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# Some properties of Ising automata

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**Abstract**

In this work, we shall present some arithmetical and topological properties of Ising automata. More precisely, we shall study many different notions, such as faithful and strictly faithful automata, factor and product automata, irreducible and weakly irreducible automata, prime automata, homogeneous automata, minimal automata, invertible automata, etc., and discuss their related properties. We shall also define and study three different topologies over the set of all minimal automata, and discuss the topological closure property of automatic sequences. As application, we shall use the obtained results to give a somewhat detailed analysis of Ising automata.
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## 1. Introduction

The *Ising model* introduced by Ising in the early 1920s concerns the physics of phase transitions, which occur when a small change in a parameter such as temperature or pressure causes a large-scale qualitative change in the state of a system. Phase transitions are common in physics and familiar in our daily life: for instance, it is well known that a phase transition occurs to water whenever the temperature drops below 0°C or rises to 100°C. There are also many other examples such as the formation of binary alloys, and the phenomenon of ferromagnetism. The latter also has a historical interest: originally Ising recurred to his model for a good understanding of ferromagnetism, and especially spontaneous magnetization (see [21]). To know more on this subject, the reader can consult for example the excellent survey [12] and its references.

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In this work, we shall only concentrate our attention to a very special case of the above model: the one-dimensional inhomogeneous Ising chain (see for example [16,15,4]). It occurs that this chain is tightly related to the so-called Ising automata discovered by Mendès France [26] (see also [3,27]), and many properties of it can be studied via the latter (see [4,22]). In other words, any property of the latter reflects that of the first, and it is our main purpose here to exhibit some arithmetical and topological properties of these automata. More precisely, we shall study many different notions, such as faithful and strictly faithful automata, factor and product automata, irreducible and weakly irreducible automata, prime automata, homogeneous automata, minimal automata, invertible automata, etc., and discuss their related properties. We shall also define and study three different topologies over the set of all minimal automata, and discuss the topological closure property of automatic sequences. As application, we shall use the obtained results to give a somewhat detailed analysis of Ising automata. As the careful reader can observe, our study is strongly inspired and influenced by that of Kamae and Mendès France [22].

The paper is organized as follows. After having recalled in Sections 2 and 3 some basic definitions and notations, we give in Section 4 a characterization of strictly faithful automata which is similar to that of faithful automata in [26]. Then we introduce in Section 5 the notions of irreducible automata, weakly irreducible automata, and prime automata, and discuss the problem of factorization of a finite automaton in these "simple" automata. In Section 6, we characterize homogeneous automata by their factor properties, and show that an irreducible (thus weakly irreducible) automaton is not necessarily prime although a prime automaton is always weakly irreducible. In Section 7, we show a necessary condition for automatic sequences to be linearly dependent, and we give in Section 8 a detailed study on invertible automata. We discuss the topological aspect of finite automata in Section 9, and give a sufficient condition for the weak limit of automatic sequences to be still automatic. The main results of this work are presented in Section 10 where we analyze the factor structure of Ising automata, and discuss their continuity. Finally we list some open problems in the last section.

## 2. Words over an alphabet

Let $A$ be a finite nonempty set. We call it an *alphabet* and denote by $\mathrm{Card}(A)$ the number of elements in $A$. Each member of $A$ is called a *letter*, and we fix $\varepsilon$ an element not in $A$, called the *empty word* over $A$.

Let $\mathbb{N}=\{0,1,\ldots\}$ be the set of all natural numbers and let $n\in\mathbb{N}$. If $n=0$, we define $A^0:=\{\varepsilon\}$ and in the contrary case, we denote by $A^n$ the set of all finite sequences with terms in $A$ of length $n$. Finally we set

$$A^* := \bigcup_{n=0}^{+\infty} A^n \quad \text{and} \quad \bar{A} := A^* \cup A^{\mathbb{N}}.$$

An element $w$ in $\bar{A}$ is called a *finite word* if $w\in A^*$ and an *infinite word* if $w\in A^{\mathbb{N}}$, and the length of $w$ is denoted by $|w|$. More precisely, we have $|w|=n$ if $w\in A^n$, and $|w|=+\infty$ if $w\in A^{\mathbb{N}}$. In particular, we have $|\varepsilon|=0$.

Let $w=(w(n))_{0\leqslant n<|w|}\in A^*$ and $v=(v(n))_{0\leqslant n<|v|}\in\bar{A}$ be two words over $A$. The *concatenation* or *product* between $w$ and $v$, denoted by $w*v$ (or more simply by $wv$), is again a word of length $|w|+|v|$ over $A$, defined as follows:

$$(w*v)(n)=\begin{cases} w(n) & \text{if } 0\leqslant n<|w|,\\ v(n-|w|) & \text{if } |w|\leqslant n<|w|+|v|.\end{cases}$$

Therefore, $w\varepsilon=\varepsilon w=w$ for all $w\in A^*$. Clearly $(A^*,*)$ is a monoid with $\varepsilon$ as the identity element, and by induction, we can also define the product of a finite or even an infinite number of words over $A$. Thus every $w=(w(n))_{0\leqslant n<|w|}\in\bar{A}$ can be represented by a finite or an infinite product

$$w=\prod_{n=0}^{|w|-1}w(n):=w(0)w(1)\cdots$$

and every *prefix* of $w$ can be written as $w[0,n]:=w(0)\cdots w(n)$, with $0\leqslant n<|w|$.

Let $w=(w(n))_{0\leqslant n<|w|}$ and $v=(v(n))_{0\leqslant n<|v|}$ be two words over $A$. We define

$$\mathbf{d}_A(w,v)=2^{-\max\{0\leqslant n\leqslant\min(|w|,|v|)\,|\,w(j)=v(j),0\leqslant j<n\}}$$

if $w\neq v$, and $\mathbf{d}_A(w,v)=0$ if $w=v$. Clearly $\mathbf{d}_A$ is a metric over $\bar{A}$. Endowed with this metric, $\bar{A}$ becomes a compact metric space and contains $A^*$ as a dense subset. Finally we remark that $A^{\mathbb{N}}$ is a compact subspace of $\bar{A}$.

## 3. Finite automata and automatic sequences

We begin with the definition of finite automaton (see for example [17,13]).

Let $\Sigma$ be an alphabet which contains at least two elements. A *finite automaton* over $\Sigma$ (called $\Sigma$-*automaton*) is a quadruple $\mathscr{A}=(S,i,\Sigma,t)$ which consists of

- an alphabet $S$ of states; one of the states, say $i$, is distinguished and called the *initial state*.
- a mapping $t:S\times\Sigma\to S$, called the *transition function*.

For all $s\in S$, put $t(s,\varepsilon)=s$. Then extend $t$ over $S\times\Sigma^*$ (denoted still by $t$) such that $t(s,\sigma\eta):=t(t(s,\sigma),\eta)$, for all $s\in S$ and $\sigma,\eta\in\Sigma^*$. The finite $\Sigma$-automaton $\mathscr{A}$ also induces a mapping (denoted again by $\mathscr{A}$) from $\bar{\Sigma}$ to $\bar{S}$ defined by

$$(\mathscr{A}\eta)(m):=t(i,\eta[0,m])=t(i,\eta(0)\cdots\eta(m))$$

for all $\eta\in\bar{\Sigma}$ and $m\in\mathbb{N}$ $(0\leqslant m<|\eta|)$.

It is useful to give a pictorial representation of $\mathscr{A}=(S,i,\Sigma,t)$. States will be represented by points or nodes or vertices. For all $s\in S$ and $\sigma\in\Sigma$, we link $s$ to $t(s,\sigma)$ by a (directed) arrow, labelled $\sigma$. This arrow (also called edge) is said of *type* $\sigma$ and denoted by $(s,\sigma,t(s,\sigma))$ (i.e., treated as an element in $S\times\Sigma\times S$) where $s$ is the starting-point, $\sigma$ is the label or type of the arrow and $t(s,\sigma)$ is the endpoint. In the following, we shall constantly identify $\mathscr{A}$ with its graph (and we use a vertical incident arrow to mark the initial state). Then $S$ becomes the set of vertices and $\Sigma$ becomes the set of labels or types of arrows.

Let $r,s$ be two states of $\mathscr{A}=(S,i,\Sigma,t)$. We say that $s$ is *accessible from* $r$ if there exists $\sigma\in\Sigma^*$ such that $s=t(r,\sigma)$. So $s$ is accessible from itself for $t(s,\varepsilon)=s$. A state of $\mathscr{A}$ is said *accessible* if it is accessible from the initial state $i$, and we call $\mathscr{A}$ an *accessible* (resp. *strictly accessible*) automaton if every state of $\mathscr{A}$ is accessible (resp. for all states $a$ and $b$, $a$ is accessible from $b$ and vice versa). From now on, all finite automata in discussion will be supposed (implicitly) accessible.

Let $o$ be a mapping defined on $S$ with values in some given set. We shall call the couple $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ a *finite $\Sigma$-automaton with output* and $o$ the *output function* of $\mathscr{A}$. Just like the finite $\Sigma$-automaton $\mathscr{A}$, this couple also induces a mapping (denoted still by $(\mathscr{A},o)$) from $\bar{\Sigma}$ to $\overline{o(S)}$ such that

$$\forall\eta\in\bar{\Sigma} \text{ and } \forall m\in\mathbb{N}\ (0\leqslant m<|\eta|) \quad \text{we have } (\mathscr{A},o)(\eta)(m):=o((\mathscr{A}\eta)(m)).$$

These two mappings $\mathscr{A}$ and $(\mathscr{A},o)$ will be the kernel of our present study. Finally we denote by Card$(\mathscr{A})$ the number of states of $\mathscr{A}$ (i.e., Card$(S)$), by AUT$(\Sigma)$ the set of all finite $\Sigma$-automata, and by AUTO$(\Sigma)$ the set of all finite $\Sigma$-automata with output. Remark that every finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ can be looked as a finite $\Sigma$-automaton with output (the output function is the identity mapping $id_S$), so we can always treat AUT$(\Sigma)$ as a subset of AUTO$(\Sigma)$.

Let $p\geqslant 2$ be an integer. Let $\Sigma_p:=\{0,1,\ldots,p-1\}$. A sequence $u=(u(n))_{n\geqslant 0}$ will be called a *$p$-automatic sequence* if there exists a finite $\Sigma_p$-automaton with output $(\mathscr{A},o)=(S,i,\Sigma_p,t,o)$ such that $u(0)=o(i)$, and $u(n)=o(t(i,n_k\cdots n_0))$ for all integer $n\geqslant 1$ with standard $p$-adic expansion $n=\sum_{j=0}^{k}n_j p^j$. In this case, we also say that $u$ is generated by $(\mathscr{A},o)$, and we denote by AUTS$(\Sigma_p)$ the set of all $p$-automatic sequences.

Now we give several examples to illustrate the above definitions and notations.

**Example 1** (one-state automaton). Let $S=\{i\}$. For all $\sigma\in\Sigma$, put $t(i,\sigma)=i$. The finite $\Sigma$-automaton $\mathscr{I}_\Sigma=(S,i,\Sigma,t)$ is strictly accessible and generates the constant sequence $iii\cdots$, if $\Sigma=\Sigma_p$ for some integer $p\geqslant 2$.

**Example 2** (identity automaton). Let $S=\{a,b\}$, $i=a$, $\Sigma=\{0,1\}$, and define the transition function $t$ by $t(a,0)=a$, $t(b,0)=a$, $t(a,1)=b$, and $t(b,1)=b$. The finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is strictly accessible, and if we define $o(a)=0$ and $o(b)=1$, then $(\mathscr{A},o)(\eta)=\eta$, for all $\eta\in\bar{\Sigma}$. Moreover the 2-automatic sequence generated by $\mathscr{A}$ is just the periodic sequence $abab\cdots$ (Fig. 1).

**Example 3** (Thue–Morse automaton). Let $S=\{a,b\}$, $i=a$, $\Sigma=\{0,1\}$, and let $t$ be defined by $t(a,0)=a$, $t(b,0)=b$, $t(a,1)=b$, and $t(b,1)=a$. The finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is strictly accessible and generates the well-known Thue–Morse sequence in $a,b$ (see for example [2]) (Fig. 2).

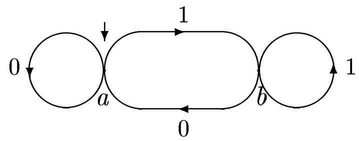



Fig. 1. Identity automaton.

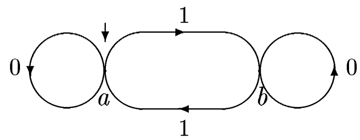



Fig. 2. Thue–Morse automaton.

## 4. Faithfulness and strict faithfulness

Let $u=(u(n))_{n\geqslant 0}$ and $v=(v(n))_{n\geqslant 0}$ be two sequences. They are said ultimately equal (and written as $u\sim v$) if there exists $d\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ $(n\geqslant d)$, we have $u(n)=v(n)$.

Let $\mathscr{A}=(S,i,\Sigma,t)$ be a finite $\Sigma$-automaton. We endow $\Sigma^{\mathbb{N}}$ with the uniform Bernoulli product measure $\mu_\Sigma$ determined by

$$\mu_\Sigma([w]) = |\Sigma|^{-|w|}$$

for all $w\in\Sigma^*$, where $[w]$ denotes the set of all elements in $\Sigma^{\mathbb{N}}$ with $w$ as prefix. Then it is meaningful to speak of "almost all". We say that $\mathscr{A}$ is *faithful* (see [26]) if for almost all $\sigma\in\Sigma^{\mathbb{N}}$, $\sigma\sim\sigma'$ implies $\mathscr{A}\sigma\sim\mathscr{A}\sigma'$. If this is true for all $\sigma\in\Sigma^{\mathbb{N}}$, i.e., if for all $\sigma\in\Sigma^{\mathbb{N}}$, $\sigma\sim\sigma'$ implies $\mathscr{A}\sigma\sim\mathscr{A}\sigma'$, we say that $\mathscr{A}$ is *strictly faithful*.

Obviously a finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is faithful if and only if for all $r,r'\in S$ and almost all $\sigma\in\Sigma^{\mathbb{N}}$, $\sigma\sim\sigma'$ implies $t(r,\sigma[0,n])=t(r',\sigma'[0,n])$, for all large integer $n$. Similarly a finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is strictly faithful if and only if the above property holds for all $r,r'\in S$ and all $\sigma\in\Sigma^{\mathbb{N}}$.

The following result characterizes faithful $\Sigma$-automata (cf. [26]).

**Proposition 1.** *A finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is faithful if and only if there exist $s\in S$ and $\eta\in\Sigma^*$ such that $t(r,\eta)=s$ for all $r\in S$.*

**Proof.** For completeness, we reformulate below the original proof of [26].

Assume that there exist $s\in S$ and $\eta\in\Sigma^*$ such that $t(r,\eta)=s$ for all $r\in S$. A classical result of Borel (see for example [25, p. 71] or [28, p. 29]) asserts that almost all $\sigma\in\Sigma^{\mathbb{N}}$ contains $\eta$ infinitely many times. For such a $\sigma=(\sigma(n))_{n\geqslant 0}\in\Sigma^{\mathbb{N}}$, if $\sigma'=(\sigma'(n))_{n\geqslant 0}\in\Sigma^{\mathbb{N}}$ satisfying $\sigma'\sim\sigma$, then from some rank on, the word $\eta$ appears at the same position in $\sigma$ and in $\sigma'$, and indeed infinitely often.

Let $n$ be a large integer. Write $\sigma[0,n]=\alpha\eta\gamma$, and $\sigma'[0,n]=\alpha'\eta\gamma$, with $\alpha,\alpha'$ and $\gamma$ in $\Sigma^*$. Then we have

$$(\mathscr{A}\sigma)(n)=t(t(t(i,\alpha),\eta),\gamma)=t(s,\gamma)=t(t(t(r',\alpha'),\eta),\gamma)=(\mathscr{A}\sigma')(n).$$

Conversely, suppose that for at least one $\sigma=(\sigma(n))_{n\geqslant 0}\in\Sigma^{\mathbb{N}}$, and all $r,r'\in S$, the relation $\sigma'\sim\sigma$ ($\sigma'\in\Sigma^{\mathbb{N}}$) implies $t(r,\sigma[0,n])=t(r',\sigma'[0,n])$, for all sufficiently large integer $n$. Write $\delta_r=(\delta_r(n))_{n\geqslant 0}:=(t(r,\sigma[0,n]))_{n\geqslant 0}$ ($r\in S$). Then $\delta_r\sim\delta_i$, for all $r\in S$. Hence for each $r\in S$, we can find $d(r)\in\mathbb{N}$ such that $\delta_r(n)=\delta_i(n)$ for all integer $n\geqslant d(r)$. Set

$$d=\max_{r\in S} d(r).$$

It is clear that $\delta_r(d)=\delta_i(d)$ for all $r\in S$. So if we define $\eta:=\sigma[0,d]$ and $s:=\delta_i(d)$, then we have $t(r,\eta)=\delta_r(d)=s$, for all $r\in S$. □

For strictly faithful $\Sigma$-automata, we have a similar result below (see also [31]).

**Proposition 2.** *A finite $\Sigma$-automaton $\mathscr{A}=(S,i,\Sigma,t)$ is strictly faithful if and only if there exists an integer $k\geqslant 1$ such that for all $\eta\in\Sigma^k$ and all $r\in S$, the state $t(r,\eta)$ only depends on $\eta$ but not on $r$.*

**Proof.** Assume that there exists an integer $k\geqslant 1$ satisfying the condition of our proposition. We should show that if $\sigma=(\sigma(n))_{n\geqslant 0}$ and $\sigma'=(\sigma'(n))_{n\geqslant 0}$ are two elements in $\Sigma^{\mathbb{N}}$ such that $\sigma\sim\sigma'$, then $\mathscr{A}\sigma\sim\mathscr{A}\sigma'$.

Indeed if $\sigma\sim\sigma'$, then for all large integer $n$, we can write

$$\sigma[0,n]=\alpha\eta \quad\text{and}\quad \sigma'[0,n]=\alpha'\eta,$$

with $\alpha,\alpha'\in\Sigma^*$ and $\eta\in\Sigma^k$. Put $r=t(i,\alpha)$ and $r'=t(i,\alpha')$. Then we have

$$(\mathscr{A}\sigma)(n)=t(i,\alpha\eta)=t(r,\eta)=t(r',\eta)=t(i,\alpha'\eta)=(\mathscr{A}\sigma')(n).$$

Now we show the converse by absurdity. Suppose that for all integer $n\geqslant 1$, there exists $\eta_n=(\eta_n(j))_{0\leqslant j<n}\in\Sigma^n$ such that for all $s\in S$, we can find $r_s^{(n)}\in S$ satisfying $t(r_s^{(n)},\eta_n)\neq s$. Since $(\bar{\Sigma},d_\Sigma)$ is compact, then $(\eta_n)_{n\geqslant 1}$ possesses a limit point $\sigma$. Remark that $\mathscr{A}$ is strictly faithful. So for this $\sigma$ and for all $r\in S$, we can find $d(r)\in\mathbb{N}$ such that $t(i,\sigma[0,n])=t(r,\sigma[0,n])$, for all $n\geqslant d(r)$. Put

$$d=\max_{r\in S} d(r).$$

Then $t(i,\sigma[0,d])=t(r,\sigma[0,d])$, for all $r\in S$. So if $n$ is a sufficiently large integer such that $n>d$ and $\eta_n[0,d]=\sigma[0,d]$, then for all $r\in S$, we have

$$\begin{aligned}t(r,\eta_n)&=t(t(r,\sigma[0,d]),\eta_n[d+1,n-1])\\&=t(t(i,\sigma[0,d]),\eta_n[d+1,n-1])\\&=t(i,\eta_n).\end{aligned}$$

Set $s=t(i,\eta_n)$. Then $s=t(r,\eta_n)$ for all $r\in S$, so for $r_s^{(n)}$. This is absurd. □

We remark that the one-state automaton and the identity automaton are both strictly faithful, but the Thue–Morse automaton is not faithful.

Let $\mathscr{A} = (S, i, \Sigma, t)$ be a finite $\Sigma$-automaton. An element $\sigma \in \Sigma^*$ is called a *synchronizing word* of $\mathscr{A}$ if $t(s, \sigma)$ is independent of $s \in S$. Obviously, $\mathscr{A}$ is faithful if and only if it has a synchronizing word, and it is strictly faithful if and only if there exists $k \in \mathbb{N}$ such that all the elements in $\Sigma^k$ are synchronizing words of $\mathscr{A}$. It is not difficult to see that the existence of a synchronizing word can be decided by computing the finite automaton in question. The existence of an integer $k$ such that all words of length $k$ are synchronizing is also easily decidable.

Synchronizing words are also important for symbolic dynamical systems. For example, it was shown in [14, p. 233] (where the author discussed two-sided instead of one-sided sequences) that the symbolic dynamical system associated to a $p$-automatic sequence, generated by a primitive $\Sigma_p$-automaton $\mathscr{A} = (S, i, \Sigma_p, t)$ (i.e., there exists an integer $k \geqslant 1$ such that $[s]_k := \{t(s, \sigma) \mid \sigma \in \Sigma_p^k\} = S$ for all $s \in S$) satisfying $t(i, 0) = i$, has discrete spectrum if and only if $\mathscr{A}$ has a synchronizing word, i.e., it is faithful. For more discussions on this subject, see [1,30].

Proposition 2 tells us that the structure of a strictly faithful $\Sigma$-automaton is very simple. This point can be specified further.

**Corollary 1.** *Let $p \geqslant 2$ be an integer. If $\mathscr{A} = (S, i, \Sigma_p, t)$ is a strictly faithful $\Sigma_p$-automaton, then the $p$-automatic sequence generated by $\mathscr{A}$ is ultimately periodic, and it becomes (purely) periodic if in addition holds $t(i, 0) = i$.*

**Proof.** Let $\mathscr{A} = (S, i, \Sigma_p, t)$ be a strictly faithful $\Sigma_p$-automaton. Then there exists an integer $k \geqslant 1$ such that for all $\eta \in \Sigma_p^k$, the state $t(r, \eta)$ is independent of $r \in S$. Let $u = (u(n))_{n \geqslant 0}$ be the $p$-automatic sequence generated by $\mathscr{A}$. For all $n \in \mathbb{N}$ with standard $p$-adic expansion $n = \sum_{j=0}^{h-1} n_j p^j$, define

$$w(n) = \begin{cases} \overbrace{0 \cdots 0}^{k-h} n_{h-1} n_{h-2} \cdots n_0 & \text{if } h < k, \\ n_{k-1} n_{k-2} \cdots n_0 & \text{otherwise.} \end{cases}$$

Obviously $w(n + p^k) = w(n)$, and thus $u(n) = t(i, w(n))$ if $n \geqslant p^{k-1}$ or $t(i, 0) = i$. As a result, we obtain $u(n + p^k) = u(n)$, if $n \geqslant p^{k-1}$ or $t(i, 0) = i$. $\square$

In the next section, we shall recall some definitions and results which will be implicitly used later. For more detailed discussions, see for example [5,13,17–20,22,11], and their references.

## 5. Factor and product automata

Let $\mathscr{A} = (S, i, \Sigma, t)$ and $\mathscr{A}' = (S', i', \Sigma, t')$ be two finite $\Sigma$-automata. We call $\mathscr{A}'$ a *factor* of $\mathscr{A}$ (see e.g. [22]) if there exists a surjective mapping $\lambda$ defined on $S$ with values in $S'$ such that $i' = \lambda(i)$, and

$$t'(\lambda(s), \sigma) = \lambda(t(s, \sigma))$$

for all $s \in S$ and all $\sigma \in \Sigma$. In this case, we call $\lambda$ a *$\Sigma$-automaton homomorphism of $\mathscr{A}$*, and write $\mathscr{A}' = \lambda(\mathscr{A})$.

A finite $\Sigma$-automaton $\mathscr{A}$ has at least two factors: the one-state $\Sigma$-automaton $\mathscr{I}_\Sigma$ and $\mathscr{A}$ itself. These factors will be referred to as the *trivial factors* of $\mathscr{A}$, and the set of all factors of $\mathscr{A}$ will be denoted by FAC($\mathscr{A}$). One shall see later that FAC($\mathscr{A}$) is closed under the multiplication of finite automata defined below.

Let $\lambda$ be a $\Sigma$-automaton homomorphism of $\mathscr{A}$. If $\lambda$ is also injective, then its inverse mapping $\lambda^{-1}$ is a $\Sigma$-automaton homomorphism of $\lambda(\mathscr{A})$. In this case, we call $\lambda$ a *$\Sigma$-automaton isomorphism of $\mathscr{A}$*, and say that $\mathscr{A}$ and $\mathscr{A}'$ are *isomorphic*, noted $\mathscr{A} \simeq \mathscr{A}'$. Intuitively two finite $\Sigma$-automata are isomorphic if and only if, up to the notations of states, they have the same graph. Clearly this isomorphism defines over AUT($\Sigma$) an equivalence relation. From now on, we shall always identify isomorphic $\Sigma$-automata and use, if no confusion possible, the same symbols $\mathscr{A}$, $\mathscr{B}$, and so on, for finite $\Sigma$-automata and for classes of isomorphic $\Sigma$-automata. In particular, up to isomorphism, there exists only one one-state $\Sigma$-automaton $\mathscr{I}_\Sigma$.

The following result is well known.

**Proposition 3.** *Let $\mathscr{A}$ and $\mathscr{A}'$ be two finite $\Sigma$-automata. If $\mathscr{A}'$ is a factor of $\mathscr{A}$, and $\mathscr{A}$ is a factor of $\mathscr{A}'$, then $\mathscr{A} \simeq \mathscr{A}'$.*

Let $\mathscr{A} = (S, i, \Sigma, t)$ be a finite $\Sigma$-automaton, and let $\pi$ be a partition of $S$. We shall use the same symbol $\pi$ to represent the equivalence relation over $S$ defined by $\pi$. For all $s \in S$, we denote by $\pi(s)$ the class of $s$, and write $r \equiv s \pmod{\pi}$ if $r \in \pi(s)$ (or equivalently $s \in \pi(r)$). We call $\pi$ an *automaton partition* of $\mathscr{A}$ if for all $r, s \in S$, the relation $r \equiv s \pmod{\pi}$ implies $t(r,\sigma) \equiv t(s,\sigma) \pmod{\pi}$ for all $\sigma \in \Sigma$. For example, the trivial partitions $(S)$ and $(\{s\})_{s\in S}$ are automaton partitions of $\mathscr{A}$. Finally we remark that the automaton partitions are just those partitions which have substitution property (cf. [19, p. 22]), and in the literature, they are also called *right regular partitions* (see for example [5, p. 18]).

Let $\pi$ be an automaton partition of $\mathscr{A}$. From $\pi$, we can deduce a new finite $\Sigma$-automaton $\mathscr{A}/\pi := (\pi, \pi(i), \Sigma, t')$ (called *quotient $\Sigma$-automaton of $\mathscr{A}$*), where $t'$ is defined by $t'(\pi(s), \sigma) = \pi(t(s, \sigma))$, for all $s \in S$ and for all $\sigma \in \Sigma$. One can remark that $\mathscr{A}/\pi$ is a factor of $\mathscr{A}$, where the corresponding $\Sigma$-automaton homomorphism is just $s \mapsto \pi(s) (s \in S)$.

Conversely if $\mathscr{A}' = (S', i', \Sigma, t')$ is a factor of $\mathscr{A}$, then there is a $\Sigma$-automaton homomorphism $\lambda$ of $\mathscr{A}$, defined from $S$ onto $S'$, such that $\mathscr{A}' = \lambda(\mathscr{A})$. Put

$$\pi(\lambda) := \{\lambda^{-1}(\lambda(s)) \mid s \in S\}.$$

Then it is clear that $\pi(\lambda)$ is an automaton partition of $\mathscr{A}$ and $\mathscr{A}/\pi(\lambda) \simeq \mathscr{A}'$. So the factors and the quotient $\Sigma$-automata of $\mathscr{A}$ coincide.

In other words, we have just established the following result (cf. [19]).

**Proposition 4.** *Let $\mathscr{A}=(S,i,\Sigma,t)$ be a finite $\Sigma$-automaton. There is a bijection between the factors and the automaton partitions of $\mathscr{A}$. In particular, the trivial factor $\mathscr{I}_\Sigma$ (resp. $\mathscr{A}$) corresponds to the trivial partition $(S)$ (resp. $(\{s\})_{s\in S}$).*

We remark that if $\pi$ is an automaton partition of $\mathscr{A}$, and if $\Pi$ is an automaton partition of $\mathscr{A}/\pi$, then $(\mathscr{A}/\pi)/\Pi \simeq \mathscr{A}/\pi'$, where $\pi'=\{\bigcup_{\boldsymbol{\sigma}\in\mathbf{E}}\boldsymbol{\sigma} \mid \mathbf{E}\in\Pi\}$.

From Proposition 4, we deduce immediately

**Corollary 2.** *Let $\lambda$ be a mapping defined on $S$. It is a $\Sigma$-automaton homomorphism of $\mathscr{A}$ if and only if $\pi(\lambda):=\{\lambda^{-1}(\lambda(s)) \mid s\in S\}$ is an automaton partition of $\mathscr{A}$.*

**Corollary 3.** *Every finite $\Sigma$-automaton only has a finite number of factors.*

Of course, in the last result, we should identify all isomorphic factors, just as we have already conventionalized in this section.

Now we define and study the products of finite $\Sigma$-automata. For more general definitions and discussions, the reader can consult for example [17–19,22], and their references.

Let $\mathscr{A}=(S,i,\Sigma,t)$ and $\mathscr{A}'=(S',i',\Sigma,t')$ be two finite $\Sigma$-automata. The *product $\Sigma$-automaton* $\mathscr{A}\times\mathscr{A}'$ (also written as $\mathscr{A}\mathscr{A}'$) is defined as follows (cf. [22]):
- the set of states is $S\otimes S':=\{(t(i,\sigma),\ t'(i',\sigma)) \mid \sigma\in\Sigma^*\}$, with $(i,i')$ as the initial state,
- the transition function is $t\otimes t'$, where $t\otimes t'(s,\sigma):=(t(s_1,\sigma),\ t'(s_2,\sigma))$, for all $s=(s_1,s_2)\in S\otimes S'$ and all $\sigma\in\Sigma$.

The binary operation $\times$ is associative, commutative and idempotent, i.e., for all finite $\Sigma$-automata $\mathscr{A}$, $\mathscr{A}'$, and $\mathscr{A}''$, we have

$$(\mathscr{A}\times\mathscr{A}')\times\mathscr{A}''\simeq\mathscr{A}\times(\mathscr{A}'\times\mathscr{A}''),\quad \mathscr{A}\times\mathscr{A}'\simeq\mathscr{A}'\times\mathscr{A},\ \text{and}\ \mathscr{A}\times\mathscr{A}\simeq\mathscr{A}.$$

Endowed with $\times$, the set $\mathrm{AUT}(\Sigma)$ becomes a commutative monoid with $\mathscr{I}_\Sigma$ as the identity element. Moreover $\mathscr{I}_\Sigma$ is also the unique invertible element in $\mathrm{AUT}(\Sigma)$.

The following result justify the definition of factor $\Sigma$-automaton.

**Proposition 5.** *Let $\mathscr{A}$ and $\mathscr{A}'$ be two finite $\Sigma$-automata. Then $\mathscr{A}$ is a factor of $\mathscr{A}\times\mathscr{A}'$, and if $\mathscr{A}'$ is a factor of $\mathscr{A}$, then $\mathscr{A}\simeq\mathscr{A}\times\mathscr{A}'$.*

Let $\mathscr{A}=(S,i,\Sigma,t)$ be a finite $\Sigma$-automaton. If $\pi_1$ and $\pi_2$ are two automaton partitions of $\mathscr{A}$, then $\pi_1\cap\pi_2=\{\mathbf{s}\cap\mathbf{s}' \mid \mathbf{s}\in\pi_1,\ \mathbf{s}'\in\pi_2, \text{ and } \mathbf{s}\cap\mathbf{s}'\neq\emptyset\}$ is a new automaton partition of $\mathscr{A}$, and $\pi_1\cap\pi_2=\{\pi_1(s)\cap\pi_2(s) \mid s\in S\}$. Moreover

$$\mathscr{A}/\pi_1\times\mathscr{A}/\pi_2\simeq\mathscr{A}/\pi_1\cap\pi_2. \tag{1}$$

So the bijection between $\mathrm{FAC}(\mathscr{A})$ and the set of all automaton partitions of $\mathscr{A}$ keeps the product, i.e., the product of two factors of $\mathscr{A}$ corresponds to the intersection of the corresponding automaton partitions of $\mathscr{A}$.

Let $\mathscr{A}$ and $\mathscr{A}'$ be two finite $\Sigma$-automata. We say that *$\mathscr{A}$ is divisible by $\mathscr{A}'$* or *$\mathscr{A}$ is a multiple of $\mathscr{A}'$* (denoted by $\mathscr{A}' \mid \mathscr{A}$), if there exists a finite $\Sigma$-automaton $\mathscr{A}''$

such that $\mathscr{A} \simeq \mathscr{A}' \times \mathscr{A}''$. It is clear that $\mathscr{A}$ is divisible by $\mathscr{A}'$ if and only if $\mathscr{A}'$ is a factor of $\mathscr{A}$. It is also clear that the divisibility defines a partial order over $\mathrm{AUT}(\Sigma)$, and endowed with this partial order, $\mathrm{AUT}(\Sigma)$ becomes a lattice. Indeed, for any two finite $\Sigma$-automata $\mathscr{A}'$ and $\mathscr{A}''$, $\mathscr{A}' \times \mathscr{A}''$ is the least common multiple of $\mathscr{A}'$ and $\mathscr{A}''$ (i.e., $\mathscr{A}' \times \mathscr{A}''$ divides every common multiple of $\mathscr{A}'$ and $\mathscr{A}''$), and the product of all common factors of $\mathscr{A}'$ and $\mathscr{A}''$ is the greatest common factor of $\mathscr{A}'$ and $\mathscr{A}''$ (i.e., every common factor of $\mathscr{A}'$ and $\mathscr{A}''$ divides this product).

A finite $\Sigma$-automaton $\mathscr{A}$ is said *irreducible* if it only has trivial factors. By Proposition 4, we know that a finite $\Sigma$-automaton is irreducible if and only if it only has two automaton partitions (i.e., the two trivial partitions). Consequently every two-states $\Sigma$-automaton is irreducible. In general, a finite $\Sigma$-automaton cannot be decomposed into the product of irreducible $\Sigma$-automata (see Section 10). This encourages us to introduce a new notion, namely weakly irreducible automaton. A finite $\Sigma$-automaton $\mathscr{A}$ is said *weakly irreducible* if $\mathscr{A} \simeq \mathscr{A}' \times \mathscr{A}''$ implies that $\mathscr{A}'$ or $\mathscr{A}''$ is a trivial factor of $\mathscr{A}$. Obviously, every irreducible $\Sigma$-automaton is weakly irreducible, however the converse is not always true (see Section 10). Finally we remark that by induction on the number of states, we can easily show that every finite $\Sigma$-automaton can be factorized into the product of weakly irreducible $\Sigma$-automata.

Now consider prime $\Sigma$-automata, i.e., those finite $\Sigma$-automata $\mathscr{A}$ such that the divisibility $\mathscr{A} \mid \mathscr{A}' \times \mathscr{A}''$ implies $\mathscr{A} \mid \mathscr{A}'$ or $\mathscr{A} \mid \mathscr{A}''$. Although we do not know yet whether there exist or not prime $\Sigma$-automata, we can still prove the following result.

**Proposition 6.** *Every prime $\Sigma$-automaton is weakly irreducible.*

**Proof.** By absurdity, we suppose that there would exist a prime but not weakly irreducible $\Sigma$-automaton $\mathscr{A}$. Then we could find proper factors $\mathscr{A}'$, $\mathscr{A}''$ of $\mathscr{A}$ such that $\mathscr{A} \simeq \mathscr{A}' \times \mathscr{A}''$. Thus $\mathscr{A} \mid \mathscr{A}'$ or $\mathscr{A} \mid \mathscr{A}''$ for $\mathscr{A}$ is prime. Now by Proposition 3, we have $\mathscr{A} \simeq \mathscr{A}'$ or $\mathscr{A} \simeq \mathscr{A}''$, from which we obtain the desired contradiction. □

The converse of Proposition 6 is false, and this explains why the factorization in weakly irreducible $\Sigma$-automata is not always unique (see Example 4).

## 6. Homogeneous automata

Let $\mathscr{A} = (S, i, \Sigma, t)$ be a finite $\Sigma$-automaton, and let $s$ be a state of $\mathscr{A}$. We call $s$ a *homogeneous state* of type $\sigma$ ($\sigma \in \Sigma$) of $\mathscr{A}$ if over the graph of $\mathscr{A}$, all the incident arrows into $s$ are of type $\sigma$. In other words, if there exist $r \in S$ and $\rho \in \Sigma$ such that $t(r, \rho) = s$, then $\rho = \sigma$. Finally we call $\mathscr{A}$ a *homogeneous $\Sigma$-automaton* if all states of $\mathscr{A}$ are homogeneous.

We remark that homogeneous automata defined above are called *pure automata* in [22], which are objects quite different from those studied in [7], although they have the same name. We remark also that this notion can be generalized to more general automata (not necessarily deterministic), and that Glushkov automata are homogeneous (see for example [10]).

It is clear that the identity automaton defined in Example 2 is homogeneous. More generally, for each $\rho \in \Sigma$, if we define $\mathscr{P}_\rho = (\Sigma, \rho, \Sigma, t_\rho)$, where $t_\rho(s, \sigma) = \sigma$, for all $s \in \Sigma$ and all $\sigma \in \Sigma$, then $\mathscr{P}_\rho$ is homogeneous. Remark that for every homogeneous $\Sigma$-automaton $\mathscr{A}$, we have necessarily $\mathrm{Card}(\mathscr{A}) \geqslant \mathrm{Card}(\Sigma)$. So in a certain sense, $\mathscr{P}_\rho$ ($\rho \in \Sigma$) is a "minimal" homogeneous $\Sigma$-automaton. The following result specifies this point of view, generalizes and improves a result in [22].

**Proposition 7.** *A finite $\Sigma$-automaton $\mathscr{A} = (S, i, \Sigma, t)$ is homogeneous if and only if there exists $\rho \in \Sigma$ such that $\mathscr{P}_\rho$ divides $\mathscr{A}$.*

**Proof.** Let $\mathscr{A} = (S, i, \Sigma, t)$ be a homogeneous $\Sigma$-automaton. For all $s \in S$, we denote by $\lambda(s)$ the type of $s$ (in the case that there is not any arrow incident into the initial state $i$, we fix in advance an element in $\Sigma$, and then take it as $\lambda(i)$). Clearly $\lambda$ is a $\Sigma$-automaton homomorphism of $\mathscr{A}$, and we have $\lambda(\mathscr{A}) = \mathscr{P}_{\lambda(i)}$.

Reciprocally, assume that there exists $\rho \in \Sigma$ such that $\mathscr{P}_\rho$ is a factor of $\mathscr{A}$. Then we can find a $\Sigma$-automaton homomorphism $\lambda$ of $\mathscr{A}$ such that $\mathscr{P}_\rho = \lambda(\mathscr{A})$. Let $s$ be a state of $\mathscr{A}$. If over $\mathscr{A}$, one of the incident arrows into $s$ is of type $\sigma$, then over $\mathscr{P}_\rho$, there exists also an arrow of type $\sigma$ incident into $\lambda(s)$. But $\mathscr{P}_\rho$ is homogeneous, so over $\mathscr{P}_\rho$, all the incident arrows into $\lambda(s)$ are of type $\sigma$. Thus over $\mathscr{A}$, all the incident arrows into $s$ are of the same type. Consequently $\mathscr{A}$ is homogeneous. $\square$

We remark that homogeneity is a "product property", i.e., the product with a homogeneous $\Sigma$-automaton gives again a homogeneous $\Sigma$-automaton.

Homogeneous automata are also important in opacity theory of finite automata. In fact, it has been shown in [33] (see also [31]) that a strictly accessible automaton is transparent if and only if it is homogeneous.

Let $\rho \in \Sigma$, and let $\pi$ be a partition of $\Sigma$. Since the transition function $t_\rho$ of $\mathscr{P}_\rho$ is independent of its first variable, so $\pi$ is an automaton partition of $\mathscr{P}_\rho$. As a result, we obtain that $\mathrm{FAC}(\mathscr{P}_\rho)$ is in bijection with the set of all partitions of $\Sigma$.

For all $\sigma \in \Sigma$, define $\pi_\sigma = \{\{\sigma\}, \Sigma \backslash \{\sigma\}\}$. Then the $\Sigma$-automaton $\mathscr{P}_\rho/\pi_\sigma$ is irreducible for it consists of two states. Moreover by formula (1), we have also

$$\mathscr{P}_\rho \simeq \prod_{\sigma \in \Sigma} \mathscr{P}_\rho/\pi_\sigma,$$

i.e., $\mathscr{P}_\rho$ can be decomposed into irreducible $\Sigma$-automata.

Now we can give an example to show that an irreducible (thus weakly irreducible) $\Sigma$-automaton may not be prime.

**Example 4.** Let $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ and $\pi = \{\{\alpha, \beta\}, \{\gamma, \delta\}\}$. Then $\mathscr{P}_\alpha/\pi$ is an irreducible factor of $\mathscr{P}_\alpha$, and different from all $\mathscr{P}_\alpha/\pi_\sigma$ ($\sigma \in \Sigma$), so it is not prime.

In other words, the product of two finite automata may contain a factor which is not the product of any factors of the original two automata!

Let $\mathscr{A} = (S, i, \Sigma, t)$ be a finite $\Sigma$-automaton, and let $\pi$ be a partition of $\Sigma$. A state $s \in S$ is said $\pi$-*homogeneous* over $\mathscr{A}$ if over $\mathscr{A}$, all the incident arrows into $s$ take their types in a same class of $\pi$, i.e. there exists $\mathbf{s}\in\pi$ such that if $\sigma\in\Sigma$ and $r\in S$ satisfying $t(r,\sigma)=s$, then $\sigma\in\mathbf{s}$. Finally we call $\mathscr{A}$ a *$\pi$-homogeneous $\Sigma$-automaton* if all states of $\mathscr{A}$ are $\pi$-homogeneous. Clearly homogeneity is a special $\pi$-homogeneity with $\pi=\{\{\sigma\}\mid\sigma\in\Sigma\}$. We remark also that $\pi$-homogeneity appeared already in [22] in the case that $\pi$ only contains two elements.

Using the same argument as above, we can easily show the following result.

**Proposition 8.** *Let $\mathscr{A}$ be a finite $\Sigma$-automaton, and $\pi$ a partition of $\Sigma$. Then $\mathscr{A}$ is $\pi$-homogeneous if and only if there exists $\rho\in\Sigma$ such that $\mathscr{P}_\rho/\pi$ divides $\mathscr{A}$.*

## 7. Minimal automata

Let $(\mathscr{A},o)$ and $(\mathscr{A}',o')$ be two finite $\Sigma$-automata with output. If for all $\eta\in\bar{\Sigma}$, we have $(\mathscr{A},o)(\eta)=(\mathscr{A}',o')(\eta)$, then we call them *equivalent* and write $(\mathscr{A},o)\approx(\mathscr{A}',o')$. If in addition $\mathscr{A}\simeq\mathscr{A}'$, then we call them *isomorphic* and write $(\mathscr{A},o)\cong(\mathscr{A}',o')$. Such an isomorphic relation between finite $\Sigma$-automata with output, induces an equivalence relation over AUTO($\Sigma$), and henceforth, we shall always identify all isomorphic elements in AUTO($\Sigma$).

Let $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ be a finite $\Sigma$-automaton with output. Two states $r,s$ of $\mathscr{A}$ are said *indistinguishable* if $o(t(r,\sigma))=o(t(s,\sigma))$ for all $\sigma\in\Sigma^*$. Otherwise we call them *distinguishable*. If all distinct states of $\mathscr{A}$ are distinguishable, then $(\mathscr{A},o)$ is called *minimal*. Clearly every finite $\Sigma$-automaton is minimal since its output function is the identity mapping. Finally we remark that two equivalent minimal $\Sigma$-automata are isomorphic.

It is well known that every finite $\Sigma$-automaton with output is equivalent to some minimal $\Sigma$-automaton (see for example [13]). This result can be slightly specified by the following one, which shows that a minimal $\Sigma$-automaton is in fact the least common factor of all finite $\Sigma$-automata with output, which are equivalent to it.

**Proposition 9.** *For each finite $\Sigma$-automaton with output $(\mathscr{A},o)$, there exists a unique minimal $\Sigma$-automaton $(\mathscr{A}',o')$ such that $(\mathscr{A},o)\approx(\mathscr{A}',o')$ and $\mathscr{A}'\mid\mathscr{A}$.*

Indeed $\mathscr{A}'$ is just the factor of $\mathscr{A}$ which corresponds to the classical Nerode partition of $\mathscr{A}$ associated with $(\mathscr{A},o)$.

From now on, we can (and shall) identify AUTO($\Sigma$) with the set of all minimal $\Sigma$-automata, i.e., we shall always suppose implicitly that all the studied $\Sigma$-automata with output are minimal. Such an identification is acceptable in the present context, although in general it is mathematically difficult to be justified, notably because it evacuates all kinds of interesting representation or algorithmic problems.

Let $p\geqslant 2$ be an integer. Let $\mathscr{A}=(S,i,\Sigma_p,t)$ be a finite $\Sigma_p$-automaton. It is said *normalized* if $t(i,0)=i$. Obviously being normalized is a "factor property", i.e., a factor of a normalized $\Sigma$-automaton is still normalized. The set of all normalized $\Sigma_p$-automata with output will be denoted by NAUTO($\Sigma_p$).

Every $p$-automatic sequence $u$ can be generated by a normalized $\Sigma_p$-automaton with output. In fact, if $u$ is generated by $(\mathscr{A},o)=(S,i,\Sigma_p,t,o)$ with $t(i,0)\neq i$, then by adding a new state $i'$ to $S$ and defining $S'=S\cup\{i'\}$, $t'|_{S\times\Sigma_p}=t$ and $t'(i',0)=i'$, $t'(i',k)=t(i,k)$ for all $k\in\Sigma_p\backslash\{0\}$, $o'|_S=o$ and $o'(i')=o(i)$, we obtain a normalized $\Sigma_p$-automaton with output $(\mathscr{A}',o')=(S',i',\Sigma_p,t',o')$, which generates $u$. If $u$ is generated by two normalized $\Sigma_p$-automata with output, then they are equivalent, and since we only consider minimal $\Sigma$-automata, we know that they are indeed isomorphic. Thus we establish a bijection $\Theta$ from AUTS$(\Sigma_p)$ onto NAUTO$(\Sigma_p)$, and every property of NAUTO$(\Sigma_p)$ can be transferred via this mapping to AUTS$(\Sigma_p)$ and vice versa. This point is important for our study.

**Proposition 10.** *Let $u_1,u_2,\ldots,u_k$ be complex-valued $p$-automatic sequences. If they are linearly dependent over $\mathbb{C}$, then one of $\mathscr{A}_j$ divides the product of all the other* $\mathscr{A}_j$'s, *where $\Theta(u_j)=(\mathscr{A}_j,o_j)$ $(1\leqslant j\leqslant k)$.*

**Proof.** If $u_1,u_2,\ldots,u_k$ are linearly dependent over $\mathbb{C}$, then up to the notations of indices, we can find complex numbers $c_2,c_3,\ldots,c_k$ such that

$$u_1 = c_2u_2 + c_3u_3 + \cdots + c_ku_k$$

from which we deduce immediately

$$\left(\prod_{j=2}^{k}\mathscr{A}_j,\sum_{j=1}^{k}c_jo_j\right)\approx(\mathscr{A}_1,o_1)\quad\text{and}\quad \mathscr{A}_1\left|\prod_{j=2}^{k}\mathscr{A}_j\right.$$

for the finite $\Sigma_p$-automaton with output $(\mathscr{A}_1,o_1)$ is minimal. $\square$

As application, we obtain that the Thue–Morse sequence, the Rudin–Shapiro sequence, the Baum–Sweet sequence, the paperfolding sequence, and the constant sequence 1 are linearly independent over $\mathbb{C}$ (consult for example [2] for the definition of all above-mentioned sequences). Of course this result can also be proved directly and easily by definition.

## 8. Invertible automata

Let $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ be a finite $\Sigma$-automaton with output. We shall say that $(\mathscr{A},o)$ is *left-invertible* (resp. *invertible*) if the mapping $\sigma\mapsto(\mathscr{A},o)(\sigma)$ $(\sigma\in\bar{\Sigma}\backslash\{\varepsilon\})$ is injective (resp. bijective). Clearly $(\mathscr{A},o)$ is left-invertible if and only if for every $s\in S$, all the output values $o(t(s,\sigma))$ $(\sigma\in\Sigma)$ are different. Remark that in this case, we have also Card$(\Sigma)\leqslant$Card$(o(S))$. So all homogeneous $\Sigma$-automata, in particular, the identity automaton, are left-invertible, but the one-state automaton is not, although it is invertible for multiplication.

The following result justifies the above definition of left-invertibility.

**Proposition 11.** *A finite $\Sigma$-automaton with output $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ is left-invertible if and only if there exists a finite $o(S)$-automaton with output $(\mathscr{A}',o')$ such that for all $\sigma\in\bar{\Sigma}\backslash\{\varepsilon\}$, we have $(\mathscr{A}',o')\circ(\mathscr{A},o)(\sigma)=\sigma$.*

**Proof.** The sufficiency is quite evident. So we need only show the necessity.

Suppose that $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ is a left-invertible $\Sigma$-automaton with output. Then for every $s\in S$, all the output values $o(t(s,\sigma))$ $(\sigma\in\Sigma)$ are different. Based on this property, we shall construct in the following a finite $o(S)$-automaton with output $(\mathscr{A}',o')=(S\times S,(i,i),o(S),t',o')$, which satisfies our need.

First we define $t'$. Let $(r,s)\in S\times S$ and $\delta\in o(S)$. If $o(t(s,\sigma))\neq\delta$ for all $\sigma\in\Sigma$, then we define $t'((r,s),\delta)=(s,i)$. In the contrary case, we can find a unique $\sigma\in\Sigma$ such that $o(t(s,\sigma))=\delta$, and we define in this case $t'((r,s),\delta)=(s,t(s,\sigma))$.

Second we define $o'$. For all couple $(r,s)\in S\times S$, we define $o'((r,s))=\sigma$ if there exists $\sigma\in\Sigma$ such that $t(r,\sigma)=s$ (remark here that such a $\sigma$ is unique). Otherwise we put $o'((r,s))=\sigma_0$, where $\sigma_0\in\Sigma$ is an element fixed in advance.

Finally by induction on the length of the input word, we can easily verify that for all $\sigma\in\bar{\Sigma}\backslash\{\varepsilon\}$, we have $(\mathscr{A}',o')\circ(\mathscr{A},o)(\sigma)=\sigma$. $\square$

For invertible $\Sigma$-automata with output, we have the following similar result.

**Proposition 12.** *A finite $\Sigma$-automaton with output $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ is invertible if and only if there exists a finite $o(S)$-automaton with output $(\mathscr{A}',o')$ such that for all $\sigma\in\bar{\Sigma}\backslash\{\varepsilon\}$ and all $\delta\in\overline{o(S)}\backslash\{\varepsilon\}$, we have*

$$(\mathscr{A}',o')\circ(\mathscr{A},o)(\sigma)=\sigma \quad \text{and} \quad (\mathscr{A},o)\circ(\mathscr{A}',o')(\delta)=\delta.$$

**Proof.** The sufficiency is quite evident. So we shall only show the necessity.

Let $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ be an invertible $\Sigma$-automaton with output. Then by Proposition 11, we can find a finite $o(S)$-automaton with output $(\mathscr{A}',o')$ such that $(\mathscr{A}',o')\circ(\mathscr{A},o)(\sigma)=\sigma$ for all $\sigma\in\bar{\Sigma}\backslash\{\varepsilon\}$. Now $\eta\mapsto(\mathscr{A},o)(\eta)$ $(\eta\in\bar{\Sigma}\backslash\{\varepsilon\})$ is bijective, so $\delta\mapsto(\mathscr{A}',o')(\delta)$ $(\delta\in\overline{o(S)}\backslash\{\varepsilon\})$ is its inverse mapping, and then for all $\delta\in\overline{o(S)}\backslash\{\varepsilon\}$, we have $(\mathscr{A},o)\circ(\mathscr{A}',o')(\delta)=\delta$. $\square$

We can also give a simple characterization of invertible $\Sigma$-automata with output.

**Proposition 13.** *A finite $\Sigma$-automaton with output $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ is invertible if and only if it is left-invertible and* $\mathrm{Card}(o(S))=\mathrm{Card}(\Sigma)$.

**Proof.** First suppose that $(\mathscr{A},o)$ is an invertible $\Sigma$-automaton with output. Then necessarily $\mathrm{Card}(\Sigma)\leqslant\mathrm{Card}(o(S))$, and we can find a finite $o(S)$-automaton with output $(\mathscr{A}',o')=(S',i',o(S),t',o')$ such that for $\sigma\in\bar{\Sigma}\backslash\{\varepsilon\}$ and $\delta\in\overline{o(S)}\backslash\{\varepsilon\}$, we have $(\mathscr{A}',o')\circ(\mathscr{A},o)(\sigma)=\sigma$ and $(\mathscr{A},o)\circ(\mathscr{A}',o')(\delta)=\delta$. Clearly $(\mathscr{A}',o')$ is also invertible and $o'(S')=\Sigma$, thus $\mathrm{Card}(o(S))\leqslant\mathrm{Card}(o'(S'))=\mathrm{Card}(\Sigma)$, and we obtain finally $\mathrm{Card}(o(S))=\mathrm{Card}(\Sigma)$.

Now suppose that $(\mathscr{A},o)$ is a left-invertible $\Sigma$-automaton with output such that $\mathrm{Card}(o(S))=\mathrm{Card}(\Sigma)$. Then by Proposition 11, we can construct a finite $o(S)$-automaton with output $(\mathscr{A}', o') = (S \times S, (i,i), o(S), t', o')$ such that for all $\sigma \in \bar{\Sigma}\backslash\{\varepsilon\}$, we have $(\mathscr{A}', o') \circ (\mathscr{A}, o)(\sigma) = \sigma$. In particular, we can remark that the mapping $\delta \mapsto (\mathscr{A}', o')(\delta)$ $(\delta \in \overline{o(S)}\backslash\{\varepsilon\})$ is surjective. So to obtain the desired conclusion, we need only show that this mapping is also injective, i.e., to show that for every $(r,s) \in S \times S$, all the output values $o'(t'((r,s),\delta))$ $(\delta \in o(S))$ are different. Let $\delta$ and $\delta'$ be two different elements in $o(S)$. Since $\mathrm{Card}(o(S)) = \mathrm{Card}(\Sigma)$ and all the output values $o(t(s,\sigma))$ $(\sigma \in \Sigma)$ are different, the mapping $\sigma \mapsto o(t(s,\sigma))$ is bijective, and there exists $\sigma, \sigma' \in \Sigma$ such that $o(t(s,\sigma)) = \delta$ and $o(t(s,\sigma')) = \delta'$. Hence $o'(t'((r,s),\delta)) = \sigma \neq \sigma' = o'(t'((r,s),\delta'))$, and the result is established. □

Hence the Thue–Morse automaton, and the identity automaton or more generally the "minimal" homogeneous $\Sigma$-automaton $\mathscr{P}_\rho$ $(\rho \in \Sigma)$ are invertible. Incidentally, one can remark that all invertible $\Sigma$-automata are strictly accessible.

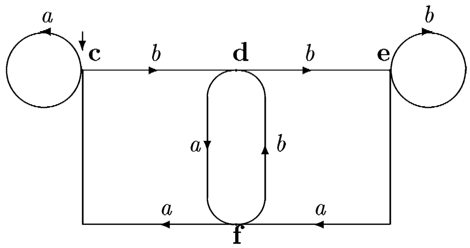
Let $(\mathscr{A}, o) = (S, i, \Sigma, t, o)$ be a finite $\Sigma$-automaton with output. Let $(\mathscr{A}', o')$ be a finite $o(S)$-automaton with output. If $(\mathscr{A}', o') \circ (\mathscr{A}, o)(\sigma) = \sigma$ for all $\sigma \in \bar{\Sigma}\backslash\{\varepsilon\}$, then we call $(\mathscr{A}', o')$ *a left-inverse* of $(\mathscr{A}, o)$, and $(\mathscr{A}, o)$ *a right-inverse* of $(\mathscr{A}', o')$. Finally we call $(\mathscr{A}', o')$ *a bilateral-inverse* of $(\mathscr{A}, o)$ if $(\mathscr{A}', o')$ is a left-inverse and also a right-inverse of $(\mathscr{A}, o)$. A finite $\Sigma$-automaton with output may have many different left-inverses or right-inverses, but it has at most one bilateral-inverse.

As an illustrative example, we give below the bilateral-inverse $(\mathscr{A}', o')$ of the Thue–Morse automaton (Fig. 3). Remark that if we redefine the output function $o'$ by

$$o'(\mathbf{c}) = o'(\mathbf{f}) = 0 \quad \text{and} \quad o'(\mathbf{d}) = o'(\mathbf{e}) = 1,$$

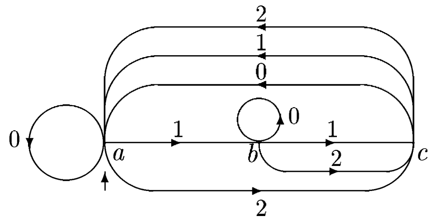
then the new $(\mathscr{A}', o')$ is the bilateral-inverse of the identity automaton. In general, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two invertible $\Sigma$-automata with the same set of states, then their bilateral-inverses only differ in the output functions.

Now Propositions 11 and 12 can be reformulated as follows: a finite automaton with output has a left-inverse (resp. bilateral-inverse) if and only if it is left-invertible (resp. invertible). Thus we obtain an exact analog of the well-known result that a mapping $g$ is injective (resp. bijective) if and only if there exists a mapping $h$ such that $h \circ g$ is the identity mapping (resp. $h \circ g$ and $g \circ h$ are the corresponding identity mappings).




$o'(\mathbf{c}) = o'(\mathbf{e}) = 0$, and $o'(\mathbf{d}) = o'(\mathbf{f}) = 1$

Fig. 3. The bilateral-inverse of the Thue–Morse automaton.




$o(a) = +$, and $o(b) = o(c) = -$

Fig. 4. A counter-example.

Let $(\mathscr{A}, o)$ be a finite $\Sigma$-automaton with output. Clearly if it has a right-inverse, then the mapping $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective. Inspired by the above results, one can ask whether the converse is also true. Unfortunately, just as we can see below, the answer to this question is negative in general.

**Example 5.** Let $\Sigma = \{0, 1, 2\}$. We denote by $(\mathscr{A}, o) = (S, a, \Sigma, t, o)$ the finite $\Sigma$-automaton with output defined in Fig. 4, where $S = \{a, b, c\}$. Obviously the mapping $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective. However $(\mathscr{A}, o)$ does not have a right-inverse. By absurdity, assume that $(\mathscr{A}, o)$ has a right inverse $(\mathscr{A}', o')$. Then we have $(\mathscr{A}, o) \circ (\mathscr{A}', o')(\delta) = \delta$ for all $\delta \in \overline{o(S)} \backslash \{\varepsilon\}$. Thus $\mathscr{A} \circ (\mathscr{A}', o')(+-+) = aca$ and $\mathscr{A} \circ (\mathscr{A}', o')(+--) = abc$ or $abb$. Hence $\mathscr{A} \circ (\mathscr{A}', o')(+-)$ takes two values!

Let $(\mathscr{A}, o) = (S, i, \Sigma, t, o)$ be a finite $\Sigma$-automaton with output. Obviously when the mapping $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective, we have necessarily

$$\mathrm{Card}(o(S)) \leqslant \mathrm{Card}(\Sigma).$$

This property is not sufficient, and until now, we do not know how to characterize a finite $\Sigma$-automaton with output $(\mathscr{A}, o)$ such that $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective, and we do not know either when $(\mathscr{A}, o)$ can have a right-inverse. However we can still remark that if the output function $o$ is injective, then the mapping $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective if and only if for all state $s \in S$, we have $\{t(s, \sigma) \mid \sigma \in \Sigma\} = S$, and by the same construction as in the proof of Proposition 11, we can also show that $\sigma \mapsto (\mathscr{A}, o)(\sigma)$ $(\sigma \in \bar{\Sigma} \backslash \{\varepsilon\})$ is surjective if and only if $(\mathscr{A}, o)$ has a right-inverse.

## 9. Some topological properties of finite automata

Finite automata are discrete objects. It is somehow surprising to talk of the limit of finite automata. How can a four-state finite automaton, say, become a five-state finite automaton in a continuous fashion? We remark that such a phenomenon exists abundantly, and the Ising automata in the next section will offer us a good example.

To explain how this is possible, Kamae and Mendès France have introduced in [22] an intuitive notion of continuity. In this section, we shall treat this problem systematically and thoroughly via another approach. Indeed we shall define three different but natural topologies so that we can obtain some finer results which cannot be attained via the above-mentioned notion of continuity. Then we use these results to study in the next section the continuity of Ising automata. By the way, we also obtain a sufficient condition such that the weak limit of a sequence of automatic sequences is still automatic. This is the first approach towards the difficult problem about the topological closure property of automatic sequences.

Let $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ be the set of all minimal $\Sigma$-automata with complex-valued output function. Without loss of generality, we shall also suppose that $\Sigma \subseteq \mathbb{C}$ and all the output functions are defined on the whole $\mathbb{C}$.

Let $(\mathscr{A}_1, o_1)$ and $(\mathscr{A}_2, o_2)$ be two elements in $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. For all $a, b \in \mathbb{C}$, and all state $(r, s)$ of $\mathscr{A}_1 \times \mathscr{A}_2$, we define

$$o_1o_2((r,s)) = o_1(r)o_2(s) \quad \text{and} \quad (ao_1 + bo_2)((r,s)) = ao_1(r) + bo_2(s)$$

and we denote by $(\mathscr{A}_1, o_1) \times (\mathscr{A}_2, o_2)$ (resp. $a(\mathscr{A}_1, o_1) + b(\mathscr{A}_2, o_2)$) the unique minimal $\Sigma$-automaton which is equivalent to $(\mathscr{A}_1 \times \mathscr{A}_2, o_1o_2)$ (resp. $(\mathscr{A}_1 \times \mathscr{A}_2, ao_1 + bo_2)$). Clearly endowed with these binary operations, $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ becomes a $\mathbb{C}$-algebra. Finally we remark that $(\mathscr{I}_\Sigma, 0)$ is the zero element, $(\mathscr{I}_\Sigma, 1)$ is the identity element for multiplication, and an element $(\mathscr{A}, o) = (S, i, \Sigma, t, o) \in \mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ is invertible for multiplication if and only if $0 \notin o(S)$.

Now we define three topologies over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. The reader can consult for example [24] for all the basic notions and results in topology cited in this section.

Let $(\mathscr{A}, o) \in \mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. For all $\sigma \in \Sigma^*$, define $f_\sigma((\mathscr{A}, o)) = o(\mathscr{A}\sigma)$. Then $f_\sigma$ is a complex-valued mapping defined on $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. Let $\mathscr{W}_{\mathbb{C}}(\Sigma)$ be the weakest topology (called the *weak topology*) over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ such that all the $f_\sigma$ $(\sigma \in \Sigma^*)$ are continuous. Clearly $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \mathscr{W}_{\mathbb{C}}(\Sigma))$ is a Hausdorff space (see [8, Chapter 10]). But unfortunately, it is not complete.

Let $\Phi$ be the mapping defined on $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ which associates all element $(\mathscr{A}, o)$ in $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ with the mapping $\sigma \mapsto f_\sigma((\mathscr{A}, o))$. Obviously $\Phi$ is a homeomorphism from $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \mathscr{W}_{\mathbb{C}}(\Sigma))$ onto $\Phi(\mathrm{AUTO}_{\mathbb{C}}(\Sigma)) \subseteq \mathbb{C}^{\Sigma^*}$, endowed with the product topology. Moreover $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \mathscr{W}_{\mathbb{C}}(\Sigma))$ is metrizable (i.e., we can define a metric over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ which induces $\mathscr{W}_{\mathbb{C}}(\Sigma)$) and $\Phi(\mathrm{AUTO}_{\mathbb{C}}(\Sigma))$ is dense in $\mathbb{C}^{\Sigma^*}$, which is complete and metrizable.

Let $(\mathscr{A}, o) = (S, i, \Sigma, t, o)$ be an element in $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. Define

$$\|(\mathscr{A}, o)\| = \sup_{s \in S} |o(s)|.$$

Then $\|\cdot\|$ is a norm on $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$, and the topology induced by it over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$, called the *uniform topology*, is stronger than the weak topology $\mathscr{W}_{\mathbb{C}}(\Sigma)$.

The normed algebra $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \|\cdot\|)$ is not complete. Hence the topological vector space $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \mathscr{W}_{\mathbb{C}}(\Sigma))$ is not complete neither. For the proof, we first consider the case $\Sigma = \Sigma_p$, where $p \geqslant 2$ is an integer. Let $u = (u(n))_{n \geqslant 0}$ be a complex-valued sequence almost periodic in the sense of Bohr (see [23]). We also suppose that $u$ takes an infinite number of values. So it is not $p$-automatic, but it can be approached uniformly by periodic sequences $u_k=(u_k(n))_{n\geqslant 0}$ ($k\in\mathbb{N}$), i.e.,

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}|u_k(n)-u(n)|=0.$$

For all $k\in\mathbb{N}$, write $\Theta(u_k)=(\mathscr{A}_k,o_k)$. Then $((\mathscr{A}_k,o_k))_{k\geqslant 0}$ is a family of Cauchy sequences in $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\|\cdot\|)$, since for all $k,l\in\mathbb{N}$, we have

$$\|(\mathscr{A}_k,o_k)-(\mathscr{A}_l,o_l)\|=\sup_{n\in\mathbb{N}}|u_k(n)-u_l(n)|.$$

However it is not convergent in $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\|\cdot\|)$ for $u$ is not $p$-automatic. Now let $\Sigma$ be an alphabet with $p$ elements. Then we can find a bijection between $\Sigma$ and $\Sigma_p$, which induces in turn a bijection between $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ and $\mathrm{AUTO}_{\mathbb{C}}(\Sigma_p)$, and the desired conclusion that $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\|\cdot\|)$ is not complete can be deduced from that of $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma_p),\|\cdot\|)$.

**Proposition 14.** *Let* $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ *be a sequence in* $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ *which converges uniformly to* $(\mathscr{A},o)\in\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$. *Then there exists an integer* $d\geqslant 0$ *such that* $\mathscr{A}$ *divides* $\mathscr{A}_n$ *for all* $n\geqslant d$.

**Proof.** Write $(\mathscr{A},o)=(S,i,\Sigma,t,o)$ and $(\mathscr{A}_n,o_n)=(S_n,i_n,\Sigma,t_n,o_n)$ ($n\in\mathbb{N}$), and let $\delta>0$ be a real number sufficiently small so that all the open discs

$$\mathbb{D}(o(s),\delta)=\{x\in\mathbb{C}\mid |x-o(s)|<\delta\}\quad(s\in S)$$

are disjoint. Since $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ converges uniformly to $(\mathscr{A},\ o)$, then there exists an integer $d\geqslant 0$ such that $\|(\mathscr{A}_n,o_n)-(\mathscr{A},o)\|<\delta$, for all $n\geqslant d$. Therefore

$$o_n(S_n)\subseteq\mathbb{D}:=\bigcup_{s\in S}\mathbb{D}(o(s),\delta)$$

for all $n\geqslant d$. Let $o'$ be the mapping defined on $\mathbb{D}$ such that $o'(x)=o(s)$, for all $x\in\mathbb{D}(o(s),\delta)$. Then for all $n\geqslant d$, we have $(\mathscr{A}_n,o'\circ o_n)\approx(\mathscr{A},o)$. Thus $\mathscr{A}$ divides $\mathscr{A}_n$ by Proposition 9, for $(\mathscr{A},o)$ is supposed to be minimal. □

Below we shall define the strong topology $\mathscr{S}_{\mathbb{C}}(\Sigma)$ over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ such that the topological vector space $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{S}_{\mathbb{C}}(\Sigma))$ is locally convex (i.e., $\mathscr{S}_{\mathbb{C}}(\Sigma)$ has a fundamental system of the origin composed of convex sets) and complete.

Let $n\geqslant 1$ be an integer. We denote by $\mathrm{AUTO}^{(n)}_{\mathbb{C}}(\Sigma)$ the vector subspace of $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$, generated by all minimal $\Sigma$-automata $(\mathscr{A},o)$ satisfying $\mathrm{Card}(\mathscr{A})\leqslant n$, where $\mathrm{Card}(\mathscr{A})$ is the number of states of $\mathscr{A}$, and by $\iota_n$ the canonical injection of the vector subspace $\mathrm{AUTO}^{(n)}_{\mathbb{C}}(\Sigma)$ into $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$.

The dimension of $\mathrm{AUTO}^{(n)}_{\mathbb{C}}(\Sigma)$ is finite. Indeed there is only a finite number of non isomorphic $\Sigma$-automata $\mathscr{A}$ with $\mathrm{Card}(\mathscr{A})\leqslant n$. So if we denote by $\mathscr{B}=(S,i,\Sigma,t)$ the product of all these finite $\Sigma$-automata, then for every $(\mathscr{A},o)\in\mathrm{AUTO}^{(n)}_{\mathbb{C}}(\Sigma)$, we can find an output function $o'$ such that $(\mathscr{A},o)\approx(\mathscr{B},o')$. Hence $\mathrm{AUTO}^{(n)}_{\mathbb{C}}(\Sigma)$ can be generated by a finite number of minimal $\Sigma$-automata which are equivalent to one of these $(\mathscr{B},\rho_s)$, where $\rho_s$ is the characteristic function of $s\in S$, i.e.,

$$\rho_s(s)=1 \text{ and } \rho_s(r)=0 \quad \text{for all } r\in S\backslash\{s\}.$$

Now that $\mathbb{C}$ is complete for the usual absolute value, so over $\mathrm{AUTO}_{\mathbb{C}}^{(n)}(\Sigma)$, all the Hausdorff topologies which are compatible with its $\mathbb{C}$-vector space structure coincide (see [9, EVT I, p.14]), a fortiori, the weak topology and the uniform topology coincide. Let $\mathscr{S}_{\mathbb{C}}^{(n)}(\Sigma)$ be the restriction of $\mathscr{W}_{\mathbb{C}}(\Sigma)$ over $\mathrm{AUTO}_{\mathbb{C}}^{(n)}(\Sigma)$. We denote by $\mathscr{S}_{\mathbb{C}}(\Sigma)$ the strongest locally convex topology over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ such that all the following mappings

$$\imath_n : (\mathrm{AUTO}_{\mathbb{C}}^{(n)}(\Sigma),\mathscr{S}_{\mathbb{C}}^{(n)}(\Sigma)) \to (\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{S}_{\mathbb{C}}(\Sigma))$$

are continuous. It is well known that $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{S}_{\mathbb{C}}(\Sigma))$ is locally convex and complete (see for example [9, EVT II, p. 35]). Finally, we remark that the algebraic dual of $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ coincides with the topological dual of $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{S}_{\mathbb{C}}(\Sigma))$, i.e. every linear form $\mu$ over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ is continuous for $\mathscr{S}_{\mathbb{C}}(\Sigma)$. In fact, by the result in [29, p. 58], a linear form $\mu$ over $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ is continuous for $\mathscr{S}_{\mathbb{C}}(\Sigma)$ if and only if its restriction on $\mathrm{AUTO}_{\mathbb{C}}^{(n)}(\Sigma)$ is continuous for $\mathscr{S}_{\mathbb{C}}^{(n)}(\Sigma)$, which is trivially true for $\mathrm{AUTO}_{\mathbb{C}}^{(n)}(\Sigma)$ has a finite dimension.

The topology $\mathscr{S}_{\mathbb{C}}(\Sigma)$ is surely stronger than the uniform topology. In particular, Proposition 14 also holds for $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{S}_{\mathbb{C}}(\Sigma))$.

**Proposition 15.** *A sequence $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ is convergent in $\mathrm{AUTO}_{\mathbb{C}}(\Sigma)$ for the strong topology if and only if there exists an integer $d\geqslant 1$ such that the same sequence is convergent in $(\mathrm{AUTO}_{\mathbb{C}}^{(d)}(\Sigma),\mathscr{S}_{\mathbb{C}}^{(d)}(\Sigma))$.*

**Proof.** This is just a reformulation of the result in [29, p. 62]. □

**Proposition 16.** *Let $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ be a Cauchy sequence in $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma),\mathscr{W}_{\mathbb{C}}(\Sigma))$. If $(\mathrm{Card}(\mathscr{A}_n))_{n\geqslant 0}$ is bounded, then $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ converges for the strong topology.*

**Proof.** In fact, since $(\mathrm{Card}(\mathscr{A}_n))_{n\geqslant 0}$ is bounded, then we can find an integer $d\geqslant 1$ such that $(\mathscr{A}_n,o_n)\in \mathrm{AUTO}_{\mathbb{C}}^{(d)}(\Sigma)$ for all $n\geqslant 0$. But the weak topology and the strong topology coincide over $\mathrm{AUTO}_{\mathbb{C}}^{(d)}(\Sigma)$, so $((\mathscr{A}_n,o_n))_{n\geqslant 0}$ is also a Cauchy sequence for $\mathscr{S}_{\mathbb{C}}(\Sigma)$, thus convergent. □

Let $p\geqslant 2$ be an integer. We denote by $\mathrm{AUTS}_{\mathbb{C}}(\Sigma_p)$ (resp. $\mathrm{NAUTO}_{\mathbb{C}}(\Sigma_p)$) the set of all complex-valued $p$-automatic sequences (resp. the $\mathbb{C}$-vector subspace of all normalized minimal $\Sigma_p$-automata in $\mathrm{AUTO}_{\mathbb{C}}(\Sigma_p)$). Then $\mathrm{AUTS}_{\mathbb{C}}(\Sigma_p)$ is a $\mathbb{C}$-vector space and the restriction of $\Theta$ on $\mathrm{AUTS}_{\mathbb{C}}(\Sigma_p)$ (denoted by $\Theta_{\mathbb{C}}$) is an isomorphism of $\mathbb{C}$-vector spaces. So all topological property over $\mathrm{NAUTO}_{\mathbb{C}}(\Sigma_p)$ can be transferred via $\Theta_{\mathbb{C}}$ over $\mathrm{AUTS}_{\mathbb{C}}(\Sigma_p)$, and we say that a family $(u_k)_{k\geqslant 0}$ of $p$-automatic sequences converges weakly (resp. uniformly or strongly) to a $p$-automatic sequence $u$, if the family $(\Theta(u_k))_{k\geqslant 0}$ converges weakly (resp. uniformly or strongly) to $\Theta(u)$.

Clearly the family $(u_k)_{k\geqslant 0}$ converges weakly to $u$ if and only if for all $n\in\mathbb{N}$, we have

$$\lim_{k\to\infty} u_k(n) = u(n).$$

Similarly $(u_k)_{k\geqslant 0}$ converges uniformly to $u$ if and only if we have

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} |u_k(n) - u(n)| = 0.$$

One can remark here that even if $u$ is not $p$-automatic, it is still meaningful to say whether $(u_k)_{k\geqslant 0}$ converges weakly (resp. uniformly) or not to $u$.

The characterization of strongly convergent $p$-automatic sequences is a little more complicated and needs some new notion, namely $p$-kernel. For all complex-valued sequence $u=(u(n))_{n\geqslant 0}$, we define

$$\mathscr{N}_p(u) := \{(u(p^b n + a))_{n\geqslant 0} \mid 0 \leqslant a < p^b, \text{and } a,b\in\mathbb{N}\}$$

and call it the *$p$-kernel* of $u$. It is well known that $u$ is $p$-automatic if and only if $\mathscr{N}_p(u)$ is finite, and in this case, $\mathrm{Card}(\mathscr{N}_p(u)) = \mathrm{Card}(\Theta_{\mathbb{C}}(u))$ (cf. [2]). With this notation, Proposition 15 can be reformulated as follows: a family $(u_n)_{n\geqslant 0}$ of $p$-automatic sequences converges strongly to a $p$-automatic sequence $u$ if and only if the family $(\mathrm{Card}(\mathscr{N}_p(u_n)))_{n\geqslant 0}$ is bounded and $(u_n)_{n\geqslant 0}$ converges weakly to $u$.

An important problem in the study of $p$-automatic sequences concerns their topological closure property. More precisely, we would like to know when the limit of a family of $p$-automatic sequences is also $p$-automatic. Clearly the topology in question plays a capital role in this problem. For example, the limit of a weakly or uniformly convergent sequence of $p$-automatic sequences is not $p$-automatic in general (the counter-example, which serves to show that $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \|\cdot\|)$ is not complete, can also be used here). However Proposition 16 tells us that we have the following topological closure property.

**Proposition 17.** *Let $(u_n)_{n\geqslant 0}$ be a family of complex-valued $p$-automatic sequences which converges weakly to a sequence $u$. If the sequence $(\mathrm{Card}(\mathscr{N}_p(u_n)))_{n\geqslant 0}$ is bounded, then $u$ is $p$-automatic.*

**Proof.** Since $(u_n)_{n\geqslant 0}$ converges weakly to $u$, so $(\Theta_{\mathbb{C}}(u_n))_{n\geqslant 0}$ is a Cauchy sequence in $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma_p), \mathscr{W}_{\mathbb{C}}(\Sigma_p))$, and by Proposition 16, it converges strongly to an element of $\mathrm{AUTO}_{\mathbb{C}}(\Sigma_p)$, which obviously generates $u$. $\square$

## 10. Ising automata

Since the discovery of quasicrystals, automata theory has found a good place in the study of theoretical physics to describe some nonperiodic but ordered phenomena. A typical example is the one-dimensional inhomogeneous Ising chain, which contains $N+1$ particles of spins $\pm 1$ ranged on a line. For each $q\in\mathbb{N}$ $(0\leqslant q\leqslant N)$, we denote by $\sigma(q)$ the spin of the $q$th particle, and call the finite word $\sigma=(\sigma(q))_{0\leqslant q\leqslant N}$ a configuration of the system. Let $\eta \in \{-1,+1\}^N$ be a finite word which represents for example the distribution of two different substances or some impurities in an alloy. The Hamiltonian of this system at the configuration $\sigma=(\sigma(q))_{0\leqslant q\leqslant N}$ is defined as

$$\mathscr{H}_\eta(\sigma) = -J\sum_{q=0}^{N-1}\eta(q)\sigma(q)\sigma(q+1) - H\sum_{q=0}^{N}\sigma(q),$$

where $J>0$ is the coupling constant and $H\geqslant 0$ is the external magnetic field.

Given two parameters $J$ and $H$, an important problem in statistical mechanics is to determine the system's equilibrium state, i.e., the configuration $\hat{\sigma}$ which may minimize the Hamiltonian $\mathscr{H}_\eta(\sigma)$. Kamae and Mendès France [22] showed that such an equilibrium configuration $\hat{\sigma}$ must satisfy the recurrent relation:

$$\hat{\sigma}(N) = \operatorname{sgn}(\delta(N)) \quad \text{and} \quad \hat{\sigma}(q) = \operatorname{sgn}(\delta(q) + 2\eta(q)\hat{\sigma}(q+1)) \quad \text{for } 0 \leqslant q < N,$$

where the finite word $\delta=(\delta(q))_{0\leqslant q\leqslant N}$ is defined by

$$\delta(q+1) = \alpha + \eta(q)\operatorname{sgn}(\delta(q))\min\{2, |\delta(q)|\} \ (0 \leqslant q < N),$$

with $\alpha=2H/J$ and $\delta(0)$ fixed beforehand (by convention, here sgn(0) may take the two values $+1$ and $-1$ arbitrarily. This corresponds to the fact that there are probably more than one equilibrium states under the same conditions).

In this work, we shall only restrict our attention on the case $\delta(0)=\alpha+2$. Then the study of our system is reduced to that of the following recurrent relation:

$$\begin{aligned} &\delta(0) = \alpha + 2, \\ &\delta(q+1) = \alpha + \eta(q)\operatorname{sgn}(\delta(q))\min\{2, |\delta(q)|\}. \end{aligned} \tag{2}$$

The finite word $\delta$ depends on $\alpha$ and on $\eta$, so we can denote it by $\delta_\alpha(\eta)$. It was shown in [22,27] that the mapping $\eta \mapsto \delta_\alpha(\eta)$, defined from $\Sigma^*\backslash\{\varepsilon\}$ to $O_\alpha^*$, where $\Sigma=\{-1,+1\}$ and $O_\alpha$ is a finite subset of $[\alpha-2,\alpha+2]$, can be defined by a finite automaton with output $(\mathscr{A}_\alpha, o_\alpha)=(S_\alpha, i_\alpha, \Sigma, t_\alpha, o_\alpha)$. In other words, for all $\eta=(\eta(j))_{0\leqslant j<|\eta|}\in\Sigma^*$ and $q\in\mathbb{N}$ $(0\leqslant q<|\eta|)$, we have $\delta_\alpha(\eta)(0)=\alpha+2$, and

$$\delta_\alpha(\eta)(q+1) = o_\alpha(t_\alpha(i_\alpha, \eta[0,q])).$$

For $\alpha=0$, relation (2) becomes $\delta(q+1)=2\eta(0)\eta(1)\cdots\eta(q)$, and gives the finite $\Sigma$-automaton with output defined in Fig. 5. Remark here that if we replace $+1$ by 0

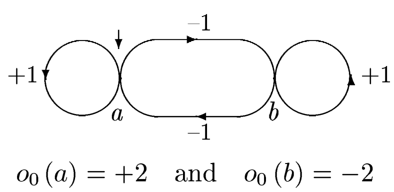



Fig. 5. Ising automaton $\mathscr{A}_0$.

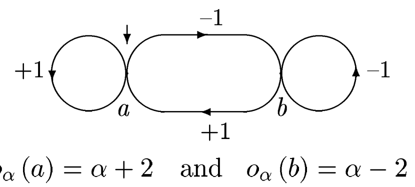



Fig. 6. Ising automaton $\mathscr{A}_\alpha$ with $\alpha \geqslant 4$.

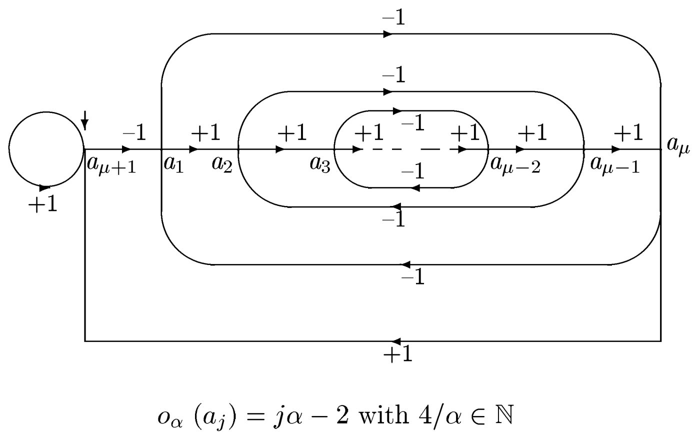



Fig. 7. Ising automaton $\mathscr{A}_\alpha$ (denoted by $\mathscr{N}_\mu$).

and $-1$ by 1, then we obtain the Thue–Morse automaton. So $\mathscr{A}_0$ is irreducible and invertible, but it is neither faithful (cf. [26]) nor homogeneous.

If $\alpha \geqslant 4$, relation (2) gives $\delta(q+1) = \alpha + 2\eta(q)$, and the finite $\Sigma$-automaton with output defined by Fig. 6 satisfies our need. Once again, if we replace $+1$ by 0 and $-1$ by 1, we get the identity automaton. So $\mathscr{A}_\alpha$ is strictly faithful, irreducible, homogeneous, and invertible.

When $0 < \alpha < 4$, we obtain two types of finite automata with output. To distinguish them, we shall denote $\mathscr{A}_\alpha$ by $\mathscr{N}_\mu$ or $\mathscr{L}_\mu$ according to $4/\alpha \in \mathbb{N}$ or not, where $\mu = [4/\alpha]$ is the integral part of $4/\alpha$ (see Figs. 7 and 8), and give below a somewhat detailed analysis of $\mathscr{A}_\alpha$.

First we examine the Ising automaton $\mathscr{N}_\mu$ ($\mu \geqslant 1$).

Clearly $\mathscr{N}_1$ is just the finite $\Sigma$-automaton defined in Fig. 6, so it is strictly faithful, irreducible, homogeneous, and invertible.

In the following, we shall show that for all $\mu \geqslant 2$, the Ising automaton $\mathscr{N}_\mu$ is weakly irreducible, but not irreducible. More precisely, we have

$$\mathrm{FAC}(\mathscr{N}_\mu) = \{\mathscr{I}_\Sigma, \mathscr{N}'_\mu, \mathscr{N}_\mu\}, \tag{3}$$

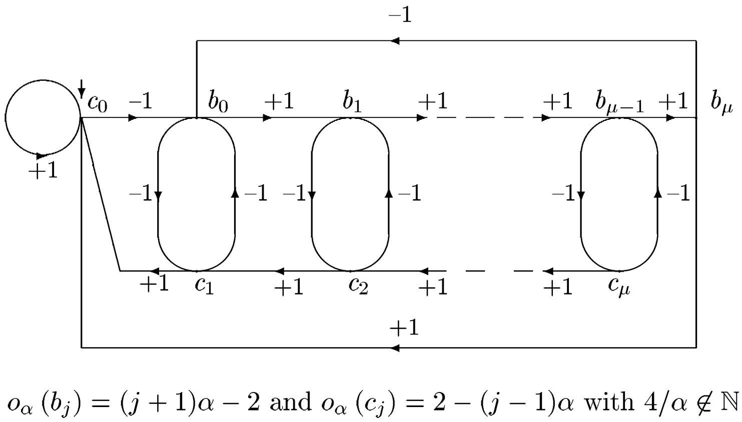



Fig. 8. Ising automaton $\mathscr{A}_\alpha$ (denoted by $\mathscr{L}_\mu$).

where $\mathscr{N}'_\mu$ is the factor which corresponds to the automaton partition

$$\pi_\mu = \{\{a_{\mu+1}, a_\mu\}, \{a_1\}, \ldots, \{a_{\mu-1}\}\}$$

of $\mathscr{N}_\mu$. By the way, we remark that $\mathscr{N}_\mu$ is left-invertible but not invertible, it is also faithful (since $t_\alpha(s, 1^\mu) = i_\alpha$, for all $s \in S_\alpha$) but not strictly faithful (cf. [26]).

Now show (3) by absurdity. Suppose that there exists a proper automaton partition $\pi = \{\mathbf{D}_j\}_{0 \leqslant j \leqslant n}$ of $\mathscr{N}_\mu$, different from $\pi_\mu$, and such that $a_{\mu+1} \in \mathbf{D}_0$.

We shall distinguish two cases.

*Case* 1: $\mathbf{D}_0$ contains at least two states. In this case, we shall show that there exists $j \in \mathbb{N}$ $(1 \leqslant j < \mu)$ such that $a_j \in \mathbf{D}_0$. Indeed if this were false, then we would have $\mathbf{D}_0 = \{a_{\mu+1}, a_\mu\}$. But $\pi \neq \pi_\mu$, so we can find two integers $k, l$ $(1 \leqslant k < l < \mu)$ such that $a_k \equiv a_l \pmod{\pi}$. Applying $\mu - l + 1$ times $t^{(\mu)}(\cdot, +1)$ over $a_k$ and $a_l$, where $t^{(\mu)}$ is the transition function of $\mathscr{N}_\mu$, then $a_{\mu+1+k-l} \equiv a_{\mu+1} \pmod{\pi}$. Thus we obtain $a_{\mu+1+k-l} = a_\mu$, and $l = k+1$. Consequently for all $h \in \mathbb{N}$ $(k \leqslant h \leqslant \mu)$, we have $a_h = a_{\mu+1}$, for $a_{h+1} = t^{(\mu)}(a_h, +1)$. This is absurd.

Now let $j$ be the least integer such that $1 \leqslant j < \mu$ and $a_j \in \mathbf{D}_0$. Since

$$t^{(\mu)}(a_{\mu+1}, +1) = a_{\mu+1} \quad \text{and} \quad t^{(\mu)}(a_j, +1) = a_{j+1},$$

so $a_{j+1} \in \mathbf{D}_0$ too. By recurrence, we obtain $a_k \in \mathbf{D}_0$ for all $k \in \mathbb{N}$ $(j \leqslant k \leqslant \mu)$. But for all integer $k \geqslant j$, we also have $a_1 \equiv a_{\mu-k+1} \pmod{\pi}$ for $a_1 = t^{(\mu)}(a_{\mu+1}, -1)$ and $a_{\mu-k+1} = t^{(\mu)}(a_k, -1)$. In particular, we have $a_1 \equiv a_2 \pmod{\pi}$, and then by the same argument as above, we obtain $a_1 \equiv a_k \pmod{\pi}$ for all $k \in \mathbb{N}$ $(1 \leqslant k \leqslant \mu)$. Hence $a_{\mu+1} \equiv a_1 \pmod{\pi}$, and $\pi$ is trivial.

*Case* 2: $\mathbf{D}_0 = \{a_{\mu+1}\}$. Then there exists two integers $k$, $l$ $(1 \leqslant k < l \leqslant \mu)$ such that $a_k \equiv a_l \pmod{\pi}$. Remark that $a_{k+1} = t^{(\mu)}(a_k, +1)$ and $a_{l+1} = t^{(\mu)}(a_l, +1)$, so $a_{k+1} \equiv a_{l+1} \pmod{\pi}$, and by recurrence, we obtain $a_{\mu+1+k-l} \equiv a_{\mu+1} \pmod{\pi}$. Whence $k = l$, and it contradicts our original hypothesis on $k, l$. $\square$

The case $\mathscr{L}_\mu$ ($\mu \geqslant 1$) is much more complicated. Clearly it is left-invertible but not invertible. It is also faithful (since for all $s \in S_\alpha$, we have $t_\alpha(s, 1^{\mu+1}) = i_\alpha$) but not strictly faithful (cf. [26]). Moreover we also have $\mathscr{L}_\mu = \mathscr{N}_\mu \times \mathscr{N}_{\mu+1}$ (see [22]). Below we shall specify all the elements in FAC($\mathscr{L}_\mu$).

For $\mu = 1$, FAC($\mathscr{L}_1$) consists of seven elements:

$$\begin{aligned}
&\pi_0 = \{c_0, c_1, b_0, b_1\}\ (\mathscr{I}_\Sigma),\\
&\pi_1 = \{\{c_0\}, \{b_0\}, \{c_1, b_1\}\}\ (\mathscr{N}_2),\\
&\pi_2 = \{\{c_0, c_1\}, \{b_0\}, \{b_1\}\}\ (\mathscr{L}'_1),\\
&\pi_3 = \{\{c_0, c_1, b_1\}, \{b_0\}\}\ (\mathscr{N}'_2),\\
&\pi_4 = \{\{c_0, b_1\}, \{c_1\}, \{b_0\}\}\ (\mathscr{N}_1 \times \mathscr{N}'_2),\\
&\pi_5 = \{\{c_0, b_1\}, \{c_1, b_0\}\}\ (\mathscr{N}_1),\\
&\pi_6 = \{\{c_0\}, \{c_1\}, \{b_0\}, \{b_1\}\}\ (\mathscr{L}_1).
\end{aligned}$$

Hence, FAC($\mathscr{L}_1$) is not generated by $\mathscr{N}_1$, $\mathscr{N}'_2$, and $\mathscr{N}_2$, although $\mathscr{L}_1 = \mathscr{N}_1 \times \mathscr{N}_2$. Indeed we also have $\mathscr{L}_1 = \mathscr{N}_1 \times \mathscr{L}'_1$. Moreover, $\mathscr{L}'_1$ is weakly irreducible and $\mathscr{N}'_2$ is its unique proper factor. Incidentally, $\mathscr{L}_1$ is also homogeneous for $\mathscr{N}_1 \mid \mathscr{L}_1$. Here, we shall not give the proof of all these results. In fact, the reader can check them directly or just use the same argument as follows.

For $\mu \geqslant 2$, FAC($\mathscr{L}_\mu$) is composed of 10 elements:

$$\begin{aligned}
&\pi_0 = \{c_j, b_j;\ 0 \leqslant j \leqslant \mu\}\ (\mathscr{I}_\Sigma),\\
&\pi_1 = \{\{c_0\}, \{c_j, b_{\mu-j+1}\}, \{b_0\};\ 1 \leqslant j \leqslant \mu\}\ (\mathscr{N}_{\mu+1}),\\
&\pi_2 = \{\{c_0\}, \{c_1, b_\mu\}, \{c_j\}, \{b_{j-1}\}, \{b_0\};\ 2 \leqslant j \leqslant \mu\}\ (\mathscr{N}'_\mu \times \mathscr{N}_{\mu+1}),\\
&\pi_3 = \{\{c_0, c_1\}, \{c_j\}, \{b_0\}, \{b_1\}, \{b_j\};\ 2 \leqslant j \leqslant \mu\}\ (\mathscr{L}'_\mu),\\
&\pi_4 = \{\{c_0, c_1, b_{\mu-1}, b_\mu\}, \{c_j, b_{\mu-j}\};\ 2 \leqslant j \leqslant \mu\}\ (\mathscr{N}'_\mu),\\
&\pi_5 = \{\{c_0, c_1, b_\mu\}, \{c_{j+1}\}, \{b_j\};\ 0 \leqslant j \leqslant \mu - 1\}\ (\mathscr{N}'_\mu \times \mathscr{N}'_{\mu+1}),\\
&\pi_6 = \{\{c_0, c_1, b_\mu\}, \{c_j, b_{\mu+1-j}\}, \{b_0\};\ 2 \leqslant j \leqslant \mu\}\ (\mathscr{N}'_{\mu+1}),\\
&\pi_7 = \{\{c_0, b_\mu\}, \{c_j\}, \{b_{j-1}\};\ 1 \leqslant j \leqslant \mu\}\ (\mathscr{N}_\mu \times \mathscr{N}'_{\mu+1}),\\
&\pi_8 = \{\{c_j, b_{\mu-j}\};\ 0 \leqslant j \leqslant \mu\}\ (\mathscr{N}_\mu),\\
&\pi_9 = \{\{c_j\}, \{b_j\};\ 0 \leqslant j \leqslant \mu\}\ (\mathscr{L}_\mu).
\end{aligned}$$

From the above formulas and also Relation (1), we obtain immediately

$$\mathrm{FAC}(\mathscr{L}'_\mu) = \{\mathscr{I}_\Sigma, \mathscr{N}'_\mu, \mathscr{N}'_{\mu+1}, \mathscr{N}'_\mu \times \mathscr{N}'_{\mu+1}, \mathscr{L}'_\mu\},$$

thus $\mathscr{L}'_\mu$ is weakly irreducible, and

$$\mathscr{L}_\mu = \mathscr{N}_\mu \times \mathscr{N}_{\mu+1} = \mathscr{L}'_\mu \times \mathscr{N}_{\mu+1} = \mathscr{N}_\mu \times \mathscr{L}'_{\mu+1},$$

where the first equality existed already in [22].

Now let $\pi$ be a proper automaton partition of $\mathscr{L}_\mu$, and we shall show that it must be one of the above eight nontrivial elements.

*Case* I: $\pi(c_0)$ only contains one state. Since $\pi$ is proper, then there exists an integer $j$ $(0\leqslant j\leqslant\mu)$ such that $\pi(b_j)$ contains at least two elements, or there exists an integer $k$ $(1\leqslant k\leqslant\mu)$ such that $\pi(c_k)$ contains at least two states.

If there exists an integer $j$ $(0\leqslant j\leqslant\mu)$ such that $\pi(b_j)$ contains at least two elements, then $b_k\notin\pi(b_j)$ for all $k\neq j$. In fact if $b_k\in\pi(b_j)$ with $0\leqslant k\neq j\leqslant\mu$, then $b_{\mu+1-|k-j|}\in\pi(c_0)$, and it contradicts $\pi(c_0)=\{c_0\}$. So there must exist an integer $l$ $(1\leqslant l\leqslant\mu)$ such that $c_l\in\pi(b_j)$. Necessarily we should have $l=\mu-j+1$. Indeed if $\mu-j>l-1$, then $b_{j+l}\in\pi(c_0)$ with $1\leqslant j+l\leqslant\mu$, and we obtain again a contradiction with $\pi(c_0)=\{c_0\}$. If $\mu-j<l-1$, then $c_{l-\mu+j-1}\in\pi(c_0)$ and it contradicts also $\pi(c_0)=\{c_0\}$, for we have $l-\mu+j-1>0$.

Likewise if there exists an integer $k$ $(1\leqslant k\leqslant\mu)$ such that $\pi(c_k)$ contains at least two elements, then $c_h\notin\pi(c_k)$ for $1\leqslant h\neq k\leqslant\mu$, and we fall into the above case.

According to the value of the integer $j$, we should distinguish two possibilities.

(a) If $0\leqslant j<\mu$, then $c_{\mu-j+1}\equiv b_j \pmod{\pi}$, and thus $c_{\mu-k+1}\equiv b_k \pmod{\pi}$ for $j\leqslant k\leqslant\mu$. In particular $c_2\equiv b_{\mu-1} \pmod{\pi}$, and so $b_1\equiv c_\mu \pmod{\pi}$. In other words, we can suppose $j=1$. Hence $c_{\mu-k+1}\equiv b_k \pmod{\pi}$ for $1\leqslant k\leqslant\mu$, and consequently we have $\pi=\pi_1$.

(b) Now suppose that $j=\mu$ and there is not any other integer $j$ satisfying the above property. Then $c_1\in\pi(b_\mu)$, and we must have $\pi=\pi_2$.

*Case* II: $\pi(c_0)$ contains at least two states.

A. Suppose $c_1\in\pi(c_0)$. Then we have the following two possibilities.

(a) If $\pi(c_0)=\{c_0,c_1\}$, then for all integer $j$ $(0\leqslant j\leqslant\mu)$, we have $\pi(b_j)=\{b_j\}$.

First $b_k\notin\pi(b_j)$ for $0\leqslant k\neq j\leqslant\mu$, since $b_k\in\pi(b_j)$ implies $b_{\mu+1-|k-j|}\in\pi(c_0)$.

Now we show by absurdity $c_k\notin\pi(b_j)$ for $2\leqslant k\leqslant\mu$. Suppose that there exists an integer $k$ $(2\leqslant k\leqslant\mu)$ such that $c_k\in\pi(b_j)$. If $\mu-j>k-1$, then $b_{j+k}\in\pi(c_0)$ with $2\leqslant j+k\leqslant\mu$, and it contradicts $\pi(c_0)=\{c_0,c_1\}$. If $\mu-j<k-2$, then we have $c_{k-\mu+j-1}\in\pi(c_0)$ with $k-\mu+j-1>1$, and we obtain a similar contradiction as above. Hence $j+k-\mu=1$ or 2. In the first case, we would have $c_1\equiv b_\mu (\mathrm{mod}\,\pi)$, just as in Case I, and it contradicts again $\pi(c_0)=\{c_0,c_1\}$. In the second case, we would have $c_{\mu-j+2}\equiv b_j \pmod{\pi}$, and then $c_{\mu-l+2}\equiv b_l \pmod{\pi}$ for $j\leqslant l\leqslant\mu$. In particular $c_2\equiv b_\mu \pmod{\pi}$, and thus $b_1\equiv b_0$ $(\mathrm{mod}\,\pi)$, which is impossible.

Similarly we also have $\pi(c_j)=\{c_j\}$ for $2\leqslant j\leqslant\mu$. To see this, we only need check $c_k\notin\pi(c_j)$ for $3\leqslant k\neq j$. However if $c_k\in\pi(c_j)$, then $c_1\equiv c_{|k-j|+1} \pmod{\pi}$ with $|k-j|+1\geqslant 2$, and we obtain a contradiction with $\pi(c_0)=\{c_0,c_1\}$.

In conclusion, we have just obtained $\pi=\pi_3$.

(b) Now assume that $\pi(c_0)$ contains at least three elements. Clearly $b_0\notin\pi(c_0)$, for $b_0\in\pi(c_0)$ will imply $\pi=\pi_0$. Also we have $c_j\notin\pi(c_0)$ for $2\leqslant j\leqslant\mu$. In fact, if there exists an integer $j$ $(2\leqslant j\leqslant\mu)$ such that $c_j\in\pi(c_0)$, then for $0\leqslant k\leqslant j$, we must have $c_k\in\pi(c_0)$. Thus $b_0\equiv b_{k-1} \pmod{\pi}$ for $2\leqslant k\leqslant j$. In particular, we obtain $b_0\equiv b_1$ $(\mathrm{mod}\,\pi)$, which implies directly $b_0\equiv b_k \pmod{\pi}$ for $1\leqslant k\leqslant\mu$, and thus $b_0\in\pi(c_0)$. This is absurd. Hence there must exist an integer $j$ $(2\leqslant j\leqslant\mu)$ such that $b_j\in\pi(c_0)$. Again we need distinguish two different situations.

(i) Assume $1\leqslant j\leqslant\mu-1$. Then $c_1\equiv b_j \pmod{\pi}$ and $c_0\equiv b_j \pmod{\pi}$, from which we deduce $b_0\equiv c_{j+1} \pmod{\pi}$ and $b_k\in\pi(c_0)$ for $j\leqslant k\leqslant\mu$. Then from the latter, we obtain immediately $c_{k+1}\in\pi(b_0)$ for $j\leqslant k\leqslant\mu-1$, and thus $c_k\in\pi(b_1)$. If $j<\mu-1$, then we have $b_1\equiv c_{j+1} \pmod{\pi}$, and therefore $b_1\equiv b_0 \pmod{\pi}$. But this is impossible, as we have already shown above. So we must have $j=\mu-1$, and then $\pi(c_0)=\{c_0,c_1,b_{\mu-1},b_\mu\}$, $c_\mu \in \pi(b_0)$. Thus $c_k \in \pi(b_{\mu-k})$ for $0 \leqslant k \leqslant \mu$. Now remark that if $b_g \equiv b_h \pmod{\pi}$ for $0 \leqslant g < h < \mu-1$, then $b_{\mu-1-h+g} \in \pi(c_0)$ with $0 \leqslant \mu-1-h+g < \mu-1$, and this is absurd. Hence we have $\pi=\pi_4$.

(ii) Assume $j=\mu$ and $b_k \notin \pi(c_0)$ for $2 \leqslant k \leqslant \mu-1$. Then $\pi(c_0)=\{c_0,c_1,b_\mu\}$.

(1) If $\pi(b_k)=\{b_k\}$ for $0 \leqslant k \leqslant \mu-1$, then $\pi(c_{k+1})=\{c_{k+1}\}$ for $1 \leqslant k \leqslant \mu-1$, and thus $\pi=\pi_5$. In fact if $c_g \equiv c_h \pmod{\pi}$ with $2 \leqslant g < h \leqslant \mu$, then we would have $b_{g-1} \equiv b_{h-1} \pmod{\pi}$ with $1 \leqslant g-1 < h-1 \leqslant \mu-1$, and this contradicts our hypothesis $\pi(b_{g-1})=\{b_{g-1}\}$.

(2) Assume that there exists an integer $k$ $(0 \leqslant k \leqslant \mu-1)$ such that $\pi(b_k)$ contains at least two elements. Clearly $b_l \notin \pi(b_k)$ for $0 \leqslant l \neq k \leqslant \mu-1$. Indeed if there exists an integer $l$ $(0 \leqslant l \neq k \leqslant \mu-1)$ such that $b_l \in \pi(b_k)$, then we must have $b_{\mu-1-|l-k|} \in \pi(c_0)$ with $0 \leqslant \mu-1-|l-k| < \mu-1$, and this is absurd. So there exists an integer $g$ $(2 \leqslant g \leqslant \mu)$ such that $c_g \equiv b_k \pmod{\pi}$. If $\mu-k>g$, then we have $b_{k+g} \in \pi(c_0)$ with $2 \leqslant k+g < \mu$, which is impossible. If $\mu-k<g-1$, then we have $c_{g+k-\mu} \in \pi(c_0)$ with $2 \leqslant k+g-\mu$, and this is absurd again. Thus we must have $k+g-\mu=0$ or 1. If $k+g-\mu=0$, then $c_{\mu-k} \equiv b_k \pmod{\pi}$, and we obtain thus $c_{\mu-h} \equiv b_h \pmod{\pi}$ for $k \leqslant h \leqslant \mu$. In particular $c_1 \equiv b_{\mu-1} \pmod{\pi}$, and this is absurd. Therefore we must have $k+g-\mu=1$, and then $c_{\mu+1-k} \equiv b_k \pmod{\pi}$. Hence for $k \leqslant h \leqslant \mu$, we have $c_{\mu+1-h} \equiv b_h \pmod{\pi}$, and $b_{\mu-h} \equiv c_{h+1} \pmod{\pi}$. In particular $b_1 \equiv c_\mu \pmod{\pi}$, and thus $b_h \equiv c_{\mu+1-h} \pmod{\pi}$ for $1 \leqslant h \leqslant \mu$. Since for $0 \leqslant l \neq k \leqslant \mu-1$, we have $b_l \notin \pi(b_k)$, so $\pi=\pi_6$.

B. Now suppose $c_1 \notin \pi(c_0)$. Then $c_j \notin \pi(c_0)$ for $1 \leqslant j \leqslant \mu$. In fact, if there exists an integer $j \geqslant 2$ such that $c_j \in \pi(c_0)$, then $c_k \in \pi(c_0)$ for all integer $k \leqslant j$, thus $c_1 \in \pi(c_0)$ and it is absurd. Hence there exists an integer $j$ $(1 \leqslant j \leqslant \mu)$ such that $b_j \in \pi(c_0)$, which implies directly $b_k \in \pi(c_0)$ for $j \leqslant k \leqslant \mu$. If $1 \leqslant j \leqslant \mu-1$, then $c_{k+1} \in \pi(b_0)$ for $j \leqslant k \leqslant \mu-1$. In particular, we obtain $c_\mu \in \pi(b_0)$, and it gives $b_k \equiv c_{\mu-k} \pmod{\pi}$ for $0 \leqslant k \leqslant \mu$. Therefore $c_{\mu-k} \in \pi(c_0)$ for $j \leqslant k \leqslant \mu$, and so $c_1 \in \pi(c_0)$, which is absurd. Thus $j=\mu$, and $\pi(c_0)=\{c_0,b_\mu\}$.

(a) If $\pi(b_k)=\{b_k\}$ for $0 \leqslant k \leqslant \mu-1$, then we have $\pi(c_k)=\{c_k\}$ for $1 \leqslant k \leqslant \mu$, and consequently $\pi=\pi_7$. In fact if there exists integers $g,h$ $(1 \leqslant g < h \leqslant \mu)$ such that $c_g \equiv c_h \pmod{\pi}$, then $b_{g-1} \equiv b_{h-1} \pmod{\pi}$ with $0 \leqslant g-1 < h-1 \leqslant \mu-1$, and this contradicts our hypothesis $\pi(b_{g-1})=\{b_{g-1}\}$.

(b) Assume that there exists an integer $k$ $(0 \leqslant k \leqslant \mu-1)$ such that $\pi(b_k)$ contains at least two elements. Clearly $b_l \notin \pi(b_k)$ for $0 \leqslant l \neq k \leqslant \mu-1$. Indeed if there exists an integer $l$ $(0 \leqslant l \neq k \leqslant \mu-1)$ such that $b_l \in \pi(b_k)$, then we should have $b_{\mu-1-|l-k|} \in \pi(c_0)$ with $0 \leqslant \mu-1-|l-k| < \mu-1$, and this is absurd. So there exists an integer $g$ $(1 \leqslant g \leqslant \mu)$ such that $c_g \equiv b_k \pmod{\pi}$. If $\mu-k>g$, then we have $b_{k+g} \in \pi(c_0)$ with $1 \leqslant k+g < \mu$, and we obtain a contradiction with the fact $\pi(c_0)=\{c_0,b_\mu\}$. If $\mu-k<g$, then $c_{g+k-\mu} \in \pi(c_0)$ with $1 \leqslant k+g-\mu$, and this is also absurd. Consequently $g=\mu-k$, and $c_{\mu-k} \equiv b_k \pmod{\pi}$, from which we obtain $c_{\mu-h} \equiv b_h \pmod{\pi}$ for $k \leqslant h \leqslant \mu$. In particular $c_1 \equiv b_{\mu-1} \pmod{\pi}$, and thus $b_0 \equiv c_\mu \pmod{\pi}$. Then $c_{\mu-h} \equiv b_h \pmod{\pi}$ for $0 \leqslant h \leqslant \mu$. Since $b_l \notin \pi(b_k)$ for $0 \leqslant l \neq k \leqslant \mu-1$, thus $\pi(b_h)=\{b_h,c_{\mu-h}\}$ for $0 \leqslant k \leqslant \mu-1$, and $\pi=\pi_8$. □

In conclusion, we have just established the following result about the factor structure of Ising automata.

**Theorem 1.** *Let $\alpha \geqslant 0$ be a real number.*

(1) $\mathscr{A}_0$ *is irreducible and invertible, but it is neither faithful nor homogeneous;*

(2) $\mathscr{A}_\alpha$ $(\alpha \geqslant 4)$ *is strictly faithful, irreducible, homogeneous, and invertible;*

(3) *If* $0 < \alpha < 4$, *we denote* $\mathscr{A}_\alpha$ *by* $\mathscr{N}_\mu$ *or* $\mathscr{L}_\mu$ *according to* $4/\alpha \in \mathbb{N}$ *or not, where* $\mu = [4/\alpha]$ *is the integral part of* $4/\alpha$:

  (a) $\mathscr{N}_1$ *is strictly faithful, irreducible, homogeneous, and invertible;*

  (b) $\mathscr{N}_\mu$ $(\mu \geqslant 2)$ *is faithful but not strictly faithful, left-invertible but not invertible, weakly irreducible but not irreducible. Indeed it has three factors:* $\mathscr{I}_\Sigma$, $\mathscr{N}'_\mu$, *and* $\mathscr{N}_\mu$, *where* $\mathscr{N}'_\mu$ *is the factor corresponding to the automaton partition* $\pi_\mu = \{\{a_{\mu+1}, a_\mu\}, \{a_1\}, \ldots, \{a_{\mu-1}\}\}$ *of* $\mathscr{N}_\mu$;

  (c) $\mathscr{L}_\mu$ $(\mu \geqslant 1)$ *is left-invertible but not invertible. It is also faithful but not strictly faithful. Moreover* FAC$(\mathscr{L}_1)$ only contains seven elements: $\mathscr{I}_\Sigma$, $\mathscr{N}_2$, $\mathscr{L}'_1$, $\mathscr{N}'_2$, $\mathscr{N}_1 \times \mathscr{N}'_2$, $\mathscr{N}_1$, *and* $\mathscr{L}_1$; *while* FAC$(\mathscr{L}_\mu)$ $(\mu \geqslant 2)$ *contains ten elements:* $\mathscr{I}_\Sigma$, $\mathscr{N}_{\mu+1}$, $\mathscr{N}'_\mu \times \mathscr{N}_{\mu+1}$, $\mathscr{L}'_\mu$, $\mathscr{N}'_\mu$, $\mathscr{N}'_\mu \times \mathscr{N}'_{\mu+1}$, $\mathscr{N}'_{\mu+1}$, $\mathscr{N}_\mu \times \mathscr{N}'_{\mu+1}$, $\mathscr{N}_\mu$, *and* $\mathscr{L}_\mu$.

Now we discuss the topological properties of $(\mathscr{A}_\alpha, o_\alpha)$ (see also [22]). About this subject, we have the following theorem which is based heavily on the results obtained in the preceding section.

**Theorem 2.** *The Ising mapping* $\alpha \mapsto (\mathscr{A}_\alpha, o_\alpha)$ *is uniformly continuous from the interval* $[\alpha_0, +\infty)$ *to the topological vector space* $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \mathscr{S}_{\mathbb{C}}(\Sigma))$, *where* $\alpha_0 > 0$. *However it is only weakly but not strongly continuous at* $\alpha = 0$.

**Proof.** Let $\alpha_0 > 0$ be a real number. For all $\alpha, \beta \in [\alpha_0, +\infty)$, we can easily show that we have (see Theorem 4 in [22])

$$\| (\mathscr{A}_\alpha, o_\alpha) - (\mathscr{A}_\beta, o_\beta) \| \leqslant \left(1 + \left[\frac{4}{\alpha_0}\right]\right) |\alpha - \beta|,$$

hence the mapping $\alpha \mapsto (\mathscr{A}_\alpha, o_\alpha)$ is uniformly continuous from $[\alpha_0, +\infty)$ to the topological vector space $(\mathrm{AUTO}_{\mathbb{C}}(\Sigma), \| \cdot \|)$. Remark that for all $\alpha \geqslant \alpha_0$, we have

$$\mathrm{Card}(\mathscr{A}_\alpha) \leqslant 2[4/\alpha] + 1 \leqslant 2[4/\alpha_0] + 1,$$

thus by Proposition 16, the mapping $\alpha \mapsto (\mathscr{A}_\alpha, o_\alpha)$ is also uniformly continuous in $[\alpha_0, +\infty)$ for the strong topology $\mathscr{S}_{\mathbb{C}}(\Sigma)$, whence continuous over $(0, +\infty)$ for the same topology. This mapping is clearly weakly continuous at $\alpha = 0$. Whereas it is not strongly continuous at that point, since $\mathscr{A}_0$ cannot divide $\mathscr{A}_\alpha$ even when $\alpha$ is very close to 0 (see Proposition 14). Indeed at the neighborhood of $\alpha = 0$, the function $\alpha \mapsto \mathrm{Card}(\mathscr{A}_\alpha)$ is unbounded (see Proposition 15). $\square$

The above theorem, although simple in appearance, reveals a surprising fact: a discrete model such as Ising chain can imply underground the continuity! However its meaning in physics is rather self-evident (see [22]): the so-called Ising automata describe the induced field of the inhomogeneous Ising chain which depends on the external field. The induced field varies continuously with the external field, and therefore the family of Ising automata must be continuous!

To conclude this section, we indicate in the following a remarkable property of the above Ising family $(\mathscr{A}_\alpha, o_\alpha)_{\alpha\geqslant 0}$, pointed out to the author by Kamae in response to a question put forward by Mendès France.

For all $n\in\mathbb{N}$ with standard binary expansion $n=\sum_{j=0}^{k} n_j 2^j$, define

$$u_\alpha(n) = o_\alpha(t_\alpha(i_\alpha, (-1)^{n_k}(-1)^{n_{k-1}}\cdots(-1)^{n_0})).$$

The sequence $u_\alpha = (u_\alpha(n))_{n\geqslant 0}$ is clearly 2-automatic. We remark also that $u_0$ is just the Thue–Morse sequence in $\pm 2$, hence its correlative measure is singular continuous (see for example [4] or [32]), and $u_0$ itself is pseudo-random (cf. [6]). Now let $\alpha>0$. Since the Ising automaton $\mathscr{A}_\alpha$ is normalized, primitive, and faithful, so the symbolic dynamical system associated to $u_\alpha$ has discrete spectrum, in particular the correlative measure of $u_\alpha$ is discrete, and $u_\alpha$ itself is almost-periodic in the sense of Bertrandias (see [4, p. 336]). When $\alpha\geqslant 4$, the conclusion becomes much more precise: $u_\alpha$ is indeed periodic of period 2 (see Example 2). So when $\alpha$ drops from 4 to 0, the sequence $u_\alpha$ is at first periodic, then almost-periodic in the sense of Bertrandias, and finally pseudo-random. It marks a "phase transition" at the point $\alpha=0$. In physics, this means that when $\alpha>0$, the external magnetic field is present, so our system is ordered, and when $\alpha\geqslant 4$, i.e. $H\geqslant 2J$, the external magnetic field dominates the internal interaction, and thus the system becomes highly ordered. But when $\alpha=0$, the external magnetic field disappears, and the system is only governed by the internal interaction, so it becomes totally chaotic, i.e. pseudo-random.

For more details on Ising automata, see for example [3,4,22,26,27,34].

## 11. Further studies

As the reader can remark, many problems remains open. For example, we do not know whether prime automata exist, and although we know how to characterize irreducible automata, but until now we have no idea about the characterization of weakly irreducible automata. We have already pointed out that homogeneity or more general $\pi$-homogeneity is a product property, and being normalized is a factor property, and clearly these properties merit a more thorough and deep discussion. We have already characterized faithful and strictly faithful automata. It is also possible to study and characterize faithful and strictly faithful automata with output. The open problems listed here are certainly far from completeness, and we shall discuss and study all of them in a later work.

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