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Schur–Weyl dualities for symmetric inverse semigroups

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Abstract

We obtain Schur–Weyl dualities in which the algebras, acting on both sides, are semigroup algebras of various symmetric inverse semigroups and their deformations.

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1. Introduction and description of results

Let $V = \mathbb{C}^n$ be the natural n-dimensional representation of the group $\mathbf{GL}(n)$. Then for every k the group $\mathbf{GL}(n)$ acts diagonally on the k-fold tensor product $V^{\otimes k}$. At the same time the symmetric group S_k acts on $V^{\otimes k}$ by permuting the factors of a k-tensor. These two actions obviously commute. Moreover, the classical *Schur–Weyl duality* from [17,18,25] states that $\mathbf{GL}(n)$ and S_k generate full centralizers of each other on $V^{\otimes k}$. In particular, $\mathrm{End}_{\mathbf{GL}(n)}(V^{\otimes k}) = \mathbb{C}[S_k]$ if $n \geq k$.

There are various generalizations of the Schur–Weyl duality. In [2] the above action of $\mathbf{GL}(n)$ was restricted to the orthogonal subgroup $\mathbf{O}(n)$ of $\mathbf{GL}(n)$. The corresponding centralizer algebra, obtained on the right-hand side, is what is now known as the *Brauer algebra*. Further restriction of the $\mathbf{GL}(n)$ action to the subgroup S_n, which was considered in [8,14], gives on the right-hand side the so-called *partition algebra*. Some other generalizations are discussed in [3], see also the references of the latter paper.

Both Brauer algebras and partition algebras are deformations of semigroup algebras of certain finite semigroups, which have been also intensively studied, see for example [12,11,15] and references therein. Finite semigroups clearly entered the game after the paper [19] of L. Solomon. In the latter paper it was shown that the representation V of $\mathbf{GL}(n)$ can be slightly modified such that the centralizing object obtained on the right-hand side is the *symmetric inverse semigroup* $\mathcal{IS}_n$, introduced in [24] and also known as the *rook monoid*, see [20]. This idea of modification of

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V was recently used in [6] to obtain a Schur–Weyl duality between S_n and a generalization of the partition algebra, called in [6] the *rook partition algebra*.

For a "finite semigroup theorist" there is a slight feeling of dissatisfaction in the Schur–Weyl dualities listed above, which is explained by the fact that the objects, appearing on the different sides of a Schur–Weyl duality, although closely connected to finite semigroups, still at least one of them has different nature. The aim of the present paper is to establish two Schur–Weyl dualities, where on each side one has an action of a finite inverse semigroup.

For the first Schur–Weyl duality we have an action of the symmetric inverse semigroup $\mathcal{IS}_n$ mentioned above on the left and an action of the *dual symmetric inverse semigroup* $\mathcal{I}_n^*$ on the right. The semigroup $\mathcal{I}_n^*$ was introduced in [4] as a kind of a "categorical dual" for $\mathcal{IS}_n$ (see also [13] for more details on the categorical approach). It is remarkable that in the present paper the semigroup $\mathcal{I}_n^*$ again appears as the dual of $\mathcal{IS}_n$, but now with respect to a Schur–Weyl duality. The connection between these two types of dualities is not yet clear. This first Schur–Weyl duality is considered in Section 2.

For the second Schur–Weyl duality we have an action of the semigroup $\mathcal{IS}_n$ on the left and an action of the *partial analogue* $\mathcal{PI}_n^*$ of the semigroup $\mathcal{I}_n^*$ on the right. The latter semigroup was introduced in [10] via the usual semigroup-theoretic "partialization" philosophy. This philosophy is rather similar to the philosophy used to construct "rook" algebras. Our second Schur–Weyl duality shows, in particular, an explicit connection between these philosophies. This second Schur–Weyl duality is considered in Section 3. In Section 4 we show that the action on the right-hand side of the latter Schur–Weyl duality can be "deformed" such that it becomes an action of another inverse semigroup, recently constructed in [23].

For a semigroup S we denote by $\mathbb{C}[S]$ the *semigroup algebra* of S over complex numbers. If S has the zero element 0, we denote by $\overline{\mathbb{C}[S]}$ the *contracted semigroup algebra* $\mathbb{C}[S]/(0)$.

2. A Schur–Weyl duality for $\mathcal{IS}_n$ and $\mathcal{I}_k^*$

Throughout the paper n and k are fixed positive integers. The semigroup $\mathcal{IS}_n$ is the semigroup of all partial injections from the set $\mathbf{N} = \{1, 2, \ldots, n\}$ to itself with respect to the usual composition of partial maps, see for example [5, Section 2]. One can also consider the elements of $\mathcal{IS}_n$ as bijections between different subsets of $\mathbf{N}$ (this will be important to understand the dual nature of $\mathcal{I}_n^*$). A standard notation for elements of $\mathcal{IS}_n$ and the multiplication rule in this semigroup is best understood on the following example for $\mathcal{IS}_5$ (note that we understand the elements of $\mathcal{IS}_n$ as maps and hence compose them from the right to the left):

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & \varnothing & 3 & 5 & \varnothing\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 4 & 1 & \varnothing & \varnothing\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ \varnothing & 5 & 2 & \varnothing & \varnothing\end{pmatrix}.$$

Each element $\alpha \in \mathcal{IS}_n$ is uniquely determined by a subset $A \subset \mathbf{N}$ and an injective map $A \to \mathbf{N}$. Abusing notation we will denote the latter map by α. The set A is called the *domain* of α, the set $\alpha(A)$ is called the *image* of α and the number $|A|$ is called the *rank* of α, see [5] for details. For each subset $A \subset \mathbf{N}$ we denote by ε_A the idempotent of $\mathcal{IS}_n$, which corresponds to the natural inclusion $A \hookrightarrow \mathbf{N}$. The element $\varepsilon_\varnothing$ is the zero element of $\mathcal{IS}_n$.

Further, the semigroup $\mathcal{IS}_n$ can also be realized as the semigroup of all $n \times n$ matrices with entries from $\{0, 1\}$ satisfying the condition that each row and each column of the matrix contains at most one non-zero entry (such matrices are called *rook matrices* in e.g. [19]). The operation in the latter semigroup is the usual matrix multiplication. This realization defines on $V = \mathbb{C}^n$ the structure of a $\mathbb{C}[\mathcal{IS}_n]$-module in the natural way. It is easy to see that this module is irreducible. We call it the *natural representation* of $\mathcal{IS}_n$. For each k the semigroup $\mathcal{IS}_n$ acts diagonally on the k-fold tensor product $V^{\otimes k}$. This is the left-hand side of our first Schur–Weyl duality.

Consider the sets $\mathbf{K} = \{1, 2, \ldots, k\}$ and $\mathbf{K}' = \{1', 2', \ldots, k'\}$. We consider $' : \mathbf{K} \to \mathbf{K}'$ as the *natural bijection* between these two sets and, abusing notation, denote its inverse also by $'$ (thus $(2')' = 2$). The semigroup $\mathcal{I}_k^*$ is defined in [4] as the semigroup of all bijections between different quotient sets of $\mathbf{K}$. Hence we can view the elements of $\mathcal{I}_k^*$ as all possible partitions of the set $\tilde{\mathbf{K}} = \{1, 2, \ldots, k, 1', 2', \ldots, k'\}$ into disjoint unions of subsets (these subsets will be called *blocks*), satisfying the condition that each block intersects both, $\mathbf{K}$ and $\mathbf{K}'$, non-trivially. If one drops the latter condition out, one obtains the list of elements of the *composition semigroup* $\mathfrak{C}_k$, see [15]. The multiplication on $\mathcal{I}_k^*$ is much more complicated than that on $\mathcal{IS}_n$, and is in fact obtained by restricting the multiplication from $\mathfrak{C}_k$. For an explicit formal definition of the latter we refer the reader to [6,7,15]. Informally, to multiply two partitions α and β of $\tilde{\mathbf{K}}$ one identifies the elements $1', 2', \ldots, k'$ of α with the corresponding elements $1, 2, \ldots, k$ of β and thus forms a new

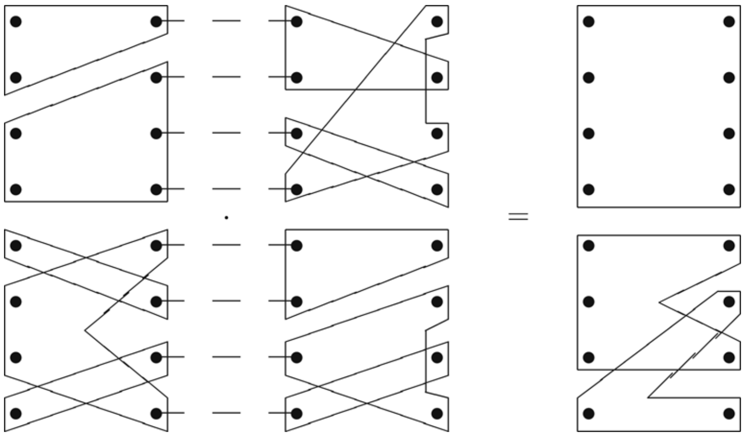

Fig. 1. Elements of $\mathcal{I}_8^*$ and their multiplication.

partition $\alpha\beta$ of $\tilde{\mathbf{K}}$ (in the latter set the elements $1, 2, \dots, k$ are taken from α and the elements $1', 2', \dots, k'$ are taken from β), possibly deleting some "garbage" which does not contain any elements from $\mathbf{K}$ for α and any element from $\mathbf{K}'$ for β. The partition algebra $\mathcal{P}_k(q)$ of [8,14] is a deformation of the semigroup algebra of $\mathfrak{C}_k$, in which the number of garbage components, which appear during the above procedure, is taken into account in terms of a multiplicative parameter q. If both α and β are elements from $\mathcal{I}_k^*$, then, in fact, no garbage appears. In particular, the algebra $\mathbb{C}[\mathcal{I}_k^*]$ is a subalgebra of both $\mathbb{C}[\mathfrak{C}_k]$ and $\mathcal{P}_k(q)$. An example of multiplication of two elements from $\mathcal{I}_8^*$ is given on Fig. 1. Since the elements of $\mathcal{I}_k^*$ are defined as certain partitions of $\tilde{\mathbf{K}}$, the set $\mathcal{I}_k^*$ is partially ordered with respect to inclusions in the natural way. For $\alpha, \beta \in \mathcal{I}_k^*$ we will write $\alpha \preceq \beta$ provided that each block of the partition β is a union of some blocks of the partition α.

Now let us define an action of $\mathcal{I}_k^*$ on $V^{\otimes k}$. Denote by $\mathbf{e} = (e_1, \dots, e_n)$ the standard basis of V. For $\mathbf{i} = (i_1, \dots, i_k) \in \mathbf{N}^k$ set

$$v_{\mathbf{i}} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}.$$

Then the set $\mathbf{B} = \{v_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^k\}$ is a distinguished basis of $V^{\otimes k}$. For all $\alpha \in \mathfrak{C}_n$ (in particular, for all $\alpha \in \mathcal{I}_k^*$) and $\mathbf{i} \in \mathbf{N}^k$ let $M(\alpha, \mathbf{i})$ denote the set of all $\mathbf{l} \in \mathbf{N}^k$ such that for each block $\{a_1, \dots, a_p, b_1', \dots, b_q'\}$ of α (here if $\alpha \notin \mathcal{I}_k^*$ it may happen that $p = 0$ or $q = 0$) we have

$$i_{a_1} = i_{a_2} = \cdots = i_{a_p} = l_{b_1} = l_{b_2} = \cdots = l_{b_q}.$$

Note that $|M(\alpha, \mathbf{i})| \leq 1$ for all $\alpha \in \mathcal{I}_k^*$. Now for $\alpha \in \mathfrak{C}_n$ (in particular, for all $\alpha \in \mathcal{I}_k^*$) we define that α acts on $V^{\otimes k}$ as the unique linear operator such that for all $\mathbf{i} \in \mathbf{N}^k$ we have

$$\alpha(v_{\mathbf{i}}) = \begin{cases} \sum_{\mathbf{l} \in M(\alpha, \mathbf{i})} v_{\mathbf{l}}, & M(\alpha, \mathbf{i}) \neq \varnothing; \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

According to [8,14] this gives an action of $\mathcal{P}_k(n)$ on $V^{\otimes k}$. By restriction, we thus also obtain an action of $\mathcal{I}_k^*$ on $V^{\otimes k}$. Now we are ready to formulate our first result:

Theorem 1. (i) *The actions of $\mathcal{IS}_n$ and $\mathcal{I}_k^*$ on $V^{\otimes k}$ commute.*
(ii) $\mathrm{End}_{\mathcal{IS}_n}(V^{\otimes k})$ *coincides with the image of* $\mathbb{C}[\mathcal{I}_k^*]$.
(iii) $\mathrm{End}_{\mathcal{I}_k^*}(V^{\otimes k})$ *coincides with the image of* $\mathbb{C}[\mathcal{IS}_n]$.
(iv) *The representation of $\mathcal{IS}_n$ on $V^{\otimes k}$ is faithful.*
(v) *The representation of $\mathcal{I}_k^*$ on $V^{\otimes k}$ is faithful if and only if $n \geq 2$ or $k = 1$.*
(vi) *The representation of $\overline{\mathbb{C}[\mathcal{IS}_n]}$ on $V^{\otimes k}$ is faithful if and only if $k \geq n$.*
(vii) *The representation of $\mathbb{C}[\mathcal{I}_k^*]$ on $V^{\otimes k}$ is faithful if and only if $k \leq n$.*

Proof. The left action of the group $S_n \subset \mathcal{IS}_n$ commutes with the right action of $\mathcal{P}_k(n)$ on $V^{\otimes k}$ by [8,14]. Hence the left action of S_n commutes with the right action of $\mathcal{I}_k^*$ as the latter action is obtained from the action of $\mathcal{P}_k(n)$ by restriction. To shorten our notation, set $\varepsilon_n = \varepsilon_{\mathbf{N}\setminus\{n\}}$. The action of ε_n on V is given by

$$\varepsilon_n(e_i) = \begin{cases} e_i, & i \neq n; \\ 0, & i = n. \end{cases}$$

Hence for any $\mathbf{i} \in \mathbf{N}^k$ we have

$$\varepsilon_n(v_{\mathbf{i}}) = \begin{cases} v_{\mathbf{i}}, & n \notin \{i_1, \ldots, i_k\}; \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Now let $\alpha \in \mathcal{I}_k^*$ and $\mathbf{i} \in \mathbf{N}^k$. Assume first that $n \notin \{i_1, \ldots, i_k\}$. As each block of the partition α intersects both $\mathbf{K}$ and $\mathbf{K}'$, from the formula (1) we have that $\alpha(v_{\mathbf{i}}) = v_{\mathbf{j}}$, where $\mathbf{j} \in \mathbf{N}^k$ is such that $n \notin \{j_1, \ldots, j_k\}$. Applying (2) we obtain $\varepsilon_n\alpha(v_{\mathbf{i}}) = \alpha\varepsilon_n(v_{\mathbf{i}})$. Assume now that $n \in \{i_1, \ldots, i_k\}$. Then $\varepsilon_n(v_{\mathbf{i}}) = 0$ by (2). However, as each block of the partition α intersects both $\mathbf{K}$ and $\mathbf{K}'$, from the formula (1) we have that $\alpha(v_{\mathbf{i}}) = v_{\mathbf{j}}$, where $\mathbf{j} \in \mathbf{N}^k$ is such that $n \in \{j_1, \ldots, j_k\}$. Hence $\varepsilon_n(v_{\mathbf{j}}) = 0$ and we again have $\varepsilon_n\alpha(v_{\mathbf{i}}) = 0 = \alpha\varepsilon_n(v_{\mathbf{i}})$. Therefore $\varepsilon_n\alpha = \alpha\varepsilon_n$. By [5, Theorem 3.1.4], the semigroup $\mathcal{IS}_n$ is generated by its subgroup S_n and the element ε_n. The statement (i) follows.

For $j \in \{1, 2, \ldots, 2k\}$ let $\mathfrak{C}_k^j$ denote the set of all elements form $\mathfrak{C}_k$, which are partitions of $\tilde{\mathbf{K}}$ into at most j blocks. From e.g. [1, 2.1] it follows that the vector space $\mathrm{End}_{S_n}(V^{\otimes k})$ is generated by $\alpha \in \mathfrak{C}_k^n$. Let

$$\mathbf{u} = \sum_{\alpha \in \mathfrak{C}_k^n \setminus \mathcal{I}_k^*} a_\alpha \alpha$$

be a linear combination of operators acting on $V^{\otimes k}$ and

$$X = \{\alpha \in \mathfrak{C}_k^n \setminus \mathcal{I}_k^* : a_\alpha \neq 0\}.$$

We would like to show that the condition $\mathbf{u}\varepsilon_n = \varepsilon_n\mathbf{u}$ implies $X = \varnothing$. Assume $X \neq \varnothing$. Let $\alpha \in X$ be a minimal element with respect to $\preceq$, that is a partition, which does not properly contain any other partition from X. As $\alpha \notin \mathcal{I}_k^*$, the element α contains some block, say B, which is contained in either $\mathbf{K}$ or $\mathbf{K}'$. Consider some map $f : \tilde{\mathbf{K}} \to \mathbf{N}$, which satisfies the following conditions:

- f is constant on blocks of α;
- f has different values on elements from different blocks;
- f has value n on elements from the block B.

Such map exists because $\alpha \in \mathfrak{C}_k^n$. Consider now the elements

$$v = e_{f(1)} \otimes e_{f(2)} \otimes \cdots \otimes e_{f(k)} \quad \text{and} \quad w = e_{f(1')} \otimes e_{f(2')} \otimes \cdots \otimes e_{f(k')}. \tag{3}$$

Assume first that $B \subset \mathbf{K}$. Then $\varepsilon_n(v) = 0$ as n occurs among $f(1), \ldots, f(k)$ and hence $\mathbf{u}\varepsilon_n(v) = 0$ as well. On the other hand, the element $\alpha(v)$, when expressed as a linear combination of elements from $\mathbf{B}$, has a non-zero coefficient at w because of (1) and the definition of f. Further, for any $\alpha' \in X$ different from α the coefficient of $\alpha'(v)$ at w (again when $\alpha'(v)$ is expressed as a linear combination of the elements from $\mathbf{B}$) is zero because of (1), the minimality of the partition α with respect to $\preceq$ and the definition of f. Hence the element $\mathbf{u}(v)$, when expressed as a linear combination of the elements from $\mathbf{B}$, has a non-zero coefficient at w. But n does not occur among $f(1'), \ldots, f(k')$, which means $\varepsilon_n(w) = w$. As the action of ε_n is diagonal with respect to the basis $\mathbf{B}$, we get that $\varepsilon_n\mathbf{u}(v) \neq 0$.

In the case $B \subset \mathbf{N}'$ by similar arguments we get $\mathbf{u}\varepsilon_n(v) \neq 0$ while $\varepsilon_n\mathbf{u}(v) = 0$. Hence $\mathbf{u}\varepsilon_n \neq \varepsilon_n\mathbf{u}$, which means that $\mathrm{End}_{\mathcal{IS}_n}(V^{\otimes k})$ is already generated by $\mathfrak{C}_k^n \cap \mathcal{I}_k^*$. The statement (ii) follows.

The statement (iii) is now a standard double-centralizer property. As the semigroup $\mathcal{IS}_n$ is an inverse semigroup (see e.g. [5, Theorem 2.6.7]), the semigroup algebra $\mathbb{C}[\mathcal{IS}_n]$ is semisimple by [16, Theorem 4.4]. Hence the image of $\mathbb{C}[\mathcal{IS}_n]$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ is semisimple as well. The double-centralizer property for semisimple algebras is a trivial case of Tachikawa's theory of dominance dimension, see [22] (the idea of the above argument is taken from [1, Theorem 2.3] and [9, Theorem 2.8]). The statement (iii) follows.

The representation of $\mathcal{IS}_n$ on V is faithful by the definition. Hence for any $\pi, \tau \in \mathcal{IS}_n$, $\pi \neq \tau$, there exists $i \in \mathbf{N}$ such that $\pi e_i \neq \tau e_i$. But then $\pi(e_i \otimes \cdots \otimes e_i) \neq \tau(e_i \otimes \cdots \otimes e_i)$ and hence the actions of π and τ on $V^{\otimes k}$ are different. This proves (iv).

As $|\mathcal{I}_1^*| = 1$, any representation of $\mathcal{I}_1^*$ is faithful. If $n = 1$ and $k > 1$, then $|\mathcal{I}_k^*| > 1$. However, the formula (1) says that all elements of $\mathcal{I}_k^*$ are represented by the identity operator on $V^{\otimes k}$. Hence the representation of $\mathcal{I}_k^*$ on $V^{\otimes k}$ is not faithful in the case $n = 1$ and $k > 1$. Finally, assume that $n > 1$. Let $\alpha, \beta \in \mathcal{I}_k^*$ and assume that the actions of α and β on $V^{\otimes k}$ coincide. Let B be some block of α. Consider the map $f : \tilde{\mathbf{K}} \to \mathbf{N}$ defined as follows:

$$f(x) = \begin{cases} 1, & x \in B; \\ 2, & x \notin B, \end{cases}$$

and consider the corresponding elements v and w given by (3). By (1) and our construction of f, we have $\alpha(v) = w$. As $\alpha(v) = \beta(v)$, from (1) it follows that B should be a union of blocks of β. Hence each block of α is a union of some blocks of β. Analogously, each block of β is a union of some blocks of α. This implies $\alpha = \beta$ and proves (v).

To prove (vi) we first note that the zero element $\varepsilon_\varnothing$ of $\mathcal{IS}_n$ acts as the zero operator on $V^{\otimes k}$. Hence $V^{\otimes k}$ is even an $\overline{\mathbb{C}[\mathcal{IS}_n]}$-module. If $k < n$ then from e.g. [7, Theorem 3.22(a)] it follows that already the restriction of the $\overline{\mathbb{C}[\mathcal{IS}_n]}$-action to $\mathbb{C}[S_n]$ does not give a faithful representation of $\mathbb{C}[S_n]$, as not all simple $\mathbb{C}[S_n]$-modules occur in the decomposition of $V^{\otimes k}$. Hence for $k < n$ the action of $\overline{\mathbb{C}[\mathcal{IS}_n]}$ on $V^{\otimes k}$ is not faithful.

Let now $k \geq n$. Consider some linear combination

$$\mathbf{u} = \sum_{\alpha \in \mathcal{IS}_n \setminus \{\varepsilon_\varnothing\}} a_\alpha \alpha$$

and assume that $\mathbf{u}$ annihilates $V^{\otimes k}$. Then, in particular, $\mathbf{u}v = 0$, where

$$v = e_1 \otimes e_2 \otimes \cdots \otimes e_{n-1} \otimes e_n \otimes e_n \otimes \cdots \otimes e_n.$$

However, the element v is annihilated by all elements $\alpha \in \mathcal{IS}_n$ of rank at most $n - 1$. At the same time the elements of rank n map v to linearly independent elements of $V^{\otimes k}$. This implies that $a_\alpha = 0$ for all α of rank n. Applying now exactly the same arguments to the vector

$$v' = e_1 \otimes e_2 \otimes \cdots \otimes e_{n-2} \otimes e_{n-1} \otimes e_{n-1} \otimes \cdots \otimes e_{n-1}$$

we obtain that $a_\alpha = 0$ for all α with domain $\{1, 2, \ldots, n-1\}$. Analogously one shows that $a_\alpha = 0$ for all α of rank $n - 1$. Proceeding by induction on the rank of α we thus get $a_\alpha = 0$ for all α. This proves the statement (vi).

Finally, let us prove the statement (vii). If $k > n$, then we recall that during the proof of (ii) we saw that the image of $\mathbb{C}[\mathcal{I}_k^*]$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ is generated already by the image of $\mathfrak{C}_k^n \cap \mathcal{I}_k^*$. Hence for $k > n$ the representation of $\mathbb{C}[\mathcal{I}_k^*]$ on $V^{\otimes k}$ is not faithful.

Let $k \leq n$. Consider some linear combination

$$\mathbf{u} = \sum_{\alpha \in \mathcal{I}_k^*} a_\alpha \alpha$$

and assume that $\mathbf{u}$ annihilates $V^{\otimes k}$. Let $\alpha \in \mathcal{I}_k^*$ be minimal with respect to the partial order $\preceq$. Consider some map $f : \tilde{\mathbf{K}} \to \mathbf{N}$, which satisfies the following conditions:

- f is constant on blocks of α;
- f has different values on elements from different blocks.

Such a map exists as α has at most k blocks and $k \leq n$. Consider now the corresponding elements v and w given by (3). From (1) we get that $\alpha(v) = w$ while the coefficient of $\beta(v)$ at w (when expressed with respect to $\mathbf{B}$) is zero for all $\beta \in \mathcal{I}_k^*$, $\beta \neq \alpha$, because of the minimality of α and the choice of f. Since $\mathbf{u}(v) = 0$, we thus must have $a_\alpha = 0$. Proceeding in the same way with respect to the partial order $\preceq$ on $\mathcal{I}_k^*$ we obtain $a_\alpha = 0$ for all $\alpha \in \mathcal{I}_k^*$ and the statement (vii) follows. This completes the proof of the theorem. $\square$

3. A Schur–Weyl duality for $\mathcal{IS}_n$ and $\mathcal{PI}_k^*$

For the second Schur–Weyl duality we consider the trivial $\mathcal{IS}_n$-module $\mathbb{C}$ on which all elements of $\mathcal{IS}_n$ (including the zero element $\varepsilon_\varnothing$) act via the identity transformation. We denote by e_0 some basis element of $\mathbb{C}$. Consider now the $\mathcal{IS}_n$-module $U = V \oplus \mathbb{C}$ and the vector space $U^{\otimes k}$ as an $\mathcal{IS}_n$-module with respect to the diagonal action of $\mathcal{IS}_n$. This is the left-hand side.

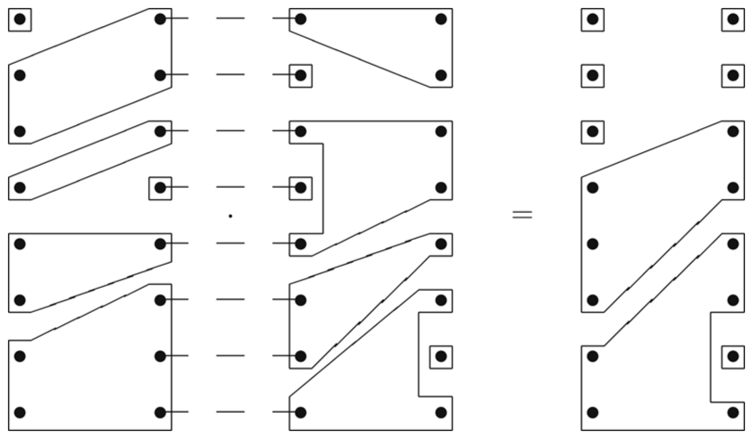

Fig. 2. Elements of $\mathcal{PI}_8^*$ and their multiplication.

To describe the right-hand side we have to define another semigroup, namely the *partial dual inverse symmetric semigroup* $\mathcal{PI}_k^*$. This semigroup was introduced in [10]. The elements of $\mathcal{PI}_k^*$ are all possible partitions α of subsets $A \subset \tilde{\mathbf{K}}$, which satisfy the condition that each block of α has a non-trivial intersection with both $\mathbf{K}$ and $\mathbf{K}'$. We can consider $\mathcal{PI}_k^*$ as a subset of $\mathfrak{C}_k$ extending each $\alpha \in \mathcal{PI}_k^*$ to a partition of $\tilde{\mathbf{K}}$ as follows: if α is a partition of some $A \subset \tilde{\mathbf{K}}$, then we add to this partition all elements from $\tilde{\mathbf{K}} \setminus A$ as separate one-element blocks. In this way $\mathcal{PI}_k^*$ becomes a subset, but not a subsemigroup of $\mathfrak{C}_k$ (an example, illustrating that $\mathcal{PI}_k^*$ is not closed with respect to the multiplication on $\mathfrak{C}_k$, can be found on [10, Figure 2]). To make $\mathcal{PI}_k^*$ into a subsemigroup the multiplication should be changed as follows: Let $\alpha, \beta \in \mathcal{PI}_k^*$. Identify $\mathbf{K}'$-elements of α with the corresponding $\mathbf{K}$-elements of β. Now those blocks, which do not contain any one-element blocks from α or β survive, and all other blocks break down into one-element blocks. We refer the reader to [10, 2.1] for the formal definition. An example of multiplication of two elements from $\mathcal{PI}_8^*$ is given on Fig. 2. The algebra $\mathbb{C}[\mathcal{PI}_k^*]$ is a subalgebra of the rook partition algebra from [6, 2.1] in the natural way. This follows immediately by comparing the definitions.

Now let us define an action of $\mathcal{PI}_k^*$ on $U^{\otimes k}$. This follows closely [19, Section 5] and [6, Section 2]. The set $\mathbf{B}' = \{v_{\mathbf{i}} : \mathbf{i} \in (\mathbf{N} \cup \{0\})^k\}$ is a distinguished basis of $U^{\otimes k}$. For $\alpha \in \mathcal{PI}_k^*$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$ let $M(\alpha, \mathbf{i})$ denote the set of all $\mathbf{l} \in (\mathbf{N} \cup \{0\})^k$ such that the following two conditions are satisfied:

- for each block $\{a_1, \dots, a_p, b_1', \dots, b_q'\}$ of α we have
$$i_{a_1} = i_{a_2} = \dots = i_{a_p} = l_{b_1} = l_{b_2} = \dots = l_{b_q};$$
- for any $a \in \mathbf{K}$ and $b \in \mathbf{K}'$ which do not belong to any block of α we have $i_a = 0 = l_b$.

Again note that $|M(\alpha, \mathbf{i})| \leq 1$ for all $\alpha \in \mathcal{PI}_k^*$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$. Now for $\alpha \in \mathcal{PI}_k^*$ we define the action of α on $V^{\otimes k}$ via the formula (1). This action is in fact the restriction of the action of the rook partition algebra, constructed in [6, 2.1]. In particular, we automatically obtain an action of the semigroup $\mathcal{PI}_k^*$ on $U^{\otimes k}$. Now we are ready to formulate our next result.

Theorem 2. (i) *The actions of $\mathcal{IS}_n$ and $\mathcal{PI}_k^*$ on $U^{\otimes k}$ commute.*
(ii) $\mathrm{End}_{\mathcal{IS}_n}(U^{\otimes k})$ *coincides with the image of* $\mathbb{C}[\mathcal{PI}_k^*]$.
(iii) $\mathrm{End}_{\mathcal{PI}_k^*}(U^{\otimes k})$ *coincides with the image of* $\mathbb{C}[\mathcal{IS}_n]$.
(iv) *The representation of $\mathcal{IS}_n$ on $U^{\otimes k}$ is faithful.*
(v) *The representation of $\mathcal{PI}_k^*$ on $U^{\otimes k}$ is faithful.*
(vi) *The representation of $\mathbb{C}[\mathcal{IS}_n]$ on $U^{\otimes k}$ is faithful if and only if $k \geq n$.*
(vii) *The representation of $\mathbb{C}[\mathcal{PI}_k^*]$ on $U^{\otimes k}$ is faithful if and only if $k \leq n$.*

Proof. By [6, Theorem 18], the right action of $\mathcal{PI}_k^*$ on $U^{\otimes k}$ commutes with the left action of S_n. So, it is enough to check that the right action of $\mathcal{PI}_k^*$ commutes with the left action of ε_n. This is a straightforward calculation using definitions, which is similar to one in the proof of Theorem 1(i). This proves the statement (i).

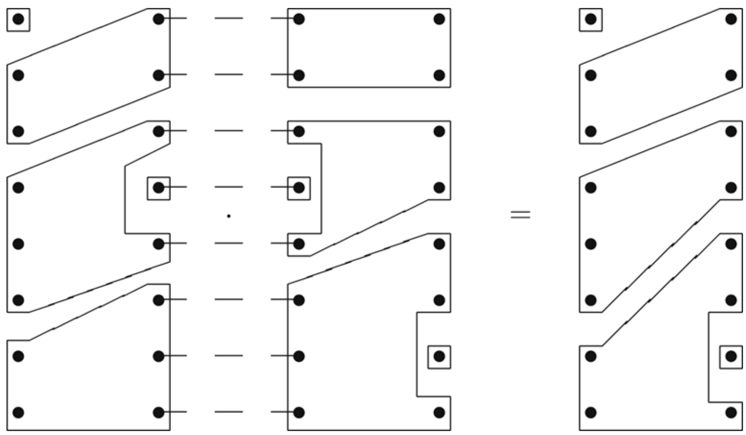

Fig. 3. An example of a non-trivial product in $\widehat{\mathcal{PI}}_8^*$.

Analogously, because of [6, Theorem 18] the proof of the statement (ii) is similar to that of Theorem 1(ii). The proof of the statement (iii) is exactly the same as one of Theorem 1(iii). As $V^{\otimes k}$ is a submodule of $U^{\otimes k}$, the statement (iv) follows from Theorem 1(iv). The statement (v) is proved analogously to Theorem 1(v).

As $\mathbb{C}$ is the trivial $\mathcal{IS}_n$-module, by the additivity of the tensor product the module $U^{\otimes k}$ decomposes into a direct sum of $\mathcal{IS}_n$-modules, each of which is isomorphic to some $V^{\otimes r}$, where $r \leq k$. Hence the fact that the representation of $\mathbb{C}[\mathcal{IS}_n]$ on $U^{\otimes k}$ is not faithful for $k < n$ follows from Theorem 1(vi). If $k \geq n$, then $V^{\otimes k}$ is a direct summand of $U^{\otimes k}$ and hence the representation of $\overline{\mathbb{C}[\mathcal{IS}_n]}$ on $U^{\otimes k}$ is faithful by Theorem 1(vi). At the same time the action of $\varepsilon_\varnothing$ on $U^{\otimes k}$ is obviously non-zero as the trivial $\mathcal{IS}_n$-module is a direct summand of $U^{\otimes k}$ as well. This implies (vi).

Finally, for $k > n$ the fact that the representation of $\mathbb{C}[\mathcal{PI}_k^*]$ on $U^{\otimes k}$ is not faithful follows from [6, Theorem 9]. For $k \leq n$ the fact that the representation of $\mathbb{C}[\mathcal{PI}_k^*]$ on $U^{\otimes k}$ is faithful is proved analogously to the corresponding part of Theorem 1(vii). $\square$

4. Deformations of the second Schur–Weyl duality

There exist at least two different ways to deform the multiplication on the semigroup $\mathcal{PI}_k^*$. As we will now work with different multiplications, to distinguish them we denote by $\cdot$ the usual multiplication in $\mathcal{PI}_k^*$. The first "naive" deformation can be constructed for any inverse semigroups (see e.g. [21, 4.1]). For the semigroup $\mathcal{PI}_k^*$ this works as follows: Set $\widehat{\mathcal{PI}}_k^* = \mathcal{PI}_k^* \cup \{\mathbf{0}\}$. For $\alpha, \beta \in \mathcal{PI}_k^*$ consider the following condition:

$$\text{If } A \text{ is a block of } \alpha \text{ and } B \text{ is a block of } \beta \text{ such that} (A \cap \mathbf{K}') \cap (B \cap \mathbf{K})' \neq \varnothing, \text{ then } A \cap \mathbf{K}' = (B \cap \mathbf{K})'. \tag{4}$$

Define a new operation $\star$ on $\widehat{\mathcal{PI}}_k^*$ as follows:

$$\alpha \star \beta = \begin{cases} \alpha \cdot \beta, & \alpha, \beta \in \mathcal{PI}_k^* \text{ and (4) is satisfied}, \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

For example, the $\star$-product of the two elements on the left-hand side of Fig. 2 equals $\mathbf{0}$. An example of a non-trivial product in $\widehat{\mathcal{PI}}_k^*$ is shown on Fig. 3.

From the general theory (see e.g. [21, 4.1]) it follows that $\widehat{\mathcal{PI}}_k^*$ is a semigroup. By [21, Lemma 4.1] the map

$$\begin{aligned} \varphi : \mathbb{C}[\mathcal{PI}_k^*] &\longrightarrow \overline{\mathbb{C}[\widehat{\mathcal{PI}}_k^*]}, \\ \alpha &\longmapsto \sum_{\beta \succeq \alpha} \beta \end{aligned}$$

is an algebra isomorphism. The map φ allows one to reformulate Theorem 2 in terms of the right action on $U^{\otimes k}$ of the semigroup $\widehat{\mathcal{PI}}_k^*$. This is fairly straightforward. What we would like to do is to give an explicit combinatorial description of the action of $\widehat{\mathcal{PI}}_k^*$ on $U^{\otimes k}$, which is induced by the action of $\mathcal{PI}_k^*$. For $\alpha \in \widehat{\mathcal{PI}}_k^* \setminus \{\mathbf{0}\}$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$ let $\hat{M}(\alpha, \mathbf{i})$ denote the set of all $\mathbf{l} \in (\mathbf{N} \cup \{0\})^k$ such that

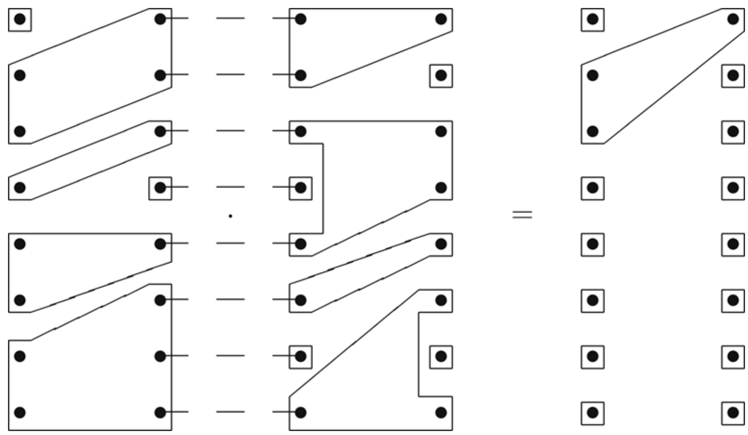

Fig. 4. Elements of $\widetilde{\mathcal{PI}}_8^*$ and their multiplication.

- for each block $\{a_1, \dots, a_p, b'_1, \dots, b'_q\}$ of α we have
  $$i_{a_1} = i_{a_2} = \cdots = i_{a_p} = l_{b_1} = l_{b_2} = \cdots = l_{b_q} \neq 0$$
  (we will say that this common value is the *block number*, corresponding to this block);
- block numbers of different blocks of α are different;
- for any $a \in \mathbf{K}$ and $b \in \mathbf{K}'$ which do not belong to any block of α we have $i_a = 0 = l_b$.

If $\alpha = \mathbf{0}$, we set $\hat{M}(\alpha, \mathbf{i}) = \varnothing$. As before, it is easy to see that $|\hat{M}(\alpha, \mathbf{i})| \leq 1$ for all α and $\mathbf{i}$. Set

$$\alpha \star v_{\mathbf{i}} = \begin{cases} \displaystyle\sum_{\mathbf{l} \in \hat{M}(\alpha,\mathbf{i})} v_{\mathbf{l}}, & \hat{M}(\alpha, \mathbf{i}) \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Proposition 3. (i) *Let $\alpha \in \mathcal{PI}_k^*$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$. Then $\alpha(v_{\mathbf{i}}) = 0$ implies $\beta \star v_{\mathbf{i}} = 0$ for all $\beta \succeq \alpha$.*
(ii) *Let $\alpha \in \mathcal{PI}_k^*$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$. Then $\alpha(v_{\mathbf{i}}) \neq 0$ implies that there exists a unique $\beta \succeq \alpha$ such that $\beta \star v_{\mathbf{i}} \neq 0$.*
(iii) *For any $\alpha \in \widehat{\mathcal{PI}}_k^*$ we have $\alpha \star v_{\mathbf{i}} = \varphi^{-1}(\alpha)(v_{\mathbf{i}})$.*

Proof. By (1), the condition $\alpha(v_{\mathbf{i}}) = 0$ means that there exists some block A of α and $a, b \in A$ such that $i_a \neq i_b$. If $\beta \succeq \alpha$, then there exists a block B of β, which contains A. We still have $i_a \neq i_b$ for $a, b \in B$. Hence $\beta \star v_{\mathbf{i}} = 0$ by (5). This proves (i).

By (1), the condition $\alpha(v_{\mathbf{i}}) \neq 0$ means that $i_a = i_b$ for all a, b from the same block of α. There is a unique way to unite blocks of A into bigger blocks such that the latter property still holds for these bigger blocks, but $i_a \neq i_b$ if a and b are form different bigger blocks. By (5), this defines a unique $\beta \succeq \alpha$ such that $\beta \star v_{\mathbf{i}} \neq 0$.

The statement (iii) follows from (i), (ii) and the definition of φ. $\square$

We remark that an explicit formula for $\varphi^{-1}(\alpha)$ can be obtained using the Möbius inversion formula with respect to the partial order $\preceq$ on the set $\mathcal{PI}_k^*$.

Another deformation of $\mathcal{PI}_k^*$ was proposed in [23] and studied in [10]. The idea is rather similar to the "naive" deformation $\widehat{\mathcal{PI}}_k^*$, however, the deformation of the multiplication in this second case is more subtle than in the "naive" case. Set $\widetilde{\mathcal{PI}}_k^* = \mathcal{PI}_k^*$ (as a set). For $\alpha, \beta \in \widetilde{\mathcal{PI}}_k^*$ define the element $\alpha \bullet \beta \in \widetilde{\mathcal{PI}}_k^*$ in the following way: A block C belongs to $\alpha \bullet \beta$ if and only if there exists a block A of α and a block B of β such that $A \cap \mathbf{K}' = (B \cap \mathbf{K})'$ and $C = (A \cap \mathbf{K}) \cup (B \cap \mathbf{K}')$. An example of multiplication for two elements from $\widetilde{\mathcal{PI}}_8^*$ is shown on Fig. 4.

For $\alpha, \beta \in \widetilde{\mathcal{PI}}_k^*$ we will write $\beta \vdash \alpha$ provided that each block of β is a block of α. By [21, Lemma 4.1] the mapping

$$\psi : \mathbb{C}[\widetilde{\mathcal{PI}}_k^*] \longrightarrow \overline{\mathbb{C}[\widehat{\mathcal{PI}}_k^*]},$$
$$\alpha \longmapsto \sum_{\beta \vdash \alpha} \beta$$

is an algebra isomorphism. The maps ψ and φ allow one to reformulate Theorem 2 in terms of the right action on $U^{\otimes k}$ of the semigroup $\widetilde{\mathcal{PI}}_k^*$. This is again fairly straightforward, so we just give an explicit combinatorial description of the action of $\widetilde{\mathcal{PI}}_k^*$ on $U^{\otimes k}$, which is induced by the action of $\mathcal{PI}_k^*$. For $\alpha \in \widetilde{\mathcal{PI}}_k^*$ and $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$ let $\tilde{M}(\alpha, \mathbf{i})$ denote the set of all $\mathbf{l} \in (\mathbf{N} \cup \{0\})^k$ such that

- for each block $\{a_1, \dots, a_p, b'_1, \dots, b'_q\}$ of α we have
$$i_{a_1} = i_{a_2} = \cdots = i_{a_p} = l_{b_1} = l_{b_2} = \cdots = l_{b_q};$$
- non-zero block numbers of different blocks of α are different;
- for any $a \in \mathbf{K}$ and $b \in \mathbf{K}'$ which do not belong to any block of α we have $i_a = 0 = l_b$.

Again we have $|\tilde{M}(\alpha, \mathbf{i})| \leq 1$ for all α and $\mathbf{i}$. Set

$$\alpha \bullet v_{\mathbf{i}} = \begin{cases} \sum\limits_{\mathbf{l} \in \tilde{M}(\alpha, \mathbf{i})} v_{\mathbf{l}}, & \tilde{M}(\alpha, \mathbf{i}) \neq \varnothing; \\ 0, & \text{otherwise}. \end{cases}$$

Proposition 4. *For any $\alpha \in \widetilde{\mathcal{PI}}_k^*$ and any $\mathbf{i} \in (\mathbf{N} \cup \{0\})^k$ we have $\alpha \bullet v_{\mathbf{i}} = \psi(\alpha) \star v_{\mathbf{i}}$.*

Proof. This follows immediately from the definitions. □

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References

[1] M. Bloss, G-colored partition algebras as centralizer algebras of wreath products, J. Algebra 265 (2) (2003) 690–710.
[2] R. Brauer, On algebras which are connected with the semisimple continuous groups, Ann. of Math. (2) 38 (4) (1937) 857–872.
[3] S. Doty, New versions of Schur–Weyl duality, in: Finite Groups 2003, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 59–71.
[4] D. FitzGerald, J. Leech, Dual symmetric inverse monoids and representation theory, J. Austral. Math. Soc. Ser. A 64 (3) (1998) 345–367.
[5] O. Ganyushkin, V. Mazorchuk, Introduction to classical finite transformation semigroups, in: Algebra and its Application Series, Springer Verlag (in press). Available at http://www.math.uu.se/˜mazor/.
[6] C. Grood, The rook partition algebra, J. Combin. Theory Ser. A 113 (2) (2006) 325–351.
[7] T. Halverson, A. Ram, Partition algebras, European J. Combin. 26 (6) (2005) 869–921.
[8] V. Jones, The Potts model and the symmetric group, in: Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 259–267.
[9] S. König, I. Slungård, C. Xi, Double centralizer properties, dominant dimension, and tilting modules, J. Algebra 240 (1) (2001) 393–412.
[10] G. Kudryavtseva, V. Maltcev, On the structure of two generalizations of the full inverse symmetric semigroup. math.GR/0602623.
[11] G. Kudryavtseva, V. Maltcev, V. Mazorchuk, $\mathcal{L}$- and $\mathcal{R}$-cross-sections in the Brauer semigroup, Semigroup Forum 72 (2) (2006) 223–248.
[12] G. Kudryavtseva, V. Mazorchuk, On presentations of Brauer-type monoids, Cent. Eur. J. Math. 4 (3) (2006) 413–434.
[13] G. Kudryavtseva, V. Mazorchuk, Partialization of categories and inverse braid-permutation monoid. math.GR/0610730.
[14] P. Martin, Representations of graph Temperley–Lieb algebras, Publ. Res. Inst. Math. Sci. 26 (3) (1990) 485–503.
[15] V. Mazorchuk, Endomorphisms of $\mathfrak{B}_n$, $\mathcal{P}\mathfrak{B}_n$, and $\mathfrak{C}_n$, Comm. Algebra 30 (7) (2002) 3489–3513.
[16] W. Munn, Matrix representations of semigroups, Proc. Cambrdige Philos. Soc. 53 (1957) 5–12.
[17] I. Schur, Über eine Klasse von Matrixen die sich einer gegebenen Matrix zuordnen lasses. Thesis, Berlin, 1901. Reprinted in: I. Schur, Gesammelte Abhandlungen I. Springer, Berlin, 1973, pp. 1–70.
[18] I. Schur, Über die rationalen Darstellungen der allgemeinen linearen Gruppe, 1927. Reprinted in: I. Schur, Gesammelte Abhandlungen III, Springer, Berlin, 1973, pp. 68–85.
[19] L. Solomon, Representations of the rook monoid, J. Algebra 256 (2) (2002) 309–342.
[20] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, Geom. Dedicata 36 (1) (1990) 15–49.
[21] B. Steinberg, Möbius Functions and Semigroup Representation Theory II: Character formulas and multiplicities Adv. Math. (in press) math.CO/0607564.
[22] H. Tachikawa, Quasi-Frobenius rings and generalizations. QF − 3 and QF − 1 rings, in: Notes by Claus Michael Ringel, in: Lecture Notes in Mathematics, vol. 351, Springer-Verlag, Berlin, New York, 1973.
[23] A. Vernitskii, A generalization of symmetric inverse semigroups, Semigroup Forum 75 (2) (2007) 417–426.
[24] V. Wagner, Generalized groups, Dokl. Akad. Nauk SSSR 84 (1952) 1119–1122.
[25] H. Weyl, The Classical Groups. Their Invariants and Representations, Princeton University Press, Princeton, NJ, 1939.