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Sinc-Galerkin Method for Solving Nonlinear Boundary-Value Problems

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Abstract—The sinc-Galerkin method is used to approximate solutions of nonlinear problems involving nonlinear second-, fourth-, and sixth-order differential equations with homogeneous and nonhomogeneous boundary conditions. The scheme is tested on four nonlinear problems. The results demonstrate the reliability and efficiency of the algorithm developed.

Keywords—Sinc-Galerkin, Sinc function, Nonlinear differential equations, Numerical solutions, Newton's method.

1. INTRODUCTION

Linear two-point boundary-value problems can be readily solved by many methods, e.g., shooting, band matrix, parallel shooting, collocation, Ritz-Galerkin [1], sinc-Galerkin [2–5]. Even singular linear two-point boundary-value problems can be handled by the Ritz-Galerkin method, as was shown by Jespersen [6]. Nonlinear problems result in a nonlinear system of equations to solve, and the typical suggestion [1] is that this system of equations be solved by a quasi-Newton method, or by embedding if a good initial approximation is not known.

Broyden's, Newton, and Steffensen's methods for solving a nonlinear system of equations are local in nature and may fail if the starting point is not close to the solution. Embedding is an attempt to overcome this difficulty, but unfortunately embedding also fails frequently due to "singular points" [1]. There are conditions, somewhat restrictive though, which preclude the existence of "singular points" [7].

Accurate and fast numerical solution of two-point boundary value ordinary differential equations is necessary in many important scientific and engineering applications, e.g., boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology.

The sinc-Galerkin methods for ordinary differential equations have many salient features due to the properties of the basis functions and the manner in which the problem is discretized. Of

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equal practical significance is the fact that the method's implementation requires no modification in the presence of singularities. The approximating discrete system depends only on parameters of the differential equation regardless of whether it is singular or nonsingular.

In this paper, we consider nonlinear differential equations of order $2m$, $m = 1, 2, 3$,

$$Lu = u^{(2m)} + \tau(x)uu' + \kappa(x)H(u) = f(x), \qquad 0 \le x \le 1, \tag{1.1}$$

subject to boundary conditions

$$u^{(j)}(0) = 0, \qquad u^{(j)}(1) = 0, \qquad 0 \le j \le m-1, \tag{1.2}$$

where $H(u)$ may be a polynomial or a rational function, or exponential. Due to the large number of different possibilities, our work will be focused mainly on the following forms $H(u)$:

- $H(u) = u^n$, $n > 1$,
- $H(u) = \exp(\pm u), \cos(u), \sin(u), \sinh(u), \cosh(u), \dots,$
- $H(u) = 1/(1 \pm u)^n, 1/(1 \pm u^2)^n, 1/(u^2 \pm 1)^n$, $n \neq 0$,

or any analytic function of u which has a power series expansion.

Agarwal and Akrivis [8] have discussed in detail the existence and uniqueness of (1.1),(1.2). Throughout this paper in keeping with Stenger [9], we shall assume that $u(x)$, $\tau(x)$, $\kappa(x)$, and $f(x)$ are analytic with respect to x in a neighborhood of $[0, 1]$.

The sinc-Galerkin method utilizes a modified Galerkin scheme to discretize (1.1),(1.2). The basis elements that are used in this approach are the sinc function composed with a suitable conformal map. A thorough description of the sinc function properties may be found in [9].

The outline of the paper is as follows. In Section 2, we review some of the main properties of sinc-Galerkin that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the sinc-Galerkin method may be used to replace equation (1.1) by an explicit system of nonlinear algebraic equations that is solved by *Newton's method*. Section 4 presents appropriate techniques to treat nonhomogeneous boundary conditions. Finally, some numerical examples are presented in Section 5, where the scheme is tested on four nonlinear problems. The results demonstrate the reliability and efficiency of the algorithm developed.

2. SINC FUNCTION PRELIMINARIES

The sinc function is defined on the whole real line by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \qquad -\infty < x < \infty. \tag{2.1}$$

For $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x - kh}{h}\right), \qquad k = 0 \pm 1, \pm 2, \dots. \tag{2.2}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(hk) \operatorname{sinc}\left(\frac{x - hk}{h}\right) \tag{2.3}$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of (2.3) have been extensively studied. A comprehensive survey of these approximation properties is found in [10].

To construct approximations on the interval $(0, 1)$, which are used in this paper, consider the conformal maps

$$\phi(z) = \ln\left(\frac{z}{1 - z}\right). \tag{2.4}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \le \frac{\pi}{2} \right\}, \tag{2.5}$$

onto the infinite strip

$$D_d = \left\{ \zeta = \xi + i\eta : |\eta| < d \le \frac{\pi}{2} \right\}. \tag{2.6}$$

The composition

$$S_j(x) = S(h,j) \circ \phi(x) = \operatorname{sinc}\left(\frac{\phi(x) - jh}{h}\right) \tag{2.7}$$

defines the basis element for equation (1.1) on the interval $(0,1)$. The "mesh size" h is the mesh size in D_d for the uniform grids $\{kh\}$, $-\infty < k < \infty$. The sinc grid points $z_k \in (0,1)$ in D_E will be denoted by x_k because they are real. The inverse images of the equispaced grids are

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}. \tag{2.8}$$

DEFINITION 2.1. *Let D_E be a simply connected domain in the complex plane $\mathbb{C}$, and let ∂D_E denote the boundary of D_E. Let a, b ($a \neq b$) be points on ∂D_E, and ϕ be a conformal map D_E onto D_d such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by ψ, define*

$$\Gamma = \{\psi(u) : -\infty < u < \infty\},$$

and $z_k = \psi(kh)$, $k = 0, \pm 1, \pm 2, \dots$.

DEFINITION 2.2. *Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy*

$$\int_{\psi(L+u)} |F(z)\,dz| \to 0, \qquad \text{as } u = \pm\infty, \tag{2.9}$$

where

$$L = \left\{ iy : |y| < d \le \frac{\pi}{2} \right\}, \tag{2.10}$$

and on the boundary of D_E (denoted ∂D_E) satisfy

$$T(F) = \int_{\partial D_E} |F(z)\,dz| < \infty. \tag{2.11}$$

The importance of the class $B(D_E)$ with regard to numerical integration is summarized in the following theorems [10].

THEOREM 2.1. *Let Γ be $(0,1)$, if $F \in B(D_E)$, then for $h > 0$ sufficiently small*

$$\int_\Gamma F(z)\,dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2} \int_{\partial D} \frac{F(z) k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}\,dz \equiv I_F, \tag{2.12}$$

where

$$|k(\phi,h)|_{z\in\partial D} = \left| \exp\left[\frac{i\pi\phi(z)}{h} \operatorname{sgn}(\operatorname{Im}\phi(z))\right] \right|_{z\in\partial D} = e^{-\pi d/h}. \tag{2.13}$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which exponential convergence results.

THEOREM 2.2. *If there exist positive constants α, β, and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C \begin{cases} \exp(-\alpha|\phi(x)|), & x \in \psi((-\infty,0)), \\ \exp(-\beta|\phi(x)|), & x \in \psi((0,\infty)), \end{cases} \tag{2.14}$$

then the error bound for the quadrature rule (2.12) is

$$\left|\int_\Gamma F(x)\,dx - h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha M h}}{\alpha} + \frac{e^{-\beta N h}}{\beta}\right) + |I_F|. \tag{2.15}$$

The infinite sum in (2.12) is truncated with the use of (2.14) to arrive at this inequality (2.15). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \tag{2.16}$$

and

$$N \equiv \left[\left|\frac{\alpha}{\beta} M + 1\right|\right], \tag{2.17}$$

where $[x]$ is the integer part of x, then

$$\int_\Gamma F(x)\,dx = h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha d M)^{1/2}}\right). \tag{2.18}$$

Theorems 2.1 and 2.2 are used to approximate the integrals that arise in the formulation of the discrete systems corresponding to equations (1.1),(1.2).

3. SINC-GALERKIN METHOD

We start with the case $H(u) = u^n$, where n is a nonnegative integer, and assume an approximate solution of the form

$$u_Q(x) = \sum_{j=-M}^{N} c_j S_j(x), \qquad Q = M + N + 1, \tag{3.1}$$

where $S_j(x)$ is the function $S(j,h) \circ \phi(x)$ for some fixed step size h. The unknown coefficients $\{c_j\}_{-M}^{N}$ in (3.1) are determined by orthogonalizing the residual $Lu_Q - f$ with respect to the functions $\{S_k\}_{k=-M}^{N}$. This yields the discrete system

$$\langle Lu_Q - f, S_k \rangle = 0, \tag{3.2}$$

for $k = -M, -M+1, \dots, N$. The weighted inner product $\langle\,,\rangle$ is taken to be

$$\langle g(x), f(x) \rangle = \int_0^1 g(x) f(x) w(x)\,dx. \tag{3.3}$$

Here, $w(x)$ plays the role of a weight function which is chosen depending on the boundary conditions, the domain, and the differential equation. For the case of $2m$-order boundary value problems, it is convenient to take

$$w(x) = \frac{1}{(\phi'(x))^m}. \tag{3.4}$$

A complete discussion on the choice of the weight function can be found in [3,9]. The most direct development of the discrete system for equation (3.1) is obtained by substituting (3.1) into (1.1). The system can then be expressed in integral form via (3.3). This approach, however, obscures the analysis which is necessary for applying sinc quadrature formulae to (3.2). An alternative approach is to analyze instead

$$\left\langle u^{(2m)}, S_k \right\rangle + \langle \tau u u', S_k \rangle + \langle \kappa u^n, S_k \rangle = \langle f, S_k \rangle, \qquad k = -M, \dots, N. \tag{3.5}$$

The method of approximating the integrals in (3.5) begins by integrating by parts to transfer all derivatives from u to S_k. The approximation of the last inner products on the right-hand side of (3.5)

$$\langle f, S_k \rangle = h \frac{f(x_k) w(x_k)}{\phi'(x_k)}. \tag{3.6}$$

We need the following two theorems.

THEOREM 3.1. *The following relations hold:*

$$\left\langle u^{(2m)}, S_k \right\rangle = h \sum_{j=-M}^{N} \sum_{i=0}^{2m} \frac{u(x_j)}{\phi'(x_j) h^i} \delta_{kj}^{(i)} g_{2m,i}(x_j), \tag{3.7}$$

for some functions $g_{2m,i}$ to be determined.

PROOF. The inner product with sinc basis element is given by

$$\left\langle u^{(2m)}, S_k \right\rangle = \int_0^1 u^{(2m)}(x) S_k(x) w(x)\, dx. \tag{3.8}$$

This expression contains $2m$ derivative of u but the desired result is the variable u with no derivatives. Integrating by parts to remove $2m$ derivatives from the dependent variable u leads to the equality

$$\left\langle u^{(2m)}(x), S_k(x) \right\rangle = B_x + \int_0^1 u(x) (S_k(x) w(x))^{(2m)}\, dx, \tag{3.9}$$

where the boundary term

$$B_x = \left[\sum_{i=0}^{2m-1} (-1)^i u^{(2m-1-i)} (S_k w)^{(i)} \right]_{x=0}^{1} = 0. \tag{3.10}$$

Setting

$$\frac{d^n}{d\phi^n}[S_k(x)] = S_k^{(n)}(x), \qquad 0 \le n \le 2m,$$

and noting that

$$\frac{d}{dx}[S_k(x)] = S_k^{(1)}(x) \phi'(x),$$

we obtain by expanding the derivatives under the integral in (3.9)

$$\left\langle u^{(2m)}, S_k(x) \right\rangle = \int_0^1 \left(\sum_{i=0}^{2m} u(x) S_k^{(i)}(x) g_{2m,i} \right) dx, \tag{3.11}$$

where $g_{2m,1}$ are given as the following.

CASE $m = 1$.

$$g_{2,2}(x) = w(\phi')^2, \qquad g_{2,1}(x) = w(\phi)'' + 2w'\phi', \qquad g_{2,0}(x) = w''. \tag{3.12}$$

CASE $m = 2$.

$$g_{4,0}(x) = w^{(4)}, \qquad g_{4,4}(x) = w(\phi')^4, \qquad g_{4,3}(x) = 6w(\phi')^2\phi'' + 4w'(\phi')^3, \tag{3.13}$$

$$g_{4,2}(x) = 3w(\phi'')^2 + 4w\phi'\phi''' + 12w'\phi'\phi'' + 6w''(\phi')^2, \tag{3.14}$$

$$g_{4,1}(x) = w(\phi)^4 + 4w'(\phi)''' + 6w''\phi'' + 4w'''\phi'. \tag{3.15}$$

CASE $m = 3$.

$$g_{6,0} = w^{(6)}, \qquad g_{6,6} = w(\phi')^6, \qquad g_{6,5} = 15w(\phi')^4\phi'' + 6w'(\phi')^5, \tag{3.16}$$

$$g_{6,4} = 20w\phi^{(3)}(\phi')^{(3)} + 45w(\phi')^2(\phi'')^2 + 60w'(\phi')^3\phi'' + 15w''(\phi')^{(4)}, \tag{3.17}$$

$$\begin{aligned} g_{6,3} &= 15w(\phi'')^3 + 15w(\phi')^2(\phi)^{(4)} + 60w\phi'\phi''\phi''' + 60w'(\phi')^2\phi''' \\ &\quad + 90w'\phi'(\phi'')^2 + 80w''\phi''(\phi')^2 + 20w'''(\phi')^3, \end{aligned} \tag{3.18}$$

$$\begin{aligned} g_{6,2} &= 10w(\phi''')^2 + 6w\phi'\phi^{(5)} + 15w\phi''\phi^{(4)} + 30w'\phi'\phi^{(4)} + 60w'\phi''\phi''' \\ &\quad + 60w''\phi'\phi''' + 45w''(\phi'')^2 + 60w'''\phi'\phi'' + 15w^{(4)}(\phi')^2, \end{aligned} \tag{3.19}$$

$$g_{6,1} = \phi^{(6)}w + 6\phi^{(5)}w' + 15\phi^{(4)}w^{(2)} + 20\phi^{(3)}w^{(3)} + 15\phi^{(2)}w^{(4)} + 6\phi'w^{(5)}. \tag{3.20}$$

Applying the sinc quadrature rule to the right-hand side of (3.11) and deleting the error terms yields (3.7). ∎

THEOREM 3.2. *The following relations hold:*

$$\langle \tau(x)uu', S_k\rangle = -\frac{h}{2}\sum_{j=-M}^{N}\frac{u^2(x_j)}{\phi'(x_j)}\left[\frac{1}{h}\delta_{kj}^{(1)}(\phi'\tau w)(x_j) + \delta_{kj}^{(0)}(\tau w)'(x_j)\right], \tag{3.21}$$

$$\langle \kappa(x)u^n, S_k\rangle = h\frac{w(x_k)u^n(x_k)\kappa(x_k)}{\phi'(x_k)}. \tag{3.22}$$

PROOF. For $\tau(x)uu'$, the inner product with sinc basis elements is given by

$$\langle \tau uu', S_k\rangle = \int_0^1 uu'(S_k\tau w)\,dx. \tag{3.23}$$

Integrating by parts to remove the first derivative from the dependent variable u leads to the equality

$$\langle \tau uu', S_k\rangle = B_1 - \frac{1}{2}\int_0^1 u^2(S_k\tau w)'\,dx, \tag{3.24}$$

where the boundary term is

$$B_1 = \left[\frac{1}{2}\left(u^2 S_k\tau w\right)\right]_{x=0}^{1} = 0, \tag{3.25}$$

and expanding the derivatives under the integral in (3.24) yields

$$\langle \tau(x)uu', S_k\rangle = -\frac{1}{2}\int_0^1 u^2(x)\left[S_k^{(1)}\phi'(\tau w) + S_k^{(0)}(\tau w)'\right]dx. \tag{3.26}$$

Applying the sinc quadrature rule to the right-hand side of (3.26) and deleting the error term yields (3.21).

For $\kappa(x)u^n$, the inner product with sinc basis elements can be evaluated directly by application of (2.18) and deleting the error term to yield (3.22). ∎

Replacing each term of (3.5) with the approximation defined in (3.7), (3.21), (3.22), and (3.6), respectively, and replacing $u(x_j)$ by c_j, and dividing by h, we obtain the following theorem.

THEOREM 3.3. *If the assumed approximate solution of the boundary-value problem (1.1),(1.2) is (3.1), then the discrete sinc-Galerkin system for the determination of the unknown coefficients* $\{c_j\}_{j=-M}^{N}$ *is given, for* $k = -M, \dots, N$, *by*

$$\begin{aligned} \sum_{j=-M}^{N}\sum_{i=0}^{2m}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2m,i}(x_j)}{\phi'(x_j)}c_j - \frac{1}{2}\left[\sum_{j=-M}^{N}\frac{1}{h}\delta_{kj}^{(1)}(\tau w)(x_j)c_j^2 + \frac{(\tau w)'(x_k)}{\phi'(x_k)}c_k^2\right] \\ + \frac{\kappa(x_k)w(x_k)}{\phi'(x_k)}c_k^n = \frac{f(x_k)w(x_k)}{\phi'(x_k)}. \end{aligned} \tag{3.27}$$

The following notation will be necessary for writing down the system. Let $D(g)$ be the $Q \times Q$ diagonal matrix:

$$D(g) = \begin{pmatrix} g(x_{-M}) & & & \\ & g(x_{-M+1}) & & \\ & & \cdots & \\ & & & g(x_N) \end{pmatrix}. \tag{3.28}$$

We need the following two lemmas.

LEMMA 3.1. *(See [5].) Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (2.4). Then,*

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{3.29}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{3.30}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & j = k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \tag{3.31}$$

$$\delta_{jk}^{(3)} = h^3\frac{d^3}{d\phi^3}[S(j,h) \circ \phi(x)]_{x=x_k} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{k-j}}{(k-j)^3}\left[6 - \pi^2(k-j)^2\right], & j \neq k, \end{cases} \tag{3.32}$$

and

$$\delta_{jk}^{(4)} = h^4\frac{d^4}{d\phi^4}[S(j,h) \circ \phi(x)]_{x=x_k} = \begin{cases} \dfrac{\pi^4}{5}, & j = k, \\ \dfrac{-4(-1)^{k-j}}{(k-j)^4}\left[6 - \pi^2(k-j)^2\right], & j \neq k. \end{cases} \tag{3.33}$$ ∎

With some computations, one can prove the following lemma.

LEMMA 3.2. *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (2.4). Then,*

$$\delta_{jk}^{(5)} = h^5\frac{d^5}{d\phi^5}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \kappa_{jk}, & j \neq k, \end{cases} \tag{3.34}$$

where $\kappa_{jk} = ((-1)^{k-j}/(k-j)^5)[120 - 20\pi^2(k-j)^2 + \pi^4(k-j)^4]$,

$$\delta_{jk}^{(6)} = h^6\frac{d^6}{d\phi^6}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} \dfrac{\pi^6}{7}, & j = k, \\ \mu_{jk}, & j \neq k, \end{cases} \tag{3.35}$$

where

$$\mu_{jk} = \frac{-6(-1)^{k-j}}{(k-j)^6}\left[120 - 20\pi^2(k-j)^2 + \pi^4(k-j)^4\right].$$ ∎

Define the $Q \times Q$ matrices $\mathbf{I}^{(p)}$ (see [11]) for $0 \leq p \leq 2m$ by

$$I^{(p)} = \left[\delta_{jk}^{(p)}\right], \qquad j,k = -M,\dots,N. \tag{3.36}$$

Let $\mathbf{c}$ be the Q-vector with j^{th} component given by c_j, and $\mathbf{c}^n$ be the Q-vector with j^{th} component given by c_j^n, and $\mathbf{1}$ is an Q-vector each of whose components are 1. In this notation, the system in (3.27) takes the matrix form

$$\mathbf{A}\begin{pmatrix} c_{-M} \\ c_{-M+1} \\ \vdots \\ c_N \end{pmatrix} + \mathbf{B}\begin{pmatrix} c_{-M}^2 \\ c_{-M+1}^2 \\ \vdots \\ c_N^2 \end{pmatrix} + \mathbf{E}\begin{pmatrix} c_{-M}^n \\ c_{-M+1}^n \\ \vdots \\ c_N^n \end{pmatrix} = \Theta, \tag{3.37}$$

where

$$\mathbf{B} = \frac{-1}{2}\left[\frac{1}{h}I^{(1)}D(\tau w) + I^{(0)}D\left(\frac{(\tau w)'}{\phi'}\right)\right], \tag{3.38}$$

$$\mathbf{E} = D\left(\frac{\kappa w}{\phi'}\right), \tag{3.39}$$

$$\Theta = D\left(\frac{wf}{\phi'}\right)\mathbf{1}, \tag{3.40}$$

and

$$\mathbf{A} = \sum_{j=0}^{2m} \frac{1}{h^j}\mathbf{I}^{(j)}\mathbf{D}\left(\frac{g_{2m,j}}{\phi'}\right). \tag{3.41}$$

Now, we have a nonlinear system of $Q = M + N + 1$ equations of the Q unknown coefficients, namely, $\{c_j\}_{j=-M}^{N}$. We can obtain the coefficients of the approximate solution by solving this nonlinear system by *Newton's method* [12–17]. The solution $\mathbf{c} = (c_{-M}, \dots, c_N)^\top$ gives the coefficients in the approximate sinc-Galerkin solution $u_m(x)$ of $u(x)$.

Newton's Method

To solve the system of equations (3.37), we write it in the form

$$\mathbf{F}(\mathbf{c}) = \begin{pmatrix} F_{-M}(c_{-M}, c_{-M+1}, \dots, c_N) \\ F_{-M+1}(c_{-M}, c_{-M+1}, \dots, c_N) \\ \vdots \\ F_N(c_{-M}, c_{-M+1}, \dots, c_N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.42}$$

where $\mathbf{c}$ is the column vector of independent variables and $\mathbf{F}$ is the column vector of the functions F_j, with $F_j(\mathbf{c}) = F_j(c_{-M}, c_{-M+1}, \dots, c_N)$, $-M \le j \le N$. The number of functions that are set equal to zero is equal to the number of independent variables. A very good method for solving equation (3.42) is Newton's method.

Let $\mathbf{c}^{(i)}$ be the guess at the solution for iteration i. Let $\mathbf{F}^{(i)}$ denote the value of $\mathbf{F}$ at the i^{th} iteration. Assuming that $\|\mathbf{F}^{(i)}\|$ is not too small, we seek update vectors $\Delta\mathbf{c}^{(i)}$

$$\mathbf{c}^{(i+1)} = \mathbf{c}^{(i)} + \Delta\mathbf{c}^{(i)} \Longleftrightarrow \begin{pmatrix} c_{-M}^{(i+1)} \\ c_{-M+1}^{(i+1)} \\ \vdots \\ c_N^{(i+1)} \end{pmatrix} = \begin{pmatrix} c_{-M}^{(i)} \\ c_{-M+1}^{(i)} \\ \vdots \\ c_N^{(i)} \end{pmatrix} + \begin{pmatrix} \Delta c_{-M}^{(i)} \\ \Delta c_{-M+1}^{(i)} \\ \vdots \\ \Delta c_N^{(i)} \end{pmatrix}, \tag{3.43}$$

such that $\mathbf{F}(\mathbf{c}^{(i+1)}) = 0$. Using the multidimensional extension of Taylor's theorem to approximate the variation of $\mathbf{F}(\mathbf{c})$ in the neighborhood of $\mathbf{c}^{(i)}$ gives

$$\mathbf{F}\left(\mathbf{c}^{(i)} + \Delta\mathbf{c}^{(i)}\right) = \mathbf{F}\left(\mathbf{c}^{(i)}\right) + \mathbf{F}'\left(\mathbf{c}^{(i)}\right)\Delta\mathbf{c}^{(i)} + O\left(\left\|\Delta\mathbf{c}^{(i)}\right\|^2\right), \tag{3.44}$$

where $\mathbf{F}'(\mathbf{c}^{(i)})$ is the *Jacobian* of the system of equations

$$\mathbf{F}'(\mathbf{c}) \equiv \mathbf{J}(\mathbf{c}) = \begin{pmatrix} \dfrac{\partial F_{-M}}{\partial c_{-M}}(\mathbf{c}) & \dfrac{\partial F_{-M}}{\partial c_{-M+1}}(\mathbf{c}) & \cdots & \dfrac{\partial F_{-M}}{\partial c_N}(\mathbf{c}), \\ \dfrac{\partial F_{-M+1}}{\partial c_{-M}}(\mathbf{c}) & \dfrac{\partial F_{-M+1}}{\partial c_{-M+1}}(\mathbf{c}) & \cdots & \dfrac{\partial F_{-M+1}}{\partial c_N}(\mathbf{c}), \\ \vdots & & \ddots & \\ \dfrac{\partial F_N}{\partial c_{-M}}(\mathbf{c}) & \dfrac{\partial F_N}{\partial c_{-M+1}}(\mathbf{c}) & \cdots & \dfrac{\partial F_N}{\partial c_N}(\mathbf{c}). \end{pmatrix}. \tag{3.45}$$

Neglecting higher order terms and designating $J^{(i)}$ as the Jacobian evaluated at $\mathbf{c}^{(i)}$. We can rearrange equation (3.44)

$$\mathbf{F}\left(\mathbf{c}^{(i)} + \Delta\mathbf{c}^{(i)}\right) = \mathbf{F}\left(\mathbf{c}^{(i)}\right) + \mathbf{J}^{(i)}\Delta\mathbf{c}^{(i)}. \tag{3.46}$$

The goal of Newton iterations is to make $\mathbf{F}(\mathbf{c}^{(i)} + \Delta\mathbf{c}^{(i)}) = 0$, so setting that term to zero in the preceding equation gives

$$\mathbf{J}^{(i)}\Delta\mathbf{c}^{(i)} = -\mathbf{F}\left(\mathbf{c}^{(i)}\right). \tag{3.47}$$

Equation (3.47) is a system of Q linear equations in the Q unknown $\Delta\mathbf{c}^{(i)}$. Each Newton iteration step involves evaluation of the vector $\mathbf{F}^{(i)}$, the matrix $\mathbf{J}^{(i)}$, and the solution to equation (3.47). A common numerical practice is to stop the Newton iteration whenever the distance between two iterates is less than a given tolerance, i.e., when $\|\mathbf{c}^{(i+1)} - \mathbf{c}^{(i)}\| \le \varepsilon$.

Algorithm

- initialize $\mathbf{c} = \mathbf{c}^{(0)}$,
- for $i = 0, 1, 2, \dots$, $\mathbf{F}^{(i)} = \mathbf{A}\mathbf{c}|^{(\mathrm{i})} + \mathbf{B}\mathbf{c}^2|^{(\mathrm{i})} + \mathbf{E}\mathbf{c}^n|^{(\mathrm{i})} - \mathbf{\Theta}$,
- if $\|\mathbf{F}^{(i)}\|$ is small enough, stop,
- compute $\mathbf{J}^{(i)}$,
- solve $\mathbf{J}^{(i)}\Delta\mathbf{c}^{(i)} = -\mathbf{F}(\mathbf{c}^{(i)})$,
- $\mathbf{c}^{(i+1)} = \mathbf{c}^{(i)} + \Delta\mathbf{c}^{(i)}$,
- end.

Also, some of the well-known techniques we can use in solving equation (3.37) are the quasi-Newton and secant methods; for more detail, see [18–21].

4. TREATMENT OF THE BOUNDARY CONDITION

In the previous section the development of the sinc-Galerkin technique for homogeneous boundary conditions provided a practical approach since the sinc function composed with various conformal mappings, $S(j,h)\circ\phi$, are zero at the endpoints of the interval. If the boundary conditions are nonhomogeneous, then these conditions need be converted to homogeneous ones via an interpolation by a known function. For example, consider

$$u^{(2m)} + \tau(x)uu' + \kappa(x)u^n = f(x), \qquad 0 \le x \le 1, \tag{4.1}$$

subject to boundary conditions

$$u^{(i)}(0) = R_i, \quad u^{(i)}(1) = T_i, \qquad 0 \le i \le m-1. \tag{4.2}$$

The nonhomogeneous boundary conditions in (4.2) can be transformed to homogeneous boundary conditions by the change of dependent variable

$$W(x) = u(x) - \Lambda(x), \tag{4.3}$$

where $\Lambda(x)$ is the interpolating polynomial that satisfies $\Lambda^{(i)}(0) = R_i$ and $\Lambda^{(i)}(1) = T_i$, $0 \le i \le m-1$

$$\Lambda(x) = \sum_{i=0}^{2m-1} \mu_i x^i. \tag{4.4}$$

It is easy to see the following.

CASE $m = 1$. $\mu_0 = R_0,\ \mu_1 = T_0 - R_0$.

CASE $m = 2$. $\mu_0 = R_0,\ \mu_1 = R_1,\ \mu_2 = 3T_0 - T_1 - 2R_1 - 3R_0,\ \mu_3 = T_1 - 2T_0 + R_1 + 2R_0$.

CASE $m = 3$. $\mu_0 = R_0,\ \mu_1 = R_1,\ \mu_2 = R_2/2$,

$$\begin{aligned}
\mu_3 &= \frac{1}{2}[(20T_0 - 8T_1 + T_2) - (20R_0 + 12R_1 + 3R_2)],\\
\mu_4 &= \left[(-15T_0 + 7T_1 - T_2) + \left(15R_0 + 8R_1 + \frac{3}{2}R_2\right)\right],\\
\mu_5 &= \frac{1}{2}[(12T_0 - 6T_1 + T_2) - (12R_0 + 6R_1 + R_2)].
\end{aligned}$$

The new problem with homogeneous boundary conditions is then

$$W^{(2m)} + \tau(x)[\Lambda + W]W^{(1)} + \tau(x)\Lambda^{(1)}W + \kappa(x)\sum_{k=0}^{n-1}\binom{n}{k}W^{n-k}\Lambda^k = \tilde{f}(x), \qquad 0 \le x \le 1, \tag{4.5}$$

subject to the boundary conditions

$$W^{(i)}(0) = 0, \quad W^{(i)}(1) = 0, \qquad 0 \le i \le m-1, \tag{4.6}$$

where

$$\tilde{f}(x) = f(x) - \tau(x)\Lambda\Lambda^{(1)} - \kappa(x)\Lambda^n. \tag{4.7}$$

Now, apply the standard sinc-Galerkin method to (4.5). We define an approximate solution of (4.5) via the formula

$$W_Q(x) = \sum_{j=-M}^{N} c_j S_j(x), \qquad Q = M + N + 1. \tag{4.8}$$

Then, the approximate solution of (4.1) is

$$u_Q(x) = \sum_{j=-M}^{N} c_j S_j(x) + \Lambda(x). \tag{4.9}$$

5. NUMERICAL RESULTS

In this section, four nonlinear problems will be tested by using the sinc Galerkin method discussed above. For comparison reasons, the problems have homogeneous and nonhomogeneous boundary conditions and known solutions. As will be demonstrated by the numerical results, the boundary singularities have no adverse effect on the performance of the method. All the experiments were performed in MATLAB. In our tests, the zero vector is the initial guess and the stopping criterion is $\|\mathbf{c}^{(j+1)} - \mathbf{c}^{(j)}\| < 10^{-6}$.

In all the examples we take $d = \pi/2$. Once M is chosen, the step size and remaining summation limit can be determined as follows:

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \qquad N = \left[\left|\frac{\alpha M}{\beta}\right|\right],$$

where $[x]$ is the integer part of x. Note that if α/β is an integer, it suffices to choose $N = (\alpha/\beta)M$. We use absolute relative error which is defined as

$$\text{absolute relative error} = \frac{|U_{\text{exact solution}} - U_{\text{sinc-Galerkin}}|}{|U_{\text{exact solution}}|}. \tag{5.1}$$

For the sake of comparison only, we will discuss the first examples that were investigated by Chawla and Katti [22], Agarwal [8], and Twizell and Tirmizi [23].

EXAMPLE 1. (See [8,22,23].) Consider the boundary value problem

$$u^{(4)} = 6\exp(-4u) - 12(1+x)^{-4}, \qquad 0 < x < 1, \tag{5.2}$$

subject to boundary conditions

$$u(0) = 0, \qquad u(1) = \ln 2, \qquad u'(0) = 1, \qquad u'(1) = 0.5, \tag{5.3}$$

which has the exact solution given by $u(x) = \ln(1+x)$.

The parameters are selected so that $\alpha = \beta = 1/2$ and $M = 60$. The exact and approximate solutions and the absolute relative error are displayed in Table 1.

In Table 2, we compare the results obtained by the Sinc-Galerkin method with those obtained by Chawla and Katta, using a fourth-order finite difference method, Agarwal and Akrivis, using the finite difference method, and Twizell and Tirmizi [23], using a fourth-order multiderivative method.

EXAMPLE 2. Consider the boundary value problem

$$u'' + uu' + u^3 = \frac{1}{x} + x\ln x(1+\ln x) + (x\ln x)^3, \qquad 0 \le x \le 1, \tag{5.4}$$

subject to boundary conditions

$$u(0) = 0, \qquad u(1) = 0, \tag{5.5}$$

which has the exact solution given by $u(x) = x\ln x$.

Table 1.

x	Exact Solution	Sinc-Galerkin	Relative Error 1.0e − 10
0.0	0.0	0.0	—
0.08065	0.077568262040	0.077568262046	0.06
0.16488	0.152623517296	0.152623517297	0.1
0.22851	0.205803507218	0.205803507212	0.04
0.39997	0.336452906454	0.336452906455	0.01
0.5	0.405465108108	0.405465108103	0.04
0.69235	0.526121481267	0.526121481263	0.03
0.77148	0.571819991855	0.571819991858	0.06
0.88369	0.633234913798	0.633234913793	0.04
0.94474	0.665133248137	0.665133248135	0.02
1.0	0.693147180559	0.693147180559	0.0

Table 2. Error norms.

Sinc-Galerkin	Chawla and Katti [22]	Agarwal and Akrivis [8]	Twizell and Tirmizi [23]
0.5E−8	2.9E−7	5.4E−8	0.26E−7

Table 3.

x	Exact Solution	Sinc-Galerkin	Absolute Relative Error 1.0e − 06
0.0	0.0	0.0	—
0.07701	− 0.19744378	− 0.19744377	0.06
0.12058	− 0.25508370	− 0.25508365	0.20
0.27022	− 0.35359087	− 0.35359081	0.15
0.37830	− 0.36773296	− 0.36773296	0.02
0.5	− 0.34657359	− 0.34657353	0.14
0.62169	− 0.29549755	− 0.29549756	0.02
0.72977	− 0.22989603	− 0.22989600	0.16
0.87941	− 0.11300194	− 0.11300192	0.20
0.97002	− 0.02951702	− 0.02951703	0.23
1.0	0.0	0.00	—

In this problem the function $f(x)$ has a singularity at $x = 0$. The parameters $M = 40$ and $\alpha = \beta = 1/2$ are used. The exact, the approximate solutions, and absolute relative error are displayed in Table 3.

EXAMPLE 3. Consider the boundary value problem

$$u^{(4)} + \frac{x^2}{1+u^2} = -72\left(1-5x+5x^2\right) + \frac{x^2}{1+\left(x-x^2\right)^6}, \qquad 0 < x < 1, \tag{5.6}$$

subject to boundary conditions

$$u(0) = 0, \qquad u(1) = 0, \qquad u'(0) = 0, \qquad u'(1) = 0, \tag{5.7}$$

which has the exact solution given by $u(x) = x^3(1-x)^3$.

By writing

$$\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + u^8 + \cdots,$$

equation (5.6) becomes

$$u^{(4)} + x^2\left(1 - u^2 + u^4 - u^6 + u^8 + \cdots\right) = -72\left(1-5x+5x^2\right) + \frac{x^2}{1+\left(x-x^2\right)^6}, \quad 0 < x < 1. \tag{5.8}$$

When equation (5.8) is solved by the sinc-Galerkin method, we get a discrete system of the form

$$A\mathbf{c} - E\mathbf{c}^2 + E\mathbf{c}^4 - E\mathbf{c}^6 + \cdots = \Theta, \tag{5.9}$$

where A, E, and Θ are defined by equations (3.39)–(3.41) but f changes to

$$f(x) = -x^2 - 72\left(1-5x+5x^2\right) + \frac{x^2}{1+\left(x-x^2\right)^6}.$$

The parameters selected are $\alpha = \beta = 1/2$ and $M = 30$. The exact, the approximate solutions, and absolute relative error are displayed in Table 4.

In the following example, we have nonhomogeneous boundary conditions.

EXAMPLE 4. Consider the boundary value problem

$$u^{(6)} + e^{-x}u^2 = e^{-x} + e^{-3x}, \qquad 0 \le x \le 1, \tag{5.10}$$

Table 4.

x	Exact Solution	Sinc-Galerkin	Relative Error 1.0e − 03
0.0	0.0	0.0	—
0.0537	0.0001316	0.0001316	0.0
0.0915	0.0005760	0.0005759	0.21
0.2410	0.0061203	0.0061173	0.48
0.3604	0.0122489	0.0122509	0.20
0.5	0.0156250	0.0156245	0.02
0.7589	0.0061203	0.0061222	0.30
0.9084	0.0005760	0.0005759	0.21
0.9462	0.0001316	0.0001316	0.0
0.9822	0.0000052	0.0000052	0.0
1.0	0.0	0.00	—

Table 5.

x	Exact Solution	Sinc-Galerkin	Relative Error 1.0e − 03
0.0	1.0	1.0	0.0
0.0089	0.99113	0.99113	0.0
0.0414	0.95942	0.95942	0.0
0.1721	0.84189	0.84189	0.0
0.3131	0.73113	0.73114	0.01
0.5	0.60653	0.60655	0.04
0.6868	0.50316	0.50320	0.08
0.8278	0.43696	0.43701	0.09
0.9134	0.40114	0.40118	0.1
0.9585	0.38343	0.38347	0.1
1.0	0.36787	0.36787	0.0

subject to boundary conditions

$$u(0) = 1, \qquad u'(0) = -1, \qquad u''(0) = 1,$$
$$u(1) = \frac{1}{e}, \qquad u'(1) = -\frac{1}{e}, \qquad u''(1) = \frac{1}{e},$$

which has the exact solution given by $u(x) = e^{-x}$.

The parameters selected are $\alpha = \beta = 1/2$ and $M = 16$. The exact, the approximate solutions, and absolute relative error are displayed in Table 5.

6. CONCLUSION

The results of the previous section indicate that our procedure can be used to obtain accurate numerical solutions of nonlinear boundary value problem with very little computational effort. The accuracy of our methods depends on the magnitude of M. The results of Example 2 clearly indicate that our methods are accurate even when singularities occur at the boundaries.

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