimensional sphere $\mathbb{S}^m$ with canonical metric has $R_{\bar{p}}(\mathfrak{S}) \equiv 6$ for any base point $\bar{p}$ and any subspace $\mathfrak{S}$, so the assumption is that our ambient manifold has at least one point $\bar{p}$ and at least three directions spanning $\mathfrak{S}$ where the manifold is "more positively curved" than $\mathbb{S}^m$. Let us make a remark about the regularity in the branch points.

Remark 1.7. The removability of point singularities for Willmore surfaces in Euclidean space has been studied in [16,17,32]; recently Y. Bernard and the second author, in [4], proved that the parametrization is smooth also in the branch points if two residues vanish. Analogous statements should hold for branched Willmore immersions in manifolds. □

Remark 1.8. It is always possible to minimize F_1 by forcing the immersion to pass through a fixed family of points. For an arbitrary choice of points sufficiently close to the minimizers we found in Theorem 1.4, this should generate a Willmore sphere passing through these points but satisfying the Willmore equation only away from these points. Since in the variational argument these points cannot be moved the corresponding residues obtained in [16,32,4] have no reason to vanish and the conformal parametrization $\vec{\Phi}$ of a minimizer should be at most $C^{1,\alpha}$ in general. This should contrast presumably with the situation at the branched points of the minimizers obtained in Theorem 1.4. Since these points are left free during the minimization procedure, the first residue $\vec{\gamma}_0$ (see [4]) should vanish and the conformal map $\vec{\Phi}$ should be at least $C^{2,\alpha}$ at these points. □

Remark 1.9. Another interesting question to deepen concerns the analysis of energy identities/loss of energy in the possible neck regions. Such a study was performed in [3] for the Willmore functional in Euclidean space; we expect that similar results hold for immersions of spheres (or of higher genus surfaces under the assumption that the conformal classes do not degenerate in the moduli space) in Riemannian manifolds. □

Now let us consider the problem of minimizing the functional $F = \int |\mathbb{I}|^2$. In codimension one, E. Kuwert, J. Schygulla and the first author, in [15], proved the existence of a smooth immersion of $\mathbb{S}^2$ *without branched points* minimizing the functional F under curvature conditions on the compact ambient 3-manifold (see also [27] for *non compact* ambient 3-manifolds); notice that the topological argument employed for excluding the branch points crucially depends on the codimension one assumption. Therefore, in higher codimension, it makes sense to look for minimizers of F among *branched* immersions, as done in the following theorem.

Theorem 1.5. *Let (M^m, h) be a compact Riemannian manifold. Assume there is a minimizing sequence for the functional $F = \frac{1}{2}\int |\mathbb{I}|^2$ in $\mathcal{F}_{\mathbb{S}^2}$ (among weak possibly branched immersions with finite total curvature), $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{S}^2}$, with area bounded by positive constants from below and above:*

$$0 < \frac{1}{C} \le A(\vec{\Phi}_k) \le C < \infty.$$

Then there exists a branched conformal immersion $\vec{\Phi}$ of $\mathbb{S}^2$ into (M^m, h) with finitely many branched points $b^1, \dots, b^N$, smooth on $\mathbb{S}^2 \setminus \{b^1, \dots, b^N\}$, minimizing the functional F in $\mathcal{F}_{\mathbb{S}^2}$, i.e. among weak branched immersions with finite total curvature. □

Remark 1.10. By analogous arguments to the proof of Theorem 1.4, the lower bound on the area is ensured if $R_{\bar{p}}(\mathfrak{S}) > 0$ for some point $\bar{p}$ and 3-dimensional subspace $\mathfrak{S} < T_{\bar{p}}M$.

Notice that a uniform upper bound on the areas of the minimizing sequence is a crucial information for compactness issues; moreover generally this is not a trivial property in view of the possibility of totally geodesic laminations (a similar constraint appears in [23]). □

Up to here we studied the existence of minimizers of curvature functionals under *curvature* conditions on the ambient manifold. Now we move to consider the existence of area-constrained Willmore spheres under *topological* conditions on the ambient manifold.

For any $x_0 \in M^m$ we denote respectively by $\pi_2(M^m, x_0)$ the *homotopy groups of based maps* from $\mathbb{S}^2$ into M^m sending the south pole to x_0 and by $\pi_0(C^0(\mathbb{S}^2, M^m))$ the free homotopy classes. It is well known that the group $\pi_2(M^m, x_0)$ for different x_0 are isomorphic to each other and $\pi_2(M)$ denotes any of the $\pi_2(M^m, x_0)$ modulo isomorphisms. Recall that, in [38], J. Sacks and K. Uhlenbeck proceeded to the minimization of the Dirichlet energy

$$E(\vec{\Phi}) = \frac{1}{2}\int_{\mathbb{S}^2} |d\vec{\Phi}|^2 \, d\mathrm{vol}_{\mathbb{S}^2}$$

among mappings $\vec{\Phi}$ of the two sphere $\mathbb{S}^2$ into M^m within a fixed based homotopy class in $\pi_2(M^m, x_0)$ in order to generate area minimizing, possibly branched, immersed spheres realizing this homotopy class.

Even if the paper had a great impact in mathematics, the program of Sacks and Uhlenbeck was only partially successful. Indeed the possible loss of compactness arising in the minimization process can generate a union of immersed spheres realizing the corresponding *free homotopy class* but for which the underlying component in the *homotopy group* $\pi_2(M^m)$ may have been forgotten (for more details see also the Introduction to [26]). It is very hard in the Sacks–Uhlenbeck's work to distinguish the classes which are realized by minimal conformal immersions from the somehow not satisfying classes. At least Sacks and Uhlenbeck could prove that the set of *satisfying classes* generates, as a π_1-module, the homotopy group $\pi_2(M^m)$.

To overcome this difficulty, we minimize a curvature functional – corresponding to $A + W$ in the absence of branched points – under homotopy constraint and we prove that, even if we still have a bubbling phenomenon, the limit object must be *connected*. More precisely we show that for every non trivial 2-homotopy group of M^m there is a canonical representative given by a Lipschitz map from $\mathbb{S}^2$ to M realizing the *connected* union of conformal branched area-constrained Willmore spheres which are smooth outside the branched points. Notice that this is a natural generalization of Sacks–Uhlenbeck's procedure in a sense that, if a class γ in $\pi_2(M^m)$ possesses an area minimizing immersion $\vec{\Phi}$ then $\vec{H}_{\vec{\Phi}} \equiv 0$, in particular $\vec{\Phi}$ is an area-constrained Willmore sphere minimizing $A + W$ in its homotopy class.

Before stating the theorem let us recall that for any Lipschitz mapping $\vec{a}$ from $\mathbb{S}^2$ into M^m, $(\vec{a})_*[\mathbb{S}^2]$ denotes the current given by the push-forward by $\vec{a}$ of the current of integration over $\mathbb{S}^2$: for any smooth two-form ω on M^m

$$\left\langle (\vec{a})_*[\mathbb{S}^2], \omega \right\rangle := \int_{\mathbb{S}^2} (\vec{a})^*\omega.$$

Moreover we denote with $[\vec{a}] \in \pi_2(M^m)$ the 2-homotopy class corresponding to the continuous map $\vec{a} : \mathbb{S}^2 \to M^m$.

Theorem 1.6. *Let (M^m, h) be a compact Riemannian manifold and fix $0 \neq \gamma \in \pi_2(M^m)$. Then there exist finitely many branched conformal immersions $\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{\mathbb{S}^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ with $[\vec{f}] = \gamma$ satisfying*

$$\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^{N} \vec{\Phi}^i(\mathbb{S}^2) \quad \text{and} \tag{1.20}$$

$$\vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^{N} \vec{\Phi}^i_*[\mathbb{S}^2]. \tag{1.21}$$

Moreover for every i, the map $\vec{\Phi}_i$ is a conformal branched area-constrained Willmore immersion which is smooth outside the finitely many branched points $b^1, \dots, b^{N_i}$. More precisely we mean that, outside the branched points, every $\vec{\Phi}^i$ is a smooth solution to the Willmore equation with the Lagrange multiplier $2\vec{H}$:

$$\frac{1}{2} D_g^{*_g}\left[D_g \vec{H} - 3\pi_{\vec{n}}(D_g \vec{H}) + \star_h\left((*_g\, D_g \vec{n}) \wedge_M \vec{H}\right)\right] = 2\vec{H} + \tilde{R}(\vec{H}) - R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \quad (1.22)$$

where $\pi_{\vec{n}}$ is the projection onto the normal space to $\vec{\Phi}$, $\star_h$ and $*_g$ are respectively the Hodge operator on (M, h) and $(\mathbb{S}^2, g := \vec{\Phi}^* h)$; $\tilde{R}$ and $R^{\perp}$ are the curvature endomorphisms defined respectively in (1.5) and (1.7). The operators D_g, $D_g^{*_g}, \dots$ are defined above (see also more explicit expressions in Section 3). □

Remark 1.11. With the same proof, the analogous theorem about the existence of a connected family of smooth branched conformal immersions of $\mathbb{S}^2$ which are area-constrained critical points for the functional F and are realizing a fixed homotopy class holds. □

Remark 1.12. It might be interesting to investigate whether the minimizer in a fixed homotopy class is really obtained by a Lipschitz realization of *more than one* smooth branched immersions of spheres or it is realized by *exactly one* smooth branched immersion of $\mathbb{S}^2$. The asymptotic behavior of the solutions at possible connection points of 2 distinct spheres in relation with the cancellation of the first residue $\vec{\gamma}_0$ mentioned in Remark 1.8 (which should also hold in the situation of Theorem 1.6) is a starting point for studying the possibility to have such connection points while considering an absolute minimizer. □

Let us give here an idea of the proof of Theorem 1.6. Consider the following Lagrangian L defined on $\mathcal{F}_{\mathbb{S}^2}$

$$L(\vec{\Phi}) := \int_{\mathbb{S}^2}\left(\frac{1}{4}|\mathbb{I}|^2 - \frac{1}{2}\bar{K}(T\vec{\Phi}) + 1\right) d\mathrm{vol}_g, \quad (1.23)$$

where $\bar{K}(T\vec{\Phi})$ is the sectional curvature of the ambient manifold (M^m, h) evaluated on the tangent space to $\vec{\Phi}(\mathbb{S}^2)$ and observe that, by the Gauss equation, outside the branch points it holds

$$\frac{1}{4}|\mathbb{I}|^2 - \frac{1}{2}\bar{K}(T\vec{\Phi}) + 1 = |\vec{H}|^2 + 1 - \frac{1}{2}K_{\vec{\Phi}}, \quad (1.24)$$

where $K_{\vec{\Phi}}$ is the Gauss curvature of $\vec{\Phi}$, i.e. the sectional curvature of the metric $g = \vec{\Phi}^*(h)$ on $\mathbb{S}^2$. Notice that, since the Gauss curvature integrated on compact subsets away the branch points gives a null Lagrangian (i.e. a Lagrangian with null first variation with respect to compactly supported variations), the Euler–Lagrange equation of L coincides with the Euler–Lagrange equation of $\int(|\vec{H}|^2 + 1)$ outside the branched points; therefore the critical points of L satisfy the area-constrained Willmore equation (1.31) outside the branched points.

Our approach is then to minimize L; the space on which the minimization procedure is performed is the set $\mathcal{T}$ of $N+1$-tuples $\vec{T} = (\vec{f}, \vec{\Phi}^1, \dots, \vec{\Phi}^N)$, where N is an arbitrary positive integer, where $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ and $\vec{\Phi}^i \in \mathcal{F}_{\mathbb{S}^2}$ satisfy (1.20) and (1.21); naturally we define

$$L(\vec{T}) = \sum_{i=1}^{N} L(\vec{\Phi}^i). \quad (1.25)$$

Observe that, up to rescaling the ambient metric h by a positive constant, we can always assume that $\bar{K} \leq 1$ on all M (or equivalently choose in (1.23), instead of 1, a large positive constant $C > \max_M \bar{K}$). On a minimizing sequence $\vec{T}_k$ under the constraint that the map $\vec{f}_k \in W^{1,\infty}(\mathbb{S}^2, M^m)$ is in the fixed homotopy class $0 \neq \gamma \in \pi_2(M^m)$, both the areas and the L^2 norms of the second fundamental forms are clearly equibounded; therefore, using results from [26] we construct a minimizer $\vec{T}_\infty = (\vec{f}_\infty, \vec{\Phi}^1_\infty, \ldots, \vec{\Phi}^{N_\infty}_\infty) \in \mathcal{T}$ such that $\vec{f}_\infty \in W^{1,\infty}(\mathbb{S}^2, M^2)$ is still in the homotopy class γ. Using the regularity theory developed in Section 6 we conclude with the smoothness of the $\vec{\Phi}^i_\infty$ outside the finitely many branched points.

Observe that, for *small* values of the area, smooth (contractible in M) area constrained-Willmore spheres have been constructed in [18] (see also [8,19,20,24,25]) as perturbations of small geodesic spheres using perturbative methods; notice that instead Theorem 1.6 deals with the global situation when the topology of the ambient manifold plays a crucial role. Moreover in the next theorem we produce area-constrained Willmore spheres for *any* value of the area. More precisely consider the Lagrangian W_K defined on $\mathcal{F}_{\mathbb{S}^2}$ as follows

$$W_K(\vec{\Phi}) := \int_{\mathbb{S}^2} \left(\frac{1}{4}|\mathbb{I}|^2 - \frac{1}{2}\bar{K}(T\vec{\Phi}) \right) d\mathrm{vol}_g. \tag{1.26}$$

Using the Gauss equation, one has

$$\frac{1}{4}|\mathbb{I}|^2 - \frac{1}{2}\bar{K}(T\vec{\Phi}) = |\vec{H}|^2 - \frac{1}{2}K_{\vec{\Phi}}, \tag{1.27}$$

and, as before, this implies that the critical points of W_K satisfy exactly the Willmore equation outside the branch points. Notice moreover that, if one considers just *non branched* immersions then W_K is exactly the Willmore functional W up to an additive topological constant by the Gauss–Bonnet Theorem, so minimizing W_K under area constraint among *branched* immersions is the natural generalization of minimizing W under area constraint among *non branched* immersions; moreover the possibility of having a branched minimal sphere (for the existence of branched minimal spheres in Riemannian manifolds see for example [38]) for a fixed value of the area suggests that the correct setting, for the global problem of minimizing the Willmore functional under area constraint for *not necessarily small* values of the area, is the one of branched immersions.

Theorem 1.7. *Let (M^m, h) be a compact Riemannian manifold and fix* any $\mathcal{A} > 0$. *Then there exist finitely many branched conformal immersions $\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{\mathbb{S}^2}$ and a Lipschitz map $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ with*

$$\sum_{i=1}^{N} A(\vec{\Phi}_i) = \mathcal{A}, \tag{1.28}$$

$$\vec{f}(\mathbb{S}^2) = \bigcup_{i=1}^{N} \vec{\Phi}(\mathbb{S}^2) \quad \textit{and} \tag{1.29}$$

$$\vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^{N} \vec{\Phi}^i_*[\mathbb{S}^2], \tag{1.30}$$

such that for every i, the map $\vec{\Phi}_i$ is a conformal branched area-constrained Willmore immersion which is smooth outside the finitely many branched points $b^1, \ldots, b^{N_i}$. More precisely we mean *that, outside the branched points, every* $\vec{\Phi}^i$ *is a smooth solution to the Willmore equation with the Lagrange multiplier* $\mathfrak{a}\vec{H}$ *(for some* $\mathfrak{a} \in \mathbb{R}$*)*

$$\frac{1}{2}D_g^{*_g}\left[D_g\vec{H} - 3\pi_{\vec{n}}(D_g\vec{H}) + \star_h\left((*_g\, D_g\vec{n}) \wedge_M \vec{H}\right)\right] = \mathfrak{a}\vec{H} + \tilde{R}(\vec{H}) - R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \quad (1.31)$$

with the same notation as in Theorem 1.6. Moreover the $N+1$*-tuple* $\vec{T} = (\vec{f}, \vec{\Phi}^1, \ldots, \vec{\Phi}^N)$ *minimizes the functional* W_K *in the set of tuples* $\mathcal{T}$ *(defined above) having area* $\mathcal{A}$*, where the area and the* W_K *functional of an element* $\vec{T} = (\vec{f}, \vec{\Phi}^1, \ldots, \vec{\Phi}^N) \in \mathcal{T}$ *are defined in a natural way as* $A(\vec{T}) = \sum_{i=1}^N A(\vec{\Phi}^i)$ *and* $W_K(\vec{T}) = \sum_{i=1}^N W_K(\vec{\Phi}^i)$ *respectively.* □

With the same proof, the analogous theorem about the existence of a connected family of smooth branched conformal immersions of $\mathbb{S}^2$ which are area-constrained critical points for the functional F and whose total area is an arbitrary $\mathcal{A} > 0$ holds; this connected family moreover minimize the functional F in $\mathcal{T}$ under the area constraint $A(T) = \mathcal{A}$.

Remark 1.13. For small area $\mathcal{A} < \varepsilon_0$, by the monotonicity formula (the monotonicity formula is a crucial tool introduced in [39], for the proof in this context see Lemma 7.2) the minimizer has also small diameter and thanks to the estimates contained in [19,18], the minimum of the functional W_K is close to 2π. With arguments analogous to [18] (using [21]), one checks that, for small area $\mathcal{A} < \varepsilon_0$, the minimizer produced in Theorem 1.7 is made of *just one* smooth *non branched* area-constrained Willmore immersion of the 2-sphere. Therefore Theorem 1.7 is the natural generalization of the main theorem in [18], where Lamm and Metzger minimize the Willmore functional under *small area* constraint among *non branched* little spheres. □

Notations and conventions.

For the Riemann curvature tensor Riem^h of (M^m, h) we use the convention of [45] (notice that other authors, like [9], adopt the opposite sign convention): for any $\vec{X}, \vec{Y}, \vec{Z} \in T_xM$ define

$$\mathrm{Riem}^h(\vec{X}, \vec{Y})\vec{Z} := D_{\vec{X}}D_{\vec{Y}}\vec{Z} - D_{\vec{Y}}D_{\vec{X}}\vec{Z} - D_{[\vec{X},\vec{Y}]}\vec{Z}.$$

The *Hodge operator* on $\mathbb{R}^m$ or more generally on the tangent space T_xM of an oriented Riemannian manifold (M^m, h) is the linear map from $\wedge^p T_xM$ into $\wedge^{m-p} T_xM$ which to a p-vector α assigns the $m-p$-vector $\star_h\alpha$ on T_xM such that for any p-vector β in $\wedge^p T_xM$ the following identity holds:

$$\beta \wedge \star_h \alpha = \langle \beta, \alpha \rangle_h \vec{E}_1 \wedge \cdots \wedge \vec{E}_m \quad (1.32)$$

where $(\vec{E}_1, \ldots, \vec{E}_m)$ is an orthonormal positively oriented basis of T_xM and $\langle \cdot, \cdot \rangle_h$ is the scalar product on $\wedge^p T_xM$ induced by h. Notice that even if M^m is not orientable, we can still define $\star_h$ locally.

We will also need the concept of *interior multiplication* $\llcorner$ between p- and q-vectors, $p \geq q$, producing a $p-q$-vector such that (see [10] 1.5.1 combined with 1.7.5): for every choice of p-, q- and $p-q$-vectors, respectively α, β and γ the following holds:

$$\langle \alpha \llcorner \beta, \gamma \rangle = \langle \alpha, \beta \wedge \gamma \rangle. \quad (1.33)$$

We call $\bullet$ the following contraction operation which to a pair of p- and q-vectors (α, β) assigns the $p+q-2$-vector $\alpha \bullet \beta$ such that:

– if $q = 1$, $\alpha \bullet \beta := \alpha \llcorner \beta$,

– if $\alpha \in \wedge^p T_xM$, $\beta \in \wedge^q T_xM$ and $\gamma \in \wedge^s T_xM$ then

$$\alpha \bullet (\beta \wedge \gamma) := (\alpha \bullet \beta) \wedge \gamma + (-1)^{rs}(\alpha \bullet \gamma) \wedge \beta. \tag{1.34}$$

2. The conservative form of the Willmore surface equation in 3-dimensional manifolds

Let Σ^2 be an abstract closed surface, (M, h) a 3 dimensional Riemannian manifold and $\vec{\Phi} : \Sigma^2 \hookrightarrow (M, h)$ a smooth immersion. Since the following results are local, we can work locally in a disc-neighborhood of a point and use isothermal coordinates on this disc. This means that we can assume $\vec{\Phi}$ to be a conformal immersion from the unit disc $D^2 \subset \mathbb{R}^2$ into (M, h).

Let us introduce some notations. Given the conformal immersion $\vec{\Phi} : D^2 \hookrightarrow (M, h)$ we call $g := \vec{\Phi}^* h = e^{2\lambda}(dx_1^2 + dx_2^2)$ the induced metric; denote $(\vec{e}_1, \vec{e}_2)$ the orthonormal basis of $\vec{\Phi}_*(T\Sigma^2)$ given by

$$\vec{e}_i := e^{-\lambda}\, \frac{\partial \vec{\Phi}}{\partial x_i},$$

where $e^\lambda = |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$. The unit normal vector $\vec{n}$ to $\vec{\Phi}(\Sigma)$ is then given by

$$\vec{n} = \star_h(\vec{e}_1 \wedge \vec{e}_2).$$

Denoted with D the covariant derivative of (M, h) we have the second fundamental form

$$\vec{\mathbb{I}}(\vec{X}, \vec{Y}) := -\langle D_{\vec{X}}\vec{n}, \vec{Y}\rangle \vec{n}$$

and the mean curvature

$$\vec{H} := \frac{1}{2}\left[\vec{\mathbb{I}}(\vec{e}_1, \vec{e}_1) + \vec{\mathbb{I}}(\vec{e}_2, \vec{e}_2)\right].$$

Introduce moreover the *Weingarten operator* expressed in conformal coordinates (x_1, x_2) as:

$$\vec{H}_0 := \frac{1}{2}\left[\vec{\mathbb{I}}(e_1, e_1) - \vec{\mathbb{I}}(e_2, e_2) - 2\,i\,\vec{\mathbb{I}}(e_1, e_2)\right].$$

In [32] an alternative form to the Euler–Lagrange equation of Willmore functional in euclidean setting was proposed; our goal is to do the same for immersions in a Riemannian manifold.

Theorem 2.1. *Let $\vec{\Phi}$ be a smooth immersion of a two dimensional manifold Σ^2 into a 3-dimensional Riemannian manifold (M^3, h); restricting the immersion to a small disc neighborhood of a point where we consider local conformal coordinate, we can see $\vec{\Phi}$ as a conformal immersion of D^2 into (M, h). Then the following identity holds*

$$\begin{aligned} &-2e^{2\lambda}\Delta_g H\,\vec{n} - 4e^{2\lambda}\,\vec{H}\,(H^2 - (K^g - K^h)) + 2e^{2\lambda} R^\perp_{\vec{\Phi}}(T\vec{\Phi}) \\ &\quad = D^*\left[-2\nabla H \vec{n} + H D\vec{n} - H \star_h(\vec{n} \wedge D^\perp \vec{n})\right], \end{aligned} \tag{2.1}$$

where $\vec{H}$ is the mean curvature vector of the immersion $\vec{\Phi}$, Δ_g is the negative Laplace–Beltrami operator, $\star_h$ is the Hodge operator associated to metric h, $D\cdot := (D_{\partial_{x_1}\vec{\Phi}}\cdot, D_{\partial_{x_2}\vec{\Phi}}\cdot)$ and $D^\perp\cdot := (-D_{\partial_{x_2}\vec{\Phi}}\cdot, D_{\partial_{x_1}\vec{\Phi}}\cdot)$ and D^* is an operator acting on couples of vector fields $(\vec{V}_1, \vec{V}_2)$ along $\vec{\Phi}_*(T\Sigma)$ defined as

$$D^*(\vec{V}_1, \vec{V}_2) := D_{\partial_{x_1}\vec{\Phi}}\vec{V}_1 + D_{\partial_{x_2}\vec{\Phi}}\vec{V}_2.$$

Finally recall the definition (1.7) *of* $R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) := (\mathrm{Riem}(\vec{e}_1,\vec{e}_2)\vec{H})^{\perp} = \star_h\big(\vec{n}\wedge \mathrm{Riem}^h(\vec{e}_1,\vec{e}_2)\vec{H}\big)$. $\square$

A straightforward but important consequence of Theorem 2.1 is the following conservative form of Willmore surfaces equations.

Corollary 2.1. *A conformal immersion* $\vec{\Phi}$ *of a* 2*-dimensional disc* D^2 *is Willmore if and only if*

$$2e^{2\lambda}[R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) + \vec{H}\mathrm{Ric}_h(\vec{n},\vec{n})] = D^*\left[-2D\vec{H} + 3HD\vec{n} - \star_h(\vec{H}\wedge D^{\perp}\vec{n})\right]. \quad \square \tag{2.2}$$

Now recall that an immersion $\vec{\Phi}$ is said to be *constrained-conformal Willmore* if and only if it is a critical point of the Willmore functional under the constraint that the conformal class is fixed. In [6] is derived the Willmore equation under conformal constraint for immersions of surfaces in a 3-dimensional Riemannian manifold, which, matched with Theorem 2.1, gives the following corollary.

Corollary 2.2. *A conformal immersion* $\vec{\Phi}$ *of a* 2*-dimensional disc* D^2 *is constrained-conformal Willmore if and only if there exists an holomorphic function* $f(z)$ *such that*

$$\begin{aligned} &2e^{2\lambda}\left[R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) + \vec{H}\mathrm{Ric}_h(\vec{n},\vec{n}) + e^{-2\lambda}\Im(f(z)\overline{\vec{H}_0})\right] \\ &\quad = D^*\left[-2D\vec{H} + 3HD\vec{n} - \star_h(\vec{H}\wedge D^{\perp}\vec{n})\right]. \quad \square \end{aligned} \tag{2.3}$$

We proceed in the following way: first we prove a general lemma for conformal immersions of the 2-disc in (M^3,h), then we pass to the proof of the theorem and of its corollaries.

Lemma 2.1. *Let* $\vec{\Phi}$ *be a conformal immersion from* D^2 *into* (M,h)*. Denote by* $\vec{n}$ *the unit normal vector* $\vec{n} = *_h(\vec{e}_1\wedge\vec{e}_2)$ *of the conformal immersion* $\vec{\Phi}$ *and denote by* H *the mean curvature. Then the following identity holds*

$$-2H\,\nabla\vec{\Phi} = D\vec{n} + \star_h(\vec{n}\wedge D^{\perp}\vec{n}) \tag{2.4}$$

where $D\cdot := (D_{\partial_{x_1}\vec{\Phi}}\cdot, D_{\partial_{x_2}\vec{\Phi}}\cdot)$ *and* $D^{\perp}\cdot := (-D_{\partial_{x_2}\vec{\Phi}}\cdot, D_{\partial_{x_1}\vec{\Phi}}\cdot)$. $\square$

Proof of Lemma 2.1. Denote $(\vec{e}_1,\vec{e}_2)$ the orthonormal basis of $\vec{\Phi}_*(T\Sigma^2)$ given by

$$\vec{e}_i := e^{-\lambda}\,\frac{\partial\vec{\Phi}}{\partial x_i},$$

where $e^{\lambda} = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. The unit normal vector $\vec{n}$ is then given by

$$\vec{n} = \star_h(\vec{e}_1\wedge\vec{e}_2).$$

We have

$$\begin{cases} \langle\vec{e}_1, \star_h(\vec{n}\wedge D^{\perp}\vec{n})\rangle = -\langle D^{\perp}\vec{n},\vec{e}_2\rangle \\ \langle\vec{e}_1, \star_h(\vec{n}\wedge D^{\perp}\vec{n})\rangle = \langle D^{\perp}\vec{n},\vec{e}_1\rangle. \end{cases}$$

From which we deduce

$$\begin{cases} -\star_h(\vec{n}\wedge D_{\partial_{x_2}\vec{\Phi}}\vec{n}) = \langle D_{\partial_{x_2}\vec{\Phi}}\vec{n},\vec{e}_2\rangle\vec{e}_1 - \langle D_{\partial_{x_2}\vec{\Phi}}\vec{n},\vec{e}_1\rangle\,\vec{e}_2 \\ \star_h(\vec{n}\wedge D_{\partial_{x_1}\vec{\Phi}}\vec{n}) = -\langle D_{\partial_{x_1}\vec{\Phi}}\vec{n},\vec{e}_2\rangle\vec{e}_1 + \langle D_{\partial_{x_1}\vec{\Phi}}\vec{n},\vec{e}_1\rangle\,\vec{e}_2. \end{cases}$$

Thus, by the symmetry of the second fundamental form,

$$\begin{cases} D_{\partial_{x_1}\vec{\Phi}}\vec{n} - \star_h(\vec{n}\wedge D_{\partial_{x_2}\vec{\Phi}}\vec{n}) = [\langle D_{\partial_{x_2}\vec{\Phi}}\vec{n}, \vec{e}_2\rangle + \langle D_{\partial_{x_1}\vec{\Phi}}\vec{n}, \vec{e}_1\rangle]\vec{e}_1 \\ D_{\partial_{x_2}\vec{\Phi}}\vec{n} + \star_h(\vec{n}\wedge D_{\partial_{x_1}\vec{\Phi}}\vec{n}) = [\langle D_{\partial_{x_2}\vec{\Phi}}\vec{n}, \vec{e}_2\rangle + \langle D_{\partial_{x_1}\vec{\Phi}}\vec{n}, \vec{e}_1\rangle]\,\vec{e}_2. \end{cases}$$

Since $H = -e^{-\lambda}\ 2^{-1}[\langle D_{\partial_{x_2}\vec{\Phi}}\vec{n}, \vec{e}_2\rangle + \langle D_{\partial_{x_1}\vec{\Phi}}\vec{n}, \vec{e}_1\rangle]$ we deduce (2.4) and Lemma 2.1 is proved. □

Proof of Theorem 2.1 and its corollaries. First let us introduce the operator D^* acting on couples of vector fields $(\vec{V}_1, \vec{V}_2)$ along $\vec{\Phi}_*(T\Sigma)$ defined as

$$D^*(\vec{V}_1, \vec{V}_2) := D_{\partial_{x_1}\vec{\Phi}}\vec{V}_1 + D_{\partial_{x_2}\vec{\Phi}}\vec{V}_2.$$

We can again assume that $\vec{\Phi}$ is conformal. First apply the operator D^* to (2.4) and multiply by H. This gives

$$-2H^2 D^* D\vec{\Phi} - 2H\nabla H\cdot D\vec{\Phi} = H\,D^*\left[D\vec{n} + \star_h(\vec{n}\wedge D^{\perp}\vec{n})\right]. \tag{2.5}$$

We replace $-2H\,D\vec{\Phi}$ in (2.5) by the expression given by (2.4), moreover we also use the expression of the mean curvature vector in terms of $\vec{\Phi}$:

$$D^* D\vec{\Phi} = 2e^{2\lambda}\ \vec{H}. \tag{2.6}$$

So (2.5) becomes

$$-4H^2\ \vec{H}\ e^{2\lambda} + \nabla H\cdot\left[D\vec{n} + \star_h(\vec{n}\wedge D^{\perp}\vec{n})\right] = H\,D^*\left[D\vec{n} + \star_h(\vec{n}\wedge D^{\perp}\vec{n})\right]. \tag{2.7}$$

By the Gauss equations, called K^g the Gauss curvature of Σ and $K^h = K^h(\vec{\Phi}_*(T\Sigma))$ the sectional curvature of (M, h) evaluated on the tangent plane to $\vec{\Phi}(\Sigma)$ we have

$$(K^g - K^h)\vec{n} = -\frac{1}{2}\star_h(D\vec{n}\wedge D^{\perp}\vec{n})e^{-2\lambda}.$$

Using that the Hodge duality $\star_h$ commutes with the covariant differentiation D we get

$$D^*[\star_h(\vec{n}\wedge D^{\perp}\vec{n})] = \star_h(D\vec{n}\wedge D^{\perp}\vec{n}) + \star_h[\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi}, \partial_{x_1}\vec{\Phi})\vec{n}]$$

where we use the convention that $\mathrm{Riem}^h(\vec{X}, \vec{Y})\vec{Z} := D_{\vec{X}}\vec{Y} - D_{\vec{Y}}\vec{X} - D_{[\vec{X},\vec{Y}]}\vec{Z}$; putting together the last two equations we obtain

$$D^*[\star_h(\vec{n}\wedge D^{\perp}\vec{n})] = -2e^{2\lambda}(K^g - K^h)\vec{n} + \star_h[\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi}, \partial_{x_1}\vec{\Phi})\vec{n}]. \tag{2.8}$$

Computing (2.7) $- 2H$(2.8) we get

$$\begin{aligned} &-4e^{2\lambda}\ \vec{H}\ (H^2 - (K^g - K^h)) + \nabla H\cdot\left[D\vec{n} + \star_h(\vec{n}\wedge D^{\perp}\vec{n})\right] \\ &\quad -2\star_h\left(\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi}, \partial_{x_1}\vec{\Phi})\vec{H}\right) = H\,D^*\left[D\vec{n} - \star_h(\vec{n}\wedge D^{\perp}\vec{n})\right]. \end{aligned} \tag{2.9}$$

Since $D^*(\nabla H\vec{n}) = e^{2\lambda}\Delta_g H\vec{n} + \nabla H\cdot D\vec{n}$ we have

$$\begin{aligned} HD^*(D\vec{n} - \star_h(\vec{n}\wedge D^{\perp}\vec{n})) = {}& 2e^{2\lambda}\Delta_g H\vec{n} + \nabla H\star_h(\vec{n}\wedge D^{\perp}\vec{n}) \\ & + D^*[-2\nabla H\vec{n} + HD\vec{n} - H\star_h(\vec{n}\wedge D^{\perp}\vec{n})] \\ & + \nabla HD\vec{n}. \end{aligned} \tag{2.10}$$

Plugging (2.10) into (2.9) we obtain

$$\begin{aligned}&-4e^{2\lambda}\,\vec{H}\,(H^2-(K^g-K^h))-2\star_h\left(\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi})\vec{H}\right)-2e^{2\lambda}\Delta_g H\,\vec{n}\\ &\quad= D^*\left[-2\nabla H\vec{n}+HD\vec{n}-H\star_h(\vec{n}\wedge D^{\perp}\vec{n})\right].\end{aligned}\tag{2.11}$$

Now observe that

$$\begin{cases}\langle\star_h(\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi})\vec{H}),\vec{e_1}\rangle=-\langle \mathrm{Riem}^h(\vec{e_2},\vec{e_1})\vec{H},\vec{e_2}\rangle\\ \langle\star_h(\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi})\vec{H}),\vec{e_2}\rangle=\langle \mathrm{Riem}^h(\vec{e_2},\vec{e_1})\vec{H},\vec{e_1}\rangle\end{cases}$$

and the normal component is null; hence

$$\star_h\left(\vec{n}\wedge \mathrm{Riem}^h(\partial_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi})\vec{H}\right)=-e^{2\lambda}(\mathrm{Riem}(\vec{e_1},\vec{e_2})\vec{H})^{\perp}=:-e^{2\lambda}R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})$$

where $\cdot^{\perp}$ denotes the rotation in the plane $\vec{\Phi}_*(T\Sigma)$ of $\frac{\pi}{2}$ in the sense from $\vec{e_1}$ to $\vec{e_2}$. Therefore we finally write the relation (2.11) as

$$\begin{aligned}&-2e^{2\lambda}\Delta_g H\,\vec{n}-4e^{2\lambda}\,\vec{H}\,(H^2-(K^g-K^h))+2e^{2\lambda}R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})\\ &\quad= D^*\left[-2\nabla H\vec{n}+HD\vec{n}-H\star_h(\vec{n}\wedge D^{\perp}\vec{n})\right].\end{aligned}\tag{2.12}$$

Now recall that the immersion $\vec{\Phi}$ is Willmore if and only if

$$(\Delta_g H)\vec{n}+2\vec{H}(H^2-(K^g-K^h))+\vec{H}\mathrm{Ric}_h(\vec{n},\vec{n})=0$$

so, using Eq. (2.12), $\vec{\Phi}$ is a Willmore immersion if and only if

$$2e^{2\lambda}R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})+2e^{2\lambda}\vec{H}\mathrm{Ric}_h(\vec{n},\vec{n})=D^*\left[-2\nabla H\vec{n}+HD\vec{n}-H\star_h(\vec{n}\wedge D^{\perp}\vec{n})\right].\tag{2.13}$$

Observing that $D\vec{H}=D(H\vec{n})=HD\vec{n}+\nabla H\vec{n}$ we can rewrite the last relation as

$$2e^{2\lambda}[R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})+\vec{H}\mathrm{Ric}_h(\vec{n},\vec{n})]=D^*\left[-2D\vec{H}+3HD\vec{n}-\star_h(\vec{H}\wedge D^{\perp}\vec{n})\right]\tag{2.14}$$

which is the desired identity.

In [6] is derived the Willmore equation under conformal constraint for immersions of surfaces in a 3-dimensional Riemannian manifold; more precisely, by [6, Proposition 2], $\vec{\Phi}$ is a conformal constrained Willmore immersion if and only if there exists an holomorphic quadratic differential $q\in H^0(K^2)$ such that $W'(\vec{\Phi})=\delta^*(q)$, which is equivalent to ask that there exists an holomorphic function $f(z)$ such that $W'(\vec{\Phi})=e^{-2\lambda}\Im(f(z)\overline{\vec{H}_0})$. Hence $\vec{\Phi}$ is conformal constrained Willmore if and only if

$$-(\Delta_g H)\vec{n}-2\vec{H}(H^2-(K^g-K^h))-\vec{H}\mathrm{Ric}_h(\vec{n},\vec{n})=e^{-2\lambda}\Im(f(z)\overline{\vec{H}_0})\tag{2.15}$$

for some holomorphic function $f(z)$; we conclude using the relation (2.12). □

3. Conservative form of the Willmore equation in manifold in arbitrary codimension

Let us start introducing some notation. Let $\vec{\Phi}$ be a smooth immersion of the disc D^2 into a Riemannian manifold (M^m,h) of dimension $m\geq 3$. We stress that at this point $\vec{\Phi}$ is *not* assumed to be *conformal*. Let us denote with $g=g_{\vec{\Phi}}:=\vec{\Phi}^{\star}h$ the pull back metric on D^2 by $\vec{\Phi}$. Call $\star_h$ and $*_g$ the Hodge duality operators, defined in (1.32) for p-vectors tangent respectively to M and to D^2. Consider a positively oriented orthonormal frame $\vec{f}_1, \vec{f}_2$ of TD^2 endowed with the metric g and let $\vec{e}_1 := \vec{\Phi}_*(\vec{f}_1), \vec{e}_2 := \vec{\Phi}_*(\vec{f}_2)$ be the corresponding orthonormal frame of $\vec{\Phi}_*(TD^2)$, called D the covariant derivative in (M, h) we define

$$\begin{aligned} D_g : &\Gamma_{D^2}(T_{\vec{\Phi}}M \otimes \wedge^p TD^2) \to \Gamma_{D^2}(T_{\vec{\Phi}}M \otimes \wedge^{p+1} TD^2) \\ \vec{X} \otimes \vec{\alpha} &\mapsto D_{\vec{e}_1}\vec{X} \otimes (\vec{f}_1 \wedge \vec{\alpha}) + D_{\vec{e}_2}\vec{X} \otimes (\vec{f}_2 \wedge \vec{\alpha}) \\ &= g^{ij}\left[D_{\partial_{x^i}\vec{\Phi}}\vec{X} \otimes \left(\frac{\partial}{\partial x^j} \wedge \vec{\alpha}\right)\right], \end{aligned} \tag{3.1}$$

where, in the last line we used coordinates (x^1, x^2) on D^2, Γ_{D^2} denotes the set of the sections of the corresponding bundle, and $T_{\vec{\Phi}}M$ is the tangent bundle of M along $\vec{\Phi}(D^2)$. Notice that the definition does not depend on the choice of coordinates chosen on D^2, i.e. it is intrinsic. Observe we defined D_g on a generating family, so the definition extends to the whole space.

Next extend the definition of $*_g$ to $T_{\vec{\Phi}}M \otimes \wedge^p TD^2$ as

$$\begin{aligned} *_g : &T_{\vec{\Phi}}M \otimes \wedge^p TD^2 \to T_{\vec{\Phi}}M \otimes \wedge^{(2-p)} TD^2 \\ &\vec{X} \otimes \vec{\alpha} \mapsto \vec{X} \otimes (*_g \vec{\alpha}). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2) let us define

$$D_g^{*_g} := (-1) *_g D_g *_g . \tag{3.3}$$

We also need to extend the definitions of $\star_h$, $\wedge_M$, scalar product and the projection $\pi_{\vec{n}}$ onto the normal space to $\vec{\Phi}$ as follows

$$\star_h : \wedge^p T_{\vec{\Phi}}M \otimes \wedge^q TD^2 \to \wedge^{(m-p)} T_{\vec{\Phi}}M \otimes \wedge^q TD^2$$

$$\vec{\eta} \otimes \vec{\alpha} \mapsto (\star_h \vec{\eta}) \otimes \vec{\alpha} \tag{3.4}$$

$$\wedge_M : \wedge^p T_{\vec{\Phi}}M \otimes \wedge^q TD^2 \times \wedge^s T_{\vec{\Phi}}M \to \wedge^{p+s} T_{\vec{\Phi}}M \otimes \wedge^q TD^2$$

$$(\vec{\eta} \otimes \vec{\alpha}, \vec{\tau}) \mapsto (\vec{\eta} \wedge_M \vec{\tau}) \otimes \vec{\alpha}. \tag{3.5}$$

$$\langle \cdot, \cdot \rangle : (\wedge^p T_{\vec{\Phi}}M \otimes \wedge^q TD^2) \times (\wedge^p T_{\vec{\Phi}}M \otimes \wedge^q TD^2) \to \mathbb{R}$$

$$(\vec{\eta} \otimes \vec{\alpha}, \vec{\tau} \otimes \vec{\beta}) \mapsto \langle \vec{\eta}, \vec{\tau} \rangle_h \langle \vec{\alpha}, \vec{\beta} \rangle_g \tag{3.6}$$

$$\pi_{\vec{n}} : T_{\vec{\Phi}}M \otimes \wedge^q TD^2 \to T_{\vec{\Phi}}M \otimes \wedge^q TD^2$$

$$\vec{X} \otimes \vec{\alpha} \mapsto (\pi_{\vec{n}}(\vec{X})) \otimes \vec{\alpha}. \tag{3.7}$$

Define also $\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi})$ to be

$$\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) := \sum_{j=1}^{2} \Big(\langle \mathrm{Riem}^h(\mathbb{I}_{1j}, \vec{e}_2)\vec{e}_1, \vec{e}_2 \rangle \vec{e}_j + \langle \mathrm{Riem}^h(\vec{e}_1, \mathbb{I}_{2j})\vec{e}_1, \vec{e}_2 \rangle \vec{e}_j \Big) \tag{3.8}$$

and

$$(DR)(T\vec{\Phi}) := \sum_{i=1}^{m} \langle (D_{\vec{E}_i}\mathrm{Riem}^h)(\vec{e}_1, \vec{e}_2)\vec{e}_1, \vec{e}_2 \rangle \vec{E}_i. \tag{3.9}$$

The goal of this section is to prove Theorem 1.1. Observe that, now, all the terms appearing in the statement have been defined. Let us summarize the arguments of the proof.

Proof of Theorem 1.1. The proof is almost given in the subsection below; more precisely it follows by combining (3.28) of Theorem 3.1 with (3.29) in Remark 3.1; indeed with some straightforward computation following the definitions (3.1)–(3.5) and (3.7) one checks that the left hand side of (1.8) and the left hand side of (3.29) coincide. □

Before passing to the proof of the corollaries, some comments have to be done.

In order to exploit analytically Eq. (1.8) we will need a more explicit expression of $\pi_{\vec{n}}(D_g\vec{H})$. Recall the definition of $\llcorner$ given in (1.33), let as before $(\vec{e}_1, \vec{e}_2)$ be an orthonormal basis of $\vec{\Phi}_*(TD^2)$, call $\vec{n}$ the orthogonal $m-2$ plane given by

$$\star_h(\vec{e}_1 \wedge \vec{e}_2) = \vec{n}$$

and let $(\vec{n}_1 \cdots \vec{n}_{m-2})$ be a positively oriented orthonormal basis of the $m-2$-plane given by $\vec{n}$ satisfying $\vec{n} = \wedge_\alpha \vec{n}_\alpha$. One verifies easily that

$$\begin{cases} \vec{n}\llcorner\vec{e}_i = 0 \\ \vec{n}\llcorner\vec{n}_\alpha = (-1)^{\alpha-1} \wedge_{\beta\neq\alpha} \vec{n}_\beta \\ \vec{n}\llcorner(\wedge_{\beta\neq\alpha} \vec{n}_\beta) = (-1)^{m+\alpha-2}\, \vec{n}_\alpha. \end{cases}$$

We then deduce the following identity.

$$\forall \vec{w} \in T_{\vec{\Phi}(x)}M \quad \pi_{\vec{n}}(\vec{w}) = (-1)^{m-1}\, \vec{n}\llcorner(\vec{n}\llcorner\vec{w}). \tag{3.10}$$

From (3.10) we deduce in particular

$$\begin{aligned} \pi_{\vec{n}}(D_g\vec{H}) = D_g\vec{H} - (-1)^{m-1}\, D_g(\vec{n})\llcorner_M(\vec{n}\llcorner\vec{H}) \\ - (-1)^{m-1}\, \vec{n}\llcorner_M(D_g(\vec{n})\llcorner_M\vec{H}); \end{aligned} \tag{3.11}$$

where, analogously as before, we define

$$\begin{aligned} &\llcorner_M : (\wedge^p T_{\vec{\Phi}}M \otimes TD^2) \times \wedge^q T_{\vec{\Phi}}M \to \wedge^{(p-q)}\, T_{\vec{\Phi}}M \otimes TD^2 \\ &(\vec{\alpha} \otimes \vec{v}, \vec{\beta}) \mapsto (\vec{\alpha} \otimes \vec{v})\llcorner_M\vec{\beta} := (\vec{\alpha}\llcorner\vec{\beta}) \otimes \vec{v}. \end{aligned} \tag{3.12}$$

A straightforward but important consequence of Theorem 1.1 is the conservative form of Willmore surface equations given in Corollary 1.1. Let us prove it.

Proof of Corollary 1.1. Recall that the first variation of the Willmore functional in general Riemannian manifolds has been computed in [43]; equating it to zero we get the classical Willmore equation in manifolds: $\vec{\Phi}$ is a Willmore immersion if and only if

$$\Delta_\perp\vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} - \tilde{R}(\vec{H}) = 0 \tag{3.13}$$

where $\tilde{R}$ is the curvature endomorphism defined in (1.5). Collecting (3.13) and Eq. (1.8) we get the thesis. □

Recall that an immersion $\vec{\Phi}$ of Σ is said to be *constrained-conformal Willmore* if and only if it is a critical point of the Willmore functional under the constraint that the conformal class is fixed. Let us prove the conservative form of the constrained-conformal Willmore surface equation given in Corollary 1.2.

Proof of Corollary 1.2. Recall (see [35], and the notation given in the introduction before Corollary 1.2) that an immersion $\vec{\Phi}$ is a constrained-conformal Willmore immersion if and only if there exists an holomorphic quadratic differential $q \in Q(J)$ such that

$$\Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} - \tilde{R}(\vec{H}) = \Im[(q, \vec{h}_0)_{WP}] \tag{3.14}$$

where $\tilde{R}$ is the curvature endomorphism defined in (1.5). The thesis follows putting together (3.32) and Eq. (3.28). □

Now we prove that also the Euler–Lagrange equation of the functional $F = \frac{1}{2}\int |\mathbb{I}|^2$ can be written in the conservative form, being the Euler–Lagrange equation of W plus some lower order terms. Let us start with an auxiliary lemma.

Lemma 3.1. *Let $\vec{\Phi} : D^2 \hookrightarrow (M, h)$ be a smooth immersion, then the first variation of the functional $\int_{D^2} \bar{K}(\vec{\Phi}_*(TD^2))d\mathrm{vol}_g$ with respect to a smooth compactly supported variation $\vec{w}$ is given*

$$\frac{d}{dt}\int_{D^2} \bar{K}((\vec{\Phi} + t\vec{w})_*(TD^2))d\mathrm{vol}_{g_{\vec{\Phi}+t\vec{w}}}\ (t=0)$$

$$= -\int_{D^2} \langle (D\,R)(T\vec{\Phi}) + 2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) + 2\bar{K}(\vec{\Phi}_*(TD^2))\vec{H}, \vec{w}\rangle d\mathrm{vol}_g \tag{3.15}$$

where $(D\,R)(T\vec{\Phi})$ and $\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi})$ are the curvature quantities defined respectively in (3.9) and (3.8). □

Proof. Let $(\vec{e}_1, \vec{e}_2)$ be an orthonormal frame of $\vec{\Phi}_*(TD^2)$ extended in the neighborhood of $\vec{\Phi}(D^2)$ by parallel translation in the normal directions, and π_T denotes the projection on $\vec{\Phi}_*(TD^2)$ By definition $\bar{K}(\vec{\Phi}_*(TD^2)) = -\langle \mathrm{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{e}_1, \vec{e}_2\rangle$. Observe that using the orthonormality of $(\vec{e}_1, \vec{e}_2)$, the antisymmetry of $\mathrm{Riem}^h(\cdot,\cdot)$ and the fact that $D_{\pi_{\vec{n}}(\vec{w})}\vec{e}_i = 0$ we get

$$\mathrm{Riem}^h(D_{\vec{w}}\vec{e}_1, \vec{e}_2) = \mathrm{Riem}^h(\mathbb{I}(\pi_T(\vec{w}), \vec{e}_1), \vec{e}_2);$$

recall moreover that the first variation of the volume element is $-2\langle \vec{H}, \vec{w}\rangle \mathrm{vol}_g$. Collecting these informations and using the symmetry of the Riemann tensor one gets

$$-\int_{D^2}\Big[\langle (D_{\vec{w}}\mathrm{Riem}^h)(\vec{e}_1, \vec{e}_2)\vec{e}_1, \vec{e}_2\rangle + 2\langle \mathrm{Riem}^h(\mathbb{I}(\pi_T(\vec{w}), \vec{e}_1), \vec{e}_2)\vec{e}_1, \vec{e}_2\rangle$$

$$+ 2\langle \mathrm{Riem}^h(\vec{e}_1, \mathbb{I}(\pi_T(\vec{w}), \vec{e}_2))\vec{e}_1, \vec{e}_2\rangle + 2\bar{K}(\vec{\Phi}_*(TD^2))\langle \vec{H}, \vec{w}\rangle\Big]d\mathrm{vol}_g. \tag{3.16}$$

Now the thesis follows recalling the definitions (3.9) and (3.8). □

Corollary 3.1. *A smooth immersion $\vec{\Phi}$ of a 2-dimensional disc D^2 in (M^m, h) is critical for the functional $F = \frac{1}{2}\int |\mathbb{I}|^2$ if and only if*

$$D_g^{*_g}\left[D_g\vec{H} - 3\pi_{\vec{n}}(D_g\vec{H}) + \star_h\left((*_g\, D_g\vec{n}) \wedge_M \vec{H}\right)\right]$$

$$= 2\tilde{R}(\vec{H}) - 2R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) + (D\,R)(T\vec{\Phi}) + 2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) + 2\bar{K}(\vec{\Phi}_*(TD^2))\vec{H}, \tag{3.17}$$

where $\tilde{R}$ and $R^\perp$ are the curvature endomorphisms defined respectively in (1.5) and (1.7). □

Proof. The Gauss equation yields

$$\frac{1}{2}|\mathbb{I}|^2 = 2|\vec{H}|^2 + \bar{K}(\vec{\Phi}_*(TD^2)) - K_{\vec{\Phi}} \tag{3.18}$$

where $K_{\vec\Phi}$ is the Gauss curvature of the metric $g = \vec\Phi^* h$. Integrating over D^2, we get

$$F(\vec\Phi) = 2W(\vec\Phi) + \int_{D^2} \bar K(\vec\Phi_*(TD^2)) - \int_{D^2} K_{\vec\Phi} d\mathrm{vol}_g.$$

Since by the Gauss–Bonnet theorem the last integral reduces, up to an additive constant, to an integral on the boundary, if we take variations $\vec w$ compactly supported in D^2 it gives no contribution in the first variation. Therefore the thesis follows combining the first variation of W given in Corollary 1.1 and Lemma 3.1. □

Corollary 3.2. *A smooth immersion $\vec\Phi$ of a 2-dimensional disc D^2 in (M^m, h) is conformal Willmore (i.e. critical for the conformal Willmore functional $W_{\mathrm{conf}} = \int(|\vec H|^2 + \bar K)d\mathrm{vol}_g$) if and only if*

$$\begin{aligned}&\frac{1}{2}D_g^{*_g}\left[D_g\vec H - 3\pi_{\vec n}(D_g\vec H) + \star_h\left((*_g D_g\vec n)\wedge_M \vec H\right)\right]\\ &\quad = \tilde R(\vec H) - R^{\perp}_{\vec\Phi}(T\vec\Phi) + (D\,R)(T\vec\Phi) + 2\mathfrak{R}_{\vec\Phi}(T\vec\Phi) + 2\bar K(\vec\Phi_*(TD^2))\vec H. \qquad (3.19)\end{aligned}$$

Notice if (M, h) has constant sectional curvature $\bar K$ then the right hand side is null and we get

$$\frac{1}{2}D_g^{*_g}\left[D_g\vec H - 3\pi_{\vec n}(D_g\vec H) + \star_h\left((*_g D_g\vec n)\wedge_M \vec H\right)\right] = 0. \quad □ \qquad (3.20)$$

Proof. The proof of (3.25) follows combining Corollary 1.1 and Lemma 3.1.

Now assume that the sectional curvature $\bar K$ is constant; then observe that $W_{\mathrm{conf}}(\vec\Phi) = W(\vec\Phi) + \bar K A(\vec\Phi)$,

$$dW_{\mathrm{conf}} = dW - 2\bar K\vec H. \qquad (3.21)$$

Moreover, $\bar K$ constant implies that (see [9, Corollary 3.5] and recall the opposite sign convention in the Riemann tensor)

$$\begin{aligned}&\langle \mathrm{Riem}^h(\vec X, \vec Y)\vec W, \vec Z\rangle = h(\vec X, \vec Z)h(\vec Y, \vec W) - h(\vec X, \vec W)h(\vec Y, \vec Z)\\ &\quad \forall \vec X, \vec Y, \vec W, \ \vec Z \in T_x M. \qquad (3.22)\end{aligned}$$

Therefore, plugging (3.22) directly into the definitions (1.5) and (1.7) we get

$$\tilde R(\vec H) = -2\bar K\vec H, \qquad (3.23)$$

$$R^{\perp}_{\vec\Phi}(T\vec\Phi) = 0. \qquad (3.24)$$

Eq. (3.26) follows combining (3.21), (3.23), (3.24) and Corollary 1.1. □

Corollary 3.3. *Let $\vec\Phi : \Sigma \hookrightarrow M$ be a smooth immersion into the $m \geq 3$-dimensional Riemannian manifold (M^m, h) and call J the complex structure associated to $g = \vec\Phi^* h$. Then the immersion $\vec\Phi$ is constrained-conformal conformal Willmore (i.e. critical for the conformal Willmore functional $W_{\mathrm{conf}} = \int(|\vec H|^2 + \bar K)d\mathrm{vol}_g$ under the constraint of fixed conformal class) if and only if there exists an holomorphic quadratic differential $q \in Q(J)$ such that*

$$\begin{aligned}&\frac{1}{2}D_g^{*_g}\left[D_g\vec H - 3\pi_{\vec n}(D_g\vec H) + \star_h\left((*_g D_g\vec n)\wedge_M \vec H\right)\right]\\ &\quad = \Im\left[(q, \vec h_0)_{WP}\right] + \tilde R(\vec H) - R^{\perp}_{\vec\Phi}(T\vec\Phi)\end{aligned}$$

$$+(DR)(T\vec{\Phi})+2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi})+2\bar{K}(\vec{\Phi}_*(TD^2))\vec{H} \tag{3.25}$$

where $\vec{H}_0, \vec{h}_0$ *and* $(\cdot,\cdot)_{WP}$ *are defined in* (1.10)–(1.12).

Notice that if (M,h) *has constant sectional curvature* $\bar{K}$ *then the curvature terms of the right hand side vanish and we get*

$$\frac{1}{2}D_g^{*_g}\left[D_g\vec{H}-3\pi_{\vec{n}}(D_g\vec{H})+\star_h\left((*_g\,D_g\vec{n})\wedge_M\vec{H}\right)\right]=\Im\left[(q,\vec{h}_0)_{WP}\right]. \quad \Box \tag{3.26}$$

Proof. The proof is analogous to the proof of Corollary 1.2 once we have Corollary 3.2. □

3.1. Derivation of the conservative form: use of conformal coordinates and complex notation

We first introduce some complex notation that will be useful in the sequel. In this subsection $\vec{\Phi}$ is a conformal immersion into a Riemannian manifold (M^m,h) of dimension $m\geq 3$, denote $z=x_1+ix_2$, $\partial_z=2^{-1}(\partial_{x_1}-i\partial_{x_2})$, $\partial_{\bar z}=2^{-1}(\partial_{x_1}+i\partial_{x_2})$.
Moreover we denote[2]

$$\begin{cases}\vec{e}_z:=e^{-\lambda}\partial_z\vec{\Phi}=2^{-1}(\vec{e}_1-i\vec{e}_2)\\ \vec{e}_{\bar z}:=e^{-\lambda}\partial_{\bar z}\vec{\Phi}=2^{-1}(\vec{e}_1+i\vec{e}_2).\end{cases}$$

Observe that

$$\begin{cases}\langle\vec{e}_z,\vec{e}_z\rangle=0\\ \langle\vec{e}_z,\vec{e}_{\bar z}\rangle=\dfrac{1}{2}\\ \vec{e}_z\wedge\vec{e}_{\bar z}=\dfrac{i}{2}\,\vec{e}_1\wedge\vec{e}_2.\end{cases} \tag{3.27}$$

We also use the shorter notation $D_z:=D_{\partial_z\vec{\Phi}}$ and $D_{\bar z}:=D_{\partial_{\bar z}\vec{\Phi}}$ for the covariant derivative with respect to the vectors $\partial_z\vec{\Phi}$ and $\partial_{\bar z}\vec{\Phi}$. Introduce moreover the *Weingarten operator* expressed in our conformal coordinates (x_1,x_2):

$$\vec{H}_0:=\frac{1}{2}\left[\vec{\mathbb{I}}(\vec{e}_1,\vec{e}_1)-\vec{\mathbb{I}}(\vec{e}_2,\vec{e}_2)-2\,i\,\vec{\mathbb{I}}(\vec{e}_1,\vec{e}_2)\right].$$

Theorem 3.1. *Let* $\vec{\Phi}$ *be a smooth immersion of a two dimensional manifold* Σ^2 *into an* m*-dimensional Riemannian manifold* (M^m,h)*; restricting the immersion to a small disc neighborhood of a point where we consider local conformal coordinate, we can see* $\vec{\Phi}$ *as a conformal immersion of* D^2 *into* (M,h)*. Then the following identity holds*

$$\begin{aligned}&4\,e^{-2\lambda}\,\Re\left(D_{\bar z}\left[\pi_{\vec{n}}(D_z\vec{H})+\langle\vec{H},\vec{H}_0\rangle\,\partial_{\bar z}\vec{\Phi}\right]\right)\\ &\quad=\Delta_\perp\vec{H}+\tilde{A}(\vec{H})-2|\vec{H}|^2\,\vec{H}+8\Re\left(\langle\mathrm{Riem}^h(\vec{e}_{\bar z},\vec{e}_z)\vec{e}_z,\vec{H}\rangle\vec{e}_{\bar z}\right),\end{aligned} \tag{3.28}$$

where $\vec{H}$ *is the mean curvature vector of the immersion* $\vec{\Phi}$, $\Delta_\perp$ *is the negative covariant Laplacian on the normal bundle to the immersion,* $\tilde{A}$ *is the linear map given in* (1.6), $D_z\cdot:=$

---

[2] Observe that the notation has been chosen in such a way that $\overline{\vec{e}_z}=\vec{e}_{\bar z}$.

$D_{\partial_z \vec{\Phi}}\cdot$, and $D_{\bar{z}}\cdot := D_{\partial_{\bar{z}}\vec{\Phi}}\cdot$ are the covariant derivatives in (M,h) with respect to the tangent vectors $\partial_z \vec{\Phi}$ and $\partial_{\bar{z}}\vec{\Phi}$. □

Remark 3.1. Observe that using the identity (3.34) proved in Lemma 3.2, Eq. (3.28) can be written using *real* conformal coordinates as follows

$$-\frac{e^{-2\lambda}}{2} D^*\left[D\vec{H} - 3\pi_{\vec{n}}(D\vec{H}) + \star_h(D^\perp \vec{n} \wedge \vec{H})\right]$$
$$= \Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\,\vec{H} - R^\perp_{\vec{\Phi}}(T\vec{\Phi}), \tag{3.29}$$

observe we used the equation below, which follows by definition (1.7),

$$R^\perp_{\vec{\Phi}}(T\vec{\Phi}) = -8\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right) = \left(\pi_T\left[\mathrm{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H}\right]\right)^\perp.$$

Notice that identity (3.29) in codimension one gives exactly the previous (2.12). □

A straightforward but important consequence of Theorem 3.1 is the following conservative form of Willmore surfaces equation in conformal coordinates.

Corollary 3.4. *A conformal immersion $\vec{\Phi}$ of a 2-dimensional disc D^2 in (M^m,h) is Willmore if and only if*

$$4\,e^{-2\lambda}\,\Re\left(D_{\bar{z}}\left[\pi_{\vec{n}}(D_z\vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\,\partial_{\bar{z}}\vec{\Phi}\right]\right)$$
$$= \tilde{R}(\vec{H}) + 8\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right). \tag{3.30}$$

Proof. Recall that $\vec{\Phi}$ is a Willmore immersion if and only if (3.13) holds. Combining (3.13) and Eq. (3.28) we get the desired result. □

Now recall that an immersion $\vec{\Phi}$ is said to be *constrained-conformal Willmore* if and only if it is a critical point of the Willmore functional under the constraint that the conformal class is fixed.

Corollary 3.5. *A conformal immersion $\vec{\Phi}$ of a 2-dimensional disc D^2 in (M^m,h) is constrained-conformal Willmore if and only if there exists an holomorphic function $f(z)$ such that*

$$4\,e^{-2\lambda}\,\Re\left(D_{\bar{z}}\left[\pi_{\vec{n}}(D_z\vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\,\partial_{\bar{z}}\vec{\Phi}\right]\right)$$
$$= e^{-2\lambda}\,\Im(f(z)\overline{\vec{H}_0}) + \tilde{R}(\vec{H}) + 8\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right). \tag{3.31}$$

Proof of Corollary 3.5. An immersion $\vec{\Phi}$ is a constrained-conformal Willmore immersion if and only if there exists an holomorphic function f such that

$$\Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\,\vec{H} - \tilde{R}(\vec{H}) = e^{-2\lambda}\,\Im(f(z)\overline{\vec{H}_0}) \tag{3.32}$$

where $\tilde{R}$ is the curvature endomorphism defined in (1.5).

Therefore putting together (3.32) and Eq. (3.28) we get the thesis. □

In order to prove Theorem 3.1 some computational lemmas will be useful; let us start with the following.

Lemma 3.2. *Let $\vec{\Phi}$ be a conformal immersion of D^2 into (M^m, h) then*

$$\pi_T(D_z\vec{H}) - i\ \star_h(D_z\vec{n} \wedge \vec{H}) = -2\left\langle \vec{H}, \vec{H}_0\right\rangle \partial_{\bar{z}}\vec{\Phi} \tag{3.33}$$

and hence

$$D_z\vec{H} - 3\pi_{\vec{n}}(D_z\vec{H}) - i\ \star_h(D_z\vec{n} \wedge \vec{H}) = -2\left\langle \vec{H}, \vec{H}_0\right\rangle \partial_{\bar{z}}\vec{\Phi} - 2\,\pi_{\vec{n}}(D_z\vec{H}). \quad \square \tag{3.34}$$

Proof of Lemma 3.2. We denote by $(\vec{e}_1, \vec{e}_2)$ the orthonormal basis of $\vec{\Phi}_*(TD^2)$ given by

$$\vec{e}_i = e^{-\lambda}\ \frac{\partial\vec{\Phi}}{\partial x_i}.$$

With these notations the second fundamental form $\mathbf{h}$ which is a symmetric 2-form on TD^2 into $(\vec{\Phi}_*TD^2)^\perp$ is given by

$$\begin{aligned}&\mathbf{h} = \sum_{\alpha,i,j} h^\alpha_{ij}\ \vec{n}_\alpha \otimes (\vec{e}_i)^* \otimes (\vec{e}_j)^* \\ &\quad \text{with } h^\alpha_{ij} = -e^{-\lambda}\ \left(D_{\partial_{x_i}\vec{\Phi}}\vec{n}_\alpha, \vec{e}_j\right).\end{aligned} \tag{3.35}$$

We shall also denote

$$\vec{h}_{ij} := \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j) = \sum_{\alpha=1}^{m-2} h^\alpha_{ij}\ \vec{n}_\alpha.$$

In particular the mean curvature vector $\vec{H}$ is given by

$$\vec{H} = \sum_{\alpha=1}^{m-2} H^\alpha\ \vec{n}_\alpha = \frac{1}{2}\sum_{\alpha=1}^{m-2}(h^\alpha_{11} + h^\alpha_{22})\,\vec{n}_\alpha = \frac{1}{2}(\vec{h}_{11} + \vec{h}_{22}). \tag{3.36}$$

Let $\vec{n}$ be the $m-2$ vector of $T_{\vec{\Phi}(x)}M$ given by $\vec{n} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_2$. We identify vectors and $m-1$-vectors in $T_{\vec{\Phi}(x)}M$ using the Hodge operator $\star_h$ for the metric h; for the Hodge operator we use the standard notation (see for example [29, Chapter 7.9.2])

$$\langle\alpha, \beta\rangle \star_h 1 = (\alpha \wedge \star_h\, \beta)$$

for any couple of p-vectors α and β, where we set $\star_h 1 := \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{n}$; then we have for instance

$$\star_h(\vec{n} \wedge \vec{e}_1) = \vec{e}_2 \quad \text{and} \quad \star_h(\vec{n} \wedge \vec{e}_2) = -\vec{e}_1. \tag{3.37}$$

Since $\vec{e}_1, \vec{e}_2, \vec{n}_1 \cdots \vec{n}_{m-2}$ is a basis of $T_{\vec{\Phi}(x)}M$, we can write for every $\alpha = 1 \cdots m-2$

$$D\vec{n}_\alpha = \sum_{\beta=1}^{m-2}\langle D\vec{n}_\alpha, \vec{n}_\beta\rangle\,\vec{n}_\beta + \sum_{i=1}^{2}\langle D\vec{n}_\alpha, \vec{e}_i\rangle\,\vec{e}_i$$

and consequently

$$\star_h(\vec{n} \wedge D^\perp\vec{n}_\alpha) = \langle D^\perp\vec{n}_\alpha, \vec{e}_1\rangle\,\vec{e}_2 - \langle D^\perp\vec{n}_\alpha, \vec{e}_2\rangle\,\vec{e}_1. \tag{3.38}$$

Hence

$$\begin{aligned}\star_h(D^\perp\vec{n} \wedge \vec{H}) &= -\langle D^\perp\vec{H}, \vec{e}_1\rangle\,\vec{e}_2 + \langle D^\perp\vec{H}, \vec{e}_2\rangle\,\vec{e}_1 \\ &= \langle\vec{H}, \pi_{\vec{n}}(D^\perp\vec{e}_1)\rangle\,\vec{e}_2 - \langle\vec{H}, \pi_{\vec{n}}(D^\perp\vec{e}_2)\rangle\,\vec{e}_1.\end{aligned}$$

Using (3.35), we then have proved

$$\star_h(D^\perp\vec{n}\wedge\vec{H}) = \begin{pmatrix} -\langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_2}\vec{\Phi} + \langle\vec{H},\vec{h}_{22}\rangle\,\partial_{x_1}\vec{\Phi} \\ \langle\vec{H},\vec{h}_{11}\rangle\,\partial_{x_2}\vec{\Phi} - \langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_1}\vec{\Phi} \end{pmatrix}. \tag{3.39}$$

The tangential projection of $D\vec{H}$ is given by

$$\begin{aligned}\pi_T(D\vec{H}) &= \langle D\vec{H},\vec{e}_1\rangle\,\vec{e}_1 + \langle D\vec{H},\vec{e}_2\rangle\,\vec{e}_2 \\ &= -\langle\vec{H},\pi_{\vec{n}}(D\vec{e}_1)\rangle\,\vec{e}_1 - \langle\vec{H},\pi_{\vec{n}}(D\vec{e}_2)\rangle\,\vec{e}_2.\end{aligned}$$

Hence

$$\pi_T(D\vec{H}) = \begin{pmatrix} -\langle\vec{H},\vec{h}_{11}\rangle\,\partial_{x_1}\vec{\Phi} - \langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_2}\vec{\Phi} \\ -\langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_1}\vec{\Phi} - \langle\vec{H},\vec{h}_{22}\rangle\,\partial_{x_2}\vec{\Phi} \end{pmatrix}. \tag{3.40}$$

Combining (3.39) and (3.40) gives

$$-\pi_T(D\vec{H}) - \star_h(D^\perp\vec{n}\wedge\vec{H}) = \begin{pmatrix} \langle\vec{H},\vec{h}_{11}-\vec{h}_{22}\rangle\,\partial_{x_1}\vec{\Phi} + 2\langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_2}\vec{\Phi} \\ 2\langle\vec{H},\vec{h}_{12}\rangle\,\partial_{x_1}\vec{\Phi} + \langle\vec{H},\vec{h}_{22}-\vec{h}_{11}\rangle\,\partial_{x_2}\vec{\Phi} \end{pmatrix}. \tag{3.41}$$

This last identity written with the complex coordinate z is exactly (3.33) and Lemma 3.2 is proved. □

Before we move to the proof of Theorem 3.1 we shall need two more lemmas. First we have the following.

Lemma 3.3. *Let $\vec{\Phi}$ be a conformal immersion of the disc D^2 into M^m, called $z := x_1+ix_2$, $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$ denote*

$$\vec{e}_i := e^{-\lambda}\,\partial_{x_i}\vec{\Phi}, \tag{3.42}$$

and let $\vec{H}_0$ be the Weingarten operator of the immersion expressed in the conformal coordinates (x_1,x_2):

$$\vec{H}_0 := \frac{1}{2}\left[\mathbb{I}(\vec{e}_1,\vec{e}_1) - \mathbb{I}(\vec{e}_2,\vec{e}_2) - 2\,i\,\mathbb{I}(\vec{e}_1,\vec{e}_2)\right].$$

Then the following identities hold

$$D_{\bar{z}}\left[e^\lambda\,\vec{e}_z\right] = \frac{e^{2\lambda}}{2}\vec{H}, \tag{3.43}$$

and

$$D_z\left[e^{-\lambda}\vec{e}_z\right] = \frac{1}{2}\,\vec{H}_0. \quad \square \tag{3.44}$$

Proof of Lemma 3.3. The first identity (3.43) comes simply from the fact that $D_{\bar{z}}\partial_z\vec{\Phi} = \frac{1}{4}\Delta\vec{\Phi}$, from (3.48) and the expression of the mean curvature vector in conformal coordinates

$$\vec{H} = \frac{e^{-2\lambda}}{2}\,\Delta\vec{\Phi}.$$

It remains to prove the identity (3.44). One has moreover

$$D_z\left[e^\lambda\vec{e}_z\right] = D_z\partial_z\vec{\Phi} = \frac{1}{4}\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi} - D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi} - 2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]. \tag{3.45}$$

On the one hand the projection into the normal direction gives

$$\pi_{\vec{n}}\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi}-D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi}-2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]=2\,e^{2\lambda}\,\vec{H}_0. \tag{3.46}$$

On the other hand the projection into the tangent plane gives

$$\begin{aligned}&\pi_T\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi}-D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi}-2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]\\&\quad=e^{-\lambda}\left\langle\partial_{x_1}\vec{\Phi},\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi}-D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi}-2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]\right\rangle\vec{e}_1\\&\qquad+e^{-\lambda}\left\langle\partial_{x_2}\vec{\Phi},\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi}-D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi}-2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]\right\rangle\vec{e}_2.\end{aligned}$$

This implies after some computation

$$\begin{aligned}&\pi_T\left[D_{\partial_{x_1}\vec{\Phi}}\partial_{x_1}\vec{\Phi}-D_{\partial_{x_2}\vec{\Phi}}\partial_{x_2}\vec{\Phi}-2\,i\;D_{\partial_{x_1}\vec{\Phi}}\partial_{x_2}\vec{\Phi}\right]\\&\quad=2\,e^{\lambda}\,\left[\partial_{x_1}\lambda-i\partial_{x_2}\lambda\right]\vec{e}_1-2\,e^{\lambda}\,\left[\partial_{x_2}\lambda+i\partial_{x_1}\lambda\right]\vec{e}_2\\&\quad=8\,\partial_z e^{\lambda}\,\vec{e}_z.\end{aligned} \tag{3.47}$$

The combination of (3.45)–(3.47) gives

$$D_z\left[e^{\lambda}\vec{e}_z\right]=\frac{e^{2\lambda}}{2}\,\vec{H}_0+2\,\partial_z e^{\lambda}\,\vec{e}_z,$$

which implies (3.44). □

The last lemma we shall need in order to prove Theorem 3.1 is the Codazzi–Mainardi identity that we recall and prove below.

Lemma 3.4 (*Codazzi–Mainardi Identity*)**.** *Let $\vec{\Phi}$ be a conformal immersion of the disc D^2 into (M^m,h), called $z:=x_1+ix_2$, $e^{\lambda}:=|\partial_{x_1}\vec{\Phi}|=|\partial_{x_2}\vec{\Phi}|$ denote*

$$\vec{e}_i:=e^{-\lambda}\,\partial_{x_i}\vec{\Phi}, \tag{3.48}$$

and denote $\vec{H}_0$ the Weingarten operator of the immersion expressed in the conformal coordinates (x_1,x_2):

$$\vec{H}_0:=\frac{1}{2}\left[\mathbb{I}(\vec{e}_1,\vec{e}_1)-\mathbb{I}(\vec{e}_2,\vec{e}_2)-2\,i\,\mathbb{I}(\vec{e}_1,\vec{e}_2)\right].$$

Then the following identity holds

$$\begin{aligned}e^{-2\lambda}\,\partial_{\bar{z}}\left(e^{2\lambda}\,\langle\vec{H},\vec{H}_0\rangle\right)&=\langle\vec{H},D_z\vec{H}\rangle+\langle\vec{H}_0,D_{\bar{z}}\vec{H}\rangle\\&\qquad+2\langle\mathrm{Riem}^h(\vec{e}_{\bar{z}},\vec{e}_z)\partial_z\vec{\Phi},\vec{H}\rangle.\quad\square\end{aligned} \tag{3.49}$$

Proof of Lemma 3.4. Using (3.44) we obtain

$$\begin{aligned}\langle D_{\bar{z}}\vec{H}_0,\vec{H}\rangle&=2\left\langle D_{\bar{z}}\left[D_z\left(e^{-2\lambda}\,\partial_z\vec{\Phi}\right)\right],\vec{H}\right\rangle\\&=2\left\langle D_z\left[D_{\bar{z}}\left(e^{-2\lambda}\,\partial_z\vec{\Phi}\right)\right],\vec{H}\right\rangle+2\langle\mathrm{Riem}^h(\vec{e}_{\bar{z}},\vec{e}_z)\partial_z\vec{\Phi},\vec{H}\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\langle D_{\bar z}\vec{H}_0, \vec{H}\rangle &= -4\left\langle D_z\left[\partial_{\bar z}\lambda\, e^{-2\lambda}\,\partial_z\vec{\Phi}\right], \vec{H}\right\rangle + \left\langle D_z\left[\frac{e^{-2\lambda}}{2}\,\Delta\vec{\Phi}\right], \vec{H}\right\rangle \\ &\quad + 2\langle \mathrm{Riem}^h(\vec{e}_{\bar z}, \vec{e}_z)\partial_z\vec{\Phi}, \vec{H}\rangle \\ &= -2\partial_{\bar z}\lambda\,\left\langle \vec{H}_0, \vec{H}\right\rangle + \left\langle D_z\vec{H}, \vec{H}\right\rangle + 2\langle \mathrm{Riem}^h(\vec{e}_{\bar z}, \vec{e}_z)\partial_z\vec{\Phi}, \vec{H}\rangle. \end{aligned} \tag{3.50}$$

This last identity implies the Codazzi–Mainardi identity (3.49) and Lemma 3.4 is proved. □

Proof of Theorem 3.1. Due to Lemma 3.2, as explained in Remark 3.1, it suffices to prove in conformal parametrization the identity (3.29). First of all we observe that

$$4\,e^{-2\lambda}\,\Re\left(\pi_{\vec n}\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec{H})\right]\right)\right) = e^{-2\lambda}\,\pi_{\vec n}\left(D^*\left[\pi_{\vec n}(D\vec{H})\right]\right) = \Delta_\perp\vec{H}. \tag{3.51}$$

The tangential projection gives

$$\begin{aligned}4\,e^{-2\lambda}\,\pi_T\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec{H})\right]\right) &= 8\,e^{-2\lambda}\,\left\langle D_{\bar z}(\pi_{\vec n}(D_z\vec{H})), \vec{e}_z\right\rangle\,\vec{e}_{\bar z} \\ &\quad + 8\,e^{-2\lambda}\,\left\langle D_{\bar z}(\pi_{\vec n}(D_z\vec{H})), \vec{e}_{\bar z}\right\rangle\,\vec{e}_z. \end{aligned} \tag{3.52}$$

Using the fact that $\vec{e}_z$ and $\vec{e}_{\bar z}$ are orthogonal to the normal plane we have in one hand using (3.43)

$$\left\langle D_{\bar z}(\pi_{\vec n}(D_z\vec{H})), \vec{e}_z\right\rangle = -e^{-\lambda}\,\left\langle \pi_{\vec n}(D_z\vec{H}), D_{\bar z}\left[e^{\lambda}\,\vec{e}_z\right]\right\rangle = -\frac{e^{\lambda}}{2}\,\left\langle D_z\vec{H}, \vec{H}\right\rangle \tag{3.53}$$

and on the other hand using (3.44)

$$\left\langle D_{\bar z}(\pi_{\vec n}(D_z\vec{H})), \vec{e}_{\bar z}\right\rangle = -e^{\lambda}\,\left\langle \pi_{\vec n}(D_z\vec{H}), D_{\bar z}\left[e^{-\lambda}\,\vec{e}_{\bar z}\right]\right\rangle = -\frac{e^{\lambda}}{2}\,\left\langle D_z\vec{H}, \overline{\vec{H}_0}\right\rangle. \tag{3.54}$$

Combining (3.52)–(3.54) we obtain

$$4\,e^{-2\lambda}\,\pi_T\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec{H})\right]\right) = -4\,e^{-2\lambda}\left[\left\langle D_z\vec{H}, \vec{H}\right\rangle\,\partial_{\bar z}\vec{\Phi} + \left\langle D_z\vec{H}, \overline{\vec{H}_0}\right\rangle\,\partial_z\vec{\Phi}\right]. \tag{3.55}$$

Putting (3.51) and (3.55) together we obtain

$$\begin{aligned}&4\,e^{-2\lambda}\,\Re\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec{H})\right]\right) \\ &\quad = \Delta_\perp\vec{H} - 4\,e^{-2\lambda}\Re\left[\left[\left\langle D_z\vec{H}, \vec{H}\right\rangle + \left\langle D_{\bar z}\vec{H}, \vec{H}_0\right\rangle\right]\,\partial_{\bar z}\vec{\Phi}\right]. \end{aligned} \tag{3.56}$$

Using the Codazzi–Mainardi identity (3.49) and using also again identity (3.44), (3.56) becomes

$$\begin{aligned}&4\,e^{-2\lambda}\,\Re\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec{H}) + \langle\vec{H}, \vec{H}_0\rangle\,\partial_{\bar z}\vec{\Phi}\right]\right) \\ &\quad = \Delta_\perp\vec{H} + 2\,\Re\left(\left\langle\vec{H}, \vec{H}_0\right\rangle\,\overline{\vec{H}_0} + 4\langle \mathrm{Riem}^h(\vec{e}_{\bar z}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle\vec{e}_{\bar z}\right). \end{aligned} \tag{3.57}$$

The definition (1.6) of $\tilde{A}$ gives

$$\tilde{A}(\vec{H}) = \sum_{i,j=1}^{2}\langle\vec{H}, \vec{h}_{ij}\rangle\,\vec{h}_{ij};$$

hence a short elementary computation gives

$$\tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} = 2^{-1} \left\langle \vec{H}, \vec{h}_{11} - \vec{h}_{22} \right\rangle (\vec{h}_{11} - \vec{h}_{22}) + 2\langle \vec{H}, \vec{h}_{12}\rangle\, \vec{h}_{12}.$$

Using $\vec{H}_0$ this expression becomes

$$\tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} = 2\Re\left(\left\langle \vec{H}, \vec{H}_0 \right\rangle \overline{\vec{H}_0}\right). \tag{3.58}$$

Combining (3.57) and (3.58) gives

$$\begin{aligned} 4\, e^{-2\lambda}\, \Re\left(D_{\bar{z}}\left[\pi_{\vec{n}}(D_z \vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\, \partial_{\bar{z}}\vec{\Phi}\right]\right) &= \Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} \\ &\quad + 8\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right) \end{aligned} \tag{3.59}$$

which is the desired equality and Theorem 3.1 is proved. □

4. Parallel mean curvature vs. constrained-conformal conformal Willmore surfaces

As an application of the Conservative form of the Willmore equation, in this section we prove the link between parallel mean curvature surfaces and constrained-conformal conformal Willmore surfaces mentioned in the introduction; notice that Proposition 1.1 gives a lot of examples of constrained-conformal conformal Willmore surfaces.

Proof of Proposition 1.1. Observe that the proof in the Euclidean case was given by the second author in [36], here we adapt the computations to the Riemannian setting. Up to a change of coordinates, we can assume that $\vec{\Phi}$ is a smooth conformal immersion. Since (M^m, h) has constant sectional curvature $\bar{K}$, then writing Eq. (3.25) using the conformal parametrization gives that $\vec{\Phi}$ is constrained-conformal conformal Willmore if and only if there exists a holomorphic function $f(z)$ such that (see also (3.31))

$$4\, e^{-2\lambda}\, \Re\left(D_{\bar{z}}\left[\pi_{\vec{n}}(D_z \vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\, \partial_{\bar{z}}\vec{\Phi}\right]\right) = e^{-2\lambda}\, \Im(f(z)\overline{\vec{H}_0}). \tag{4.1}$$

Now assume that $\vec{H}$ is parallel, that is $\pi_{\vec{n}}(D_z \vec{H}) = \pi_{\vec{n}}(D_{\bar{z}} \vec{H}) = 0$. From the Codazzi–Mainardi identity (3.49), observing that the curvature term vanishes as showed in the proof of Corollary 3.2 (it is nothing but $R^{\perp}_{\vec{\Phi}}(T\vec{\Phi})$) we obtain

$$e^{-2\lambda}\, \partial_{\bar{z}}\left(e^{2\lambda}\, \langle \vec{H}, \vec{H}_0\rangle\right) = 0;$$

therefore $f(z) := e^{2\lambda}\, \langle \vec{H}, \vec{H}_0\rangle$ is holomorphic. Since by assumption $\pi_{\vec{n}}(D_z \vec{H}) = 0$, we can write the left hand side of (4.1) as

$$4\, e^{-2\lambda}\, \Re\left[D_{\bar{z}}\left(e^{2\lambda}\langle \vec{H}, \vec{H}_0\rangle\, \partial_{\bar{z}}\vec{\Phi}\, e^{-2\lambda}\right)\right] = 4\, e^{-2\lambda}\, \Re\left[f(z) D_{\bar{z}}(e^{-\lambda}\vec{e}_{\bar{z}})\right].$$

Now using (3.44) we write the right hand side of the last equation as $2e^{-2\lambda}\Re\left(f(z)\overline{\vec{H}_0}\right) = e^{-2\lambda}\Im\left(2if(z)\overline{\vec{H}_0}\right)$. We have just shown that $\vec{\Phi}$ satisfies the constrained-conformal conformal Willmore equation (4.1) with holomorphic function $2ie^{2\lambda}\, \langle \vec{H}, \vec{H}_0\rangle$. □

Proof of Theorem 1.3. One implication follows directly from Proposition 1.1 observing that the constraint on the conformal class is trivial on smooth immersions of spheres by the Uniformization theorem.

Let us prove the opposite implication by contradiction: we assume that the compact Riemannian 3-manifold (M^3, h) has a constant scalar curvature Scal_0 but it is not a space form and we exhibit an embedded sphere which has constant mean curvature but is not conformal Willmore.

First of all let us denote $S_{\mu\nu} := \mathrm{Ric}_{\mu\nu} - \frac{1}{3}\,\mathrm{Scal}\,h_{\mu\nu}$ the trace-free Ricci tensor of (M, h) and observe that under our assumptions

$$\mathfrak{m} := \max_{x\in M} \|S_x\|^2 > 0. \tag{4.2}$$

Indeed if $\|S\|^2 \equiv 0$ then the manifold is Einstein, but the 3-dimensional Einstein manifolds are just the space forms (for example see [31, pp. 38–41]).

Now consider the function $r : M \to \mathbb{R}$ defined as

$$r(x) := -\frac{11}{378}\|S_x\|^2 + \frac{55}{1134}\mathrm{Scal}^2(x) - \frac{1}{21}\triangle\mathrm{Scal}(x) = -\frac{11}{378}\|S_x\|^2 + \frac{55}{1134}\mathrm{Scal}_0^2 \tag{4.3}$$

where in the last equality we used that $\mathrm{Scal} \equiv \mathrm{Scal}_0$. The function r we just defined is exactly the function r defined at p. 276 in [30]; this can be seen using the irreducible decomposition of the Riemann curvature tensor which implies (notice that we are assuming M to be 3-d, so the Weil tensor vanishes)

$$\|\mathrm{Riem}^h\|^2(x) = \frac{1}{3}\mathrm{Scal}^2(x) + 4\|S_x\|^2;$$

plugging this expression in the formula in [30] and taking $m = 3$, after some straightforward computations we end up with (4.3). Let us recall Theorem 1.1 of [30].

There exists $\rho_0 > 0$ and a smooth function $\phi : M \times (0, \rho_0) \to \mathbb{R}$ such that

(i) For all $\rho \in (0, \rho_0)$, if $\bar{x}$ is a critical point of the function $\phi(\cdot, \rho)$ then, there exists an embedded hyper-surface $S^\sharp_{\bar{x},\rho}$ whose mean curvature is constant equal to $\frac{1}{\rho}$ and that is a normal graph over the geodesic sphere $S_{\bar{x},\rho}$ for some function which is bounded by a constant times ρ^3 in $C^{2,\alpha}$ topology.

(ii) For all $k > 0$, there exists $c_k > 0$ which does not depend on $\rho \in (0, \rho_0)$ such that

$$\|\phi(\cdot, \rho) - \mathrm{Scal} + \rho^2 r\|_{C^k(M)} \le C_k\rho^3. \tag{4.4}$$

Now, for ρ_0 small enough, we claim that at all points of global minimum of ϕ we have $\|S\|^2 > 0$. If it is not the case let x^ϕ_ρ be a point of global minimum of $\phi(\cdot, \rho)$ and observe that (4.4) and (4.3) yield

$$\phi(x^\phi_\rho, \rho) \ge \mathrm{Scal}_0 + \frac{55}{1134}\mathrm{Scal}_0^2\,\rho^2 - C_0\rho^3; \tag{4.5}$$

on the other hand, at a maximum point x^S for $\|S\|^2$, we have analogously

$$\phi(x^S, \rho) \le \mathrm{Scal}_0 + \frac{55}{1134}\mathrm{Scal}_0^2\,\rho^2 - \frac{11}{378}\mathfrak{m}\rho^2 + C_0\rho^3; \tag{4.6}$$

now (4.5) and (4.6) together with the crucial fact that $\mathfrak{m} > 0$ (ensured by the fact that (M, h) is not space form) imply that for ρ small enough $\phi(x^S, \rho) < \phi(x^\phi_\rho, \rho)$ contradicting the minimality of x^ϕ_ρ. Collecting Theorem 1.1 of [30] and what we have just proved we conclude the following: for ρ_0 small enough, for every $\rho \le \rho_0$ consider a minimum x_ρ a point for $\phi(\cdot, \rho)$, then

(a) $\|S_{x_\rho}\|^2 > 0$

(b) there exists an embedded CMC sphere $S^\sharp_{x_\rho,\rho}$ whose mean curvature given by a normal graph over the geodesic sphere $S_{\bar{x},\rho}$ for some function which is bounded by a constant times ρ^3 in $C^{2,\alpha}$ topology. Observe that, since the graph function satisfies the mean curvature equation, bootstrapping the $C^{2,\alpha}$ bound using Schauder estimates, one gets the graph function is bounded in $C^{4,\alpha}$ norm by a constant times ρ^3.

But now, since (a) holds, Theorem 1.4 in [25] implies that for small ρ the CMC perturbed geodesic spheres constructed in (b) cannot be conformal Willmore immersions. The proof is now complete. □

5. Conformal constrained Willmore surfaces in manifold in arbitrary codimension via a system of conservation laws

Let us start with a general lemma for surfaces.

Lemma 5.1. *Let $\vec{\Phi}$ be a conformal immersion of the disc D^2 into a Riemannian manifold (M, h) and let $\vec{X}$ be the following $L^1 + H^{-1}$ vector field*

$$\vec{X} := -2i \left\langle \vec{H}, \vec{H}_0 \right\rangle \partial_{\bar{z}} \vec{\Phi} - 2i\, \pi_{\vec{n}}(D_z \vec{H}); \tag{5.1}$$

then the following system of equations holds

$$\begin{cases} \Im\left[\left\langle \vec{e}_{\bar{z}}, \vec{X} \right\rangle\right] = 0 \quad (\text{Sys}X\text{-}1) \\ \Im\left[\vec{e}_{\bar{z}} \wedge (\vec{X} + 2i D_z \vec{H})\right] = 0 \quad (\text{Sys}X\text{-}2) \end{cases} \tag{5.2}$$

where, given two complex vectors fields $\vec{X}, \vec{Y} \in \Gamma(TM \otimes \mathbb{C}) : \vec{X} = \vec{X}_1 + i\vec{X}_2, \vec{Y} = \vec{Y}_1 + i\vec{Y}_2$ with $\vec{X}_1, \vec{X}_2, \vec{Y}_1, \vec{Y}_2 \in \Gamma(TM)$ we use the notation $\langle \vec{X}, \vec{Y} \rangle$ to denote the quantity

$$\langle \vec{X}, \vec{Y} \rangle := h(\vec{X}_1, \vec{Y}_1) - h(\vec{X}_2, \vec{Y}_2) + i\, h(\vec{X}_1, \vec{Y}_2) + i\, h(\vec{X}_2, \vec{Y}_1)$$

where, of course, $h(\cdot, \cdot)$ denotes the standard scalar product of tangent vectors in the Riemannian manifold (M, h). □

Proof of Lemma 5.1. First of all by Lemma 3.2 we can write $\vec{X}$ as

$$\vec{X} := -2i \left\langle \vec{H}, \vec{H}_0 \right\rangle \partial_{\bar{z}} \vec{\Phi} - 2i\, \pi_{\vec{n}}(D_z \vec{H}) = i D_z \vec{H} - 3i \pi_{\vec{n}}(D_z \vec{H}) + \star_h (D_z \vec{n} \wedge \vec{H}). \tag{5.3}$$

Let us start by the first equation (Sys$X$-1). Since $\vec{e}_{\bar{z}}$ is tangent and $\pi_{\vec{n}}(D_z \vec{H})$ is normal to $(\vec{\Phi})_*(TD^2)$, the scalar product simplifies as

$$\Im\left[\left\langle \vec{e}_{\bar{z}}, \vec{X} \right\rangle\right] = \Im\left[\left\langle \vec{e}_{\bar{z}}, i\pi_T(D_z \vec{H}) \right\rangle\right] + \Im\left[\left\langle \vec{e}_{\bar{z}}, \star_h(D_z \vec{n} \wedge \vec{H}) \right\rangle\right]. \tag{5.4}$$

Identity (3.33) together with (3.27) gives

$$\Im\left[\left\langle \vec{e}_{\bar{z}}, i\pi_T(D_z \vec{H}) \right\rangle\right] = -\Im\left[\left\langle \vec{e}_{\bar{z}}, \star_h(D_z \vec{n} \wedge \vec{H}) \right\rangle\right]. \tag{5.5}$$

Putting together (5.4) and (5.5) we obtain (Sys$X$-1).

Now let us prove (Sys$X$-2). Since $\vec{e}_{\bar{z}} \wedge \vec{e}_{\bar{z}} = 0$ we have

$$\begin{aligned}\Im\left[\vec{e}_{\bar{z}} \wedge \vec{X}\right] &= \Im\left[\vec{e}_{\bar{z}} \wedge \left(-2i\, \pi_{\vec{n}}(D_z \vec{H})\right)\right] \\ &= \Im\left[\vec{e}_{\bar{z}} \wedge \left(-2i\, D_z \vec{H}\right)\right] - \Im\left[\vec{e}_{\bar{z}} \wedge \left(-2i\, \pi_T(D_z \vec{H})\right)\right].\end{aligned} \tag{5.6}$$

In order to have (Sys$X$-2) it is enough to prove that $\Im\left[\vec{e}_{\bar{z}} \wedge \left(-2i\, \pi_T(D_z \vec{H})\right)\right] = 0$. Using (3.27) we write

$$\pi_T(D_z \vec{H}) = 2\langle D_z \vec{H}, \vec{e}_z\rangle \vec{e}_{\bar{z}} + 2\langle D_z \vec{H}, \vec{e}_{\bar{z}}\rangle \vec{e}_z, \tag{5.7}$$

hence, again $\vec{e}_{\bar{z}} \wedge \vec{e}_{\bar{z}} = 0$ implies that

$$\begin{aligned}\Im\left[\vec{e}_{\bar{z}} \wedge \left(-2i\, \pi_T(D_z \vec{H})\right)\right] &= \Im\left[\vec{e}_{\bar{z}} \wedge \left(-4i\, \langle D_z \vec{H}, \vec{e}_{\bar{z}}\rangle \vec{e}_z\right)\right] \\ &= 4\Re\left[\left(\langle D_z \vec{H}, \vec{e}_{\bar{z}}\rangle\right) \vec{e}_z \wedge \vec{e}_{\bar{z}}\right].\end{aligned} \tag{5.8}$$

Now use the fact that $\vec{H}$ is orthogonal to $\vec{e}_1, \vec{e}_2$ and that $\pi_{\vec{n}}(D_{\vec{e}_1} \vec{e}_2) = \mathbb{I}_{12} = \mathbb{I}_{21} = \pi_{\vec{n}}(D_{\vec{e}_2} \vec{e}_1)$ to conclude that

$$\begin{aligned}4\Re\left[\left(\langle D_z \vec{H}, \vec{e}_{\bar{z}}\rangle\right) \vec{e}_z \wedge \vec{e}_{\bar{z}}\right] &= \frac{1}{2}\left[\left(\left\langle D_{\vec{e}_2} \vec{H}, \vec{e}_1\right\rangle - \left\langle D_{\vec{e}_1} \vec{H}, \vec{e}_2\right\rangle\right) \vec{e}_1 \wedge \vec{e}_2\right] \\ &= -\frac{1}{2}\left[\left(\left\langle \vec{H}, D_{\vec{e}_2} \vec{e}_1\right\rangle - \left\langle \vec{H}, D_{\vec{e}_1} \vec{e}_2\right\rangle\right) \vec{e}_1 \wedge \vec{e}_2\right] \\ &= 0. \quad \Box\end{aligned}$$

Theorem 5.1. *Let $\vec{\Phi}$ be a conformal immersion of the disc D^2 into a Riemannian manifold (M, h), then $\vec{\Phi}$ is a conformal constrained Willmore immersion if and only if there exists an $H^{-1} + L^1$ vector field $\vec{Y}$ such that*

$$\begin{cases}\Im\left[\left\langle \vec{e}_{\bar{z}}, \vec{Y}\right\rangle\right] = 0 \quad \text{(Sys-1)} \\ \Im\left[\vec{e}_{\bar{z}} \wedge (\vec{Y} + 2i D_z \vec{H})\right] = 0 \quad \text{(Sys-2)} \\ \Im\left[D_{\bar{z}} \vec{Y}\right] = -e^{2\lambda}\left(\dfrac{1}{2}\tilde{R}(\vec{H}) + 4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right]\right) \quad \text{(Sys-3)}. \quad \Box\end{cases} \tag{5.9}$$

Proof of Theorem 5.1. Let us first prove the “only if” part: we assume that $\vec{\Phi}$ is constrained conformal Willmore and prove that there exists an $H^{-1} + L^1$ vector field $\vec{Y}$ satisfying the system of Eqs. (5.9).

Recall that $\vec{\Phi}$ is a conformal constrained Willmore immersion if and only if there exists an holomorphic function $f(z)$ such that Eq. (3.32) is satisfied, namely

$$\Delta_\perp \vec{H} + \tilde{A}(\vec{H}) - 2|\vec{H}|^2\, \vec{H} - \tilde{R}(\vec{H}) = e^{-2\lambda}\Im(f(z)\overline{\vec{H}_0}) = \langle q, \vec{h}_0\rangle_{WP}$$

where $\vec{h}_0 = \vec{H}_0\, dz \otimes dz$. We claim that the vector

$$\vec{Y} := e^{-\lambda} f \vec{e}_{\bar{z}} - 2i\left\langle \vec{H}, \vec{H}_0\right\rangle \partial_{\bar{z}} \vec{\Phi} - 2i\, \pi_{\vec{n}}(D_z \vec{H}) \tag{5.10}$$

satisfies the system. Recall the definition of the vector $\vec{X}$ given in (5.1), observe that $\vec{Y} = e^{-\lambda} f \vec{e}_{\bar{z}} + \vec{X}$. Since $\vec{X}$ satisfies the system (5.2) and since $\langle \vec{e}_{\bar{z}}, \vec{e}_{\bar{z}}\rangle = 0 = \vec{e}_{\bar{z}} \wedge \vec{e}_{\bar{z}}$ we conclude that

$\vec{Y}$ satisfies the first two equations of system (5.9). Now let us prove the third equation (Sys-3), we have

$$\begin{aligned}\Im\left[D_{\bar z}\vec{Y}\right] &= -2\Im\left[i D_{\bar z}\left(\left\langle\vec{H},\vec{H}_0\right\rangle\partial_{\bar z}\vec{\Phi}+\pi_{\vec{n}}(D_z\vec{H})\right)\right]+\Im\left[D_{\bar z}(e^{-\lambda}f\vec{e}_{\bar z})\right]\\ &= -2\Re\left[D_{\bar z}\left(\left\langle\vec{H},\vec{H}_0\right\rangle\partial_{\bar z}\vec{\Phi}+\pi_{\vec{n}}(D_z\vec{H})\right)\right]+\Im\left[f D_{\bar z}(e^{-\lambda}\vec{e}_{\bar z})\right]\\ &\quad+\Im\left[(\partial_{\bar z}f)\,e^{-\lambda}\vec{e}_{\bar z}\right].\end{aligned}$$

Recall the identity (3.44), sum and subtract $e^{2\lambda}\left(\frac{1}{2}\tilde{R}(\vec{H})+4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar z},\vec{e}_z)\vec{e}_z,\vec{H}\rangle\vec{e}_{\bar z}\right]\right)$ and get

$$\begin{aligned}\Im\left[D_{\bar z}\vec{Y}\right] &= -e^{2\lambda}\left(\frac{1}{2}\tilde{R}(\vec{H})+4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar z},\vec{e}_z)\vec{e}_z,\vec{H}\rangle\vec{e}_{\bar z}\right]\right)+\Im\left[(\partial_{\bar z}f)\,e^{-\lambda}\vec{e}_{\bar z}\right]\\ &\quad-2\Re\left[D_{\bar z}\left(\left\langle\vec{H},\vec{H}_0\right\rangle\partial_{\bar z}\vec{\Phi}+\pi_{\vec{n}}(D_z\vec{H})\right)\right]+\frac{1}{2}\Im\left[f\overline{\vec{H}_0}\right]\\ &\quad+e^{2\lambda}\left(\frac{1}{2}\tilde{R}(\vec{H})+4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar z},\vec{e}_z)\vec{e}_z,\vec{H}\rangle\vec{e}_{\bar z}\right]\right).\end{aligned}\tag{5.11}$$

Now recall that $\vec{\Phi}$ is conformal constrained Willmore if and only if the identity (3.31) holds, moreover f is holomorphic so $\partial_{\bar z}f=0$; therefore we can conclude that

$$\Im\left[D_{\bar z}\vec{Y}\right]=-e^{2\lambda}\left(\frac{1}{2}\tilde{R}(\vec{H})+4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar z},\vec{e}_z)\vec{e}_z,\vec{H}\rangle\vec{e}_{\bar z}\right]\right)$$

as desired.

For the other implication assume that there exists a vector $\vec{Y}$ satisfying the system (5.9) and write $\vec{Y}$ as

$$\vec{Y}=A\vec{e}_z+B\vec{e}_{\bar z}+\vec{V}$$

where A and B are complex numbers and $\vec{V}:=\pi_{\vec{n}}(\vec{Y})$ is a complex valued normal vector to the immersed surface. The first equation of (5.9), using (3.27), is equivalent to

$$\Im A=0.\tag{5.12}$$

Observe that if we write

$$D_z\vec{H}=C\,\vec{e}_z+D\,\vec{e}_{\bar z}+\vec{W}$$

where $\vec{W}=\pi_{\vec{n}}(D_z\vec{H})$, one has, using (3.43) and the fact that $\vec{H}$ is orthogonal to $\vec{e}_{\bar z}$

$$C=2\left\langle\vec{e}_{\bar z},D_z\vec{H}\right\rangle=-2\left\langle D_z(e^{\lambda}\,\vec{e}_{\bar z}),\vec{H}\right\rangle e^{-\lambda}=-e^{\lambda}\,|\vec{H}|^2.\tag{5.13}$$

Hence we deduce in particular

$$\Im C=0.\tag{5.14}$$

We have moreover using (3.44)

$$D=2\left\langle\vec{e}_z,D_z\vec{H}\right\rangle=-2\left\langle D_z(e^{-\lambda}\,\vec{e}_z),\vec{H}\right\rangle e^{\lambda}=-e^{\lambda}\,\langle\vec{H}_0,\vec{H}\rangle.\tag{5.15}$$

Thus combining (5.13) and (5.15) we obtain

$$D_z\vec{H}=-|\vec{H}|^2\,\partial_z\vec{\Phi}-\langle\vec{H}_0,\vec{H}\rangle\,\partial_{\bar z}\vec{\Phi}+\pi_{\vec{n}}(D_z\vec{H}).\tag{5.16}$$

Using (3.27), the second line in the conservation law (5.9) is equivalent to

$$\begin{cases} \Im(i\,A - 2C) = 0 \\ \Im\left(\vec{e}_{\bar{z}} \wedge \left[\vec{V} + 2i\,\vec{W}\right]\right) = 0. \end{cases} \tag{5.17}$$

We observe that $\vec{e}_1 \wedge \left[\vec{V} + 2i\,\vec{W}\right]$ and $\vec{e}_2 \wedge \left[\vec{V} + 2i\,\vec{W}\right]$ are linearly independent since $\left[\vec{V} + 2i\,\vec{W}\right]$ is orthogonal to the tangent plane; moreover we combine (5.14) and (5.17) and we obtain that (5.17) is equivalent to

$$\begin{cases} \Im(i\,A) = 0 \\ \vec{e}_1 \wedge \Im\left(\vec{V} + 2i\,\vec{W}\right) = 0 \\ \vec{e}_2 \wedge \Im\left(i\,\left[\vec{V} + 2i\,\vec{W}\right]\right) = 0. \end{cases} \tag{5.18}$$

Combining (5.12) and (5.18) we obtain that the first two conservation laws of (5.9) are equivalent to

$$\begin{cases} A = 0 \\ \vec{V} = -2i\,\vec{W} = -2i\,\pi_{\vec{n}}(D_z\vec{H}). \end{cases} \tag{5.19}$$

Or in other words, for a conformal immersion $\vec{\Phi}$ of the disc into $\mathbb{R}^m$, there exists a vector field $\vec{Y}$ satisfying the first two equations of the system (5.9) if and only if there exist a complex valued function B and a vector field $\vec{Y}$ such that

$$\vec{Y} = B\,\vec{e}_{\bar{z}} - 2i\,\pi_{\vec{n}}(D_z\vec{H}). \tag{5.20}$$

We shall now exploit the third equation of (5.9) by taking $D_{\bar{z}}$ of (5.20). Let

$$f := e^{\lambda} B + 2i\,e^{2\lambda}\left\langle \vec{H}, \vec{H}_0 \right\rangle. \tag{5.21}$$

With this notation (5.20) becomes

$$\vec{Y} = e^{-\lambda}\,f\,\vec{e}_{\bar{z}} - 2i\left\langle \vec{H}, \vec{H}_0 \right\rangle \partial_{\bar{z}}\vec{\Phi} - 2i\,\pi_{\vec{n}}(D_z\vec{H}) \tag{5.22}$$

which is exactly Eq. (5.10) (recall we defined a vector $\vec{Y}$ in that way starting from a conformal immersion $\vec{\Phi}$ satisfying the constrained-conformal Willmore equation). Then repeating the computations above (i.e. the ones for the "only if" implication) we get that Eq. (5.11) is still valid, but since $\vec{Y}$ satisfies (Sys-3) we get

$$\begin{aligned} 0 = {} & -2\Re\left[D_{\bar{z}}\left(\left\langle \vec{H}, \vec{H}_0 \right\rangle \partial_{\bar{z}}\vec{\Phi} + \pi_{\vec{n}}(D_z\vec{H})\right)\right] + \frac{1}{2}\Im\left[f\,\overline{\vec{H}_0}\right] \\ & + e^{2\lambda}\left(\frac{1}{2}\tilde{R}(\vec{H}) + 4\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right]\right) + \Im\left[(\partial_{\bar{z}} f)\,e^{-\lambda}\vec{e}_{\bar{z}}\right]. \end{aligned} \tag{5.23}$$

Consider the normal and the tangential projections of (5.23). The tangential projection of identity (3.28) gives

$$\begin{aligned} & \pi_T\left(-2\Re\left[D_{\bar{z}}\left(\left\langle \vec{H}, \vec{H}_0 \right\rangle \partial_{\bar{z}}\vec{\Phi} + \pi_{\vec{n}}(D_z\vec{H})\right)\right]\right) \\ & \quad = -4\,e^{2\lambda}\Re\left[\langle \mathrm{Riem}^h(\vec{e}_{\bar{z}}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar{z}}\right]; \end{aligned} \tag{5.24}$$

on the other hand, the tangential projection of (5.23) gives

$$0=\pi_T\left(-2\Re\left[D_{\bar z}\left(\left\langle\vec H,\vec H_0\right\rangle\partial_{\bar z}\vec\Phi+\pi_{\vec n}(D_z\vec H)\right)\right]\right)$$
$$+4\,e^{2\lambda}\Re\left[\langle\mathrm{Riem}^h(\vec e_{\bar z},\vec e_z)\vec e_z,\vec H\rangle\vec e_{\bar z}\right]+\Im\left[(\partial_{\bar z}f)\,e^{-\lambda}\vec e_{\bar z}\right],\tag{5.25}$$

so, combining (5.24) and (5.25) we obtain

$$0=\Im\left[(\partial_{\bar z}f)\,e^{-\lambda}\vec e_{\bar z}\right].\tag{5.26}$$

The normal projection of (5.23) gives

$$0=\pi_{\vec n}\left(-2\Re\left[D_{\bar z}\left(\left\langle\vec H,\vec H_0\right\rangle\partial_{\bar z}\vec\Phi+\pi_{\vec n}(D_z\vec H)\right)\right]+\frac{1}{2}\Im\left[f\overline{\vec H_0}\right]+\frac{e^{2\lambda}}{2}\tilde R(\vec H)\right).\tag{5.27}$$

Therefore, putting together (5.24), (5.26) and (5.27) we conclude that (5.23) implies the following system

$$\begin{cases}4\,e^{-2\lambda}\,\Re\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec H)+\langle\vec H,\vec H_0\rangle\,\partial_{\bar z}\vec\Phi\right]\right)\\ \quad=e^{-2\lambda}\,\Im(f(z)\overline{\vec H_0})+\tilde R(\vec H)+8\Re\left(\langle\mathrm{Riem}^h(\vec e_{\bar z},\vec e_z)\vec e_z,\vec H\rangle\vec e_{\bar z}\right)\\ \Im\,(\partial_{\bar z}f\ \ \vec e_{\bar z})=0.\end{cases}\tag{5.28}$$

The second line is equivalent to

$$\partial_{\bar z}f\ \ \vec e_{\bar z}-\partial_z\overline{f}\,\vec e_z=0.$$

Taking the scalar product with $\vec e_z$ and using (3.27), observe that (5.28) is equivalent to

$$\begin{cases}4\,e^{-2\lambda}\,\Re\left(D_{\bar z}\left[\pi_{\vec n}(D_z\vec H)+\langle\vec H,\vec H_0\rangle\,\partial_{\bar z}\vec\Phi\right]\right)\\ \quad=e^{-2\lambda}\,\Im(f(z)\overline{\vec H_0})+\tilde R(\vec H)+8\Re\left(\langle\mathrm{Riem}^h(\vec e_{\bar z},\vec e_z)\vec e_z,\vec H\rangle\vec e_{\bar z}\right)\\ \partial_{\bar z}f=0.\end{cases}\tag{5.29}$$

The second line gives that $f=f(z)$ is holomorphic and the equation in the first line is exactly the equation of the conformal constrained Willmore surfaces (3.31); therefore we proved that the existence of a vector field $\vec Y$ satisfying the system of conservation laws (5.9) implies that the immersion $\vec\Phi$ is conformal constrained Willmore. □

6. Regularity for Willmore immersions

We start by using the divergence structure of the constrained-conformal Willmore equation in order to construct potentials which will play a crucial role in the regularity theory.

Lemma 6.1. *Let $\vec\Phi$ be a $W^{1,\infty}$ conformal immersion of the disc D^2 taking values into a sufficiently small open subset of the Riemannian manifold (M,h), with second fundamental form in $L^2(D^2)$ and conformal factor $\lambda\in L^\infty(D^2)$. Assume $\vec\Phi$ is a constrained-conformal Willmore immersion; then there exist the following potential vector fields:*

(i) *there exists a complex vector field (i.e. a vector field with values in the complexified tangent bundle of M) $\vec L\in L^{2,\infty}(D^2)$ with $\nabla\Im\vec L\in L^{2,\infty}(D^2)$ satisfying*

$$\begin{cases}D_{\bar z}\vec L=\vec Y & \text{on } D^2\\ \Im\vec L=0 & \text{on } \partial D^2,\end{cases}\tag{6.1}$$

where $\vec{Y}$ *is the vector given in* (5.10) *in the proof of* Theorem 5.1*;*

(ii) *there exists a complex valued function* $S \in W^{1,(2,\infty)}(D^2)$ *with* $\nabla^2 \Im S \in L^q(D^2)$ *for every* $1 < q < 2$ *satisfying*

$$\begin{cases} \partial_z S = \langle \partial_z \vec{\Phi}, \overline{\vec{L}} \rangle & \text{on } D^2 \\ \Im S = 0 & \text{on } \partial D^2; \end{cases} \tag{6.2}$$

(iii) *there exists a complex valued* 2*-vector field* $\vec{R} \in W^{1,(2,\infty)}(D^2)$ *with* $\nabla^2 \Im \vec{R} \in L^q(D^2)$ *for every* $1 < q < 2$ *satisfying*

$$\begin{cases} D_z \vec{R} = \partial_z \vec{\Phi} \wedge \overline{\vec{L}} - 2i \partial_z \vec{\Phi} \wedge \vec{H} & \text{on } D^2 \\ \Im \vec{R} = 0 & \text{on } \partial D^2. \quad \Box \end{cases} \tag{6.3}$$

Proof. (i): Let $\vec{Y}$ be the vector field given by (5.10) in the proof of Theorem 5.1 and observe that, by our assumption of the immersion $\vec{\Phi}$, we have $\vec{Y} \in H^{-1} + L^1(D^2)$. Moreover, since $\vec{Y}$ satisfies Eq. (Sys-3) of (5.9), then

$$\|\Im(D_{\bar{z}} \vec{Y})\|_{L^2(D^2)} \le C \|H\|_{L^2(D^2)}. \tag{6.4}$$

Since $\vec{\Phi}$ is taking values into a small open subset $V \subset M$, by choosing Riemann normal coordinates on V centered in $\vec{\Phi}(0)$, we can assume that the functions

$$\gamma^j_k := \Gamma^j_{kl} \partial_z \Phi^l, \quad \gamma^j_k \in C^0 \cap W^{1,2}(D^2) \tag{6.5}$$

are smaller than the ϵ given in the statement of Lemmas A.1 and A.2; extend γ^j_k to the whole $\mathbb{C}$ and multiply them by a smooth cutoff function in order to obtain

$$\gamma^j_k \in C^0 \cap W^{1,2}(\mathbb{C}), \quad \operatorname{supp} \gamma^j_k \subset B_2(0), \qquad \|\gamma^j_k\|_{L^\infty(\mathbb{C})} \le \epsilon. \tag{6.6}$$

Using γ^j_k we can extend the operator D_z to complex vector fields $\vec{U} \in L^1_{\mathrm{loc}}(\mathbb{C})$ in the following way

$$D_z U^j := \partial_z U^j + \sum_{k=1}^m \gamma^j_k U^k \quad \text{in distributional sense.}$$

Analogously extend $Y^j \in H^{-1} + L^1(D^2)$ to functions $\tilde{Y}^j \in \mathring{H}^{-1} + L^1(\mathbb{C})$, where $\mathring{H}^{-1}(\mathbb{C})$ is the dual of homogeneous Sobolev space $\mathring{H}^1(\mathbb{C})$ (this is just a technical point for applying Lemma A.1) such that

$$\|\tilde{Y}^j\|_{\mathring{H}^{-1}+L^1(\mathbb{C})} \le C \|Y^j\|_{H^{-1}+L^1(D^2)} < \infty,$$
$$\|\Im(D_z \tilde{Y}^j)\|_{L^1(\mathbb{C})} \le C \|\Im(D_z Y^j)\|_{L^1(D^2)} \le C \|\Im(D_z Y^j)\|_{L^2(D^2)} < \infty.$$

For convenience, in the following we identify Y^j and its extension. Now we apply Lemma A.1 and define $\vec{L} \in L^{2,\infty}(D^2)$ to be the unique solution to the problem

$$\begin{cases} D_z \vec{L} = \vec{Y} & \text{on } D^2 \\ \Im \vec{L} = 0 & \text{on } \partial D^2. \end{cases}$$

Observe that moreover the same lemma gives that $\nabla(\Im \vec{L}) \in L^{2,\infty}(D^2)$ which implies that $\Im \vec{L} \in L^p(D^2)$ for every $1 < p < \infty$.

Proof of (ii). Let us start with a computation; from (Sys-1) of (5.9), since by (i) we have $D_z\vec{L} = \vec{Y}$ on D^2, then

$$0 = \Im\left(\langle \partial_{\bar z}\vec{\Phi}, D_z\vec{L}\rangle\right) = \Im\left(\partial_z\langle \partial_{\bar z}\vec{\Phi}, \vec{L}\rangle - \langle D_z\partial_{\bar z}\vec{\Phi}, \vec{L}\rangle\right).$$

Using identity (3.43), by complex conjugation we obtain

$$\Im(\partial_{\bar z}\langle \partial_z\vec{\Phi}, \overline{\vec{L}}\rangle) = -\frac{e^{2\lambda}}{2}\langle \vec{H}, \Im\vec{L}\rangle \in L^q(D^2) \quad \text{for every } 1 < q < 2, \tag{6.7}$$

where the L^q bound follows by the Hölder inequality observing that by (i) we have $\nabla\Im\vec{L} \in L^{2,\infty}(D^2)$ then $\Im\vec{L} \in L^p(D^2)$ for every $1 < p < \infty$; on the other hand, by assumption, $\vec{H} \in L^2(D^2)$.

By (i), we have $\langle \partial_z\vec{\Phi}, \overline{\vec{L}}\rangle \in L^{2,\infty}(D^2)$ and as before we extend it to the whole $\mathbb{C}$ keeping controlled the norms: $\langle \partial_z\vec{\Phi}, \overline{\vec{L}}\rangle \in L^1 \cap L^{2,\infty}(\mathbb{C})$ and $\Im(\partial_{\bar z}\langle \partial_z\vec{\Phi}, \overline{\vec{L}}\rangle) \in L^q(\mathbb{C})$ for every $1 < q < 2$.

Now we apply Lemma A.2, with $m = 1$ and $\gamma_k^j = 0$, and define $S \in W^{1,(2,\infty)}(D^2)$ to be the unique solution to

$$\begin{cases} \partial_z S = \langle \partial_z\vec{\Phi}, \overline{\vec{L}}\rangle & \text{on } D^2 \\ \Im S = 0 & \text{on } \partial D^2; \end{cases}$$

moreover $\nabla^2\Im S \in L^q(D^2)$ for every $1 < q < 2$ which implies, by Sobolev Embedding Theorem, $\nabla\Im S \in L^p(D^2)$ for all $1 < p < \infty$.

Proof of (iii). Since $\vec{Y} = D_z\vec{L}$, Eq. (Sys-2) in (5.9) gives

$$\begin{aligned} 0 &= \Im\left[\partial_{\bar z}\vec{\Phi} \wedge D_z\vec{L} + 2i\partial_{\bar z}\vec{\Phi} \wedge D_z\vec{H}\right] \\ &= -\Im\left[D_{\bar z}\left(\partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H}\right) - (D_{\bar z}\partial_z\vec{\Phi}) \wedge \overline{\vec{L}} + 2i(D_{\bar z}\partial_z\vec{\Phi}) \wedge \vec{H}\right], \end{aligned}$$

using (3.43) we obtain

$$\Im\left[D_{\bar z}\left(\partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H}\right)\right] = -\frac{e^{2\lambda}}{2}\vec{H} \wedge \Im\vec{L} \in L^q(D^2) \quad \text{for every } 1 < q < 2, \tag{6.8}$$

where the $L^q(D^2)$ estimate comes from the Hölder inequality since $\vec{H} \in L^2(D^2)$ and $\Im\vec{L} \in L^p(D^2)$ for every $1 < p < \infty$. As in (ii) extend the complex valued vector field $\partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H} \in L^{2,\infty}(D^2)$ to a complex valued vector field on $\mathbb{C}$ keeping the norms controlled: $\partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H} \in L^1 \cap L^{2,\infty}(\mathbb{C})$ and $\Im\left[D_{\bar z}\left(\partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H}\right)\right] \in L^q(D^2)$ for every $1 < q < 2$.

As in (ii), we apply Lemma A.2 in order to define $\vec{R} \in W^{1,(2,\infty)}(D^2)$ as the unique solution to

$$\begin{cases} D_{\bar z}\vec{R} = \partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H} & \text{on } D^2 \\ \Im\vec{R} = 0 & \text{on } \partial D^2; \end{cases}$$

moreover $\nabla^2\Im\vec{R} \in L^q(D^2)$ for every $1 < q < 2$ which implies, by the Sobolev Embedding Theorem, $\nabla\Im\vec{R} \in L^p(D^2)$ for all $1 < p < \infty$. □

Next we play with the introduced $\vec{R}$ and S in order to produce, in the following lemma, an elliptic system of Wente type involving $\vec{\Phi}$, $\vec{R}$ and S.

Lemma 6.2. *Let $\vec{\Phi}$ be a $W^{1,\infty}$ conformal immersion of the disc D^2 taking values into a sufficiently small open subset of the Riemannian manifold (M, h), with second fundamental form in $L^2(D^2)$ and conformal factor $\lambda \in L^\infty(D^2)$. Assume $\vec{\Phi}$ is a constrained-conformal Willmore immersion and let $\vec{R} \in W^{1,(2,\infty)}(D^2)$ and $S \in W^{1,(2,\infty)}(D^2)$ be given by Lemma 6.1; then $\vec{R}$ and S satisfy the following coupled system on D^2:*

$$\begin{cases} D_z\vec{R} = (-1)^{m+1} \star_h \left[\vec{n} \bullet i D_z\vec{R}\right] + (i\partial_z S) \star_h \vec{n} \\ \partial_z S = \langle -i D_z\vec{R}, \star_h \vec{n}\rangle. \quad \square \end{cases} \tag{6.9}$$

Proof. By definition, $\vec{R}$ satisfies Eq. (6.3) on D^2, i.e:

$$D_z\vec{R} = \partial_z\vec{\Phi} \wedge \overline{\vec{L}} - 2i\partial_z\vec{\Phi} \wedge \vec{H}. \tag{6.10}$$

Taking the $\bullet$ contraction defined in (1.34) between $\vec{n}$ and $D_z\vec{R}$ we obtain

$$\begin{aligned} \vec{n} \bullet D_z\vec{R} &= -(\vec{n} \llcorner \overline{\vec{L}}) \wedge \partial_z\vec{\Phi} + 2i(\vec{n} \llcorner \vec{H}) \wedge \partial_z\Phi \\ &= -[\vec{n} \llcorner \pi_{\vec{n}}(\overline{\vec{L}})]\partial_z\vec{\Phi} + 2i(\vec{n} \llcorner \vec{H}) \wedge \partial_z\Phi, \end{aligned} \tag{6.11}$$

where $\llcorner$ is the usual contraction defined in (1.33).

For a normal vector $\vec{N}$, a short computation using just the definitions of $\star_h$ and $\llcorner$ gives

$$\begin{aligned} &\star_h[(\vec{n} \llcorner \vec{N}) \wedge \vec{e}_1] = (-1)^m \ \vec{N} \wedge \vec{e}_2 \\ &\star_h[(\vec{n} \llcorner \vec{N}) \wedge \vec{e}_2] = (-1)^{m+1} \ \vec{N} \wedge \vec{e}_1, \end{aligned}$$

where, as usual, $\vec{e}_1$ and $\vec{e}_2$ are the orthonormal bases of $T\vec{\Phi}(D^2)$ given by the vectors $\partial_1\vec{\Phi}$, $\partial_2\vec{\Phi}$ normalized. Since $\partial_z\vec{\Phi} = \frac{1}{2}\left[\partial_1\vec{\Phi} - i\partial_2\vec{\Phi}\right]$, we get

$$\star_h[(\vec{n} \llcorner \vec{N}) \wedge \partial_z\vec{\Phi}] = (-1)^m \ \vec{N} \wedge (i\partial_z\vec{\Phi}). \tag{6.12}$$

Combining (6.11) and (6.12) we have

$$\star_h[\vec{n} \bullet D_z\vec{R}] = (-1)^{m+1}\pi_{\vec{n}}(\overline{\vec{L}}) \wedge (i\partial_z\vec{\Phi}) + 2i(-1)^m\vec{H} \wedge (i\partial_z\vec{\Phi}), \tag{6.13}$$

multiplying both sides with $i(-1)^m$ gives

$$(-1)^{m+1} \star_h[\vec{n} \bullet (iD_z\vec{R})] = \partial_z\vec{\Phi} \wedge \pi_{\vec{n}}(\overline{\vec{L}}) - 2i\partial_z\vec{\Phi} \wedge \vec{H}. \tag{6.14}$$

Combining (6.10) and (6.14) we obtain

$$(-1)^{m+1} \star_h[\vec{n} \bullet (iD_z\vec{R})] = D_z\vec{R} - \partial_z\vec{\Phi} \wedge \pi_T(\overline{\vec{L}}). \tag{6.15}$$

Observing that, by (3.27)

$$\pi_T(\overline{\vec{L}}) = 2\langle\overline{\vec{L}}, \vec{e}_{\bar{z}}\rangle\vec{e}_z + 2\langle\overline{\vec{L}}, \vec{e}_z\rangle\vec{e}_{\bar{z}},$$

then

$$\partial_z\vec{\Phi} \wedge \pi_T(\overline{\vec{L}}) = \partial_z\vec{\Phi} \wedge (2\langle\overline{\vec{L}}, \vec{e}_z\rangle)\vec{e}_{\bar{z}} = (2\langle\overline{\vec{L}}, \partial_z\vec{\Phi}\rangle)\vec{e}_z \wedge \vec{e}_{\bar{z}},$$

using again (3.27), and the definition of S (6.2) gives

$$\partial_z \vec{\Phi} \wedge \pi_T(\overline{\vec{L}}) = (i\partial_z S) \star_h \vec{n}. \tag{6.16}$$

The combination of (6.16) and (6.15) gives the first equation of (6.9). The second equation is obtained by taking the scalar product between the first equation and $\star_h \vec{n}$ once one have observed that

$$\langle \star_h \vec{n}, \star_h(\vec{n} \bullet D_z \vec{R})\rangle = 0.$$

This fact comes from (6.13) which implies that $\star_h(\vec{n} \bullet D_z \vec{R})$ is a linear combination of wedges of tangent and normal vectors to $T\vec{\Phi}(D^2)$. This concludes the proof. □

Proposition 6.1. *Let $\vec{\Phi}$ be a $W^{1,\infty}$ conformal immersion of the disc D^2 taking values into a sufficiently small open subset of the Riemannian manifold (M,h), with second fundamental form in $L^2(D^2)$ and conformal factor $\lambda \in L^\infty(D^2)$. Assume $\vec{\Phi}$ is a constrained-conformal Willmore immersion and let $\vec{R} \in W^{1,(2,\infty)}(D^2)$ and $S \in W^{1,(2,\infty)}(D^2)$ be given by Lemma 6.1; then the couple $(\Re\vec{R}, \Re S)$ satisfies the following system on D^2*

$$\begin{cases} \Delta(\Re\vec{R}) = (-1)^m \star_h[D\vec{n} \bullet D^\perp(\Re\vec{R})] - \star_h[D\vec{n}\ \nabla^\perp(\Re S)] + \tilde{F} \\ \Delta(\Re S) = \langle D(\star_h \vec{n}), D^\perp(\Re\vec{R})\rangle + \tilde{G}; \end{cases} \tag{6.17}$$

where $\tilde{F}$ and $\tilde{G}$ are some functions ($\tilde{F}$ is 2-vector valued) in $L^q(D^2)$ for every $1 < q < 2$. Moreover we denoted $\Delta(\Re\vec{R}) := D_{\partial_{x_1}\vec{\Phi}} D_{\partial_{x_1}\vec{\Phi}}(\Re\vec{R}) + D_{\partial_{x_2}\vec{\Phi}} D_{\partial_{x_2}\vec{\Phi}}(\Re\vec{R})$, observe this differ from the intrinsic Laplace–Beltrami operator by a factor $e^{2\lambda}$. For a more explicit shape of the equations see (6.25) and (6.27) in the end of the proof. □

Proof. Let us start by proving the first equation. Applying the $D_{\bar z}$ operator to the first equation of (6.9) we have

$$D_{\bar z} D_z \vec{R} = (-1)^{m+1} i \star_h D_{\bar z}\left[\vec{n} \bullet D_z \vec{R}\right] + i D_{\bar z}\left[\partial_z S \star_h \vec{n}\right],$$

whose real part is

$$\Re(D_{\bar z} D_z \vec{R}) = (-1)^m \star_h \Im\left(D_{\bar z}\left[\vec{n} \bullet D_z \vec{R}\right]\right) - \Im\left(D_{\bar z}\left[\partial_z S \star_h \vec{n}\right]\right). \tag{6.18}$$

Observe that

$$\begin{aligned} D_{\bar z} D_z \vec{R} &:= \frac{1}{4}\left[(D_{\partial_{x_1}\vec{\Phi}} + i D_{\partial_{x_2}\vec{\Phi}})(D_{\partial_{x_1}\vec{\Phi}} - i D_{\partial_{x_2}\vec{\Phi}})\right]\vec{R} \\ &= \frac{1}{4}\Delta\vec{R} - \frac{i}{4}\left[D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}\right]\vec{R}, \end{aligned} \tag{6.19}$$

where $\left[D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}\right] := (D_{\partial_{x_1}\vec{\Phi}} D_{\partial_{x_2}\vec{\Phi}} - D_{\partial_{x_2}\vec{\Phi}} D_{\partial_{x_1}\vec{\Phi}})$ is the usual bracket notation. An easy computation in local coordinates shows that all the derivatives appearing in $[D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}](\vec{R})$ cancel out together with all the mixed terms, giving

$$\begin{aligned} \left[D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}\right]\left(\sum_{i,j=1}^m R^{ij}\vec{E}_i \wedge \vec{E}_j\right) = \sum_{i,j=1}^m R^{ij}[(\mathrm{Riem}(\partial_{x^1}\vec{\Phi}, \partial_{x^2}\vec{\Phi})\vec{E}_i) \wedge \vec{E}_j \\ + \vec{E}_i \wedge (\mathrm{Riem}(\partial_{x^1}\vec{\Phi}, \partial_{x^2}\vec{\Phi})\vec{E}_j)], \end{aligned} \tag{6.20}$$

where as before $\{\vec{E}_i\}_{i=1,\dots,m}$ is an orthonormal frame of $T_{\vec{\Phi}(x)}M$. Putting together (6.18) and (6.19) we obtain

$$\Delta(\Re\vec{R}) = 4(-1)^m \star_h \Im\left(D_{\bar z}\left[\vec{n} \bullet D_z\vec{R}\right]\right) - 4\Im\left(D_{\bar z}\left[\partial_z S \star_h \vec{n}\right]\right) - \left[D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}\right](\Im\vec{R}). \tag{6.21}$$

Using that the $\bullet$ contraction commutes with the covariant derivative (this fact follows by the definitions and by the identity $Dh = 0$, i.e. the connection is metric) we compute

$$\begin{aligned}\Im\left[D_{\bar z}(\vec{n} \bullet D_z\vec{R})\right] &= \Im\left[\vec{n} \bullet (D_{\bar z}D_z\vec{R}) + D_{\bar z}\vec{n} \bullet D_z\vec{R}\right]\\ &= \frac{1}{4}\vec{n} \bullet [\Delta(\Im\vec{R}) - [D_{\partial_{x_1}\vec{\Phi}}, D_{\partial_{x_2}\vec{\Phi}}](\Re\vec{R})] + \Im\left[D_{\bar z}\vec{n} \bullet D_z\vec{R}\right].\end{aligned} \tag{6.22}$$

A short computation gives

$$\begin{aligned}\Im\left[D_{\bar z}\vec{n} \bullet D_z\vec{R}\right] = {}& \frac{1}{4}\left[D_{\partial_{x^1}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^1}\vec{\Phi}}(\Im\vec{R}) + D_{\partial_{x^2}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^2}\vec{\Phi}}(\Im\vec{R})\right]\\ &+ \frac{1}{4}\left[D_{\partial_{x^2}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^1}\vec{\Phi}}(\Re\vec{R}) - D_{\partial_{x^1}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^2}\vec{\Phi}}(\Re\vec{R})\right].\end{aligned} \tag{6.23}$$

Analogously, using that $\star_h$ commutes with the covariant derivative, another short computation gives

$$\begin{aligned}\Im\left[D_{\bar z}(\partial_z S \star_h \vec{n})\right] = {}& \frac{1}{4}\Delta(\Im S) \star_h \vec{n} + \frac{1}{4}\left[\partial_{x^1}(\Im S)\, D_{\partial_{x^1}\vec{\Phi}}(\star_h \vec{n}) + \partial_{x^2}(\Im S)\, D_{\partial_{x^2}\vec{\Phi}}(\star_h \vec{n})\right]\\ &+ \frac{1}{4}\left[\partial_{x^1}(\Re S)\, D_{\partial_{x^2}\vec{\Phi}}(\star_h \vec{n}) - \partial_{x^2}(\Re S)\, D_{\partial_{x^1}\vec{\Phi}}(\star_h \vec{n})\right].\end{aligned} \tag{6.24}$$

Combining (6.21), (6.20), (6.22), (6.23) and (6.24) we conclude that

$$\begin{aligned}\Delta(\Re\vec{R}) = {}& (-1)^m \star_h \left[D_{\partial_{x^2}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^1}\vec{\Phi}}(\Re\vec{R}) - D_{\partial_{x^1}\vec{\Phi}}\vec{n} \bullet D_{\partial_{x^2}\vec{\Phi}}(\Re\vec{R})\right]\\ &+ \left[\partial_{x^2}(\Re S)\, D_{\partial_{x^1}\vec{\Phi}}(\star_h \vec{n}) - \partial_{x^1}(\Re S)\, D_{\partial_{x^2}\vec{\Phi}}(\star_h \vec{n})\right] + \tilde{F}\end{aligned} \tag{6.25}$$

where $\tilde{F} \in L^q(D^2)$ for every $1 < q < 2$, and we used that $D\vec{n} \in L^2(D^2)$, $\vec{R} \in W^{1,(2,\infty)}(D^2)$, $S \in W^{1,(2,\infty)}(D^2)$, $\Im\vec{R} \in W^{2,q}(D^2)$, $\Im S \in W^{2,q}(D^2)$ for every $1 < q < 2$. This is exactly the first equation of (6.17).

The second equation of (6.17) is obtained in an analogous way: applying the $\partial_{\bar z}$ operator to the second equation of (6.9) we obtain

$$\Delta S = 4\partial_{\bar z}\partial_z S = -4i\,\partial_{\bar z}\langle D_z\vec{R}, \star_h \vec{n}\rangle.$$

A short computation gives

$$\begin{aligned}\Delta(\Re S) &= 4\Im\left[\partial_{\bar z}\langle D_z\vec{R}, \star_h \vec{n}\rangle\right]\\ &= \partial_{x_1}\langle D_{\partial_{x^1}\vec{\Phi}}\Im\vec{R}, \star_h \vec{n}\rangle + \partial_{x^2}\langle D_{\partial_{x^2}\vec{\Phi}}\Im\vec{R}, \star_h \vec{n}\rangle - \langle [D_{\partial_{x^1}\vec{\Phi}}, D_{\partial_{x^2}\vec{\Phi}}]\Re\vec{R}, \star_h \vec{n}\rangle\\ &\quad + \langle D_{\partial_{x^1}\vec{\Phi}}\Re\vec{R}, D_{\partial_{x^2}\vec{\Phi}}(\star_h \vec{n})\rangle - \langle D_{\partial_{x^2}\vec{\Phi}}\Re\vec{R}, D_{\partial_{x^1}\vec{\Phi}}(\star_h \vec{n})\rangle.\end{aligned} \tag{6.26}$$

Recalling that $\Im \vec{R} \in W^{2,q}(D^2)$ and $\vec{R} \in W^{1,(2,\infty)}(D^2)$, and using (6.20), we conclude that

$$\Delta(\Re S) = \langle D_{\partial_{x^1} \vec{\Phi}}(\Re \vec{R}), D_{\partial_{x^2} \vec{\Phi}}(\star_h \vec{n})\rangle - \langle D_{\partial_{x^2} \vec{\Phi}}(\Re \vec{R}), D_{\partial_{x^1} \vec{\Phi}}(\star_h \vec{n})\rangle + \tilde{G} \tag{6.27}$$

where $\tilde{G} \in L^q(D^2)$ for every $1 < q < 2$. This is exactly the second equation of (6.17). □

Now we are in a position to prove the C^∞ regularity of constrained-conformal Willmore immersions.

Theorem 6.1. *Let $\vec{\Phi}$ be a $W^{1,\infty}$ conformal immersion of the disc D^2 taking values into a sufficiently small open subset of the Riemannian manifold (M, h), with second fundamental form in $L^2(D^2)$ and conformal factor $\lambda \in L^\infty(D^2)$. If $\vec{\Phi}$ is a constrained-conformal Willmore immersion then $\vec{\Phi}$ is C^∞.* □

Proof. Let us call $\vec{A} := (\Re \vec{R}, \Re S) = (\Re R^{ij}, \Re S)$ the vector of the components (in local coordinates in the small neighborhood $V \subset M$) of the real parts of the potentials $\vec{R}$ and S. Using coordinates also in the domain D^2, one easily checks that the system (6.17) has the form

$$\Delta A^i = \sum_k \left[\partial_{x^1} B^i_k \, \partial_{x^2} A^k - \partial_{x^2} B^i_k \, \partial_{x^1} A^k\right] + F^i, \tag{6.28}$$

where $F^i \in L^q(D^2)$ for every $1 < q < 2$, $\nabla A^i \in L^{2,\infty}(D^2)$, $\nabla B^i_k \in L^2(D^2)$.

Step 1: $\nabla A^i \in L^2_{\text{loc}}(D^2)$. Let us write A^i as

$$A^i = \varphi^i + V^i + W^i \quad \text{on } D^2, \tag{6.29}$$

where φ^i, V^i, W^i solve the following problems

$$\begin{cases} \Delta\varphi^i = \sum_k \left[\partial_{x^1} B^i_k \, \partial_{x^2} A^k - \partial_{x^2} B^i_k \, \partial_{x^1} A^k\right] & \text{on } D^2 \\ \varphi^i = 0 \quad \text{on } \partial D^2; \end{cases} \tag{6.30}$$

$$\begin{cases} \Delta V^i = F^i & \text{on } D^2 \\ V^i = 0 & \text{on } \partial D^2; \end{cases} \tag{6.31}$$

$$\begin{cases} \Delta W^i = 0 & \text{on } D^2 \\ W^i = A^i & \text{on } \partial D^2. \end{cases} \tag{6.32}$$

Since the right hand side of (6.30) is sum of $L^{2,\infty} - L^2$-Jacobians, by a refinement of the Wente inequality obtained by Bethuel [5] as a consequence of a result by Coifman, Lions, Meyer and Semmes, we have $\nabla\varphi^i \in L^2(D^2)$.

On the other hand, since $F^i \in L^q(D^2)$, for every $1 < q < 2$, it follows that $V^i \in W^{2,q}(D^2)$, which implies by Sobolev embedding that $\nabla V^i \in L^2(D^2)$.

Finally, W^i is a harmonic $W^{1,2,\infty}(D^2)$ function; therefore the gradient $\nabla W^i \in L^2_{\text{loc}}(D^2)$.

We conclude that $\nabla A^i = \nabla\varphi^i + \nabla V^i + \nabla W^i \in L^2_{\text{loc}}(D^2)$.

Step 2: $\nabla A^i \in L^p_{\text{loc}}(D^2)$ for some $p > 2$.

We first claim that there exists $\alpha > 0$ such that

$$\sup_{x_0 \in B_{\frac{1}{2}}(0), \rho < \frac{1}{4}} \frac{1}{\rho^\alpha} \int_{B_\rho(x_0)} |\nabla A|^2 < \infty. \tag{6.33}$$

Since $\nabla B \in L^2(D^2)$, by absolute continuity of the integral, for every $\epsilon > 0$ there exists a $\rho_0 > 0$ such that

$$\sup_{x_0 \in B_{\frac{1}{2}}(0)} \int_{B_{\rho_0}(x_0)} |\nabla B|^2 < \epsilon^2. \tag{6.34}$$

Consider $\rho < \rho_0$ ($\epsilon > 0$ will be chosen later depending on universal constants) and $x_0 \in B_{\frac{1}{2}(0)}$. Analogously to Step 1 let us write

$$A^i = \varphi^i + V^i + W^i \quad \text{on } B_\rho(x_0), \tag{6.35}$$

where φ^i, V^i, W^i solve the following problems

$$\begin{cases} \Delta\varphi^i = \sum_k \left[\partial_{x^1} B^i_k \, \partial_{x^2} A^k - \partial_{x^2} B^i_k \, \partial_{x^1} A^k\right] & \text{on } B_\rho(x_0) \\ \varphi^i = 0 \quad \text{on } \partial B_\rho(x_0); \end{cases} \tag{6.36}$$

$$\begin{cases} \Delta V^i = F^i \quad \text{on } B_\rho(x_0) \\ V^i = 0 \quad \text{on } \partial B_\rho(x_0); \end{cases} \tag{6.37}$$

$$\begin{cases} \Delta W^i = 0 \quad \text{on } B_\rho(x_0) \\ W^i = A^i \quad \text{on } \partial B_\rho(x_0). \end{cases} \tag{6.38}$$

Notice that φ^i, V^i, W^i are different from the ones in Step 1 since they solve different problems, in any case for convenience of notation we call them in the same way.

Let us start analyzing φ^i solution to (6.36). Observe that the right hand side of the equation is a sum of jacobians which, by Step 1, now are in $L^2_{\mathrm{loc}}(D^2)$. By Wente estimate [44] (see also [33, Theorem III.1]) we have

$$\|\nabla\varphi^i\|_{L^2(B_\rho(x_0))} \le C\|\nabla B\|_{L^2(B_\rho(x_0))}\|\nabla A\|_{L^2(B_\rho(x_0))} \le C\epsilon\|\nabla A\|_{L^2(B_\rho(x_0))}, \tag{6.39}$$

where, in the last inequality, we used (6.34).

Now we pass to consider (6.37). Call

$$\tilde{V}^i(x) := V^i(\rho x + x_0) \qquad \tilde{F}^i(x) := \rho^2 F^i(\rho x + x_0) \tag{6.40}$$

and observe that, since V^i satisfies (6.37), then $\tilde{V}^i$ solves

$$\begin{cases} \Delta\tilde{V}^i = \tilde{F}^i \quad \text{on } D^2 \\ \tilde{V}^i = 0 \quad \text{on } \partial D^2, \end{cases} \tag{6.41}$$

which implies, by $W^{2,q}$ estimates on $\tilde{V}^i$ and Sobolev embedding, that

$$\left(\int_{D^2} |\nabla\tilde{V}^i|^2\right)^{\frac{1}{2}} \le C\left(\int_{D^2} |\tilde{F}^i|^q\right)^{\frac{1}{q}}. \tag{6.42}$$

Now, using that the left hand side is invariant under rescaling while the right hand side has a scaling factor given by the area and the definition of $\tilde{F}^i$, we obtain

$$\left(\int_{B_\rho(x_0)} |\nabla V^i|^2\right)^{\frac{1}{2}} \le C\rho^{2-\frac{2}{q}}\left(\int_{B_{\rho(x_0)}} |F^i|^q\right)^{\frac{1}{q}} \le C\rho^\alpha \quad \text{for some } \alpha > 0, \tag{6.43}$$

where in the last inequality we used that $F^i \in L^q(D^2)$ and $1 < q < 2$.

At last we study the decay of the L^2 norm of the gradient of the harmonic function W^i solving (6.38). Notice that, since W^i is harmonic, then $\Delta|\nabla W^i|^2 = 2|\nabla^2 W^i|^2 \geq 0$. An elementary calculation shows that for any non negative subharmonic function f in $\mathbb{R}^n$ one has $d/dr(r^{-n}\int_{B_r} f) \geq 0$ (see also [33, Lemma III.1]). It follows that

$$\int_{B_{\delta\rho}(x_0)} |\nabla W^i|^2 \leq \delta^2 \int_{B_\rho(x_0)} |\nabla W^i|^2 \leq C\delta^2 \int_{B_\rho(x_0)} |\nabla A|^2, \tag{6.44}$$

where, in the last inequality, we used that W^i solves (6.38).

Collecting (6.39), (6.43) and (6.44) gives

$$\int_{B_{\delta\rho}(x_0)} |\nabla A|^2 \leq C\delta^2 \int_{B_\rho(x_0)} |\nabla A|^2 + C\epsilon^2 \int_{B_\rho(x_0)} |\nabla A|^2 + C\rho^\alpha$$

where the strictly positive constants α and C are independent of ϵ, δ, x_0 and ρ. Now, in the beginning of Step 2, choose ϵ and ρ_0 such that $C\epsilon^2 < \frac{1}{4}$, moreover choose δ in (6.44) such that $C\delta^2 < \frac{1}{4}$; it follows that for every $x_0 \in B_{\frac{1}{2}}(0)$ and every $\rho < \rho_0$ we have

$$\int_{B_{\delta\rho}(x_0)} |\nabla A|^2 < \frac{1}{2} \int_{B_\rho(x_0)} |\nabla A|^2 + C\rho^\alpha \quad \text{for some } \alpha > 0.$$

It is a standard fact which follows by iterating the inequality (see for instance Lemma 5.3 in [15]) that there exist $C, \alpha > 0$ such that for every $x_0 \in B_{\frac{1}{2}}(0)$ and every $\rho < \rho_0$

$$\int_{B_\rho(x_0)} |\nabla A|^2 \leq C\rho^\alpha, \tag{6.45}$$

which implies our initial claim (6.33).

Now we easily get that there exists $\beta > 0$ such that

$$\sup_{x_0 \in B_{\frac{1}{2}}(0), \rho < \frac{1}{4}} \frac{1}{\rho^\beta} \int_{B_\rho(x_0)} |\Delta A| < \infty. \tag{6.46}$$

Indeed, by (6.33) and (6.28), for every $x_0 \in B_{\frac{1}{2}}(0)$ and $\rho < \frac{1}{4}$ we obtain

$$\begin{aligned} \int_{B_\rho(x_0)} |\Delta A| &\leq \int_{B_\rho(x_0)} |\nabla B|\,|\nabla A| + \int_{B_\rho(x_0)} |F| \\ &\leq \|\nabla B\|_{L^2(D^2)} \left[\int_{B_\rho(x_0)} |\nabla A|^2\right]^{\frac{1}{2}} + |B_\rho(x_0)|^{\frac{1}{q'}} \|F\|_{L^q(D^2)} \leq C\rho^\beta. \end{aligned}$$

By a classical result of Adams [1], (6.46) implies that $\nabla A \in L^p_{\text{loc}}(B_{\frac{1}{2}}(0))$ for some $p > 2$. With analogous arguments one gets that $\nabla A \in L^p_{\text{loc}}(D^2)$ for some $p > 2$.

Step 3: $\vec{H} \in L^p_{\text{loc}}(D^2)$ for some $p > 2$.

From Step 2 we obtain that $\nabla(\Re\vec{R})$ and $\nabla(\Re S)$ are in $L^p_{\text{loc}}(D^2)$ for some $p > 2$; recalling that, by Lemma 6.1, $\nabla^2(\Im\vec{R})$ and $\nabla^2(\Im S)$ are in $L^q(D^2)$ for every $1 < q < 2$ then, by Sobolev embedding, $\nabla\vec{R}$ and ∇S are in $L^p_{\text{loc}}(D^2)$ for some $p > 2$.

Using Eq. (6.3) and observing that $\langle \partial_z \vec{\Phi}, \partial_{\bar{z}} \vec{\Phi} \rangle = \frac{1}{2} e^{2\lambda}$, a simple computation gives

$$D_z \vec{R} \llcorner \partial_{\bar{z}} \vec{\Phi} = \frac{e^{2\lambda}}{2} \overline{\vec{L}} - \langle \overline{\vec{L}}, \partial_{\bar{z}} \vec{\Phi} \rangle \partial_z \vec{\Phi} - i e^{2\lambda} \vec{H}. \tag{6.47}$$

Using the definition of ∂_z and $\partial_{\bar{z}}$ we write

$$\begin{aligned} \Im \left[\langle \overline{\vec{L}}, \partial_{\bar{z}} \vec{\Phi} \rangle \partial_z \vec{\Phi} \right] = \frac{1}{4} \Big[& -\langle \partial_{x^1} \vec{\Phi}, \Re \vec{L} \rangle \partial_{x^2} \vec{\Phi} - \langle \partial_{x^1} \vec{\Phi}, \Im \vec{L} \rangle \partial_{x^1} \vec{\Phi} \\ & + \langle \partial_{x^2} \vec{\Phi}, \Re \vec{L} \rangle \partial_{x^1} \vec{\Phi} - \langle \partial_{x^2} \vec{\Phi}, \Im \vec{L} \rangle \partial_{x^2} \vec{\Phi} \Big]. \end{aligned} \tag{6.48}$$

On the other hand, (6.2) gives

$$\langle \partial_{x^1} \vec{\Phi}, \Re \vec{L} \rangle = 2 \Re (\partial_z S) + \langle \partial_{x^2} \vec{\Phi}, \Im \vec{L} \rangle \tag{6.49}$$

$$\langle \partial_{x^2} \vec{\Phi}, \Re \vec{L} \rangle = -2 \Im (\partial_z S) - \langle \partial_{x^1} \vec{\Phi}, \Im \vec{L} \rangle. \tag{6.50}$$

Inserting (6.49) and (6.50) in (6.48) we obtain after some elementary computations

$$\Im \left(\langle \partial_{\bar{z}} \vec{\Phi}, \overline{\vec{L}} \rangle \partial_z \vec{\Phi} \right) = \Re \left[\partial_z S (i \partial_{\bar{z}} \vec{\Phi}) \right] - 2 \Re \left[\langle \partial_z \vec{\Phi}, \Im \vec{L} \rangle \partial_{\bar{z}} \vec{\Phi} \right]. \tag{6.51}$$

Therefore, combining (6.47) and (6.51) we get that

$$e^{2\lambda} \vec{H} = -\Im \left[D_z \vec{R} \llcorner \partial_{\bar{z}} \vec{\Phi} \right] - \frac{e^{2\lambda}}{2} \Im \vec{L} - \Re \left[\partial_z S (i \partial_{\bar{z}} \vec{\Phi}) \right] + \Re \left[\langle \partial_z \vec{\Phi}, \Im \vec{L} \rangle \partial_{\bar{z}} \vec{\Phi} \right]; \tag{6.52}$$

since by Step 2 $D_z \vec{R}$ and $D_z S$ are in $L^p_{\mathrm{loc}}(D^2)$ for some $p > 2$ and by Lemma 6.1 $\nabla(\Im L) \in L^{(2,\infty)}(D^2)$, we conclude that $\vec{H} \in L^p_{\mathrm{loc}}(D^2)$ for some $p > 2$.

Step 4: Smoothness of $\vec{\Phi}$ by a bootstrap argument.

Since $\vec{\Phi}$ is a conformal parametrization, then $\Delta \vec{\Phi} = e^{2\lambda} \vec{H}$ and by Step 3 we infer that $\vec{\Phi} \in W^{2,p}_{\mathrm{loc}}(D^2)$ for some $p > 2$. Now the Willmore equation in divergence form (see (3.30) for the free problem and (3.31) for the conformal-constrained problem) becomes subcritical in $\vec{H}$: written in local coordinates it has the form

$$\Delta \vec{H} = \tilde{H} \quad \text{with } \tilde{H} \in W^{-1,\frac{p}{2}}_{\mathrm{loc}}(D^2)$$

then $\vec{H} \in W^{1,\frac{p}{2}}_{\mathrm{loc}}(D^2)$ and by Sobolev embedding $\vec{H} \in L^{\frac{2p}{4-p}}$, notice that $\frac{2p}{4-p} > p$ since $p > 2$; reinserting this information in the same equation iteratively we get $\vec{H} \in W^{1,p}_{\mathrm{loc}}(D^2)$ for every $p < \infty$; therefore $\vec{\Phi} \in W^{3,p}_{\mathrm{loc}}(D^2)$ for every $p < \infty$. Inserting this information into the same equation gives that $\vec{H} \in W^{2,p}_{\mathrm{loc}}(D^2)$ for every $p < \infty$, therefore $\vec{\Phi} \in W^{4,p}_{loc}(D^2)$ for every $p < \infty \ldots$ continuing this bootstrap argument gives that $\vec{\Phi} \in W^{k,p}_{\mathrm{loc}}(D^2)$ for every $k > 0$, $1 < p < \infty$ which implies that $\vec{\Phi} \in C^{\infty}_{\mathrm{loc}}(D^2)$. $\square$

7. A priori geometric estimates under curvature conditions

7.1. Diameter bound from below on a minimizing sequence

We start by computing the Willmore functional and the Energy functional on small geodesic 2-spheres in a Riemannian manifold (M^m, h) of arbitrary codimension; the corresponding expansions in codimension 1 were obtained by the first author in [24,25,15]. First we introduce some notation.

Let (M^m, h) be an m-dimensional Riemannian manifold. Fix a point $\bar{p}$ and a 3-dimensional subspace $\mathfrak{S} < T_{\bar{p}}M$ of the tangent space to M at $\bar{p}$. Denote with $S^{\mathfrak{S}}_{\bar{p},\rho} \subset M$ the geodesic sphere obtained by exponentiating the sphere in $\mathfrak{S}$ of center 0 and radius ρ. An equivalent way to define is the following: consider normal coordinates $(x^1, \dots, x^m)$ in M centered at $\bar{p}$ such that $(\frac{\partial}{\partial x^1}|_0, \frac{\partial}{\partial x^2}|_0, \frac{\partial}{\partial x^3}|_0)$ are orthonormal bases of $\mathfrak{S}$, then $S^{\mathfrak{S}}_{\bar{p},\rho} := \{(x^1)^2 + (x^2)^2 + (x^3)^2 = \rho^2\} \cap \{x^4 = \cdots = x^m = 0\}$. Let us denote

$$R_{\bar{p}}(\mathfrak{S}) := \sum_{i \neq j, i,j=1,2,3} \bar{K}_{\bar{p}} \left(\left.\frac{\partial}{\partial x^i}\right|_0, \left.\frac{\partial}{\partial x^j}\right|_0 \right) \tag{7.1}$$

where $\bar{K}_{\bar{p}}(\frac{\partial}{\partial x^i}|_0, \frac{\partial}{\partial x^j}|_0)$ denotes the sectional curvature of (M, h) computed on the plane spanned by $(\frac{\partial}{\partial x^i}|_0, \frac{\partial}{\partial x^j}|_0)$contained in $T_{\bar{p}}M$.

Lemma 7.1. *We have the following expansions for the Willmore functional, the Energy functional and the area for small spheres $S^{\mathfrak{S}}_{\bar{p},\rho}$ defined above:*

$$W(S^{\mathfrak{S}}_{\bar{p},\rho}) := \int_{S^{\mathfrak{S}}_{\bar{p},\rho}} |H|^2 d\mu_g = 4\pi - \frac{2\pi}{3} R_{\bar{p}}(\mathfrak{S})\rho^2 + o(\rho^2) \tag{7.2}$$

$$F(S^{\mathfrak{S}}_{\bar{p},\rho}) := \frac{1}{2} \int_{S^{\mathfrak{S}}_{\bar{p},\rho}} |\mathbb{I}|^2 d\mu_g = 4\pi - \frac{2\pi}{3} R_{\bar{p}}(\mathfrak{S})\rho^2 + o(\rho^2). \tag{7.3}$$

$$A(S^{\mathfrak{S}}_{\bar{p},\rho}) = 4\pi\rho^2 + o(\rho^2). \tag{7.4}$$

In particular, if at some point $\bar{p} \in M$ there exists a 3*-dimensional subspace $\mathfrak{S} < T_{\bar{p}}M$ such that $R_{\bar{p}}(\mathfrak{S}) > 6$ then* $\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} (W + A)(\vec{\Phi}) < 4\pi$ *and* $\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} (F + A)(\vec{\Phi}) < 4\pi$. □

Proof. Let $r < \mathrm{Inj}_{M,h}(\bar{p})$ be less than the injectivity radius of (M, h) at $\bar{p}$, then the exponential map $\mathrm{Exp}_{\bar{p}} : B_r(0) \subset T_{\bar{p}}M \to M$ is a diffeomorphism on the image. Call

$$\tau := \mathrm{Exp}_{\bar{p}}(\mathfrak{S} \cap B_r(0)),$$

the image under the exponential map of the subspace $\mathfrak{S}$. Observe that τ is a 3-dimensional submanifold which is geodesic at $\bar{p}$ (i.e. every geodesic in τ starting at $\bar{p}$ is a geodesic of M at $\bar{p}$) so the second fundamental form $\mathbb{I}_{\mathcal{S}\hookrightarrow\mathcal{M}}$ of τ as submanifold of M vanishes at $\bar{p}$ (for the easy proof see for example [9, Proposition 2.9 p. 132]). Endow τ with the metric induced by the immersion and observe that by the Gauss equations applied to $\tau \hookrightarrow M$ we get that the sectional curvatures of τ at $\bar{p}$ coincide with the corresponding sectional curvatures of M at $\bar{p}$. Therefore the scalar curvature $R^\tau(\bar{p})$ of τ at $\bar{p}$ coincide with $R_{\bar{p}}(\mathfrak{S})$ (see for example [7] p. 50 for the definition of scalar curvature via sectional curvature):

$$R^\tau(\bar{p}) = R_{\bar{p}}(\mathfrak{S}). \tag{7.5}$$

Now consider the geodesic sphere $S_{\bar{p},\rho} \hookrightarrow \tau$ in the Riemannian manifold τ and observe that the composition of the immersions $S_{\bar{p},\rho} \hookrightarrow \tau \hookrightarrow M$ coincides with $S^{\mathfrak{S}}_{\bar{p},\rho}$; call $\pi_{\vec{n}_{S\hookrightarrow M}}, \pi_{\vec{n}_{S\hookrightarrow\tau}}$ and $\pi_{\vec{n}_{\tau\hookrightarrow M}}$ the normal projections onto the normal bundles respectively of $S_{\bar{p},\rho}$ relative to M, of $S_{\bar{p},\rho}$ relative to τ and of τ relative to M (i.e. for example in the second case we mean the intersection of the normal bundle of $S_{\bar{p},\rho}$ as immersed in M with the tangent bundle of τ) then we have the orthogonal decomposition

$$\pi_{\vec{n}_{S\hookrightarrow M}} = \pi_{\vec{n}_{S\hookrightarrow\tau}} + \pi_{\vec{n}_{\tau\hookrightarrow M}}. \tag{7.6}$$

By definition of second fundamental form we get for all X, Y tangent vectors to $S_{\bar{p},\rho}$

$$\begin{aligned}\mathbb{I}_{S\hookrightarrow M}(X,Y) &:= \pi_{\vec{n}_{S\hookrightarrow M}}(D_X Y) = \pi_{\vec{n}_{S\hookrightarrow\tau}}(D_X Y) + \pi_{\vec{n}_{\tau\hookrightarrow M}}(D_X Y)\\ &=: \mathbb{I}_{S\hookrightarrow\tau}(X,Y) + \mathbb{I}_{\tau\hookrightarrow M}(X,Y).\end{aligned} \tag{7.7}$$

Therefore we obtain

$$|\mathbb{I}_{S\hookrightarrow\tau}|^2 \le |\mathbb{I}_{S\hookrightarrow M}|^2 \le |\mathbb{I}_{S\hookrightarrow\tau}|^2 + |\mathbb{I}_{\tau\hookrightarrow M}|^2, \tag{7.8}$$

and recalled that $\vec{H}_{S\hookrightarrow M} := \frac{1}{2}\sum_{i=1}^{2}\left[\mathbb{I}_{S\hookrightarrow M}(\vec{e}_i,\vec{e}_i)\right]$ where $\{\vec{e}_1,\vec{e}_2\}$ is an orthonormal frame of $T_x S_{\bar{p},\rho}$,

$$|\vec{H}_{S\hookrightarrow\tau}|^2 \le |\vec{H}_{S\hookrightarrow M}|^2 \le |\vec{H}_{S\hookrightarrow\tau}|^2 + \frac{1}{2}|\mathbb{I}_{\tau\hookrightarrow M}|^2. \tag{7.9}$$

Since $S_{\bar{p},\rho} \hookrightarrow \tau$ is a geodesic sphere in the 3-dimensional manifold τ, we can use the expansions of [24,25,15] for geodesic spheres in 3-manifolds (more precisely see Proposition 3.1 in [24] and Lemma 2.3 in [15]) and obtain that as $\rho \to 0$

$$\frac{1}{2}\int_{S_{\bar{p},\rho}} |\mathbb{I}_{S\hookrightarrow\tau}|^2 d\mu_g = 4\pi - \frac{2\pi}{3}R^{\tau}(\bar{p})\rho^2 + o(\rho^2) \tag{7.10}$$

$$\int_{S_{\bar{p},\rho}} |\vec{H}_{S\hookrightarrow\tau}|^2 d\mu_g = 4\pi - \frac{2\pi}{3}R^{\tau}(\bar{p})\rho^2 + o(\rho^2). \tag{7.11}$$

Observe that $\int_{S_{\bar{p},\rho}} d\mu_g = O(\rho^2)$ and since $\mathbb{I}_{\tau\hookrightarrow M}(\bar{p}) = 0$ we have that $|\mathbb{I}_{\tau\hookrightarrow M}|^2|_{S_{\bar{p},\rho}} \to 0$ as $\rho \to 0$. Therefore $\int_{S_{\bar{p},\rho}} |\mathbb{I}_{\tau\hookrightarrow M}|^2 d\mu_g = o(\rho^2)$ and integrating the estimates (7.8), (7.9) on $S_{\bar{p},\rho}$, using (7.12), (7.13), we conclude that

$$\frac{1}{2}\int_{S_{\bar{p},\rho}} |\mathbb{I}_{S\hookrightarrow M}|^2 d\mu_g = 4\pi - \frac{2\pi}{3}R^{\tau}(\bar{p})\rho^2 + o(\rho^2) \tag{7.12}$$

$$\int_{S_{\bar{p},\rho}} |\vec{H}_{S\hookrightarrow M}|^2 d\mu_g = 4\pi - \frac{2\pi}{3}R^{\tau}(\bar{p})\rho^2 + o(\rho^2). \tag{7.13}$$

The expansion of the area is straightforward. □

The following lemma is a variant for weak branched immersions of a lemma proved by Simon [39]; notice that a similar statement is also present in [15] in case of smooth immersions. We include it here for completeness.

Lemma 7.2. *Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ be a weak branched immersion with finite total curvature of $\mathbb{S}^2$ into the Riemannian manifold (M^m, h). Assume $W(\vec{\Phi}) + A(\vec{\Phi}) \le \Lambda$. Then there exists a constant $C = C(\Lambda, M)$ such that*

$$A(\vec{\Phi}) \le C\left[\operatorname{diam}_M(\vec{\Phi}(\mathbb{S}^2))\right]^2. \quad \square \tag{7.14}$$

Proof. By Nash's theorem, there is an isometric embedding $I : M \hookrightarrow \mathbb{R}^s$ for some $s \in \mathbb{N}$. The second fundamental forms of $\vec{\Phi}$, $I \circ \vec{\Phi}$ and I are related by the formula holding vol_g-a.e. on $\mathbb{S}^2$

$$\mathbb{I}_{I\circ\vec{\Phi}}(\cdot,\cdot) = dI|_{\vec{\Phi}} \circ \mathbb{I}_{\vec{\Phi}}(\cdot,\cdot) \oplus (\mathbb{I}_I \circ \vec{\Phi})(d\vec{\Phi}, d\vec{\Phi}).$$

Taking the trace and squaring yields for an orthonormal basis $\vec{e}_i$ of $\vec{\Phi}_*(T\mathbb{S}^2)$ that vol_g-a.e. on $\mathbb{S}^2$

$$|\vec{H}_{I\circ\vec{\Phi}}|^2 = |H_{\vec{\Phi}}|^2 + \left|\sum_{i=1}^{2} \frac{1}{2}\mathbb{I}_{I} \circ \vec{\Phi}(\vec{e}_i, \vec{e}_i)\right|^2 \leq |\vec{H}_{\vec{\Phi}}|^2 + \frac{1}{2}|\mathbb{I}_I|^2 \circ \vec{\Phi}.$$

Analogously, taking the squared norms, one gets

$$|\mathbb{I}_{I\circ\vec{\Phi}}|^2 = |\mathbb{I}_{\vec{\Phi}}|^2 + \left|\sum_{i,j=1}^{2} \mathbb{I}_I \circ \vec{\Phi}(\vec{e}_i, \vec{e}_j)\right|^2 \leq |\mathbb{I}_{\vec{\Phi}}|^2 + |\mathbb{I}_I|^2 \circ \vec{\Phi}.$$

Integrating we obtain that $\vec{\Phi}$ is a weak branched immersion with finite total curvature and

$$W(I \circ \vec{\Phi}) \leq W(\vec{\Phi}) + CA(\vec{\Phi}) \leq C_{\Lambda,M}, \tag{7.15}$$

where $C = \frac{1}{2}\max|\mathbb{I}_I|^2$.

Let $\{b^1, \ldots, b^N\}$ be the branch points of $\vec{\Phi}$ and for small $\varepsilon > 0$ let $K_\varepsilon := \mathbb{S}^2 \setminus \cup_{i=1}^N B_\varepsilon(b^i)$. Then $\vec{\Phi}|_{K_\varepsilon}$ is a weak immersion *without branch points* of the surface with smooth boundary K_ε. Recall that for a smooth vector field $\vec{X}$ on $\mathbb{R}^s$, the tangential divergence of $\vec{X}$ on $(I \circ \vec{\Phi})(\mathbb{S}^2)$ is defined by

$$\mathrm{div}_{I\circ\vec{\Phi}}\vec{X} := \sum_{i=1}^{2}\langle d\vec{X} \cdot \vec{f}_i, \vec{f}_i\rangle,$$

where $\vec{f}_i$ is an orthonormal frame on $(I \circ \vec{\Phi})_*(T\mathbb{S}^2)$. Now, from the first part of the proof of Lemma A.3 of [34], the tangential divergence theorem ((A.18) of the mentioned paper) holds for a weak immersion of a surface with boundary in $\mathbb{R}^s$ without branch points and

$$\begin{aligned}\int_{(I\circ\vec{\Phi})(K_\varepsilon)} \mathrm{div}_{I\circ\vec{\Phi}}\vec{X}\, d\mathrm{vol}_g &= \int_{\cup_{i=1}^N [I\circ\vec{\Phi}(\partial B_\varepsilon(b^i))]} \langle \vec{X}, \vec{\nu}\rangle dl \\ &\quad - 2\int_{(I\circ\vec{\Phi})(K_\varepsilon)} \langle \vec{H}_{I\circ\vec{\Phi}}, \vec{X}\rangle\, d\mathrm{vol}_g,\end{aligned} \tag{7.16}$$

where $\vec{\nu}$ is the unit limiting tangent vector to $(I \circ \vec{\Phi})(K_\varepsilon)$ on $(I \circ \vec{\Phi})(\partial K_\varepsilon)$orthogonal to it and oriented in the outward direction. Since $\vec{\Phi}$ is Lipschitz by assumption and since $\vec{X}$ and $\vec{\nu}$ are trivially bounded, it follows that

$$\int_{\cup_{i=1}^N [I\circ\vec{\Phi}(\partial B_\varepsilon(b^i))]} \langle \vec{X}, \vec{\nu}\rangle dl \to 0 \quad \text{as } \varepsilon \to 0;$$

therefore the tangential divergence theorem still holds on a weak branched immersion:

$$\int_{(I\circ\vec{\Phi})(Sp^2)} \mathrm{div}_{I\circ\vec{\Phi}}\vec{X}\, d\mathrm{vol}_g = -2\int_{(I\circ\vec{\Phi})(\mathbb{S}^2)} \langle \vec{H}_{I\circ\vec{\Phi}}, \vec{X}\rangle\, d\mathrm{vol}_g. \tag{7.17}$$

Now, as in [39], we choose $\vec{X}(\vec{x}) := \vec{x} - \vec{x}_0$ where $\vec{x}_0 \in (I \circ \vec{\Phi})(\mathbb{S}^2)$. Then, observing that $\mathrm{div}_{I\circ\vec{\Phi}}\vec{X} = 2$, by the Schwartz inequality we get

$$A(I \circ \vec{\Phi}) \leq \mathrm{diam}_{\mathbb{R}^s}[(I \circ \vec{\Phi})(\mathbb{S}^2)]\; W(I \circ \vec{\Phi})^{\frac{1}{2}}\; A(I \circ \vec{\Phi})^{\frac{1}{2}}.$$

The last inequality, together with (7.15), the fact that $A(I\circ\vec{\Phi}) = A(\vec{\Phi})$ and $\mathrm{diam}_{\mathbb{R}^s}[(I\circ\vec{\Phi})(\mathbb{S}^2)] \leq \mathrm{diam}_M(\vec{\Phi}(\mathbb{S}^2))$ ensured by the isometry I, gives that

$$A(\vec{\Phi}) \leq C_{\Lambda,M}\left[\mathrm{diam}_M(\vec{\Phi}(\mathbb{S}^2))\right]^2. \quad \square$$

In the following lemma we collect some inequalities linking the geometric quantities of two close metrics. This will be useful for working locally in normal coordinates (the analogous lemma in codimension one and for smooth immersions appears in [15]).

Lemma 7.3. *Let $h_{1,2}$ be Riemannian metrics on a manifold M^m, with norms satisfying*

$$(1+\epsilon)^{-1}\|\cdot\|_1 \leq \|\cdot\|_2 \leq (1+\epsilon)\|\cdot\|_1 \quad \textit{for some } \epsilon \in (0,1].$$

For any weak branched immersion with finite total curvature $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$, the following inequalities hold almost everywhere on Σ for a universal $C < \infty$:

- $\mathrm{vol}_{g_1} \leq (1+C\epsilon)\mathrm{vol}_{g_2}$, *where $g_{1,2} = \vec{\Phi}^*(h_{1,2})$ and $\mathrm{vol}_{g_{1,2}}$ are the associated area forms;*
- $|\mathbb{I}_1|_1^2 \leq \big(1 + C(\epsilon+\delta)\big)|\mathbb{I}_2|_2^2 + C\delta^{-1}|\Gamma|_{h_1}^2 \circ \vec{\Phi}$ *for any $\delta \in (0,1]$, where $\Gamma := D^{h_1} - D^{h_2}$ and D^{h_i} is the covariant derivative with respect to the metric h_i.*
- $|H_1|_1^2 \leq \big(1 + C(\epsilon+\delta)\big)|H_2|_2^2 + C\delta^{-1}|\Gamma|_{h_1}^2 \circ \vec{\Phi}$ *for any $\delta \in (0,1]$ and Γ defined above.* $\square$

Proof. To compare the Jacobians of $\vec{\Phi}$ with respect to $h_{1,2}$, we use $|\cdot|_{g_1} \leq (1+\varepsilon)|\cdot|_{g_2}$ and compute for $v, w \in T_p\Sigma$ with $g_2(v,w) = 0$

$$|v \wedge w|_{g_1}^2 = |v|_{g_1}^2|w|_{g_1}^2 - g_1(v,w)^2 \leq (1+\epsilon)^4|v|_{g_2}^2|w|_{g_2}^2 = (1+\epsilon)^4|v\wedge w|_{g_2}^2.$$

This proves the first inequality. Next we compare the norms for a bilinear map $B : T_p\Sigma\times T_p\Sigma \to T_{\vec{\Phi}(p)}M$ for p not a branch point. Choose a basis v_α of $T_p\Sigma$ such that $g_1(v_\alpha, v_\beta) = \delta_{\alpha\beta}$ and $g_2(v_\alpha, v_\beta) = \lambda_\alpha\delta_{\alpha\beta}$. Then

$$\lambda_\alpha = |v_\alpha|_{g_2} \leq (1+\epsilon)|v_\alpha|_{g_1} = 1+\epsilon,$$

and putting $w_\alpha = v_\alpha/\lambda_\alpha$ we obtain

$$|B|_1^2 = \sum_{\alpha,\beta=1}^{2} \lambda_\alpha^2\lambda_\beta^2|B(w_\alpha, w_\beta)|_{h_1}^2 \leq (1+C\epsilon)\sum_{\alpha,\beta=1}^{2}|B(w_\alpha,w_\beta)|_{h_2}^2 = (1+C\epsilon)|B|_2^2.$$

Now denote by $\pi_{\vec{n}_{1,2}} : T_{\vec{\Phi}(p)}M \to (\vec{\Phi}_*(T_p\Sigma))^{\perp_{h_{1,2}}}$ the orthogonal projections onto the normal spaces with respect to $h_{1,2}$. Then for any $\delta \in (0,1]$ and almost every $p \in \Sigma$ we have the following estimate (by approximation with smooth immersions locally away the branch points)

$$\begin{aligned}
|\mathbb{I}_1|_1^2 &= |\pi_{\vec{n}_1}(D^{h_1}(\nabla\vec{\Phi}))|_1^2 \\
&\leq |\pi_{\vec{n}_2}(D^{h_1}(\nabla\vec{\Phi}))|_1^2 \\
&\leq |\pi_{\vec{n}_2}\big(D^{h_2}(\nabla\vec{\Phi}) + \Gamma\circ\vec{\Phi}(\nabla\vec{\Phi},\nabla\vec{\Phi})\big)|_1^2 \\
&\leq (1+\delta)|\pi_{\vec{n}_2}D^{h_2}(\nabla\vec{\Phi})|_1^2 + C\delta^{-1}|\Gamma|_{h_1}^2\circ\vec{\Phi} \\
&\leq (1+\delta)(1+C\varepsilon)|\mathbb{I}_2|_2^2 + C\delta^{-1}|\Gamma|_{h_1}^2\circ\vec{\Phi}.
\end{aligned}$$

This proves the second inequality. The proof of the third inequality is analogous:

$$\begin{aligned}
|H_1|_1^2 &= \frac{1}{2}|\mathbb{I}_1(v_1,v_1)+\mathbb{I}_1(v_2,v_2)|_1^2 = \frac{1}{2}|\pi_{\vec{n}_1}(D^{h_1}_{v_1}(\partial_{v_1}\vec{\Phi})+D^{h_1}_{v_2}(\partial_{v_2}\vec{\Phi}))|_1^2\\
&\le \frac{1}{2}|\pi_{\vec{n}_2}(D^{h_1}_{v_1}(\partial_{v_1}\vec{\Phi})+D^{h_1}_{v_2}(\partial_{v_2}\vec{\Phi}))|_1^2\\
&\le \frac{1}{2}|\pi_{\vec{n}_2}(D^{h_2}_{v_1}(\partial_{v_1}\vec{\Phi})+D^{h_2}_{v_2}(\partial_{v_2}\vec{\Phi})\\
&\quad+\Gamma\circ\vec{\Phi}(\partial_{v_1}\vec{\Phi},\partial_{v_1}\vec{\Phi})+\Gamma\circ\vec{\Phi}(\partial_{v_2}\vec{\Phi},\partial_{v_2}\vec{\Phi}))|_1^2\\
&\le \frac{1}{2}(1+\delta)|\pi_{\vec{n}_2}(D^{h_2}_{v_1}(\partial_{v_1}\vec{\Phi})+D^{h_2}_{v_2}(\partial_{v_2}\vec{\Phi}))|_1^2+C\delta^{-1}|\Gamma|^2_{h_1}\circ\vec{\Phi}\\
&\le \frac{1}{2}(1+\delta)(1+C\epsilon)|\pi_{\vec{n}_2}(D^{h_2}_{w_1}(\partial_{w_1}\vec{\Phi})+D^{h_2}_{w_2}(\partial_{w_2}\vec{\Phi}))|_1^2+C\delta^{-1}|\Gamma|^2_{h_1}\circ\vec{\Phi}\\
&\le (1+\delta)(1+C\epsilon)|H_2|_2^2+C\delta^{-1}|\Gamma|^2_{h_1}\circ\vec{\Phi}. \quad \square
\end{aligned}$$

Since we are assuming an upper area bound, the lower diameter bound will follow combining Lemma 7.1 and the fact below (which generalizes to arbitrary codimension and non smooth immersions, Proposition 2.5 in [15], the proof is similar but we include it here for completeness).

Proposition 7.1. *Let* M^m *be a compact Riemannian* m*-manifold and consider a sequence* $\vec{\Phi}_k \in \mathcal{F}_{\mathbb{S}^2}$ *such that* $\sup_k (W+A)(\vec{\Phi}_k) \le \Lambda$. *If* $\operatorname{diam} \vec{\Phi}_k(\mathbb{S}^2) \to 0$, *then*

$$\lim_{k\to\infty} A(\vec{\Phi}_k) \to 0, \qquad \limsup_k F(\vec{\Phi}_k) \ge 4\pi \quad \textit{and} \quad \limsup_k W(\vec{\Phi}_k) \ge 4\pi. \quad \square$$

Proof. The first statement follows directly from Lemma 7.2. Let us prove the second one. After passing to a subsequence, we may assume that the $\vec{\Phi}_k(\mathbb{S}^2)$ converge to a point $\bar{p} \in M$. For given $\epsilon \in (0,1]$ we choose $\rho > 0$, such that in Riemann normal coordinates $x \in B_\rho(0) \subset \mathbb{R}^m$

$$\frac{1}{1+\epsilon}|\cdot|_{\mathrm{eucl}} \le |\cdot|_h \le (1+\epsilon)|\cdot|_{\mathrm{eucl}} \quad \text{and} \quad |\Gamma^k_{ij}(x)| \le \varepsilon,$$

where, of course, $|\cdot|_{\mathrm{eucl}}$ is the norm associated to the euclidean metric given by the coordinates and $|\cdot|_h$ is the norm in metric h. We have $\vec{\Phi}_k(\mathbb{S}^2) \subset B_\rho(x_0)$ for large k. Denoting by $\mathbb{I}^e$, g^e_k the quantities with respect to the coordinate metric, we get from Willmore's inequality and Lemma 7.3

$$4\pi \le \frac{1}{2}\int_{\mathbb{S}^2} |\mathbb{I}^e_{\vec{\Phi}_k}|_e^2\, d\mu_{g^e_k} \le (1+C\epsilon)(1+\delta)\frac{1}{2}\int_{\mathbb{S}^2} |\mathbb{I}_{\vec{\Phi}_k}|^2\, d\mu_{g_k} + C(\delta)\epsilon^2 \operatorname{Area}_{g_k}(\mathbb{S}^2).$$

Since $\operatorname{Area}_{g_k}(\mathbb{S}^2) \le C$ by assumption, we may let first $k \to \infty$, then $\epsilon \searrow 0$ and finally $\delta \searrow 0$ to obtain

$$\liminf_{k\to\infty} F(\vec{\Phi}_k) \ge 4\pi.$$

The proof for W is analogous. $\square$

8. Proof of the existence theorems

Proof of Theorem 1.4. Let $\vec{\Phi}_k \subset \mathcal{F}_{\mathbb{S}^2}$ be a minimizing sequence of $F_1 = F + A$, as before we can assume that $\vec{\Phi}_k$ are conformal; clearly there is a uniform upper bound on the areas and on the L^2 norms of the second fundamental forms $\vec{\Phi}_k$:

$$\sup_k \int_{\mathbb{S}^2} |\mathbb{I}_k|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}} \leq C < \infty, \tag{8.1}$$

$$\sup_k \mathrm{Area}_{g_{\vec{\Phi}_k}}(\mathbb{S}^2) \leq C < \infty. \tag{8.2}$$

Since we are assuming that $R_{\bar{p}}(\mathfrak{S}) > 6$ for some point $\bar{p}$ and some 3-dimensional subspace $\mathfrak{S}$, by Lemma 7.1 we have

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} F_1(\vec{\Phi}) < 4\pi. \tag{8.3}$$

Therefore, Proposition 7.1 yields

$$\liminf_k \mathrm{diam}(\vec{\Phi}_k)(\mathbb{S}^2) \geq \frac{1}{C} > 0. \tag{8.4}$$

Now, thanks to (8.1), (8.2) and (8.4), we can apply the 'Good Gauge Extraction Lemma' IV.1 in [26] and obtain that up to subsequences and up to reparametrization of $\vec{\Phi}_k$ via positive Moebius transformations of $\mathbb{S}^2$ the following holds: there exists a finite set of points $\{a^1, \dots, a^N\} \subset \mathbb{S}^2$ such that for every compact subset $K \subset\subset \mathbb{S}^2 \setminus \{a^1, \dots, a^N\}$ (it is enough for our purposes to take K with smooth boundary) there exists a constant C_K such that

$$|\log |\nabla \vec{\Phi}_k|| \leq C_K \quad \text{on } K \text{ for every } k. \tag{8.5}$$

Since the parametrization is conformal, then $|\nabla^2 \vec{\Phi}_k|^2 = e^{4\lambda_k} |\mathbb{I}_{\vec{\Phi}_k}|^2$ (where, as usual, $e^{\lambda_k} = |\partial_{x^1} \vec{\Phi}_k| = |\partial_{x^2} \vec{\Phi}_k|$), and the two estimates (8.1)–(8.5) give that $\vec{\Phi}_k|_K$ are equibounded in $W^{2,2}(K)$; therefore by the Banach–Alaoglu Theorem together with reflexivity and separability of $W^{2,2}(K)$ imply the existence of a map $\vec{\Phi}_\infty \in W^{2,2}(K)$ such that, up to subsequences,

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty \quad \text{weakly in } W^{2,2}(K). \tag{8.6}$$

Now by the Rellich–Kondrachov Theorem $\partial_{x^i} \vec{\Phi}_k \to \partial_{x^i} \vec{\Phi}_\infty$ as $k \to \infty$ strongly in $L^p(K)$ for every $1 < p < \infty$ and a.e. on K. It follows that $\vec{\Phi}_\infty$ is a $W^{1,\infty} \cap W^{2,2}$ conformal immersion of K. Moreover by the lower semicontinuity under $W^{2,2}$-weak convergence proved in Lemma A.8 we have

$$\int_K |\mathbb{I}_{\vec{\Phi}_\infty}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_\infty}} \leq \liminf_k \int_K |\mathbb{I}_{\vec{\Phi}_k}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}}, \tag{8.7}$$

and the strong $L^p(K)$ convergence of the gradients implies

$$\mathrm{Area}_{g_{\vec{\Phi}_k}}(K) \to \mathrm{Area}_{g_{\vec{\Phi}_\infty}}(K). \tag{8.8}$$

Iterating the procedure on a countable increasing family of compact subsets with smooth boundary invading $\mathbb{S}^2 \setminus \{a^1, \dots, a^N\}$, via a diagonal argument we get the existence of a $W^{1,\infty}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$ conformal immersion $\vec{\Phi}_\infty$ of $\mathbb{S}^2 \setminus \{a^1, \dots, a^N\}$ into M such that, up to subsequences,

$$\int_{\mathbb{S}^2 \setminus \{a^1,\dots,a^N\}} \left(\frac{1}{2}|\mathbb{I}_{\vec{\Phi}_\infty}|^2 + 1\right) d\mathrm{vol}g_{\vec{\Phi}_\infty} \leq \liminf_k \int_{\mathbb{S}^2 \setminus \{a^1,\dots,a^N\}} \left(\frac{1}{2}|\mathbb{I}_{\vec{\Phi}_k}|^2 + 1\right) \times d\mathrm{vol}g_{\vec{\Phi}_k} \leq C. \tag{8.9}$$

Now, thanks to the conformality of $\vec{\Phi}_\infty$ on $\mathbb{S}^2 \setminus \{a^1, \dots, a^N\}$ and the estimate (8.9), we can apply Lemma A.5 of [35] and extend $\vec{\Phi}_\infty$ to a weak conformal immersion in $\mathcal{F}_{\mathbb{S}^2}$ possibly branched in a subset of $\{a^1, \dots, a^N\}$. Since $\mathbb{I}_{\vec{\Phi}_\infty} \in L^2(\mathbb{S}^2, \mathrm{vol}_{g_{\vec{\Phi}_\infty}})$, inequality (8.9) implies

$$F_1(\vec{\Phi}_\infty) \le \liminf_k F_1(\vec{\Phi}_k) = \inf_{\vec{\Phi}\in\mathcal{F}_{\mathbb{S}^2}} F_1(\vec{\Phi}); \tag{8.10}$$

therefore $\vec{\Phi}_\infty$ is a minimizer of F_1 in $\mathcal{F}_{\mathbb{S}^2}$. By Lemma A.5, the functional F_1 is Frechét differentiable at $\vec{\Phi}_\infty$ with respect to variations $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, T_{\vec{\Phi}_\infty} M)$ with compact support in $\mathbb{S}^2 \setminus \{b^1, \dots, b^{N_\infty}\}$, where $\{b^1, \dots, b^{N_\infty}\}$ are the branched points of $\vec{\Phi}_\infty$. From the expression of the differentials given in Lemma A.5 we deduce that $\vec{\Phi}_\infty$ satisfies the following area-constrained Willmore like equation in conservative form away the branch points, and since $\vec{\Phi}_\infty$ is conformal, the equation writes

$$\begin{aligned}
&8\, e^{-2\lambda}\, \Re\left(D_{\bar z}\left[\pi_{\vec{n}}(D_z\vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\, \partial_{\bar z}\vec{\Phi}\right]\right)\\
&\quad = 2\tilde{R}(\vec{H}) + 16\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar z}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar z}\right)\\
&\qquad + 2\vec{H} + (D\,R)(T\vec{\Phi}) + 2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) + 2\bar{K}(T\vec{\Phi})\vec{H}.
\end{aligned} \tag{8.11}$$

Observe that the difference between this last equation and the Willmore equation (3.30) is just terms of the second line which are completely analogous to the curvature terms on the right hand side already appearing in (3.30) (see the definitions (3.8) and (3.9)). Therefore all the arguments of Sections 5 and 6 can be repeated including these new terms and we conclude with the smoothness of $\vec{\Phi}_\infty$ away the branched points. □

Proof of Theorem 1.5. The proof is completely analogous to the proof of Theorem 1.4 once we observe that the lower bound on the areas $A(\vec{\Phi}_k) \ge \frac{1}{C} > 0$ together with Lemma 7.2 yields a lower bound on the diameters:

$$\mathrm{diam}_M \vec{\Phi}_k(\mathbb{S}^2) \ge \frac{1}{C}. \tag{8.12}$$

Indeed we still have (8.1), (8.2) and (8.4). Therefore, as above, we obtain the existence of a minimizer $\vec{\Phi}_\infty \in \mathcal{F}_{\mathbb{S}^2}$,

$$F(\vec{\Phi}_\infty) = \inf_{\vec{\Phi}\in\mathcal{F}_{\mathbb{S}^2}} F(\vec{\Phi}),$$

satisfying the equation in conservative form

$$\begin{aligned}
&8\, e^{-2\lambda}\, \Re\left(D_{\bar z}\left[\pi_{\vec{n}}(D_z\vec{H}) + \langle \vec{H}, \vec{H}_0\rangle\, \partial_{\bar z}\vec{\Phi}\right]\right)\\
&\quad = 2\tilde{R}(\vec{H}) + 16\Re\left(\langle \mathrm{Riem}^h(\vec{e}_{\bar z}, \vec{e}_z)\vec{e}_z, \vec{H}\rangle \vec{e}_{\bar z}\right)\\
&\qquad + (D\,R)(T\vec{\Phi}) + 2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) + 2\bar{K}(T\vec{\Phi})\,\vec{H}
\end{aligned} \tag{8.13}$$

(now without the Lagrange multiplier $2\vec{H}$) outside the finitely many branched points. The smoothness of $\vec{\Phi}_\infty$ outside the branched points follows as before. □

Proof of Theorem 1.6. Recall the discussion after the statement of the Theorem, here we just formalize that idea. First of all recall the precise Definition VII.2 in [26] of a *bubble tree of weak* *immersions*; for the proof of the present Theorem we just need to recall (actually the rigorous definition is more precise and intricate) that a bubble tree of weak immersions is an $N+1$-tuple $\vec{T} := (\vec{f}, \vec{\Phi}^1 \cdots \vec{\Phi}^N)$, where N is an arbitrary integer, $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, M^m)$ and $\vec{\Phi}^i \in \mathcal{F}_{\mathbb{S}^2}$ for $i = 1 \cdots N$ such that $\vec{f}(\mathbb{S}^2) = \cup_{i=1}^N \vec{\Phi}^i(\mathbb{S}^2)$ and $\vec{f}_*[\mathbb{S}^2] = \sum_{i=1}^N \vec{\Phi}^i_*[\mathbb{S}^2]$, where for a Lipschitz map $\vec{a} \in W^{1,\infty}(\mathbb{S}^2, M)$ we denote $\vec{a}_*[\mathbb{S}^2]$ the push forward of the current of integration over $\mathbb{S}^2$. The set of bubble trees is denoted by $\mathcal{T}$ and, considered a nontrivial homotopy class $0 \neq \gamma \in \pi_2(M^m)$, the set of bubble trees such that the map $\vec{f}$ belongs to the homotopy group γ is denoted by $\mathcal{T}_\gamma$.

Consider the Lagrangian L defined in (1.23) and (1.25); up to rescaling the metric h by a positive constant we can assume that $\bar{K} \leq 1$ (or analogously instead of 1, in the definition of L, take a constant $C > \max_M \bar{K}$). Consider a minimizing sequence $\vec{T}_k \in \mathcal{T}_\gamma$, of bubble trees realizing the homotopy class γ, for the functional L. Observe that by Proposition 1.2 we can assume the $\vec{\Phi}_k$ are conformal. By the expression of L, there is a uniform bound on the F_1 functional

$$\limsup_{k\to\infty} F_1(\vec{T}_k) = \limsup_k \int_{\mathbb{S}^2} \left(1 + \frac{|\mathbb{I}|^2}{2}\right) d\mathrm{vol}_{g_k} < +\infty; \tag{8.14}$$

moreover, since $\vec{f}_k \in \gamma \neq 0$, we also have

$$\liminf_{k\to\infty} \sum_{i=1}^{N_k} \mathrm{diam}_M \left(\vec{\Phi}^i_k(\mathbb{S}^2)\right) > 0; \tag{8.15}$$

therefore we perfectly fit in the assumptions of the compactness theorem for bubble trees (Theorem VII.1 in [26]). It follows, recalling also Lemma A.8, that there exists a limit bubble tree $\vec{T}_\infty = (\vec{f}_\infty, \vec{\Phi}^1_\infty, \ldots, \vec{\Phi}^{N_\infty}_\infty)$ minimizing the Lagrangian L in $\mathcal{T}_\gamma$. By the minimality, using Lemma A.5, we have that each $\vec{\Phi}^i_\infty$ satisfies the Euler–Lagrange equation of L outside the branch points. As remarked in the introduction, the Euler–Lagrange equation of L coincides with the area-constrained Willmore equation. By the Regularity Theorem 1.2, we conclude that each $\vec{\Phi}^i_\infty$ is a branched conformal immersion of $\mathbb{S}^2$ which is smooth and satisfies the area-constrained Willmore equation outside the finitely many branched points. □

Proof of Theorem 1.7. The arguments are analogous to the proof of Theorem 1.6. Indeed observe that, fix any $\mathcal{A} > 0$, for a minimizing sequence $\vec{T}_k \in \mathcal{T}$ of the functional W_K, defined in (1.26), under the $\mathcal{A}$-area constraint

$$A(\vec{T}_k) := \mathrm{Area}(\vec{f}_k(\mathbb{S}^2)) = \mathcal{A}, \tag{8.16}$$

the bound (8.14) still holds (by the constraint on the total area and by the boundness of $\bar{K}$ ensured by the compactness of M). Moreover, by the monotonicity formula given in Lemma 7.2, the area constraint (8.16) also implies (8.15). Then, as before, we apply the compactness theorem for bubble trees and the thesis follows as above by recalling that the area constraint is preserved in the limit:

$$A(\vec{T}_\infty) := \mathrm{Area}(\vec{f}_\infty(\mathbb{S}^2)) = \lim_{k\to\infty} \mathrm{Area}(\vec{f}_k(\mathbb{S}^2)) = \mathcal{A}. \quad \square$$

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Appendix

A.1. Useful lemmas for proving the regularity

In the appendix we prove some technical lemmas used in the paper. In the following we denote by $\mathring{H}^{-1}(\mathbb{C})$ the dual of the homogeneous Sobolev space $\mathring{H}^1(\mathbb{C})$ (for the standard definition see for instance [12, Definition 6.2.5])

Lemma A.1. *For* $j, l \in \{1, \dots, m\}$ *let* $\gamma_l^j \in (C^0 \cap W^{1,2})(\mathbb{C})$ *be such that* supp $\gamma_l^j \in B_2(0)$ *and* $\|\gamma_l^j\|_{L^\infty(\mathbb{C})} \leq \epsilon$. *For every* $\vec{U} \in (L^1_{\text{loc}})(\mathbb{C})$ *denote, in distributional sense,*

$$(D_z U)^j := \partial_z U^j + \sum_{k=1}^m \gamma_k^j U^k. \tag{A.1}$$

Then for every $\vec{Y} \in (\mathring{H}^{-1} + L^1)(\mathbb{C})$ *with* $\Im(D_{\bar z}\vec{Y}) \in (\mathring{H}^{-1} + L^1)(\mathbb{C})$ *there exists a unique* $\vec{U} \in L^{2,\infty}(D^2)$ *with* $\Im(\vec{U}) \in W^{1,(2,\infty)}(D^2)$ *satisfying*

$$\begin{cases} D_z\vec{U} = \vec{Y} & \text{in } \mathcal{D}'(D^2) \\ \Im\vec{U} = 0 & \text{on } \partial D^2. \end{cases} \tag{A.2}$$

Moreover the following estimate holds:

$$\|\vec{U}\|_{L^{2,\infty}(D^2)} + \|\nabla\Im(\vec{U})\|_{L^{2,\infty}(D^2)} \leq C\left(\|\vec{Y}\|_{H^{-1}+L^1(\mathbb{C})} + \|\Im(D_{\bar z}\vec{Y})\|_{H^{-1}+L^1(\mathbb{C})}\right). \quad \square$$

Proof. Let us first construct $\tilde{U}^j \in L^{2,\infty}(\mathbb{C})$ satisfying $D_z\tilde{U}^j = Y^j$ on $\mathbb{C}$; observe this is equivalent to solving the fixed point problem in $L^{2,\infty}(\mathbb{C})$

$$\tilde{U}^j = -\frac{1}{\pi\bar z} * \left(Y^j - \sum_{k=1}^m \gamma_k^j \tilde{U}^k\right) := T(\vec{\tilde{U}}). \tag{A.3}$$

We prove that the problem has a unique solution by the contraction mapping principle in $L^{2,\infty}$. By the Young and Hölder inequalities for weak type spaces (see Theorem 1.2.13 and Exercise 1.4.19 in [11]) we have

$$\begin{aligned} \left\| -\frac{1}{\pi\bar z} * \left(\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right)\right\|_{L^{2,\infty}(\mathbb{C})} &\leq C \left\| -\frac{1}{\pi\bar z}\right\|_{L^{2,\infty}} \left\|\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right\|_{L^1(\mathbb{C})} \\ &\leq C\epsilon\|\vec{\tilde{U}}\|_{L^{2,\infty}(\mathbb{C})}. \end{aligned} \tag{A.4}$$

We choose $\epsilon > 0$ such that $C\epsilon \leq \frac{1}{2}$. Now we claim that

$$\left\| -\frac{1}{\pi\bar z} * Y^j\right\|_{L^{2,\infty}(\mathbb{C})} \leq C\|Y^j\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}. \tag{A.5}$$

Recall that $Y^j \in L^1 + \mathring{H}^{-1}(\mathbb{C})$ and $\|Y^j\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} := \inf\{\|Y_1^j\|_{L^1(\mathbb{C})} + \|Y_2^j\|_{\mathring{H}^{-1}(\mathbb{C})} : Y^j = Y_1^j + Y_2^j\}$; since we can assume $Y^j \neq 0$ otherwise trivially $-\frac{1}{\pi\bar{z}} * Y^j = 0$, we can find $Y_1^j \in L^1(\mathbb{C})$ and $Y_2^j \in \mathring{H}^{-1}(\mathbb{C})$ such that

$$\|Y_1^j\|_{L^1(\mathbb{C})} + \|Y_2^j\|_{\mathring{H}^{-1}(\mathbb{C})} \leq \frac{3}{2}\|Y^j\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}. \tag{A.6}$$

As before, by the Young inequality, we have

$$\left\| -\frac{1}{\pi\bar{z}} * Y_1^j \right\|_{L^{2,\infty}(\mathbb{C})} \leq C\|Y_1^j\|_{L^1(\mathbb{C})}. \tag{A.7}$$

On the other hand call $\hat{Y}_2^j$ the Fourier transform of Y_2^j, and observe that the Fourier transform of $-\frac{1}{\pi\bar{z}}$ is (up to a multiplicative constant) $\frac{1}{\xi}$, we have by the convolution theorem

$$\int_{\mathbb{C}} \left| -\frac{1}{\pi\bar{z}} * Y_2^j \right|^2 = C\int_{\mathbb{C}} \left| \hat{Y}_2^j(\xi)\frac{1}{\xi} \right|^2.$$

Moreover recalling that $\|h\|_{\mathring{H}^1} = \int_{\mathbb{C}} |\xi \hat{h}(\xi)|^2$ and that

$$\|Y_2^j\|_{\mathring{H}^{-1}(\mathbb{C})} = \sup_{\|h\|_{\mathring{H}^1}\leq 1} \int_{\mathbb{C}} \hat{Y}_2^j(\xi)\bar{\hat{h}}(\xi) = \sup_{\|h\|_{\mathring{H}^1}\leq 1} \int_{\mathbb{C}} \frac{\hat{Y}_2^j(\xi)}{\xi}\,\bar{\hat{h}}(\xi)\xi = \int_{\mathbb{C}} \left| \frac{\hat{Y}_2^j(\xi)}{\xi} \right|^2$$

we get

$$\left\| -\frac{1}{\pi\bar{z}} * Y_2^j \right\|_{L^{2,\infty}(\mathbb{C})} \leq \left\| -\frac{1}{\pi\bar{z}} * Y_2^j \right\|_{L^2(\mathbb{C})} \leq C\|Y_2^j\|_{\mathring{H}^{-1}(\mathbb{C})}. \tag{A.8}$$

Combining (A.6)–(A.8) we get (A.5) which was our claim. Now the estimates (A.4) and (A.5) imply that $T : L^{2,\infty}(\mathbb{C}) \to L^{2,\infty}(\mathbb{C})$ is well defined and is a contraction; the existence of a unique $\tilde{\vec{U}}$ satisfying (A.3) follows by the contraction mapping principle. Notice moreover we have the estimate

$$\|\tilde{\vec{U}}\|_{L^{2,\infty}(\mathbb{C})} \leq C\|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}. \tag{A.9}$$

Now let us consider $\Im(\tilde{\vec{U}})$. From the equation satisfied by $\tilde{\vec{U}}$ we obtain

$$\frac{1}{4}\Delta\tilde{U}^j = \partial_{\bar{z}}\partial_z\tilde{U}^j = \partial_{\bar{z}}Y^j - \partial_{\bar{z}}\left(\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right),$$

whose imaginary part gives

$$\frac{1}{4}\Delta\Im(\tilde{U}^j) = \Im(\partial_{\bar{z}}Y^j) - \Im\left(\partial_{\bar{z}}\left(\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right)\right). \tag{A.10}$$

Since by assumption $\|\gamma_k^j\|_{L^\infty\cap W^{1,2}(\mathbb{C})} \leq C$, then $\|\sum_{k=1}^m \gamma_k^j Y^k\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} \leq C\|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}$ and we have

$$\begin{aligned}\|\Im(\partial_{\bar{z}}Y^j)\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} &= \left\|\Im(D_{\bar{z}}Y^j) - \Im\left(\sum_{k=1}^m \gamma_k^j Y^k\right)\right\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} \\ &\leq \|\Im(D_{\bar{z}}Y^j)\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} + C\|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}.\end{aligned}$$

Eq. (A.10) together with (A.9) and the last estimate gives

$$\|\Delta\Im(\tilde{U}^j)\|_{\mathring{H}^{-1}+L^1+W^{-1,(2,\infty)}(\mathbb{C})} \leq \|\Im(D_{\bar{z}}Y^j)\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} + C\ \|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}$$

which implies, since $\|\Im(U)^j\|_{L^{2,\infty}(\mathbb{C})} \leq C\|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}$, that

$$\|\nabla\Im(\tilde{U}^j)\|_{L^{2,\infty}(\mathbb{C})} \leq C\left(\|\Im(D_{\bar{z}}Y^j)\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} + \|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}\right). \tag{A.11}$$

Now, since $\nabla\Im(\tilde{U}^j) \in L^{2,\infty}(\mathbb{C})$, the function $\Im(\tilde{U}^j)$ leaves a trace in $H^{\frac{1}{2},(2,\infty)}(\partial D^2)$ and we can consider the homogeneous Dirichlet problem

$$\begin{cases} \partial_z V^j + \sum_{k=1}^m \gamma_k^j V^k = 0 & \text{on } D^2 \\ \Im V^j = \Im\tilde{U}^j & \text{on } \partial D^2. \end{cases} \tag{A.12}$$

We solve it again by contraction mapping principle; given $\vec{W} \in W^{1,(2,\infty)}(D^2)$ with $\Im W^j = \Im\tilde{U}^j$ consider $\vec{V} =: S(\vec{W})$ solving

$$\begin{cases} \partial_z V^j = -\sum_{k=1}^m \gamma_k^j W^k & \text{on } D^2 \\ \Im V^j = \Im\tilde{U}^j & \text{on } \partial D^2. \end{cases}$$

Then the following estimate holds (see hand notes: bring right hand side to the left using convolution with $\frac{1}{\pi\bar{z}}$, so get homogeneous equation with different boundary data but still controlled in $H^{\frac{1}{2},(2,\infty)}$ both real and imaginary parts using Hilbert transform, the estimates then follow from the estimates for the Laplace equation, using Calderon–Zygmund theory for estimating the gradients)

$$\|\vec{V}\|_{W^{1,(2,\infty)}(D^2)} \leq C\left(\epsilon\|\vec{W}\|_{L^{2,\infty}(D^2)} + \|\Im\vec{\tilde{U}}\|_{W^{1,(2,\infty)}(\mathbb{C})}\right)$$

and

$$\|S(\vec{W}_1) - S(\vec{W}_2)\|_{W^{1,(2,\infty)}(D^2)} \leq C\epsilon\|\vec{W}_1 - \vec{W}_2\|_{L^{2,\infty}(D^2)}.$$

Therefore, for $\epsilon > 0$ small, $S : W^{1,(2,\infty)}(D^2) \to W^{1,(2,\infty)}(D^2)$ is a contraction and there exists a unique solution of problem (A.12) satisfying the estimate

$$\|\vec{V}\|_{W^{1,(2,\infty)}(D^2)} \leq C\|\Im(\vec{\tilde{U}})\|_{W^{1,(2,\infty)}(\mathbb{C})}. \tag{A.13}$$

Now we conclude observing that $U^j := \tilde{U}^j - V^j \in L^{2,\infty}(D^2)$ is the unique solution to the problem (A.2) and, combining (A.9), (A.11) and (A.13), it satisfies the estimates

$$\|\vec{U}\|_{L^{2,\infty}(D^2)} + \|\nabla\Im(\vec{U})\|_{L^{2,\infty}} \leq C\left(\|\Im(D_{\bar{z}}Y^j)\|_{L^1+\mathring{H}^{-1}(\mathbb{C})} + \|\vec{Y}\|_{L^1+\mathring{H}^{-1}(\mathbb{C})}\right)$$

as desired. □

Lemma A.2. *For* $j, l \in \{1, \dots, m\}$ *let* $\gamma_l^j \in (C^0 \cap W^{1,2})(\mathbb{C})$ *be such that* supp $\gamma_l^j \in B_2(0)$ *and* $\|\gamma_l^j\|_{L^\infty(\mathbb{C})} \leq \epsilon$. *For every* $\vec{U} \in (L^1_{\text{loc}})(\mathbb{C})$ *denote, in distributional sense,*

$$(D_z U)^j := \partial_z U^j + \sum_{k=1}^m \gamma_k^j U^k.$$

Let $\vec{Y} \in (L^1 \cap L^{2,\infty})(\mathbb{C})$ *with* $\Im(D_{\bar z}\vec{Y}) \in L^q(\mathbb{C})$ *for some* $1 < q < 2$. *Then there exists a unique* $\vec{U} \in W^{1,(2,\infty)}(D^2)$ *with* $\Im(\vec{U}) \in W^{2,q}(D^2)$ *satisfying*

$$\begin{cases} D_z\vec{U} = \vec{Y} & \text{in } \mathcal{D}'(D^2) \\ \Im\vec{U} = 0 & \text{on } \partial D^2. \end{cases} \tag{A.14}$$

Moreover the following estimate holds:

$$\begin{aligned} &\|\vec{U}\|_{L^{2,\infty}(D^2)} + \|\nabla\vec{U}\|_{L^{2,\infty}(D^2)} + \|\nabla^2\Im(U)\|_{L^q(D^2)} \\ &\quad \le C\left(\|\vec{Y}\|_{L^1\cap L^{2,\infty}(\mathbb{C})} + \|\Im(D_{\bar z}\vec{Y})\|_{L^q(\mathbb{C})}\right). \quad \square \end{aligned}$$

Proof. As in the proof of Lemma A.1 we first solve the equation $D_z\tilde{\vec{U}} = \vec{Y}$ in $\mathbb{C}$ proving the existence and uniqueness of solutions to the fixed point problem in $W^{1,(2,\infty)}(\mathbb{C})$

$$\tilde{U}^j = -\frac{1}{\pi\bar z} * \left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right) =: T(\tilde{\vec{U}}). \tag{A.15}$$

Analogously to the proof of Lemma A.1, for $\epsilon > 0$ small but depending just on universal constants, the $L^{2,\infty}(\mathbb{C})$ norm of $T(\tilde{\vec{U}})$ can be bounded as

$$\|T(\tilde{\vec{U}})\|_{L^{2,\infty}(\mathbb{C})} \le C\|\vec{Y}\|_{L^1(\mathbb{C})}, \tag{A.16}$$

and for $\tilde{\vec{U}}_1, \tilde{\vec{U}}_2 \in L^{2,\infty}(\mathbb{C})$ it holds that

$$\|T(\tilde{\vec{U}}_1) - T(\tilde{\vec{U}}_2)\|_{L^{2,\infty}(\mathbb{C})} \le C\epsilon\|\tilde{\vec{U}}_1 - \tilde{\vec{U}}_2\|_{L^{2,\infty}(\mathbb{C})}. \tag{A.17}$$

$L^{2,\infty}$*-Gradient estimate*: we have

$$\|\nabla T(\tilde{U}^j)\|_{L^{2,\infty}(\mathbb{C})} = \left\|\left(\nabla\frac{1}{\pi\bar z}\right) * \left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right)\right\|_{L^{2,\infty}(\mathbb{C})}. \tag{A.18}$$

Observe that the Fourier transform of the convolution kernel $\nabla\frac{1}{\bar z} = \nabla(\partial_{\bar z}\log|z|)$ satisfies the assumptions of Theorem 3 p. 96 in [40]; therefore

$$\left\|\left(\nabla\frac{1}{\pi\bar z}\right) * \left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right)\right\|_{L^s(\mathbb{C})} \le C_s\left\|\left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right)\right\|_{L^s(\mathbb{C})} \quad \forall 1 < s < \infty$$

and by interpolation (see for instance Theorem 3.15 in [41] of Theorem 3.3.3 in [13])

$$\begin{aligned} \left\|\left(\nabla\frac{1}{\pi\bar z}\right) * \left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right)\right\|_{L^{2,\infty}(\mathbb{C})} &\le C\left\|\left(Y^j - \sum_{k=1}^m \gamma^j_k \tilde{U}^k\right)\right\|_{L^{2,\infty}(\mathbb{C})} \\ &\le C\|Y\|_{L^{2,\infty}(\mathbb{C})} + C\epsilon\|\tilde{\vec{U}}\|_{L^{2,\infty}(\mathbb{C})}. \end{aligned} \tag{A.19}$$

Combining (A.19), (A.18) and (A.16) we get, for small $\epsilon > 0$,

$$\|T(\tilde{\vec{U}})\|_{W^{1,(2,\infty)}(\mathbb{C})} \le C\left(\|\vec{Y}\|_{L^1(\mathbb{C})} + \|\vec{Y}\|_{L^{2,\infty}(\mathbb{C})}\right). \tag{A.20}$$

So $T : W^{1,(2,\infty)}(\mathbb{C}) \to W^{1,(2,\infty)}(\mathbb{C})$ is a well defined linear operator and the same arguments imply that T is a contraction. Therefore there exists a unique $\tilde{U}^j \in W^{1,(2,\infty)}(\mathbb{C})$ satisfying (A.15) and

$$\|\tilde{\vec{U}}\|_{W^{1,(2,\infty)}(\mathbb{C})} \le C\left(\|\vec{Y}\|_{L^1(\mathbb{C})} + \|\vec{Y}\|_{L^{2,\infty}(\mathbb{C})}\right). \tag{A.21}$$

Noticing that $\Im(\tilde{U}^j)$ satisfies also

$$\begin{aligned}\Delta(\Im(\tilde{U}^j)) &= 4\Im(\partial_{\bar{z}} Y^j) - 4\Im\left[\partial_{\bar{z}}\left(\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right)\right]\\ &= 4\Im(D_{\bar{z}} Y^j) - 4\Im\left(\sum_{k=1}^m \gamma_k^j \tilde{Y}^k\right) - 4\Im\left[\partial_{\bar{z}}\left(\sum_{k=1}^m \gamma_k^j \tilde{U}^k\right)\right],\end{aligned}$$

estimate (A.21), the assumptions on γ_k^j, Hölder inequality and standard elliptic estimates imply

$$\|\nabla^2 \Im(\tilde{U}^j)\|_{L^q(D^2)} \le C\left(\|\Im(D_{\bar{z}} Y^j)\|_{L^q(D^2)} + \|\vec{Y}\|_{L^1(\mathbb{C})} + \|\vec{Y}\|_{L^{2,\infty}(\mathbb{C})}\right). \tag{A.22}$$

Now exactly as in the previous lemma it is possible to solve the corresponding homogeneous problem on D^2

$$\begin{cases}\partial_z V^j = -\sum_{k=1}^m \gamma_k^j V^k & \text{on } D^2\\ \Im V^j = \Im \tilde{U}^j & \text{on } \partial D^2\end{cases} \tag{A.23}$$

and the solution V^j satisfies the estimates

$$\|\vec{V}\|_{W^{1,(2,\infty)}(D^2)} \le C\|\Im(\tilde{\vec{U}})\|_{W^{1,(2,\infty)}(\mathbb{C})}. \tag{A.24}$$

Moreover the imaginary part $\Im(V^j)$ solves the following problem

$$\begin{cases}\Delta\Im(V^j) + 4\Im\left[\partial_{\bar{z}}\left(\sum_{k=1}^m \gamma_k^j V^k\right)\right] = 0 & \text{on } D^2\\ \Im V^j = \Im\tilde{U}^j & \text{on } \partial D^2;\end{cases} \tag{A.25}$$

then, estimate (A.22) and elliptic regularity imply

$$\|\nabla^2 \Im(V^j)\|_{L^q(D^2)} \le C\left(\|\Im(D_{\bar{z}} Y^j)\|_{L^q(D^2)} + \|\vec{Y}\|_{L^1(\mathbb{C})} + \|\vec{Y}\|_{L^{2,\infty}(\mathbb{C})}\right). \tag{A.26}$$

Now, as in the previous lemma, the function $U^j = \tilde{U}^j - V^j$ is a solution to the original problem (A.14); moreover collecting (A.21), (A.22), (A.24) and (A.26) we obtain the desired estimate

$$\begin{aligned}&\|\vec{U}\|_{W^{1,(2,\infty)}(D^2)} + \|\nabla^2\Im(\vec{U})\|_{L^q(D^2)}\\ &\quad\le C\left(\|\Im(D_{\bar{z}} Y^j)\|_{L^q(D^2)} + \|\vec{Y}\|_{L^1(\mathbb{C})} + \|\vec{Y}\|_{L^{2,\infty}(\mathbb{C})}\right). \quad \square\end{aligned}$$

A.2. Differentiability of the Willmore functional in $\mathcal{F}_{\mathbb{S}^2}$, identification of the first differential and lower semicontinuity under $W^{2,2}_{\text{loc}}$ weak convergence

Let us start with two computational lemmas whose utility will be clear later in the subsection.

Lemma A.3. *Let $\vec{\Phi}$ be a smooth immersion of the disc D^2 into the Riemannian manifold (M^n, h). Let $\vec{X} \otimes \vec{v} \in \Gamma_{D^2}(T_{\vec{\Phi}}M \otimes TD^2)$ and $\vec{w} \in \Gamma_{D^2}(T_{\vec{\Phi}}M)$, recall the notation introduced in (3.1)–(3.3) and (3.6). Then*

$$\langle D_g^{*_g}(\vec{X} \otimes \vec{v}), \vec{w}\rangle = \langle \vec{X} \otimes \vec{v}, D_g \vec{w}\rangle - \mathrm{div}_g \vec{u} + \langle \vec{X}, \vec{w}\rangle \mathrm{div}_g \vec{v}, \tag{A.27}$$

where $\vec{u} \in \Gamma_{D^2}(TD^2)$ is the vector field defined below. Let $\vec{f}_1, \vec{f}_2$ be a positive orthonormal frame of TD^2, write $\vec{v} = v^1 \vec{f}_1 + v^2 \vec{f}_2$, then define

$$\vec{u} := \langle v_1 \vec{X}, \vec{w}\rangle_h \vec{f}_1 + \langle v_2 \vec{X}, \vec{w}\rangle_h \vec{f}_2.$$

Notice that $\vec{u}$ is independent of the choice of the frame f_i, i.e. it is a well defined vector field on D^2. □

Proof. Call $\vec{e}_i := \vec{\Phi}_*(\vec{f}_i)$ the positive orthonormal frame of $\vec{\Phi}_*(TD^2)$ associated to $\vec{f}_1, \vec{f}_2$; a straightforward computation using just the definitions (3.1)–(3.3) gives

$$\langle D_g^{*_g}[\vec{X} \otimes (v^1 \vec{f}_1 + v^2 \vec{f}_2)], \vec{w}\rangle = -\langle v_1 D_{\vec{e}_1}\vec{X} + v^2 D_{\vec{e}_2}\vec{X}, \vec{w}\rangle. \tag{A.28}$$

Writing the right hand side as $-\langle D_{\vec{e}_1}(v^1\vec{X}) + D_{\vec{e}_2}(v^2\vec{X}), \vec{w}\rangle + \langle \vec{X}, \vec{w}\rangle \left(\vec{e}_1[v^1] + \vec{e}_2[v^2]\right)$, where $\vec{e}_i[v_i]$ denotes the derivative of the function v^i with respect to $\vec{e}_i$, we can express (A.28) as

$$\begin{aligned}\langle D_g^{*_g}[\vec{X} \otimes (v^1 \vec{f}_1 + v^2 \vec{f}_2)], \vec{w}\rangle = {} & \langle v^1 \vec{X}, D_{\vec{e}_1}\vec{w}\rangle + \langle v^2 \vec{X}, D_{\vec{e}_2}\vec{w}\rangle \\ & - \vec{e}_1[\langle v^1 \vec{X}, \vec{w}\rangle] - \vec{e}_2[\langle v^2 \vec{X}, \vec{w}\rangle] \\ & + \langle \vec{X}, \vec{w}\rangle \left(\vec{e}_1[v^1] + \vec{e}_2[v^2]\right).\end{aligned} \tag{A.29}$$

Notice that the first line of the right hand side is exactly $\langle \vec{X} \otimes \vec{v}, D_g \vec{w}\rangle$. Observe that, through the parametrization $\vec{\Phi}$, we can identify TD^2 and $\vec{\Phi}_*(TD^2)$; moreover noticing that for fixed $i = 1, 2$ we have $\langle D_{\vec{e}_i}\vec{e}_i, \vec{e}_i\rangle = \frac{1}{2}\vec{e}_i[\langle \vec{e}_i, \vec{e}_i\rangle] = 0$, after some easy computations we get that

$$\vec{e}_1[v^1] + \vec{e}_2[v^2] = \vec{f}_1[v^1] + \vec{f}_2[v^2] = \mathrm{div}_g(\vec{v}) + \langle \vec{v}, D_{\vec{f}_1}\vec{f}_1 + D_{\vec{f}_2}\vec{f}_2\rangle, \tag{A.30}$$

where, by definition, $\mathrm{div}_g(\vec{v}) := \sum_{i=1,2}\langle D_{\vec{f}_i}\vec{v}, \vec{f}_i\rangle$ and D is indented as the covariant derivative on TD^2 endowed with the metric $g := \vec{\Phi}^* h$ (notice that the covariant derivative in M along $\vec{\Phi}(D^2)$ projected on $\vec{\Phi}_*(TD^2)$ correspond to the covariant derivative on (D^2, g) via the identification given by the immersion $\vec{\Phi}$).

Recall that we defined $\vec{u} := \langle v_1 \vec{X}, \vec{w}\rangle_h \vec{f}_1 + \langle v_2 \vec{X}, \vec{w}\rangle_h \vec{f}_2 = u^1 \vec{f}_1 + u^2 \vec{f}_2 \in TD^2$; an easy computation gives

$$\vec{f}_1[\langle v^1 \vec{X}, \vec{w}\rangle] + \vec{f}_2[\langle v^2 \vec{X}, \vec{w}\rangle] = \mathrm{div}_g \vec{u} + \langle \vec{X}, \vec{w}\rangle \langle \vec{v}, D_{\vec{f}_1}\vec{f}_1 + D_{\vec{f}_2}\vec{f}_2\rangle. \tag{A.31}$$

Now combining (A.29)–(A.31) we get the thesis. □

Lemma A.4 (*Integration by Parts in Willmore Equation*)**.** *Let $\vec{\Phi}$ be a smooth immersion of the disc D^2 into the Riemannian manifold (M^n, h) and let $\vec{w} \in \Gamma_{D^2}(T_{\vec{\Phi}}M)$ be smooth with compact support in D^2. Then*

$$\int_{D^2}\left[\left\langle \frac{1}{2} D_g^{*_g}\left[D_g \vec{H} - 3\pi_{\vec{n}}(D_g \vec{H}) + \star_h \left((*_g\, D_g \vec{n}) \wedge_M \vec{H}\right)\right]\right.\right.$$

$$
\left.\left. - \tilde{R}(\vec{H}) + R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}),\ \vec{w}\right\rangle\right] d\mathrm{vol}_g
$$

$$
= \int_{D^2} \left(\left\langle \vec{H}, \frac{1}{2} D_g^{*_g}\left[D_g\vec{w} - 3\pi_{\vec{n}}(D_g\vec{w})\right]\right\rangle + \left\langle \star_h \left((*_g\, D_g\vec{n}) \wedge_M \vec{H}\right), D_g\vec{w}\right\rangle \right) d\mathrm{vol}_g
$$

$$
+ \int_{D^2} \langle -\tilde{R}(\vec{H}) + R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \vec{w}\rangle \, d\mathrm{vol}_g. \quad \Box \tag{A.32}
$$

Proof. Let us start considering $\int_{D^2}\langle D_g^{*_g}[D_g\vec{H}], \vec{w}\rangle d\mathrm{vol}_g$. Fix a point $x_0 \in D^2$, take normal coordinates x^i centered in x_0 with respect to the metric $g = \vec{\Phi}^* h$, and call $f_i := \frac{\partial}{\partial x^i}$ the coordinate frame; observe it is orthonormal at x_0 and $Df_i = 0$ at x_0. By Lemma A.3, we have

$$
\langle D_g^{*_g}[D_g\vec{H}], \vec{w}\rangle = \langle D_g\vec{H}, D_g\vec{w}\rangle - \mathrm{div}_g\vec{u} + \langle D_{\vec{f}_1}\vec{H}, \vec{w}\rangle \mathrm{div}_g\vec{f}_1 + \langle D_{\vec{f}_2}\vec{H}, \vec{w}\rangle \mathrm{div}_g\vec{f}_2
$$

for some vector field $\vec{u}$ compactly supported in D^2. Observe that, at x_0, the condition $D\vec{f}_i = 0$ implies

$$
\mathrm{div}_g\vec{f}_i = \langle D_{\vec{f}_1}\vec{f}_i, \vec{f}_1\rangle + \langle D_{\vec{f}_2}\vec{f}_i, \vec{f}_2\rangle = 0.
$$

Therefore taking $\vec{f}_i$ to coincide at x_0 with the frame associated to normal coordinates centered at x_0 we obtain

$$
\langle D_g^{*_g}[D_g\vec{H}], \vec{w}\rangle = \langle D_g\vec{H}, D_g\vec{w}\rangle - \mathrm{div}_g\vec{u}; \tag{A.33}
$$

since all the terms are defined intrinsically, the identity is true intrinsically at every point $x_0 \in D^2$. Now integrate (A.33) on D^2 and, observing that $\vec{u}$ is compactly supported, use the divergence theorem to infer

$$
\int_{D^2}\langle D_g^{*_g}[D_g\vec{H}], \vec{w}\rangle d\mathrm{vol}_g = \int_{D^2}\langle D_g\vec{H}, D_g\vec{w}\rangle d\mathrm{vol}_g.
$$

Repeating the same argument we have also $\int_{D^2}\langle D_g^{*_g}[D_g\vec{w}], \vec{H}\rangle d\mathrm{vol}_g = \int_{D^2}\langle D_g\vec{w}, D_g\vec{H}\rangle d\mathrm{vol}_g$, so

$$
\int_{D^2}\langle D_g^{*_g}[D_g\vec{H}], \vec{w}\rangle d\mathrm{vol}_g = \int_{D^2}\langle \vec{H}, D_g^{*_g}[D_g\vec{w}]\rangle d\mathrm{vol}_g. \tag{A.34}
$$

Using analogous arguments one checks that also

$$
\begin{aligned}
\int_{D^2}\langle D_g^{*_g}[\pi_{\vec{n}}(D_g\vec{H})], \vec{w}\rangle d\mathrm{vol}_g &= \int_{D^2}\langle \pi_{\vec{n}}(D_g\vec{H}), D_g\vec{w}\rangle d\mathrm{vol}_g \\
&= \int_{D^2}\langle D_g\vec{H}, \pi_{\vec{n}}(D_g\vec{w})\rangle d\mathrm{vol}_g \\
&= \int_{D^2}\langle \vec{H}, D_g^{*_g}[\pi_{\vec{n}}(D_g\vec{w})]\rangle d\mathrm{vol}_g.
\end{aligned} \tag{A.35}
$$

Finally, along the same lines, one has

$$
\begin{aligned}
&\int_{D^2}\left\langle D_g^{*_g}\left[\star_h\left((*_g\, D_g\vec{n})\wedge_M \vec{H}\right)\right], \vec{w}\right\rangle d\mathrm{vol}_g \\
&\quad = \int_{D^2}\left\langle \star_h\left((*_g\, D_g\vec{n})\wedge_M \vec{H}\right), D_g\vec{w}\right\rangle d\mathrm{vol}_g.
\end{aligned} \tag{A.36}
$$

The thesis follows collecting (A.34)–(A.36). $\square$

Lemma A.5 (*Differentiability of W and Identification of dW*)**.** *Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ be a weak branched immersion of $\mathbb{S}^2$ into the m-dimensional Riemannian manifold (M^m, h) with branched points $\{b^1, \ldots, b^N\}$ and let $W(\vec{\Phi}) := \int_{\mathbb{S}^2} |H_{\vec{\Phi}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}}}$ be the Willmore functional. Then W is Frechét differentiable with respect to variations $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, T_{\vec{\Phi}}M)$ with compact support in $\mathbb{S}^2 \setminus \{b^1, \ldots, b^N\}$ in the sense that*

$$W(\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]) = W(\vec{\Phi}) + t\, d_{\vec{\Phi}}W[\vec{w}] + R^{\vec{\Phi}}_{\vec{w}}[t], \tag{A.37}$$

where $\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}](x_0)$ denotes the exponential map in M centered in $\vec{\Phi}(x_0) \in M$ applied to the tangent vector $t\vec{w} \in T_{\vec{\Phi}(x_0)}M$ and where the remainder $R^{\vec{\Phi}}_{\vec{w}}[t]$ satisfies

$$\sup \Big\{ \Big| R^{\vec{\Phi}}_{\vec{w}}[t] \Big| : \|\vec{w}\|_{W^{2,2}} + \|\vec{w}\|_{W^{1,\infty}} \le 1 \\ \text{and } \operatorname{supp} \vec{w} \subset K \subset\subset \mathbb{S}^2 \setminus \{b^1, \ldots, b^{N_{\vec{\Phi}}}\} \Big\} \le C_{\vec{\Phi},K} t^2.$$

Moreover the differential $d_{\vec{\Phi}}W$ coincides with the Willmore equation in conservative form: for every $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, T_{\vec{\Phi}}M)$ with compact support in $\mathbb{S}^2 \setminus \{b^1, \ldots, b^N\}$,

$$d_{\vec{\Phi}}W[\vec{w}] = \int_{\mathbb{S}^2} \bigg(\Big\langle \vec{H}, \frac{1}{2} D_g^{*_g} \big[D_g\vec{w} - 3\pi_{\vec{n}}(D_g\vec{w}) \big] \Big\rangle + \langle \star_h((*_g D_g\vec{n}) \wedge_M \vec{H}), D_g\vec{w} \rangle \\ + \langle -\tilde{R}(\vec{H}) + R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \vec{w} \rangle \bigg) d\mathrm{vol}_g. \tag{A.38}$$

Also the functional $F(\vec{\Phi}) = \int_{\mathbb{S}^2} |\mathbb{I}_{\vec{\Phi}}|^2 d\mathrm{vol}_g$ is Frechét differentiable with respect to variations $\vec{w} \in W^{1,\infty} \cap W^{2,2}(\mathbb{S}^2, T_{\vec{\Phi}}M)$ with compact support in $\mathbb{S}^2 \setminus \{b^1, \ldots, b^N\}$ in the sense above, and

$$d_{\vec{\Phi}}F[\vec{w}] = \int_{\mathbb{S}^2} \bigg(\langle \vec{H}, D_g^{*_g}[D_g\vec{w} - 3\pi_{\vec{n}}(D_g\vec{w})] \rangle + 2\langle \star_h((*_g D_g\vec{n}) \wedge_M \vec{H}), D_g\vec{w} \rangle \\ + 2\langle -\tilde{R}(\vec{H}) + R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}) - (D\,R)(T\vec{\Phi}) \\ - 2\mathfrak{R}_{\vec{\Phi}}(T\vec{\Phi}) - 2\bar{K}(T\vec{\Phi})\vec{H}, \vec{w} \rangle \bigg) d\mathrm{vol}_g. \tag{A.39}$$

Finally also the area functional $A(\vec{\Phi}) = \mathrm{Area}_{g_{\vec{\Phi}}}(\mathbb{S}^2)$ is Frechét differentiable with respect to variations $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, T_{\vec{\Phi}}M)$ with compact support in $\mathbb{S}^2 \setminus \{b^1, \ldots, b^N\}$ in the sense above, and

$$d_{\vec{\Phi}}A[\vec{w}] = -\int_{\mathbb{S}^2} \langle 2\vec{H}, \vec{w} \rangle d\mathrm{vol}_g. \quad \square \tag{A.40}$$

Proof. Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ and observe that the mean curvature $\vec{H}_{\vec{\Phi}} \in T_{\vec{\Phi}}M^n$ is a function of $(\nabla^2\vec{\Phi}, \nabla\vec{\Phi}, \vec{\Phi})$, where $\nabla^2\vec{\Phi}$ and $\nabla\vec{\Phi}$ are respectively the Hessian and the gradient of $\vec{\Phi}$:

$$\vec{H}_{\vec{\Phi}} = \tilde{\vec{H}}(\nabla^2\vec{\Phi}, \nabla\vec{\Phi}, \vec{\Phi}), \tag{A.41}$$

where

$$\tilde{\vec{H}} : ((T\mathbb{S}^2)^2 \otimes T_{\vec{\Phi}}M, T\mathbb{S}^2 \otimes T_{\vec{\Phi}}M, M) \to T_{\vec{\Phi}}M, \quad (\vec{\xi}, \vec{q}, \vec{z}) \mapsto \tilde{\vec{H}}(\vec{\xi}, \vec{q}, \vec{z}). \tag{A.42}$$

Observe that $\tilde{\vec{H}}$ is smooth on the open set given by $|\vec{q} \wedge \vec{q}| > 0$; moreover, for every $\vec{q}_0$ and $\vec{z}_0$, the map $\vec{\xi} \mapsto \tilde{\vec{H}}(\vec{\xi}, \vec{q}_0, \vec{z}_0)$ is linear. Recall also that the area form $d\mathrm{vol}_{g_{\vec{\Phi}}}$ associated to the pullback metric $g_{\vec{\Phi}} := \vec{\Phi}^* h$ is of the form

$$d\mathrm{vol}_{g_{\vec{\Phi}}} = f(\nabla \vec{\Phi}, \vec{\Phi})\, d\mathrm{vol}_{g_0}$$

where $d\mathrm{vol}_{g_0}$ is the area form associated to the reference metric g_0 on $(\mathbb{S}^2, c_0)$, and

$$f : (T\mathbb{S}^2 \otimes T_{\vec{\Phi}}M, M) \to \mathbb{R}, \quad (\vec{q}, \vec{z}) \mapsto f(\vec{q}, \vec{z})$$

is smooth on the open subset $|\vec{q} \wedge \vec{q}| > 0$. Therefore the integrand of the Willmore functional can be written as

$$|\vec{H}_{\vec{\Phi}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}}} = |\vec{F}|^2(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) d\mathrm{vol}_{g_0}, \tag{A.43}$$

where $\vec{F}(\vec{\xi}, \vec{q}, \vec{z}) := \tilde{\vec{H}}(\vec{\xi}, \vec{q}, \vec{z})\sqrt{f}(\vec{q}, \vec{z})$; clearly $\vec{F}$ is smooth on the subset $|\vec{q} \wedge \vec{q}| > 0$ and, for every $\vec{q}_0$ and $\vec{z}_0$, the map $\vec{\xi} \mapsto \vec{F}(\vec{\xi}, \vec{q}_0, \vec{z}_0)$ is linear.

Let $\vec{w} \in W^{2,2} \cap W^{1,\infty}(\mathbb{S}^2, T_{\vec{\Phi}}M)$ be an infinitesimal perturbation supported in $\mathbb{S}^2 \setminus \{b^1, \ldots, b^{N_{\vec{\Phi}}}\}$, where $\{b^1, \ldots, b^{N_{\vec{\Phi}}}\}$ are the branch points of $\vec{\Phi}$; consider, for small $t > 0$, the perturbed weak branched immersion $\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]$, where $\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}](x_0)$ denotes the exponential map in M centered in $\vec{\Phi}(x_0) \in M$ applied to the tangent vector $t\vec{w} \in T_{\vec{\Phi}(x_0)}M$. Observe that, by definition,

$$\begin{aligned}&\int_{\mathbb{S}^2} |\vec{H}_{\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]}|^2 d\mathrm{vol}_{g_{\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]}} \\ &\quad = \int_{\mathbb{S}^2} |\vec{F}|^2(\nabla^2(\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]), \nabla(\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]), \mathrm{Exp}_{\vec{\Phi}}[t\vec{w}])\, d\mathrm{vol}_{g_0}.\end{aligned}$$

Recall that, using the construction of conformal coordinates with estimates by Chern–Heléin–Riviére, we can assume that on every compact subset $K \subset\subset \mathbb{S}^2 \setminus \{b^1, \ldots, b^{N_{\vec{\Phi}}}\}$, the immersion $\vec{\Phi}$ is conformal with $\|(\log|\nabla \vec{\Phi}|)\|_{L^\infty(K)} \le C_K$ for some constant C_K depending on K. By conformality, it follows that on every compact subset $K \subset\subset \mathbb{S}^2 \setminus \{b^1, \ldots, b^{N_{\vec{\Phi}}}\}$ there exists a positive constant c_K such that $|d\vec{\Phi} \wedge d\vec{\Phi}| \ge c_K > 0$. Since $\vec{w}$ is supported away the branch points it follows that, for t small enough, $(\nabla^2(\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]), \nabla(\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]), \mathrm{Exp}_{\vec{\Phi}}[t\vec{w}])|_{\mathrm{supp}(\vec{w})}$ is in the domain of smoothness of $\vec{F}$. By a Taylor expansion in t we get

$$\begin{aligned}&\int_{\mathbb{S}^2} |\vec{H}_{\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]}|^2 d\mathrm{vol}_{g_{\mathrm{Exp}_{\vec{\Phi}}[t\vec{w}]}} \\ &\quad = \int_{\mathbb{S}^2} |\vec{H}_{\vec{\Phi}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}}} + 2t \int_{\mathbb{S}^2} \vec{F}(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) \cdot \partial_{\xi^k_{ij}} \vec{F}(\nabla \vec{\Phi}, \vec{\Phi}) \partial^2_{x^i x^j} w^k d\mathrm{vol}_{g_0} \\ &\qquad + 2t \int_{\mathbb{S}^2} \vec{F}(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) \cdot \partial_{q^k_i} \vec{F}(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) \partial_{x^i} w^k d\mathrm{vol}_{g_0} \\ &\qquad + 2t \int_{\mathbb{S}^2} \vec{F}(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) \cdot \partial_{z^k} \vec{F}(\nabla^2 \vec{\Phi}, \nabla \vec{\Phi}, \vec{\Phi}) w^k d\mathrm{vol}_{g_0} + t^2 \int_{\mathbb{S}^2} \partial^2_{x^i x^j} w^k \partial^2_{x^r x^s}\end{aligned}$$

$$\begin{aligned}&\times w^l P^{kl}_{ijrs}(\nabla\vec{\Phi},\vec{\Phi},\nabla\vec{w},\vec{w})+t^2\int_{\mathbb{S}^2}\partial^2_{x^ix^j}w^k Q^k_{ij}(\nabla^2\vec{\Phi},\nabla\vec{\Phi},\vec{\Phi},\nabla\vec{w},\vec{w})\\ &+t^2\int_{\mathbb{S}^2}S(\nabla^2\vec{\Phi},\nabla\vec{\Phi},\vec{\Phi},\nabla\vec{w},\vec{w}),\end{aligned}\tag{A.44}$$

where, in the second line $\partial_{\xi^k_{ij}}\vec{F}$ depends just on $(\nabla\vec{\Phi},\vec{\Phi})$ since $\vec{F}$ is linear in $\vec{\xi}$, in the 5th line the function $\vec{P}$ is smooth in its arguments with $\vec{P}(\nabla\vec{\Phi},\vec{\Phi},0,0)=0$, in the 6th line the function $\vec{Q}$ is smooth in its arguments and linear in $\nabla^2\vec{\Phi}$ with $\vec{Q}(\nabla^2\vec{\Phi},\nabla\vec{\Phi},\vec{\Phi},0,0)=0$ and in the 7th line the function S is smooth in its arguments and quadratic in $\nabla^2\vec{\Phi}$ with $S(\nabla^2\vec{\Phi},\nabla\vec{\Phi},\vec{\Phi},0,0)=0$. Therefore, called $R^{\vec{\Phi}}_{\vec{w}}[t]$ the sum of the last three lines of (A.44), we have that

$$\begin{aligned}\sup\ &\Big\{\Big|R^{\vec{\Phi}}_{\vec{w}}[t]\Big| : \|\vec{w}\|_{W^{2,2}}+\|\vec{w}\|_{W^{1,\infty}}\le 1\\ &\text{and supp }\vec{w}\subset K\subset\subset\mathbb{S}^2\setminus\{b^1,\dots,b^{N_{\vec{\Phi}}}\}\Big\}\le C_{\vec{\Phi},K}t^2.\end{aligned}$$

It follows that $\int|H_{\vec{\Phi}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}}}$ is Frechét-differentiable with respect to $W^{2,2}\cap W^{1,\infty}$ variations compactly supported away from the branch points, and the first variation $dW_{\vec{\Phi}_\varepsilon}[t\vec{w}]$ is given by the sum of lines 2, 3, 4 of (A.44).

Now we identify the first order term in the expansion of $\int|H_{\vec{\Phi}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}}}$ with the conservative Willmore equation we derived before in the paper. Observe that it is not completely trivial since the conservative Willmore equation has been proved for *smooth* immersions, while now $\vec{\Phi}$ is a *weak* branched immersion. First of all, recall that if $\vec{\Psi}$ is a smooth immersion of the disc D^2 taking values in a coordinate chart of M, then for a smooth variation $\vec{w}\in C^\infty_0(D^2,\mathbb{R}^m)$ with compact support in D^2 we have that

$$\int_{D^2}|\vec{H}_{\vec{\Psi}+t\vec{w}}|^2 d\mathrm{vol}_{g_{\vec{\Psi}+t\vec{w}}}=\int_{D^2}|\vec{H}_{\vec{\Psi}}|^2 d\mathrm{vol}_{g_{\vec{\Psi}}}+t\,dW_{\vec{\Psi}}[\vec{w}]+\tilde{R}^{\vec{\Psi}}_{\vec{w}}[t];\tag{A.45}$$

where the remainder $\tilde{R}^{\vec{\Psi}}_{\vec{w}}[t]$ has the same form as the sum of the last three lines of (A.44), and where the differential $dW_{\vec{\Psi}}[\vec{w}]$, after the integration by parts procedure carried in Lemma A.4, can be written as

$$\begin{aligned}dW_{\vec{\Psi}}[\vec{w}]=&\int_{D^2}\left(\left\langle\vec{H}_{\vec{\Psi}},\frac{1}{2}D^{*_{g_{\vec{\Psi}}}}_{g_{\vec{\Psi}}}\left[D_{g_{\vec{\Psi}}}\vec{w}-3\pi_{\vec{n}_{\vec{\Psi}}}(D_{g_{\vec{\Psi}}}\vec{w})\right]\right\rangle\right)d\mathrm{vol}_{g_{\vec{\Psi}}}\\ &+\int_{D^2}\Big(\langle\star_h((*_{g_{\vec{\Psi}}}D_{g_{\vec{\Psi}}}\vec{n})\wedge_M\vec{H}_{\vec{\Psi}}),D_{g_{\vec{\Psi}}}\vec{w}\rangle\\ &+\langle-\tilde{R}(\vec{H}_{\vec{\Psi}})+R^{\perp}_{\vec{\Psi}}(T\vec{\Psi}),\vec{w}\rangle\Big)d\mathrm{vol}_{g_{\vec{\Psi}}}.\end{aligned}$$

Now let us start considering the case of $\vec{\Phi}\in W^{1,\infty}\cap W^{2,2}(D^2)$ a weak conformal immersion with finite total curvature without branch points taking values in a coordinate chart of M, and let $\vec{w}\in C^\infty_0(D^2,\mathbb{R}^m)$ be a smooth variation with compact support in D^2.

Let φ be a non negative compactly supported function of $C^\infty_0(\mathbb{R})$ such that φ is identically equal to 1 in a neighborhood of 0 and

$$2\pi\int_{\mathbb{R}}\varphi(t)\,t\,dt=1.$$

Call $\varphi_\varepsilon(t) := \varepsilon^{-2}\,\varphi(t/\varepsilon)$. Denote for $\varepsilon < 1/4$ and for any $x \in D^2_{1/2}$,

$$\vec{\Phi}_\varepsilon(x) := \varphi_\varepsilon(|x|) \star \vec{\Phi} := \int_{D^2} \varphi_\varepsilon(|x-y|)\,\vec{\Phi}(y)\,dy.$$

By Lemma A.6 there exists $0 < \varepsilon_{\vec{\Phi}} < 1/4$ such that for any $\varepsilon < \varepsilon_{\vec{\Phi}}$ the map $\vec{\Phi}_\varepsilon$ realizes a smooth immersion from $D^2_{1/2}$ into the coordinate chart; moreover we have (notice that in order to keep the notation not too heavy, in the following we replaced $D^2_{1/2}$ by D^2)

$$\vec{\Phi}_\varepsilon \to \vec{\Phi} \quad \text{strong in } W^{2,2}(D^2) \tag{A.46}$$

$$\vec{H}_\varepsilon \to \vec{H} \quad \text{strong in } L^2(D^2) \tag{A.47}$$

$$\vec{n}_\varepsilon \to \vec{n} \quad \text{strong in } W^{1,2}(D^2) \tag{A.48}$$

and

$$\sup_{0<\varepsilon\leq\varepsilon_0} \|\vec{\Phi}_\varepsilon\|_{W^{1,\infty}(D^2)} \leq C < \infty \tag{A.49}$$

$$\inf_{x\in D^2}\inf_{0<\varepsilon\leq\varepsilon_0} |d\vec{\Phi}_\varepsilon \wedge d\vec{\Phi}_\varepsilon| \geq \frac{1}{C} > 0. \tag{A.50}$$

Since $\vec{\Phi}_\varepsilon$ is smooth, the Willmore functional computed on $\vec{\Phi}_\varepsilon + t\vec{w}$ expands as in (A.45); observe that, thanks to (A.46), (A.49) and (A.50), the remainder $\tilde{R}^{\vec{\Phi}_\varepsilon}_{\vec{w}}[t]$ satisfies

$$\sup_{0<\varepsilon\leq\varepsilon_0}\ \sup_{\|\vec{w}\|_{W^{1,\infty}\cap W^{2,2}(D^2)}\leq 1} \left|\tilde{R}^{\vec{\Phi}_\varepsilon}_{\vec{w}}[t]\right| \leq C_{\vec{\Phi}}t^2. \tag{A.51}$$

Observe moreover that, by (A.46), (A.47) and (A.49), we have that $|\vec{H}_\varepsilon|^2 d\mathrm{vol}_{g_\varepsilon}$ is dominated in $L^1(D^2)$ for $\varepsilon \leq \varepsilon_0$, and converges almost everywhere on D^2 to $|\vec{H}|^2 d\mathrm{vol}_g$; therefore, by the Dominated Convergence Theorem, we have

$$\int_{D^2} |\vec{H}_\varepsilon|^2 d\mathrm{vol}_{g_\varepsilon} \to \int_{D^2} |\vec{H}|^2 d\mathrm{vol}_g. \tag{A.52}$$

Moreover, using (A.46) and (A.50) we have that, for $\|\vec{w}\|_{W^{1,\infty}\cap W^{2,2}(D^2)} \leq 1$ and t small enough, $\vec{\Phi}_\varepsilon + t\vec{w} \to \vec{\Phi} + t\vec{w}$ strongly in $W^{2,2}(D^2)$ and $\vec{H}_{\vec{\Phi}_\varepsilon + t\vec{w}} \to \vec{H}_{\vec{\Phi}+t\vec{w}}$ strongly in $L^2(D^2)$; of course it still holds $\sup_{0<\varepsilon\leq\varepsilon_0} \|\vec{\Phi}_\varepsilon + t\vec{w}\|_{W^{1,\infty}(D^2)} \leq C < \infty$. Therefore, with the same argument above, we get

$$\int_{D^2} |\vec{H}_{\vec{\Phi}_\varepsilon+t\vec{w}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_\varepsilon+t\vec{w}}} \to \int_{D^2} |\vec{H}_{\vec{\Phi}+t\vec{w}}|^2 d\mathrm{vol}_{g_{\vec{\Phi}+t\vec{w}}}. \tag{A.53}$$

Combining (A.44), (A.52) and (A.53) gives

$$\begin{aligned} dW_{\vec{\Phi}}[\vec{w}] &= \lim_{t\to 0} \frac{W(\vec{\Phi}+t\vec{w}) - W(\vec{\Phi})}{t} = \lim_{t\to 0}\lim_{\varepsilon\to 0} \frac{W(\vec{\Phi}_\varepsilon + t\vec{w}) - W(\vec{\Phi}_\varepsilon)}{t} \\ &= \lim_{t\to 0}\lim_{\varepsilon\to 0}\left(dW_{\vec{\Phi}_\varepsilon}[\vec{w}] + \frac{\tilde{R}^{\vec{\Phi}_\varepsilon}_{\vec{w}}[t]}{t}\right); \end{aligned}$$

recalling (A.51), we obtain

$$dW_{\vec{\Phi}_\varepsilon}[\vec{w}] \to dW_{\vec{\Phi}}[w] \quad \text{as } \varepsilon \to 0. \tag{A.54}$$

Therefore in order to prove that, as in the smooth situation, $dW_{\vec{\Phi}}$ is the Willmore equation in conservative form, it is sufficient to show that

$$\begin{aligned}
&\int_{D^2} \left(\left\langle \vec{H}_\varepsilon, \frac{1}{2} D^{*_{g_\varepsilon}}_{g_\varepsilon} \left[D_{g_\varepsilon}\vec{w} - 3\pi_{\vec{n}_\varepsilon}(D_{g_\varepsilon}\vec{w}) \right] \right\rangle \right) d\mathrm{vol}_{g_\varepsilon} \\
&\quad + \int_{D^2} \left(\left\langle \star_h \left((*_{g_\varepsilon} D_{g_\varepsilon}\vec{n}_\varepsilon) \wedge_M \vec{H}_\varepsilon \right), D_{g_\varepsilon}\vec{w} \right\rangle + \left\langle -\tilde{R}(\vec{H}_\varepsilon) + R^\perp_{\vec{\Phi}_\varepsilon}(T\vec{\Phi}_\varepsilon), \vec{w} \right\rangle \right) d\mathrm{vol}_{g_\varepsilon}. \\
&\to \int_{D^2} \left(\left\langle \vec{H}, \frac{1}{2} D^{*_g}_{g} \left[D_g\vec{w} - 3\pi_{\vec{n}}(D_g\vec{w}) \right] \right\rangle \right) d\mathrm{vol}_g \\
&\quad + \int_{D^2} \left(\left\langle \star_h \left((*_g D_g\vec{n}) \wedge_M \vec{H} \right), D_g\vec{w} \right\rangle + \left\langle -\tilde{R}(\vec{H}) + R^\perp_{\vec{\Phi}}(T\vec{\Phi}), \vec{w} \right\rangle \right) d\mathrm{vol}_g.
\end{aligned} \tag{A.55}$$

We are going to check the convergence term by term.

Observe that $D^{*_{g_\varepsilon}}_{g_\varepsilon}[D_{g_\varepsilon}\vec{w}] = g^{ij}_\varepsilon D_{\partial_{x^i}\vec{\Phi}_\varepsilon} D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w}$ and, using the definitions, one computes

$$\begin{aligned}
\left(g^{ij}_\varepsilon D_{\partial_{x^i}\vec{\Phi}_\varepsilon} D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w} \right)^k &= g^{ij}_\varepsilon \Big[\partial^2_{x^i x^j} w^k + (\Gamma^k_{pq} \circ \vec{\Phi}_\varepsilon)\partial_{x^j} w^p \partial_{x^i} \Phi^q_\varepsilon \\
&\quad + (\langle \mathrm{grad}_h \Gamma^k_{pq}, \partial_{x^i} \vec{\Phi}_\varepsilon \rangle_h \circ \vec{\Phi}_\varepsilon) w^p \partial_{x^j} \Phi^q \\
&\quad + (\Gamma^k_{pq} \circ \vec{\Phi}_\varepsilon)\partial_{x^i} w^p \partial_{x^j} \Phi^q + (\Gamma^k_{pq} \circ \vec{\Phi}_\varepsilon) w^p \partial^2_{x^i x^j} \Phi^q \\
&\quad + (\Gamma^k_{lm} \circ \vec{\Phi}_\varepsilon)(\Gamma^l_{pq} \circ \vec{\Phi}_\varepsilon) w^p \partial_{x^j} \Phi^q \partial_{x^i} \Phi^m \Big] \\
&= f^k_{1,\varepsilon} + f^k_{2,\varepsilon} \quad \text{with } |f^k_{1,\varepsilon}| \le F^k_1 \in L^\infty(D^2) \text{ and} \\
&\quad\ |f^k_{2,\varepsilon}| \le F^k_2 \in L^2(D^2);
\end{aligned} \tag{A.56}$$

where Γ^k_{pq} are the Christoffel symbols of (M, h) which are smooth and C^1 bounded by the compactness of M. Notice that in the last equality we used (A.46) and (A.49). Combining (A.46), (A.47) and (A.56) we get therefore that $\langle \vec{H}_\varepsilon, D^{*_{g_\varepsilon}}_{g_\varepsilon}[D_{g_\varepsilon}\vec{w}]\rangle d\mathrm{vol}_{g_\varepsilon}$ is dominated in $L^1(D^2)$ and converges almost everywhere to $\langle \vec{H}, D^{*_g}_g[D_g\vec{w}]\rangle d\mathrm{vol}_g$, then by the Dominated Convergence Theorem

$$\int_{D^2} \langle \vec{H}_\varepsilon, D^{*_{g_\varepsilon}}_{g_\varepsilon}[D_{g_\varepsilon}\vec{w}]\rangle d\mathrm{vol}_{g_\varepsilon} \to \int_{D^2} \langle \vec{H}, D^{*_g}_g[D_g\vec{w}]\rangle d\mathrm{vol}_g \quad \text{as } \varepsilon \to 0. \tag{A.57}$$

Now let us consider the second summand in the first line of (A.55). Observe that $D^{*_{g_\varepsilon}}_{g_\varepsilon}[\pi_{n_\varepsilon}(D_{g_\varepsilon}\vec{w})] = g^{ij}_\varepsilon D_{\partial_{x^i}\vec{\Phi}_\varepsilon}[\pi_{\vec{n}_\varepsilon}(D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w})]$; using (3.10) we can write

$$\begin{aligned}
&D^{*_{g_\varepsilon}}_{g_\varepsilon}[\pi_{n_\varepsilon}(D_{g_\varepsilon}\vec{w})] \\
&\quad = (-1)^{m-1} g^{ij}_\varepsilon \Big[(D_{\partial_{x^i}\vec{\Phi}_\varepsilon}\vec{n}_\varepsilon) \llcorner (\vec{n}_\varepsilon \llcorner D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w}) \\
&\qquad + \vec{n}_\varepsilon \llcorner ((D_{\partial_{x^i}\vec{\Phi}_\varepsilon}\vec{n}_\varepsilon) \llcorner D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w}) \\
&\qquad + \vec{n}_\varepsilon \llcorner (\vec{n}_\varepsilon \llcorner (D_{\partial_{x^i}\vec{\Phi}_\varepsilon} D_{\partial_{x^j}\vec{\Phi}_\varepsilon}\vec{w})) \Big].
\end{aligned} \tag{A.58}$$

Writing explicitly the right hand side as done for (A.56), one checks that

$$D^{*g_\varepsilon}_{g_\varepsilon}[\pi_{n_\varepsilon}(D_{g_\varepsilon}\vec{w})] = \vec{f}_{3,\varepsilon} + \vec{f}_{4,\varepsilon} \quad \text{with } |\vec{f}_{3,\varepsilon}| \le \vec{F}_3 \in L^\infty(D^2) \text{ and}$$
$$|\vec{f}_{4,\varepsilon}| \le \vec{F}_4 \in L^2(D^2). \tag{A.59}$$

Combining (A.46), (A.47) and (A.59) we get therefore that $\langle \vec{H}_\varepsilon, D^{*g_\varepsilon}_{g_\varepsilon}[\pi_{n_\varepsilon}(D_{g_\varepsilon}\vec{w})]\rangle d\mathrm{vol}_{g_\varepsilon}$ is dominated in $L^1(D^2)$ and converges almost everywhere to $\langle \vec{H}, D^{*g}_{g}[\pi_{\vec{n}}(D_g\vec{w})]\rangle d\mathrm{vol}_g$; then by the Dominated Convergence Theorem

$$\int_{D^2} \langle \vec{H}_\varepsilon, D^{*g_\varepsilon}_{g_\varepsilon}[\pi_{n_\varepsilon}(D_{g_\varepsilon}\vec{w})]\rangle d\mathrm{vol}_{g_\varepsilon} \to \int_{D^2} \langle \vec{H}, D^{*g}_{g}[\pi_{\vec{n}}(D_g\vec{w})]\rangle d\mathrm{vol}_g \quad \text{as } \varepsilon \to 0. \tag{A.60}$$

Now let us consider the first summand of the second line of (A.55). Observe that

$$*_{g_\varepsilon} \frac{\partial}{\partial x^j} = \sqrt{\det g_\varepsilon}\, \varepsilon_{jp}\, g^{pq}_\varepsilon \frac{\partial}{\partial x^q}, \tag{A.61}$$

where ε_{jp} is null if $j = p$ and equals the signature of the permutation $(1,2) \mapsto (j,p)$ if $j \neq p$; after some straightforward computations using the definitions (3.1), (3.2), (3.4), (3.5), (3.6), we get

$$\begin{aligned}
&\left\langle \star_h \left((*_{g_\varepsilon} D_{g_\varepsilon}\vec{n}_\varepsilon) \wedge_M \vec{H}_\varepsilon \right), D_{g_\varepsilon}\vec{w} \right\rangle \\
&\quad = \sqrt{\det g_\varepsilon}\, g^{ij}_\varepsilon\, \varepsilon_{jp}\, g^{pq}_\varepsilon \langle \star_h((D_{\partial_{x^i}\vec{\Phi}_\varepsilon}\vec{n}_\varepsilon) \wedge_M \vec{H}_\varepsilon), D_{\partial_{x^q}\vec{\Phi}_\varepsilon}\vec{w}\rangle \\
&\quad = f_{5,\varepsilon} \quad \text{with } |f_{5,\varepsilon}| \le F_5 \in L^1(D^2).
\end{aligned} \tag{A.62}$$

Using analogous arguments as before, by the Dominated Convergence Theorem we obtain

$$\begin{aligned}
&\int_{D^2} \langle \star_h((*_{g_\varepsilon} D_{g_\varepsilon}\vec{n}_\varepsilon) \wedge_M \vec{H}_\varepsilon), D_{g_\varepsilon}\vec{w}\rangle d\mathrm{vol}_{g_\varepsilon} \\
&\quad \to \int_{D^2} \langle \star_h((*_g D_g\vec{n}) \wedge_M \vec{H}), D_g\vec{w}\rangle d\mathrm{vol}_g.
\end{aligned} \tag{A.63}$$

Finally consider the last two curvature terms in (A.55). By the definition (1.5), the first one writes as

$$\langle \tilde{R}(\vec{H}_\varepsilon), \vec{w}\rangle = -\left\langle \sum_{i=1}^{2} \mathrm{Riem}^h(\vec{H}_\varepsilon, \vec{e}^{\,\varepsilon}_i)\vec{e}^{\,\varepsilon}_i, \pi_{\vec{n}_\varepsilon}(\vec{w}) \right\rangle \tag{A.64}$$

for an orthonormal frame $\vec{e}^{\,\varepsilon}_i$ of $\vec{\Phi}_{\varepsilon,*}(TD^2)$; observe that it is dominated in $L^1(D^2)$ and converges a. e. to $\langle \tilde{R}(\vec{H}), \vec{w}\rangle$ on D^2, so as before

$$\int_{D^2} \langle \tilde{R}(\vec{H}_\varepsilon), \vec{w}\rangle d\mathrm{vol}_{g_\varepsilon} \to \int_{D^2} \langle \tilde{R}(\vec{H}), \vec{w}\rangle d\mathrm{vol}_g. \tag{A.65}$$

Finally, by definition (1.7) and identity (A.61),

$$\begin{aligned}
R^\perp_{\vec{\Phi}_\varepsilon}(T\vec{\Phi}_\varepsilon) &:= \left[\pi_T\left(\mathrm{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H}\right)\right]^\perp \\
&= \sqrt{\det g_\varepsilon}\, g^{ij}_\varepsilon\, g^{kl}_\varepsilon\, \varepsilon_{lp}\, g^{pq}_\varepsilon \langle \mathrm{Riem}^h(\partial_{x^i}\vec{\Phi}_\varepsilon, \partial_{x^j}\vec{\Phi}_\varepsilon)\vec{H}_\varepsilon, \partial_{x^k}\vec{\Phi}_\varepsilon\rangle \partial_{x^q}\vec{\Phi}_\varepsilon.
\end{aligned} \tag{A.66}$$

From this explicit formula, as before, one checks that $\langle R^{\perp}_{\vec{\Phi}_\varepsilon}(T\vec{\Phi}_\varepsilon), \vec{w}\rangle d\mathrm{vol}_{g_\varepsilon}$ is dominated in $L^1(D^2)$ and converges to $\langle R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \vec{w}\rangle d\mathrm{vol}_g$ a.e. on D^2, then by the Dominated Convergence Theorem

$$\int_{D^2}\langle R^{\perp}_{\vec{\Phi}_\varepsilon}(T\vec{\Phi}_\varepsilon), \vec{w}\rangle d\mathrm{vol}_{g_\varepsilon} \to \int_{D^2}\langle R^{\perp}_{\vec{\Phi}}(T\vec{\Phi}), \vec{w}\rangle d\mathrm{vol}_g. \tag{A.67}$$

Combining (A.57), (A.60), (A.63), (A.65) and (A.67) we obtain (A.55) as desired. Let us recap what we have just proved: if $\vec{\Phi}$ is a $W^{1,\infty}\cap W^{2,2}$ immersion of the disc D^2 into a coordinate neighborhood in M and $\vec{w}\in C^\infty_0(D^2,\mathbb{R}^m)$ is a smooth variation with compact support in D^2, then the differential of the Willmore functional $d_{\vec{\Phi}}W[\vec{w}]$ coincides with the pairing between the Willmore equation in conservative form and $\vec{w}$. Now by approximation the same is true for variations in $W^{1,\infty}\cap W^{2,2}(D^2,\mathbb{R}^m)$ with compact support in D^2. By partition of unity, the same statement holds for $\vec{\Phi}\in\mathcal{F}_{\mathbb{S}^2}$ with branched points $\{b^1,\dots,b^N\}$ and any variation $\vec{w}\in W^{1,\infty}\cap W^{2,2}(D^2, T_{\vec{\Phi}}M)$ with compact support in $\mathbb{S}^2\setminus\{b^1,\dots,b^N\}$.

The proof regarding the differentiability of F is analogous since $\mathbb{I}_{\vec{\Phi}}$ is a vectorial function of $(\nabla^2\vec{\Phi}, \nabla\vec{\Phi}, \vec{\Phi})$ linear in $\nabla^2\vec{\Phi}$. Moreover for smooth immersions and smooth variations, combining Corollary 3.1 and Lemma A.4, the first variation of F is exactly (A.39). With the same approximation argument carried for W one checks that the same expression holds for a weak immersion.

The proof regarding the differentiability and the expression of the differential of the area functional is easier since $d\mathrm{vol}_g$ is a function just of $(\nabla\vec{\Phi}, \vec{\Phi})$, and can be performed along the same lines once recalled that in the smooth case the differential of the area functional is exactly (A.40). $\square$

Let us now prove the following approximation lemma used in the proof of Lemma A.5.

Lemma A.6. *Let $\vec{\Phi}$ be a conformal weak immersion in $\mathcal{F}_{D^2}$ into $\mathbb{R}^m$ without branch points. Let φ be a non negative compactly supported function of $C^\infty_0(\mathbb{R})$ such that φ is identically equal to* 1 *in a neighborhood of* 0 *and*

$$2\pi\int_{\mathbb{R}}\varphi(t)\,t\,dt = 1.$$

Denote $\varphi_\varepsilon(t) := \varepsilon^{-2}\,\varphi(t/\varepsilon)$. Denote for $\varepsilon < 1/4$ and for any $x\in D^2_{1/2}$,

$$\vec{\Phi}_\varepsilon(x) := \varphi_\varepsilon(|x|)\star\vec{\Phi} := \int_{D^2}\varphi_\varepsilon(|x-y|)\;\vec{\Phi}(y)\,dy.$$

There exists $0<\varepsilon_{\vec{\Phi}}<1/4$ such that for any $\varepsilon<\varepsilon_{\vec{\Phi}}$ the map $\vec{\Phi}_\varepsilon$ realizes a smooth immersion from $D^2_{1/2}$ into $\mathbb{R}^m$; moreover we have

$$\lim_{\varepsilon\to 0}\|g_{\vec{\Phi}_\varepsilon} - g_{\vec{\Phi}}\|_{L^\infty(D^2_{1/2})} = 0, \tag{A.68}$$

we have also

$$\lim_{\varepsilon\to 0}\|\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_\varepsilon}\|_{W^{1,2}(D^2_{1/2})} = 0, \tag{A.69}$$

and

$$\lim_{\varepsilon\to 0}\|\vec{H}_{\vec{\Phi}} - \vec{H}_{\vec{\Phi}_\varepsilon}\|_{L^2(D^2_{1/2})} = 0. \quad\square \tag{A.70}$$

Before proving Lemma A.6 we establish the following φ-Poincaré inequality.

Lemma A.7. *Let $u \in W^{1,2}(D^2)$. Let φ be a non negative compactly supported function of $C_0^\infty(\mathbb{R})$ such that φ is identically equal to 1 in a neighborhood of 0 and*

$$2\pi \int_{\mathbb{R}} \varphi(t)\, t\, dt = 1.$$

Denote $\varphi_\varepsilon(t) := \varepsilon^{-2}\, \varphi(t/\varepsilon)$. For $\varepsilon < 1/4$ and $x \in D^2_{1/2}$ denote

$$u_\varepsilon(x) := \varphi_\varepsilon(|x|) \star u := \int_{D^2} \varphi_\varepsilon(|x-y|)\, u(y)\, dy.$$

There exists a constant $C > 0$ such that for any $x \in D^2_{1/2}$

$$\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - u_\varepsilon(x)|^2\, dy \le C \int_{B_\varepsilon(x)} |\nabla u|^2(y)\, dy. \quad \Box \tag{A.71}$$

Proof of Lemma A.7. For any $x \in D^2_{1/2}$ and $0 < \varepsilon < 1/4$ we denote

$$\overline{u}^{\varepsilon,x} := \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy = \int_{D^2} \chi_\varepsilon(|x-y|)\, u(y)\, dy,$$

where $\chi_\varepsilon(t) \equiv (\pi\varepsilon^2)^{-1}$ on $[0, \varepsilon]$ and equals zero otherwise. The classical Poincaré inequality gives the existence of a universal constant such that

$$\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - \overline{u}^{\varepsilon,x}|^2\, dy \le C \int_{B_\varepsilon(x)} |\nabla u|^2(y)\, dy. \tag{A.72}$$

We have

$$\begin{aligned} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - u_\varepsilon(x)|^2\, dy &\le 2\frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - \overline{u}^{\varepsilon,x}|^2\, dy \\ &\quad + 2|\overline{u}^{\varepsilon,x} - u_\varepsilon(x)|^2; \end{aligned} \tag{A.73}$$

and

$$\overline{u}^{\varepsilon,x} - u_\varepsilon(x) = \int_{B_\varepsilon(x)} [\chi_\varepsilon(|x-y|) - \varphi_\varepsilon(|x-y|)]\, u(y)\, dy. \tag{A.74}$$

Since

$$\int_{B_\varepsilon(x)} [\chi_\varepsilon(|x-y|) - \varphi_\varepsilon(|x-y|)]\, dy = 0$$

the identity (A.74) takes the form

$$\overline{u}^{\varepsilon,x} - u_\varepsilon(x) = \int_{B_\varepsilon(x)} [\chi_\varepsilon(|x-y|) - \varphi_\varepsilon(|x-y|)]\, (u(y) - \overline{u}^{\varepsilon,x})\, dy. \tag{A.75}$$

Thus

$$\begin{aligned} |\overline{u}^{\varepsilon,x} - u_\varepsilon(x)|^2 &\le C\, \varepsilon^{-4} \left| \int_{B_\varepsilon(x)} |u(y) - \overline{u}^{\varepsilon,x}|\, dy \right|^2 \\ &\le C\, \varepsilon^{-2} \int_{B_\varepsilon(x)} |u(y) - \overline{u}^{\varepsilon,x}|^2\, dy. \end{aligned} \tag{A.76}$$

Combining (A.72), (A.73) and (A.76) gives (A.71) and this proves Lemma A.7. □

Proof of Lemma A.6. We first establish (A.68). Since $\vec{\Phi}$ is a weak conformal immersion, results from [13] imply that there exists $\lambda \in C^0(D^2)$ such that

$$g_{\vec{\Phi}} = e^{2\lambda}\ [dx_1^2 + dx_2^2],$$

and $e^\lambda = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. Then, for any $\delta > 0$ there exists ε such that

$$\forall\,\delta > 0\,\exists\,\varepsilon > 0\,\forall\,x, y \in D^2_{3/4} \quad |x-y| < \varepsilon \Longrightarrow 1-\delta < e^{\lambda(x)-\lambda(y)} \le 1+\delta. \tag{A.77}$$

Since $\vec{\Phi} \in W^{2,2}(D_2, \mathbb{R}^m)$

$$\forall\,\delta > 0\,\exists\,\varepsilon > 0\,\forall\,\varepsilon < \varepsilon_0 \quad \sup_{x\in D^2_{1/2}} \int_{B_\varepsilon(x)} |\nabla^2\vec{\Phi}|^2(y)\,dy \le \delta^2. \tag{A.78}$$

Applying Lemma A.7 to $u = \nabla\vec{\Phi}$ we deduce then that

$$\forall\,\delta > 0\,\exists\,\varepsilon_0 > 0\,\forall\,\varepsilon < \varepsilon_0 \quad \sup_{x\in D^2_{1/2}} \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} |\nabla\vec{\Phi}(y) - \nabla\vec{\Phi}_\varepsilon(x)|^2\,dy \le \delta^2. \tag{A.79}$$

Using the mean-value theorem we then deduce that

$$\begin{aligned}&\forall\,\delta > 0 \quad \exists\,\varepsilon_0 > 0 \quad \text{s.t.}\ \forall\,\varepsilon < \varepsilon_0 \quad \forall\,x \in D^2_{1/2} \quad \exists\,y_x \in B_\varepsilon(x) \quad \text{s. t.}\\ &|\nabla\vec{\Phi}(y_x) - \nabla\vec{\Phi}_\varepsilon(x)| \le \sqrt{\delta}.\end{aligned} \tag{A.80}$$

Since

$$0 < \inf_{y\in D^2_{1/2}} |\nabla\vec{\Phi}(y)|^2 = \inf_{y\in D^2_{1/2}} 2\,e^{2\lambda(y)} \le \sup_{y\in D^2_{1/2}} |\nabla\vec{\Phi}(y)|^2 \tag{A.81}$$

(A.80) implies for $i = 1, 2$

$$\begin{aligned}&\forall\,\delta > 0 \quad \exists\,\varepsilon_0 > 0 \quad \forall\,\varepsilon < \varepsilon_0 \quad \text{s.t.}\ \forall\,x \in D^2_{1/2} \quad \exists\,y_x \in B_\varepsilon(x)\\ &\text{s.t.}\ 1-\delta \le \frac{|\partial_{x_i}\vec{\Phi}_\varepsilon(x)|}{|\partial_{x_i}\vec{\Phi}(y_x)|} \le 1+\delta.\end{aligned} \tag{A.82}$$

Combining (A.77) and (A.82) we obtain for $i = 1, 2$

$$\forall\,\delta > 0 \quad \exists\,\varepsilon_0 > 0\,\forall\,\varepsilon < \varepsilon_0\,\forall\,x \in D^2_{1/2}\ 1-\delta \le \frac{|\partial_{x_i}\vec{\Phi}_\varepsilon(x)|}{|\partial_{x_i}\vec{\Phi}(x)|} \le 1+\delta. \tag{A.83}$$

Similarly, using (A.80), (A.81) and the fact that $\partial_{x_1}\vec{\Phi}(y)\cdot\partial_{x_2}\vec{\Phi}(y) \equiv 0$ we have

$$\forall\,\delta > 0\,\exists\,\varepsilon_0 > 0\,\forall\,\varepsilon < \varepsilon_0\,\forall\,x \in D^2_{1/2} \quad \frac{|\partial_{x_1}\vec{\Phi}_\varepsilon(x)\cdot\partial_{x_2}\vec{\Phi}_\varepsilon(x)|}{|\nabla\vec{\Phi}_\varepsilon(x)|^2} \le \delta. \tag{A.84}$$

It is clear that (A.83) and (A.84) imply (A.68). Finally (A.69) and (A.70) are direct consequences of the fact that (A.68) and (A.81) hold together with the fact that $\vec{\Phi}_\varepsilon \to \vec{\Phi}$ strongly in $W^{2,2}(D^2_{1/2})$. Lemma A.6 is then proved. □

Lemma A.8 (*Lower Semi Continuity Under* $W^{2,2}$*-Weak Convergence*)**.** *Let* $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_{\mathbb{S}^2}$ *and* $\vec{\Phi}_\infty$ *be weak branched conformal immersions and assume that there exist* $a^1, \ldots, a^N \in \mathbb{S}^2$ *such that for every compact subset (with smooth boundary)* $K \subset\subset \mathbb{S}^2$ *we have*

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi} \quad \text{weakly in } W^{2,2}(K) \tag{A.85}$$

$$\sup_k \sup_{x\in K} |\log|\nabla\vec{\Phi}_k||(x) \le C_K < \infty \quad \text{for some constant } c_K \text{ depending on } K. \tag{A.86}$$

Then the Willmore and the Energy functional are lower semicontinuous:

$$\begin{aligned} &\int_K |H_{\vec{\Phi}_\infty}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_\infty}} \le \liminf_k \int_K |H_{\vec{\Phi}_k}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}}, \\ &\int_K |\mathbb{I}_{\vec{\Phi}_\infty}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_\infty}} \le \liminf_k \int_K |\mathbb{I}_{\vec{\Phi}_k}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}}. \end{aligned} \qquad \square \tag{A.87}$$

Proof. Since $\vec{\Phi}_k$ are conformal, then $\vec{H}_k = \frac{1}{2}e^{-2\lambda_k}\Delta\vec{\Phi}_k$ where $\lambda_k = \log|\partial_{x^1}\vec{\Phi}_k| = \log|\partial_{x^2}\vec{\Phi}_k|$ is the conformal factor. Let us first show that

$$\vec{H}_k\sqrt{\mathrm{vol}_{g_k}} = \frac{1}{2|\partial_{x^1}\vec{\Phi}_k|}\Delta\vec{\Phi}_k \to \frac{1}{2|\partial_{x^1}\vec{\Phi}_\infty|}\Delta\vec{\Phi}_\infty = \vec{H}_\infty\sqrt{\mathrm{vol}_{g_\infty}} \quad \text{in } \mathcal{D}'(K). \tag{A.88}$$

From (A.85) and the Rellich–Kondrachov Theorem we have that $|\partial_{x^1}\vec{\Phi}_k| \to |\partial_{x^1}\vec{\Phi}_\infty|$ strongly in $L^p(K)$ for every $1 < p < \infty$; moreover assumption (A.86) guarantees that $|\partial_{x^1}\vec{\Phi}_k| \ge \frac{1}{C} > 0$ independently of k. It follows that

$$\frac{1}{|\partial_{x^1}\vec{\Phi}_k|} \to \frac{1}{|\partial_{x^1}\vec{\Phi}_\infty|} \quad \text{strongly in } L^p(K) \text{ for every } 1 < p < \infty.$$

Since, by assumption (A.85), clearly $\Delta\vec{\Phi}_k \to \Delta\vec{\Phi}_\infty$ weakly in $L^2(K)$, then (A.88) follows. In order to conclude observe that (A.85) implies that $\vec{\Phi}_k$ are uniformly bounded in $W^{2,2}(K)$, then assumption (A.87) and the conformality of $\vec{\Phi}_k$ give that $\vec{H}_k\sqrt{\mathrm{vol}_{g_k}}$ are uniformly bounded in $L^2(K)$. This last fact together with (A.88) implies that

$$\vec{H}_k\sqrt{\mathrm{vol}_{g_k}} \rightharpoonup \vec{H}_\infty\sqrt{\mathrm{vol}_{g_\infty}} \quad \text{weakly in } L^2(K).$$

The thesis then follows just by lower semicontinuity of the L^2 norm under weak convergence. The proof of the lower semicontinuity of $\int|\mathbb{I}|^2$ is analogous once observed that in conformal coordinates $|\mathbb{I}|^2 = e^{-4\lambda}|\nabla^2\vec{\Phi}|^2$. $\square$

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