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On a singular and nonhomogeneous N-Laplacian equation involving critical growth ☆

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ABSTRACT

In this paper we apply minimax methods to obtain existence and multiplicity of weak solutions for singular and nonhomogeneous elliptic equation of the form

$$-\Delta_N u = \frac{f(x,u)}{|x|^a} + h(x) \quad \text{in } \Omega,$$

where $u \in W_0^{1,N}(\Omega)$, $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian, $a \in [0,N)$, Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ containing the origin and $h \in (W_0^{1,N}(\Omega))^* = W^{-1,N'}$ is a small perturbation, $h \not\equiv 0$. Motivated by a singular Trudinger–Moser inequality, we study the case when $f(x,s)$ has the maximal growth on s which allows to treat this problem variationally in the Sobolev space $W_0^{1,N}(\Omega)$.

1. Introduction

In this paper we study the multiplicity of critical points for the functional

$$I(u) = \frac{1}{N}\int_\Omega |\nabla u|^N \,dx - \int_\Omega \frac{F(x,u)}{|x|^a}\,dx - \int_\Omega h(x)u\,dx, \tag{1.1}$$

where $u \in W_0^{1,N}(\Omega)$, $a \in [0,N)$ and Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geqslant 2$ containing the origin. We also assume that $h \in (W_0^{1,N}(\Omega))^* = W^{-1,N'}$ is a small perturbation with $N' = N/(N-1)$, $h \not\equiv 0$.

Here $W_0^{1,N}(\Omega)$ denotes the Sobolev space modeled in $L^N(\Omega)$ with the norm $\|u\| = (\int_\Omega |\nabla u|^N\,dx)^{1/N}$ and $W^{-1,N'}$ denotes the dual space of $W_0^{1,N}(\Omega)$ with the usual norm $\|\cdot\|_*$.

Our aim goal is to investigate existence of critical points of the functional I when the nonlinear term $f(x,s) = F_s(x,s)$ has the maximal growth on s for which the functional I can be studied on the $W_0^{1,N}$-setting. Such critical points are weak solutions of the associated Euler–Lagrange equation involving singular term of the form

$$\begin{cases} -\Delta_N u = \dfrac{f(x,u)}{|x|^a} + h(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian. We study (1.2) when $f(x,s)$ has subcritical or critical growth, which we define next. We say that $f(x,s)$ has *subcritical growth* at $+\infty$ if

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha|s|^{N/(N-1)}}}=0,\quad \text{uniformly on } x\in\Omega,\ \text{for all } \alpha>0 \tag{1.3}$$

and $f(x,s)$ has *critical growth* at $+\infty$ if there exists $\alpha_0>0$ such that

$$\lim_{s\to+\infty}\frac{|f(x,s)|}{e^{\alpha|s|^{N/(N-1)}}}=\begin{cases}0, & \text{uniformly on } x\in\Omega,\ \text{for all } \alpha>\alpha_0,\\ +\infty, & \text{uniformly on } x\in\Omega,\ \text{for all } \alpha<\alpha_0.\end{cases} \tag{1.4}$$

Similarly we define subcritical and critical growth at $-\infty$.

Let us introduce the precise assumptions under which our problem is studied.

(f_0) $f(x,s)\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, $f(x,0)=0$ for all $x\in\Omega$;

(f_1) there exist $\theta>N$ and $s_1>0$ such that for all $|s|\geqslant s_1$ and $x\in\Omega$,

$$0<\theta F(x,s)=\theta\int_0^s f(x,t)\,dt\leqslant sf(x,s);$$

(f_2) there exist constants $R, M>0$ such that for all $|s|\geqslant R$ and $x\in\Omega$,

$$0<F(x,s)\leqslant M\big|f(x,s)\big|.$$

(f_3) $\limsup_{s\to0}\frac{NF(x,s)}{|s|^N}<\lambda_1$,

where λ_1 is first eigenvalue of the following nonlinear eigenvalue problem

$$-\operatorname{div}\big(|\nabla u|^{N-2}\nabla u\big)=\frac{\lambda|u|^{N-2}u}{|x|^a},\quad u\in W_0^{1,N}(\Omega). \tag{1.5}$$

It is well known (cf. [5,9]) that there exists a smallest positive eigenvalue, which we denote by λ_1, and an associated eigenfunction $\psi_1>0$ in Ω that solves (1.5). Moreover λ_1 is a simple eigenvalue (that is, any two solutions u, v of (1.5) satisfy $u=cv$ for some constant c) and is variationally characterized as

$$\lambda_1=\inf\left\{\int_\Omega|\nabla u|^N\,dx:\ \int_\Omega\frac{|u|^N}{|x|^a}\,dx=1\right\}.$$

Remark 1.1. Let us briefly recall some important facts about the notion of critical growth in the Sobolev spaces $W^{1,p}$ in the case $p=N$ (Trudinger–Moser case). In this case, the notion of criticality is motivated by the so-called Trudinger–Moser inequality [13,20,21,23], which says that if $u\in W_0^{1,N}(\Omega)$ then $e^{\alpha|u|^{N/(N-1)}}\in L^1(\Omega)$, for all $\alpha>0$. Moreover, there exists a constant $C=C(N)>0$ such that

$$\sup_{\|u\|\leqslant1}\int_\Omega e^{\alpha|u|^{N/(N-1)}}\,dx\leqslant C|\Omega|\quad \text{if } \alpha\leqslant\alpha_N, \tag{1.6}$$

where $\alpha_N=N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. We would like to point out that in (1.2) we have the presence of a singular term $|x|^{-a}$ which prevents us to use the classical Trudinger–Moser inequality, so we use the following version of the Trudinger–Moser inequality due to Adimurthi–Sandeep [2]:

Proposition 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N\geqslant2)$ containing the origin and $u\in W_0^{1,N}(\Omega)$. Then for every $\alpha>0$ and $a\in[0,N)$,*

$$\int_\Omega\frac{e^{\alpha|u|^{N/(N-1)}}}{|x|^a}\,dx<\infty.$$

Moreover,

$$\sup_{\|u\|\leqslant1}\int_\Omega\frac{e^{\alpha|u|^{N/(N-1)}}}{|x|^a}\,dx<\infty\quad \textit{iff}\quad \alpha/\alpha_N+a/N\leqslant1.$$

Here, we search weak solutions of (1.2), that is, functions $u \in W_0^{1,N}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{N-2}\nabla u \nabla v \,\mathrm{d}x - \int_\Omega \frac{f(x,u)}{|x|^a} v \,\mathrm{d}x - \int_\Omega h(x) v \,\mathrm{d}x = 0, \quad \forall v \in W_0^{1,N}(\Omega). \tag{1.7}$$

Observe that if $f(x,s)$ has subcritical or critical growth, in view of Proposition 1.1, the expression in (1.7) is well defined on $W_0^{1,N}(\Omega)$ and moreover, critical points of the functional I are precisely the weak solutions of problem (1.2), for more details see Section 2.

Remark 1.2. Condition (f_3) is natural, since if $N = 2$ and $h \geqslant 0$, one can prove that the problem

$$-\Delta u = \frac{\lambda_1 u + 2ue^{u^2} - 2u}{|x|^a} + h(x) \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega$$

does not have positive solutions.

The main features of the class of problems considered in this paper, are the presence of singularity $|x|^{-a}$, critical growth and the nonlinear operator $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$. In spite of a possible failure of the Palais–Smale compactness condition, we apply minimax methods, more precisely, the mountain-pass theorem combined with minimization and the Ekeland variational principle, to obtain multiplicity of weak solutions of (1.2).

Next we state the main results of this paper which, for the sake of easy reference, we distinguish in two cases.

1.1. Subcritical case

Theorem 1.2. *Suppose (f_0), (f_1), (f_3) and that $f(x,s)$ has subcritical growth at both $+\infty$ and $-\infty$. Then there exists $\delta_1 > 0$ such that if $0 < \|h\|_* < \delta_1$, (1.2) has at least two weak solutions. One of them with positive energy, while the other one with negative energy.*

Furthermore, if $h(x)$ has defined sign, the following result holds:

Theorem 1.3. *Under the assumptions of Theorem 1.2, if $h(x) \geqslant 0$ ($h(x) \leqslant 0$) almost everywhere in Ω, then the weak solutions obtained in Theorem 1.2 are nonnegative (nonpositive, respectively).*

Remark 1.3. When $N = 2$ an example of functions satisfying assumptions (f_1), (f_3) with subcritical growth is $f(x,s) = g(x)(2s\cos(s^2) + 2se^s + s^2e^s)$, where $g : \overline{\Omega} \to \mathbb{R}$ is a continuous function with $0 < g(x) < \lambda_1/4$ in $\overline{\Omega}$. We have that $F(x,s) = g(x)(\sin(s^2) + s^2e^s)$. Note that $f(x,s)$ satisfies condition (f_1):

$$\lim_{|s|\to\infty} \frac{F(x,s)}{sf(x,s)} = \lim_{|s|\to\infty} \frac{\sin(s^2) + s^2e^s}{s(2s\cos(s^2) + 2se^s + s^2e^s)} = \lim_{|s|\to\infty} \frac{\sin(s^2)s^{-2}e^{-s} + 1}{2\cos(s^2)e^{-s} + 2 + s} = 0.$$

Furthermore, (f_3) is satisfied, since

$$\limsup_{s\to 0} \frac{2F(x,s)}{s^2} = 2g(x) \lim_{s\to 0} \frac{\sin(s^2) + s^2e^s}{s^2} = 4g(x) < \lambda_1.$$

1.2. Critical case

Theorem 1.4. *Assume (f_0), (f_2), (f_3) and that $f(x,s)$ has critical growth at both $+\infty$ and $-\infty$. Then there exists $\delta_1 > 0$ such that if $0 < \|h\|_* < \delta_1$, (1.2) has a weak solution with negative energy.*

For the next results, in the singular case, $a \in (0, N)$, we denote by r the radius of the largest open ball centered at origin and contained in Ω. In the nonsingular case, $a = 0$, we denote by r the inner radius of the set Ω, that is, $r :=$ radius of the largest open ball contained in Ω.

Theorem 1.5. *Suppose the hypotheses of Theorem 1.4. Furthermore suppose that*

(f_4^+) *there exists β_0 such that*

$$\liminf_{s\to+\infty} sf(x,s)e^{-\alpha_0|s|^{N/(N-1)}} \geqslant \beta_0 > \frac{N-a}{r^{N-a}e^{1+1/2+\cdots+1/(N-1)}}\left(\frac{N-a}{\alpha_0}\right)^{N-1}.$$

Then, there exists $\delta_2 > 0$, such that if $0 < \|h\|_* < \delta_2$, (1.2) has a second weak solution.

Furthermore, if $h(x)$ has defined sign, the following result holds:

Theorem 1.6. *Under the assumptions of Theorem* 1.5*, if* $h(x) \geqslant 0$ *almost everywhere in* Ω*, then the solutions obtained in Theorem* 1.5 *are nonnegative. Moreover, if* $h(x) \leqslant 0$ *almost everywhere in* Ω *and* $f(x,s)$ *satisfies*

(f_4^-) *there exists* β_0 *such that*

$$\liminf_{s\to-\infty} sf(x,s)e^{-\alpha_0|s|^{N/(N-1)}} \geqslant \beta_0 > \frac{N-a}{r^{N-a}e^{1+1/2+\cdots+1/(N-1)}}\left(\frac{N-a}{\alpha_0}\right)^{N-1},$$

then these solutions are nonpositive.

Remark 1.4. When $N=2$, an example of nonlinearity satisfying (f_2), (f_3), (f_4^+) and (f_4^-) with critical growth is given by $f(x,s)=g(x)(2s\cos(s^2)+2se^{s^2}-4s)$, with $\alpha_0=1$, where $g:\overline{\Omega}\to\mathbb{R}$ is a continuous function. Note that $F(x,s)=g(x)(\sin(s^2)+e^{s^2}-1-2s^2)$ and (f_2) is satisfied:

$$\lim_{|s|\to\infty}\frac{F(x,s)}{f(x,s)}=\lim_{|s|\to\infty}\frac{\sin(s^2)+e^{s^2}-1-2s^2}{2s\cos(s^2)+2se^{s^2}-4s}=0.$$

In order to show that (f_3) is satisfied, it is enough to verify that

$$\limsup_{s\to 0}\frac{2F(x,s)}{s^2}=2g(x)\lim_{s\to 0}\frac{\sin(s^2)+e^{s^2}-1-2s^2}{s^2}=0.$$

Furthermore, it is easy to see that $\liminf_{|s|\to+\infty} sf(x,s)e^{-s^2}=+\infty$, showing that (f_4^+) and (f_4^-) holds.

Remark 1.5. The assumptions on $f(x,s)$ will be altered slightly in Theorem 1.3 and in Theorem 1.6 to accommodate positives and negatives solutions. Essentially we impose symmetric constraints on $f(x,s)$. Of course, these can be lifted if we neglect interest in signs of solutions, and a remark to this effect is made later.

Remark 1.6. Condition (f_2) is stronger than (f_1), in the sense that (f_2) implies (f_1). By integrating condition (f_1), we can show that there exist positive constants C_1, C_2 such that

$$F(x,s)\geqslant C_1|s|^\theta - C_2,\quad s\in\mathbb{R}. \tag{1.8}$$

On the other hand, it follows from (f_2) that there exist positive constants C_1, C_2 such that

$$F(x,s)\geqslant C_1e^{|s|/M}-C_2,\quad s\in\mathbb{R}. \tag{1.9}$$

Moreover, there are $R_0>0$ and $\theta>N$ such that for $|s|\geqslant R_0$ and $x\in\Omega$

$$\theta F(x,s)\leqslant sf(x,s). \tag{1.10}$$

Remark 1.7. Note that if $N=2$, $\alpha_0=4\pi$, $a=0$ and r is the inner radius of Ω, assumption (f_4^+) reads

$$\liminf_{s\to+\infty} sf(x,s)e^{-4\pi s^2}\geqslant\beta_0>\frac{1}{e\pi r^2}.$$

In [11] (see also [1] and [15], for the quasilinear problems) it was used the same assumption as above with $e\pi$ replaced by 2π, where they used the Moser sequence. In order to get this improvement on the growth of the nonlinearity $f(x,s)$ at $+\infty$, it was crucial in our argument to use a new sequence introduced in [12].

Remark 1.8. In the last years, several papers have been devoted to the study of elliptic problems involving critical growth in terms of the Trudinger–Moser inequality. We refer the reader to the review article on this subject of de Figueiredo, et al. [13]. Problems with critical growth involving the Laplace operator in bounded domains of $\mathbb{R}^2$ with $a=0$ and $h\equiv 0$, have been investigated in [3,4,11,12]. Quasilinear elliptic problems with critical growth for the N-Laplacian in bounded domains of $\mathbb{R}^N$ with $a=0$ and $h\equiv 0$, have been studied in [1,15]. The case $a=0$ and $h\not\equiv 0$ was treated in [22]. Problems of this type in the whole $\mathbb{R}^N$, have been studied recently by several authors, see [16–18] and references therein. However all these papers considered nonsingular case, that is, $a=0$. Moreover, in [1,15] for $\Omega\subset\mathbb{R}^N$ a smooth bounded domain and $f(x,s)$ with critical growth, the main asymptotic hypothesis on $f(x,s)$ was of the following type:

$$\liminf_{s\to+\infty} sf(x,s)e^{-\alpha_0|s|^{N/(N-1)}}\geqslant\beta_0>\frac{1}{r^N}\left(\frac{N}{\alpha_0}\right)^{N-1}. \tag{1.11}$$

Here we are motivated by a recent paper of Adimurthi and Sandeep [2] where they proved a version the Trudinger–Moser inequality with singular weight and studied the existence of positive weak solutions for the following quasilinear and homogeneous elliptic problem

$$\begin{cases} -\Delta_N u = \dfrac{f(u)u^{N-2}}{|x|^a} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Motivated by [2], in the present paper, we improve and complement some of the results cited above for the singular and nonhomogeneous case. Moreover, the hypothesis (1.11) is improved to (f_4^+) in Theorem 1.5. The proofs of our results rely on minimization methods in combination with the mountain-pass theorem. In the subcritical case we are able to prove that the associated functional satisfies the Palais–Smale compactness condition which allow us to obtain critical points for the functional. As a consequence we can distinguish the local minimum solution from the mountain-pass solution. However, in the critical case to prove that these solutions are different is more involved, requiring fine energy level estimates. For this assumption (f_4^+) in Theorem 1.5 will be crucial in our argument to estimate the mountain-pass level.

Remark 1.9. It is well known that problems involving the p-Laplacian arises in various applications. For instance, they may be found in the study of non-Newtonian fluid, nonlinear elasticity and reaction–diffusions. For discussions on problems modelled by p-Laplacian equations, see [14].

The outline of the paper is as follows: Section 2 contains some technical results, the variational framework and we check also the geometric conditions of the associated energy functional. In Section 3, we study the Palais–Smale sequences. Finally in Section 4, we complete the proofs of our main results. In this work C, C_0, C_1, C_2, … denote positive (possibly different) constants.

2. The variational framework

By assumption (f_3), there exist $\epsilon, \delta > 0$ in such a way that $|s| \leqslant \delta$ implies

$$\left|F(x,s)\right| \leqslant \frac{(\lambda_1 - \epsilon)}{N}|s|^N. \tag{2.1}$$

Since $f(x,s)$ is continuous and has subcritical (or critical) growth at both $+\infty$ and $-\infty$, for each $q > N$ there exists a constant $C = C(q,\delta)$ such that

$$\left|F(x,s)\right| \leqslant C|s|^q e^{\alpha|s|^{N/(N-1)}} \quad \text{if } |s| \geqslant \delta. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$\left|F(x,s)\right| \leqslant \frac{(\lambda_1 - \epsilon)}{N}|s|^N + C|s|^q e^{\alpha|s|^{N/(N-1)}} \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$

Let $u \in W_0^{1,N}(\Omega)$, then by Proposition 1.1 and Hölder inequality, we see that if $\alpha > 0$ and $q > 0$,

$$\frac{e^{\alpha|u|^{N/(N-1)}}}{|x|^a}|u|^q \in L^1(\Omega) \quad \text{for all } u \in W_0^{1,N}(\Omega). \tag{2.4}$$

Consequently, we have from (2.3) and (2.4) that $F(x,u)/|x|^a \in L^1(\Omega)$. Therefore, the functional $I : W_0^{1,N}(\Omega) \to \mathbb{R}$, given by

$$I(u) = \frac{\|u\|^N}{N} - \int_\Omega \frac{F(x,u)}{|x|^a}\,dx - \int_\Omega h(x)u\,dx$$

is well defined. Furthermore, using standard arguments and Proposition 2.1, we can see that $I \in C^1(W_0^{1,N}(\Omega), \mathbb{R})$ with

$$\left\langle I'(u), v\right\rangle = \int_\Omega |\nabla u|^{N-2}\nabla u \nabla v\,dx - \int_\Omega \frac{f(x,u)}{|x|^a}v\,dx - \int_\Omega h(x)v\,dx \quad \text{for all } v \in W_0^{1,N}(\Omega).$$

Consequently, critical points of the functional I are precisely the weak solutions of (1.2).

The next proposition is a converse of the Lebesgue dominated convergence theorem in the space $W_0^{1,N}(\Omega)$.

Proposition 2.1. *Let (u_n) be a sequence in $W_0^{1,N}(\Omega)$ strongly convergent. Then there exists a subsequence (u_{n_k}) of (u_n) and $g \in W_0^{1,N}(\Omega)$ such that $|u_{n_k}(x)| \leqslant g(x)$ almost everywhere in Ω.*

Proof. See proof in [18, Proposition 1]. □

In the next lemmas we check that the functional I satisfies the geometric conditions of the mountain-pass theorem.

Lemma 2.1. *If $v \in W_0^{1,N}(\Omega)$, $\beta > 0$, $q > 0$ and $\|v\| \leqslant M$ with $\beta M^{N/(N-1)}/\alpha_N + a/N < 1$, then there exists $C > 0$ such that*

$$\int_\Omega \frac{e^{\beta |v|^{N/(N-1)}}}{|x|^a} |v|^q \, dx \leqslant C\|v\|^q.$$

Proof. We consider $r > 1$ sufficiently close to 1 such that $r\beta M^{N/(N-1)}/\alpha_N + ar/N < 1$ and $sq \geqslant 1$ where $s = r/(r-1)$. Using Hölder inequality, we have

$$\int_\Omega \frac{e^{\beta |v|^{N/(N-1)}}}{|x|^a} |v|^q \, dx \leqslant \left(\int_\Omega \frac{e^{(r\beta \|v\|^{N/(N-1)} (\frac{|v|}{\|v\|})^{N/(N-1)})}}{|x|^{ar}} \, dx \right)^{1/r} \|v\|_{qs}^q.$$

Using the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^{sq}(\Omega)$ for all $sq \geqslant 1$ and Proposition 1.1, we conclude the result. $\square$

Lemma 2.2. *Assume (f_0), (f_1) (or (f_2)), (f_3) and that $f(x,u)$ has subcritical (or critical) growth at both $+\infty$ and $-\infty$. Then there exists $\delta_1 > 0$ such that for each $h \in W^{-1,N'}$ with $\|h\|_* < \delta_1$, there exists $\rho_h > 0$ such that*

$$I(u) > 0 \quad \text{if } \|u\| = \rho_h.$$

Furthermore, ρ_h can be chosen such that $\rho_h \to 0$ as $\|h\|_* \to 0$.

Proof. Let $u \in W_0^{1,N}(\Omega)$ be such that $\alpha \|u\|^{N/(N-1)}/\alpha_N + a/N < 1$, by Lemma 2.1 and by definition of λ_1, we obtain

$$\begin{aligned} I(u) &\geqslant \frac{1}{N}\|u\|^N - \frac{(\lambda_1 - \epsilon)}{N} \int_\Omega \frac{|u|^N}{|x|^a} \, dx - C\|u\|^q - \|h\|_* \|u\| \\ &\geqslant \frac{1}{N}\left[1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right]\|u\|^N - C\|u\|^q - \|h\|_*\|u\|. \end{aligned}$$

Consequently,

$$I(u) \geqslant \|u\| \left[\frac{1}{N}\left(1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right)\|u\|^{N-1} - C\|u\|^{q-1} - \|h\|_* \right]. \tag{2.5}$$

Since $\epsilon > 0$ and $q > N$, we may choose $\rho > 0$ such that

$$\frac{1}{N}\left[1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right]\rho^{N-1} - C\rho^{q-1} > 0.$$

Thus, for $\|h\|_*$ sufficiently small there exists $\rho_h > 0$ such that $I(u) > 0$ if $\|u\| = \rho_h$ and $\rho_h \to 0$ as $\|h\|_* \to 0$. $\square$

Lemma 2.3. *Suppose that $f(x,s)$ satisfies (f_1) (or (f_2)). Then there exists $e \in W_0^{1,N}(\Omega)$ with $\|e\| > \rho_h$ such that*

$$I(e) < \inf_{\|u\| = \rho_h} I(u).$$

Proof. From (f_1) (or (f_2)) for $\theta > N$, there are positive constants C_1 and C_2 such that

$$F(x,s) \geqslant C_1 s^\theta - C_2 \quad \text{for all } s > 0.$$

Thus, for all $u \in W_0^{1,N}(\Omega) \setminus \{0\}$ and $u \geqslant 0$,

$$\begin{aligned} I(tu) &\leqslant \frac{t^N}{N}\|u\|^N - C_1 t^\theta \int_\Omega \frac{u^\theta}{|x|^a} \, dx + C_2 \int_\Omega \frac{dx}{|x|^a} - t\int_\Omega h(x)u \, dx \\ &\leqslant \frac{t^N}{N}\|u\|^N - C_1 t^\theta \int_\Omega \frac{u^\theta}{|x|^a} \, dx + t\|h\|_*\|u\| + C_3. \end{aligned}$$

Since $\theta > N$, we get $I(tu) \to -\infty$ as $t \to +\infty$. Setting $e = tu$ with t large enough, the proof is finished. $\square$

In order to find an appropriate ball to use minimization argument we need the following result:

Lemma 2.4. *If $f(x,s)$ has subcritical (or critical) growth at both $+\infty$ and $-\infty$, there exists $\eta > 0$ and $v \in W_0^{1,N}(\Omega)$ with $\|v\| = 1$ such that $I(tv) < 0$ for all $0 < t < \eta$. In particular,*

$$\inf_{\|u\| \leqslant \eta} I(u) < 0.$$

Proof. For each $h \in W^{-1,N'}$, let $v \in W_0^{1,N}(\Omega)$ be the unique solution of the problem

$$-\Delta_N v = h(x), \quad x \in \Omega \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega.$$

Thus $\int_\Omega h(x) v \, dx = \|v\|^N > 0$ for each $h \not\equiv 0$. For $t > 0$, we have

$$\frac{d}{dt} I(tv) = t^{N-1} \|v\|^N - \int_\Omega \frac{f(x,tv)}{|x|^a} v \, dx - \int_\Omega h(x) v \, dx.$$

Since $f(x,0) = 0$, by continuity, it follows that there exists $\eta > 0$ such that

$$\frac{d}{dt} I(tv) = t^{N-1} \|v\|^N - \int_\Omega \frac{f(x,tv)}{|x|^a} v \, dx - \int_\Omega h(x) v \, dx < 0,$$

for all $0 < t < \eta$. Using that $I(0) = 0$, it must hold that $I(tv) < 0$ for all $0 < t < \eta$. □

3. On Palais–Smale sequences

To show that the weak limit of a Palais–Smale sequence in $W_0^{1,N}(\Omega)$ is a weak solution of (1.2) we will use the following convergence result, which is a version of Lemma 2.1 in [11].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a continuous function. Then for any sequence (u_n) in $L^1(\Omega)$ such that*

$$u_n \to u \quad \text{in } L^1(\Omega), \qquad \frac{f(x,u_n)}{|x|^a} \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \frac{|f(x,u_n)u_n|}{|x|^a} \, dx \leqslant C_1,$$

up to a subsequence we have

$$\frac{f(x,u_n)}{|x|^a} \to \frac{f(x,u)}{|x|^a} \quad \text{in } L^1(\Omega).$$

Proof. It suffices to prove

$$\int_\Omega \frac{|f(x,u_n)|}{|x|^a} \, dx \to \int_\Omega \frac{|f(x,u)|}{|x|^a} \, dx.$$

Since u, $f(x,u)/|x|^a \in L^1(\Omega)$, given $\epsilon > 0$ there is a $\delta > 0$ such that for any measurable subsets $A \subset \Omega$,

$$\int_A |u| \, dx < \epsilon \quad \text{and} \quad \int_A \frac{|f(x,u)|}{|x|^a} \, dx < \epsilon \quad \text{if } |A| \leqslant \delta. \tag{3.1}$$

Next using the fact that $u \in L^1(\Omega)$, we find $M_1 > 0$ such that

$$\left|\left\{x \in \Omega \colon \left|u(x)\right| \geqslant M_1\right\}\right| \leqslant \delta. \tag{3.2}$$

Taking $M = \max\{M_1, C_1/\epsilon\}$, we write

$$\left| \int_\Omega \frac{|f(x,u_n)|}{|x|^a} \, dx - \int_\Omega \frac{|f(x,u)|}{|x|^a} \, dx \right| \leqslant I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$I_{1,n} = \int_{[|u_n| \geqslant M]} \frac{|f(x,u_n)|}{|x|^a} \, dx,$$

$$I_{2,n} = \left| \int_{[|u_n| < M]} \frac{|f(x,u_n)|}{|x|^a} \, dx - \int_{[|u| < M]} \frac{|f(x,u)|}{|x|^a} \, dx \right|$$

and

$$I_{3,n}=\int\limits_{[|u|\geqslant M]}\frac{|f(x,u)|}{|x|^a}\,dx.$$

Now we estimate each integral separately.

$$I_{1,n}=\int\limits_{[|u_n|\geqslant M]}\frac{|f(x,u_n)|}{|x|^a}\,dx=\int\limits_{[|u_n|\geqslant M]}\frac{|f(x,u_n)u_n|}{|u_n||x|^a}\,dx\leqslant\frac{C_1}{M}\leqslant\epsilon.$$

From (3.1) and (3.2), we have $I_{3,n}\leqslant\epsilon$.

Next we claim $I_{2,n}\to 0$ as $n\to+\infty$. Indeed,

$$I_{2,n}\leqslant\left|\int\limits_{\Omega}\frac{\mathcal{X}_{[|u_n|<M]}(|f(x,u_n)|-|f(x,u)|)}{|x|^a}\,dx\right|+\left|\int\limits_{\Omega}\frac{(\mathcal{X}_{[|u_n|<M]}-\mathcal{X}_{[|u|<M]})|f(x,u)|}{|x|^a}\,dx\right|$$

and $g_n(x)=\mathcal{X}_{[|u_n|<M]}(|f(x,u_n)|-|f(x,u)|)\to 0$ almost everywhere in Ω. Moreover

$$\left|g_n(x)\right|\leqslant\begin{cases}|f(x,u)| & \text{if } |u_n(x)|\geqslant M,\\ C+|f(x,u)| & \text{if } |u_n(x)|<M,\end{cases}$$

where $C=\sup\{|f(x,t)|\colon (x,t)\in\overline{\Omega}\times[-M,M]\}$. So, by the Lebesgue dominated convergence theorem, we get

$$\left|\int\limits_{\Omega}\frac{\mathcal{X}_{[|u_n|<M]}(|f(x,u_n)|-|f(x,u)|)}{|x|^a}\,dx\right|\to 0\quad\text{as } n\to\infty.$$

Moreover, we have

$$\left\{x\in\Omega\colon \left|u_n(x)\right|<M\right\}\setminus\left\{x\in\Omega\colon \left|u(x)\right|<M\right\}\subset\left\{x\in\Omega\colon \left|u(x)\right|\geqslant M\right\}.$$

Hence by (3.1),

$$\left|\int\limits_{\Omega}\frac{(\mathcal{X}_{[|u_n|<M]}-\mathcal{X}_{[|u|<M]})|f(x,u)|}{|x|^a}\,dx\right|\leqslant\int\limits_{[|u|\geqslant M]}\frac{|f(x,u)|}{|x|^a}\,dx<\epsilon,$$

which completes the proof. □

To prove that a Palais–Smale sequence converges to a weak solution of (1.2) we need to establish the following lemma, inspired in [15].

Lemma 3.2. *Let (u_n) be a Palais–Smale sequence for I. Then (u_n) is bounded in $W_0^{1,N}(\Omega)$. Moreover, (u_n) has a subsequence, still denoted by (u_n) and $u\in W_0^{1,N}(\Omega)$ such that*

$$\frac{f(x,u_n)}{|x|^a}\to\frac{f(x,u)}{|x|^a}\quad\text{in } L^1(\Omega),\tag{3.3}$$

$$|\nabla u_n|^{N-2}\nabla u_n\rightharpoonup|\nabla u|^{N-2}\nabla u\quad\text{weakly in } \left(L^{N/(N-1)}(\Omega)\right)^N.\tag{3.4}$$

Proof. Let $(u_n)\subset W_0^{1,N}(\Omega)$ be a Palais–Smale sequence at level c, that is,

$$\frac{1}{N}\int\limits_{\Omega}|\nabla u_n|^N\,dx-\int\limits_{\Omega}\frac{F(x,u_n)}{|x|^a}\,dx-\int\limits_{\Omega}h(x)u_n\,dx\to c\tag{3.5}$$

and

$$\int\limits_{\Omega}|\nabla u_n|^{N-2}\nabla u_n\nabla v\,dx-\int\limits_{\Omega}\frac{f(x,u_n)}{|x|^a}v\,dx-\int\limits_{\Omega}h(x)v\,dx\to 0\tag{3.6}$$

for all $v\in W_0^{1,N}(\Omega)$.

Step 1: (u_n) is bounded in $W_0^{1,N}(\Omega)$. Indeed, from (3.5) and (3.6) have that

$$\left|\left(\frac{\theta}{N}-1\right)\|u_n\|^N-\int\limits_{\Omega}\frac{(\theta F(x,u_n)-f(x,u_n)u_n)}{|x|^a}\,dx-(\theta-1)\int\limits_{\Omega}h(x)u_n\,dx\right|\leqslant C+\epsilon_n\|u_n\|$$

where $\epsilon_n\to 0$ as $n\to+\infty$. Thus,

$$\left|\left[\left(\frac{\theta}{N}-1\right)\|u_n\|^{N-1}-(\theta-1)\|h\|_*\right]\|u_n\|-\int_\Omega \frac{(\theta F(x,u_n)-f(x,u_n)u_n)}{|x|^a}\,dx\right|\leqslant C+\epsilon_n\|u_n\|$$

which together with (f_0) and (f_1), implies that (u_n) is bounded in $W_0^{1,N}(\Omega)$. Consequently, up to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W_0^{1,N}(\Omega),\\ &u_n \to u \quad \text{in } L^q(\Omega) \text{ for all } q\in[1,\infty),\\ &u_n(x)\to u(x) \quad \text{almost everywhere in } \Omega. \end{aligned} \tag{3.7}$$

Then using that (u_n) is bounded and Lemma 3.1 together with (3.6), we get that (u_n) has a subsequence such that (3.3) holds.

Step 2: (u_n) has a subsequence such that (3.4) holds.

Indeed, since $(|\nabla u_n|^{N-2}\nabla u_n)$ is bounded in $(L^{N/(N-1)}(\Omega))^N$, without loss of generality we may assume that

$$\begin{aligned} &|\nabla u_n|^N \to \mu \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and}\\ &|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup \nu \quad \text{weakly in } \left(L^{N/(N-1)}(\Omega)\right)^N, \end{aligned}$$

where μ is a nonnegative regular measure and $\mathcal{D}'(\Omega)$ are the distributions on Ω.

Let $\sigma>0$ and $\mathcal{A}_\sigma=\{x\in\overline{\Omega}\colon \forall r>0,\ \mu(B_r(x)\cap\overline{\Omega})\geqslant\sigma\}$. We claim that $\mathcal{A}_\sigma$ is a finite set. Suppose for the sake of contradiction that there exists a sequence of distinct points (x_k) in $\mathcal{A}_\sigma$. Since for all $r>0$, $\mu(B_r(x)\cap\overline{\Omega})\geqslant\sigma$, we have that $\mu(\{x_k\})\geqslant\sigma$, which implies that $\mu(\mathcal{A}_\sigma)=+\infty$, however

$$\mu(\mathcal{A}_\sigma)=\lim_{n\to+\infty}\int_{\mathcal{A}_\sigma}|\nabla u_n|^N\,dx\leqslant C.$$

Thus, $\mathcal{A}_\sigma=\{x_1,x_2,\dots,x_m\}$.

Let $u\in W^{1,N}(\mathcal{O})$, where $\mathcal{O}$ is bounded domain in $\mathbb{R}^N$. We know (cf. [6] and [8]) that there are positive constants r_1 and C_1 depending only on N such that

$$\int_{\mathcal{O}} e^{r_1\left(\frac{|u|}{\|\nabla u\|_{L^N(\mathcal{O})}}\right)^{N/(N-1)}}\,dx\leqslant C_1.$$

Consequently, there are positive constants r_2 and C_2 such that

$$\int_{\mathcal{O}}\frac{e^{r_2(|u|/\|\nabla u\|_{L^N(\mathcal{O})})^{N/(N-1)}}}{|x|^a}\,dx\leqslant C_2. \tag{3.8}$$

Indeed, let $0<r_2<r_1$ and $t>1$ be such that $r_2/r_1+at/N=1$. Using Hölder inequality, we obtain

$$\int_{\mathcal{O}}\frac{e^{r_2(|u|/\|\nabla u\|_{L^N(\mathcal{O})})^{N/(N-1)}}}{|x|^a}\,dx\leqslant\left(\int_{\mathcal{O}} e^{r_1\left(\frac{|u|}{\|\nabla u\|_{L^N(\mathcal{O})}}\right)^{N/(N-1)}}\,dx\right)^{r_2/r_1}\left(\int_{\mathcal{O}}\frac{1}{|x|^{N/t}}\,dx\right)^{at/N}\leqslant C_2.$$

Assertion 1. For any relative compact subset K of $\overline{\Omega}\setminus\mathcal{A}_\sigma$ and $\sigma>0$ such that

$$\alpha\sigma^{1/(N-1)}/r_2+a/N<1,$$

we have

$$\lim_{n\to\infty}\int_K\frac{f(x,u_n)}{|x|^a}u_n\,dx=\int_K\frac{f(x,u)}{|x|^a}u\,dx.$$

Indeed, let $x_0\in K$ and $r_0>0$ be such that $\mu(B_{r_0}(x_0)\cap\Omega)<\sigma$. Consider a function $\varphi\in C_0^\infty(\Omega,[0,1])$ such that $\varphi\equiv1$ in $B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}$ and $\varphi\equiv0$ in $\overline{\Omega}\setminus B_{r_0}(x_0)$. Thus

$$\lim_{n\to\infty}\int_{B_{r_0}(x_0)\cap\overline{\Omega}}|\nabla u_n|^N\varphi\,dx=\int_{B_{r_0}(x_0)\cap\overline{\Omega}}\varphi\,d\mu\leqslant\mu\big(B_{r_0}(x_0)\cap\overline{\Omega}\big)<\sigma.$$

Therefore, for $n \in \mathbb{N}$ sufficiently large and $\epsilon > 0$ sufficiently small, we have

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_n|^N \,\mathrm{d}x \leqslant \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_n|^N \varphi \,\mathrm{d}\mu \leqslant (1-\epsilon)\sigma,$$

which together with (3.8) implies

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \left(\frac{|f(x,u_n)|}{|x|^a}\right)^q \mathrm{d}x \leqslant C \tag{3.9}$$

if we choose $q > 1$ sufficiently close to 1 and such that $q\alpha\sigma^{1/(N-1)}/r_2 + aq/N < 1$.

Now, we estimate

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n)u_n - f(x,u)u|}{|x|^a} \,\mathrm{d}x \leqslant I_1 + I_2$$

where

$$I_1 = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n) - f(x,u)|}{|x|^a}|u| \,\mathrm{d}x \quad \text{and} \quad I_2 = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n)|}{|x|^a}|u_n - u| \,\mathrm{d}x.$$

Note that, by Hölder inequality, (3.9) and embedding Sobolev theorem,

$$I_2 = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{f(x,u_n)}{|x|^a}|u_n - u| \,\mathrm{d}x \leqslant C\left(\int_{\Omega} |u_n - u|^{q'} \,\mathrm{d}x\right)^{1/q'} \to 0 \quad \text{as } n \to \infty.$$

Now, we claim that $I_1 \to 0$. Indeed, given $\epsilon > 0$, by density we can take $\varphi \in C_0^\infty(\Omega)$ such that $\|u - \varphi\|_{q'} < \epsilon$. Thus,

$$\begin{aligned}\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n)u - f(x,u)u|}{|x|^a} \,\mathrm{d}x \leqslant{}& \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n)|}{|x|^a}|u-\varphi| \,\mathrm{d}x + \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n) - f(x,u)|}{|x|^a}|\varphi| \,\mathrm{d}x \\ &+ \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u)|}{|x|^a}|\varphi - u| \,\mathrm{d}x.\end{aligned}$$

Applying Hölder inequality and using (3.9), we have

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n)|}{|x|^a}|u-\varphi| \,\mathrm{d}x \leqslant \left(\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \left(\frac{|f(x,u_n)|}{|x|^a}\right)^q \mathrm{d}x\right)^{1/q} \|u-\varphi\|_{q'} < \epsilon.$$

Using Lemma 3.1,

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n) - f(x,u)|}{|x|^a}|\varphi| \,\mathrm{d}x \leqslant \|\varphi\|_\infty \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u_n) - f(x,u)|}{|x|^a} \,\mathrm{d}x \to 0$$

and by Proposition 1.1, we have

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} \frac{|f(x,u)|}{|x|^a}|\varphi - u| \,\mathrm{d}x \to 0.$$

To conclude Assertion 1 we use that K is a compact and we repeat the same procedure over a finite covering of balls.

To complete the proof of (3.4), we estate:

Assertion 2. Let $\epsilon_0 > 0$ be fixed and small enough such that $B_{\epsilon_0}(x_i) \cap B_{\epsilon_0}(x_j) = \emptyset$ if $i \neq j$ and $\Omega_{\epsilon_0} = \{x \in \overline{\Omega}\colon \|x - x_j\| \geqslant \epsilon_0,\ j = 1, 2, \ldots, m\}$. Then

$$\int_{\Omega_{\epsilon_0}} \big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big)(\nabla u_n - \nabla u)\,\mathrm{d}x \to 0.$$

Indeed, let $0 < \epsilon < \epsilon_0$ and $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\varphi \equiv 1$ in $B_{1/2}(0)$ and $\varphi \equiv 0$ in $\overline{\Omega} \setminus B_1(0)$. Taking

$$\psi_\epsilon(x) = 1 - \sum_{j=1}^m \varphi\left(\frac{x - x_j}{\epsilon}\right),$$

we have $0 \leqslant \psi_\epsilon \leqslant 1$, $\psi_\epsilon \equiv 1$ in $\overline{\Omega}_\epsilon = \overline{\Omega} \setminus \bigcup_{j=1}^m B(x_j, \epsilon)$, $\psi_\epsilon \equiv 0$ in $\bigcup_{j=1}^m B(x_j, \frac{\epsilon}{2})$ and $\psi_\epsilon u_n$ is bounded sequence in $W_0^{1,N}(\Omega)$. Using (3.6) with $v = \psi_\epsilon u_n$, we have

$$\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla(\psi_\epsilon u_n)\,\mathrm{d}x - \int_\Omega \frac{f(x,u_n)}{|x|^a}\psi_\epsilon u_n\,\mathrm{d}x - \int_\Omega h\psi_\epsilon u_n\,\mathrm{d}x \leqslant \epsilon_n\|\psi_\epsilon u_n\|,$$

which implies that

$$\int_\Omega |\nabla u_n|^{N-2}\nabla u_n[u_n\nabla\psi_\epsilon + \psi_\epsilon\nabla u_n]\,\mathrm{d}x - \int_\Omega \frac{f(x,u_n)}{|x|^a}\psi_\epsilon u_n\,\mathrm{d}x - \int_\Omega h\psi_\epsilon u_n\,\mathrm{d}x \leqslant \epsilon_n\|\psi_\epsilon u_n\|.$$

Hence

$$\int_\Omega \left[|\nabla u_n|^N\psi_\epsilon + u_n|\nabla u_n|^{N-2}\nabla u_n\nabla\psi_\epsilon - \psi_\epsilon\frac{f(x,u_n)}{|x|^a}u_n\right]\mathrm{d}x - \int_\Omega h\psi_\epsilon u_n\,\mathrm{d}x \leqslant \epsilon_n\|\psi_\epsilon u_n\|. \tag{3.10}$$

Now, using (3.6) with $v = -\psi_\epsilon u$, we have

$$\int_\Omega \left[-|\nabla u_n|^{N-2}\psi_\epsilon\nabla u_n\nabla u - |\nabla u_n|^{N-2}u\nabla u_n\nabla\psi_\epsilon + \psi_\epsilon\frac{f(x,u_n)}{|x|^a}u\right]\mathrm{d}x + \int_\Omega h\psi_\epsilon u\,\mathrm{d}x \leqslant \epsilon_n\|\psi_\epsilon u\|. \tag{3.11}$$

Using that the function $g : \mathbb{R}^N \to \mathbb{R}$, $g(v) = |v|^N$ is strictly convex we have that

$$0 \leqslant \big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\big)(\nabla u_n - \nabla u)$$

and consequently

$$0 \leqslant \int_{\overline{\Omega}_{\epsilon_0}} \big(|\nabla u_n|^{N-2}\nabla u_n\big) - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u)\,\mathrm{d}x$$
$$\leqslant \int_\Omega \big(|\nabla u_n|^{N-2}\nabla u_n\big) - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u)\psi_\epsilon\,\mathrm{d}x,$$

which can written as

$$0 \leqslant \int_\Omega \big[|\nabla u_n|^N\psi_\epsilon - |\nabla u_n|^{N-2}\psi_\epsilon\nabla u_n\nabla u - |\nabla u|^{N-2}\psi_\epsilon\nabla u\nabla u_n + |\nabla u|^N\psi_\epsilon\big]\,\mathrm{d}x. \tag{3.12}$$

From (3.10), (3.11) and (3.12), we obtain

$$0 \leqslant \int_\Omega \left[-|\nabla u_n|^{N-2}\psi_\epsilon + u_n|\nabla u_n|^{N-2}\nabla u_n\nabla\psi_\epsilon + \psi_\epsilon\frac{f(x,u_n)}{|x|^a}u_n + \psi_\epsilon h u_n\right]\mathrm{d}x + \epsilon_n\|\psi_\epsilon u_n\|$$
$$+ \int_\Omega \left[|\nabla u_n|^N\psi_\epsilon\nabla u_n\nabla u - u|\nabla u_n|^{N-2}\nabla u_n\nabla\psi_\epsilon - \psi_\epsilon\frac{f(x,u_n)}{|x|^a}u - \psi_\epsilon h u\right]\mathrm{d}x + \epsilon_n\|\psi_\epsilon u\|$$
$$+ \int_\Omega \big[|\nabla u_n|^N\psi_\epsilon - |\nabla u_n|^{N-2}\psi_\epsilon\nabla u_n\nabla u - |\nabla u|^{N-2}\psi_\epsilon\nabla u\nabla u_n + |\nabla u|^N\psi_\epsilon\big]\,\mathrm{d}x.$$

Therefore,

$$0 \leqslant \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla\psi_\epsilon (u_n - u)\,dx + \int_\Omega \psi_\epsilon |\nabla u|^{N-2}\nabla u(\nabla u - \nabla u_n)\,dx$$
$$+ \int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}(u_n - u)\,dx + \int_\Omega \psi_\epsilon h(u_n - u)\,dx + \epsilon_n\|\psi_\epsilon u\| + \epsilon_n\|\psi_\epsilon u_n\|. \tag{3.13}$$

Now we estimate each integral in (3.13) separately. Note that for arbitrary $\delta > 0$, using the interpolation inequality $ab \leqslant \delta a^{N/(N-1)} + C_\delta b^N$, with $C_\delta = \delta^{1-N}$, we have

$$\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla\psi_\epsilon (u - u_n)\,dx \leqslant \delta \int_\Omega |\nabla u_n|^N\,dx + C_\delta \int_\Omega |\nabla\psi_\epsilon|^N |u - u_n|^N\,dx$$
$$\leqslant \delta C + C_\delta \biggl(\int_\Omega |\nabla\psi_\epsilon|^{rN}\,dx\biggr)^{1/r}\biggl(\int_\Omega |u - u_n|^{sN}\,dx\biggr)^{1/s},$$

where $1/r + 1/s = 1$. Thus, since $u_n \to u$ in $L^{sN}(\Omega)$ and δ is arbitrary we obtain that

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla\psi_\epsilon (u - u_n)\,dx \leqslant 0. \tag{3.14}$$

Using that $u_n \rightharpoonup u$ in $W_0^{1,N}(\Omega)$, we have

$$\int_\Omega \psi_\epsilon |\nabla u|^{N-2}\nabla u(\nabla u - \nabla u_n)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.15}$$

Now, we claim

$$\int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Indeed,

$$\int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}(u_n - u)\,dx = \int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}u_n\,dx - \int_\Omega \psi_\epsilon \frac{f(x,u)}{|x|^a}u\,dx + \int_\Omega \psi_\epsilon \frac{f(x,u)}{|x|^a}u\,dx - \int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}u\,dx$$

and applying Assertion 1 with $g(x,u) = \psi_\epsilon(x)\frac{f(x,u)}{|x|^a}$ and $K = \overline{\Omega}_{\epsilon/2}$, we have that

$$\int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}u_n\,dx = \int_{\overline{\Omega}_{\epsilon/2}} \psi_\epsilon \frac{f(x,u_n)}{|x|^a}u_n\,dx \to \int_{\overline{\Omega}_{\epsilon/2}} \psi_\epsilon \frac{f(x,u)}{|x|^a}u\,dx = \int_{\overline{\Omega}} \psi_\epsilon \frac{f(x,u)}{|x|^a}u\,dx$$

and using Lemma 3.1, we obtain

$$\int_\Omega \psi_\epsilon \frac{f(x,u_n)}{|x|^a}u\,dx \to \int_\Omega \psi_\epsilon \frac{f(x,u)}{|x|^a}u\,dx \quad \text{as } n \to \infty.$$

Thus, from (3.13), (3.14), (3.15) and (3.16), we come to the conclusion that Assertion 2 holds.

Finally using Assertion 2, since ϵ_0 is arbitrary, we obtain that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{almost everywhere in } \Omega,$$

which together with the fact the sequence $(|\nabla u_n|^{N-2}\nabla u_n)$ is bounded in $L^{N/(N-1)}(\Omega)$, implies

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u \quad \text{in } L^{N/(N-1)}(\Omega).$$

up to a subsequence. Thus, we have completed the proof of Lemma 3.2. □

It follows from that

Corollary 3.3. *Let (u_n) be a Palais–Smale sequence for I. Then (u_n) has a subsequence, still denoted by (u_n) weakly convergent to a nontrivial weak solution of* (1.2).

Proof. Using Lemma 3.2, up to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$. Now, from (3.6), taking the limit and using again Lemma 3.2, we have

$$\int_\Omega |\nabla u|^{N-2}\nabla u \nabla\varphi \,dx - \int_\Omega \frac{f(x,u)}{|x|^a}\varphi \,dx - \int_\Omega h(x)\varphi \,dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,N}(\Omega)$, we conclude that u is a weak solution of (1.2). Since $h \not\equiv 0$, we conclude that $u \not\equiv 0$. □

4. Proof of the main results

In order to obtain a weak solution with negative energy, observe that by Lemma 2.4 we have

$$-\infty < c_0 \equiv \inf_{\|u\| \leqslant \eta} I(u) < 0. \tag{4.1}$$

4.1. Subcritical case

In this subsection we will give the proof of Theorem 1.2. Thus we are assuming that $f(x,s)$ satisfies (f_0), (f_1) (or (f_2)) and (f_3). To prove the existence of a local minimum solution we will use the Ekeland variational principle.

Lemma 4.1. *The functional I satisfies the Palais–Smale condition.*

Proof. Let (u_n) be a $(PS)_c$ sequence. By Lemma 3.2, (u_n) is bounded, so, up to subsequence, we may assume that $u_n = u_0 + w_n$, with $w_n \rightharpoonup 0$ weakly in $W_0^{1,N}(\Omega)$ and $w_n \to 0$ strongly in $L^q(\Omega)$ for all $q \in [1,\infty)$. By Brezis–Lieb lemma (see [7]), we have

$$\|u_n\|^N = \|u_0\|^N + \|w_n\|^N + o(1).$$

We first claim that

$$\int_\Omega \frac{f(x,u_n)}{|x|^a} u_0 \,dx \to \int_\Omega \frac{f(x,u_0)}{|x|^a} u_0 \,dx \quad \text{as } n \to \infty. \tag{4.2}$$

In fact, since $C_0^\infty(\Omega)$ is dense in $W_0^{1,N}(\Omega)$, for all $\epsilon > 0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that $\|\varphi - u_0\| < \epsilon$. Now, we write

$$\left| \int_\Omega \frac{f(x,u_n)}{|x|^a} u_0 \,dx - \int_\Omega \frac{f(x,u_0)}{|x|^a} u_0 \,dx \right| \leqslant J_1 + J_2 + J_3. \tag{4.3}$$

Since

$$\left|\left\langle I'(u_n), u_0 - \varphi\right\rangle\right| \leqslant \epsilon_n \|u_0 - \varphi\| \quad \text{with } \epsilon_n \to 0,$$

we have

$$J_1 = \left| \int_\Omega \frac{f(x,u_n)}{|x|^a}(u_0 - \varphi)\,dx \right| \leqslant \epsilon_n\|u_0-\varphi\| + \|u_n\|^{N-1}\|u_0-\varphi\| + \|h\|_*\|u_0-\varphi\| \leqslant C\|u_0-\varphi\| < C\epsilon,$$

where C is independent of n and ϵ. Similarly, using that $\langle I'(u_0), u_0 - \varphi\rangle = 0$, we can estimate

$$J_2 = \left| \int_\Omega \frac{f(x,u_0)}{|x|^a}(u_0 - \varphi)\,dx \right| < C\epsilon.$$

Using Lemma 3.2, we obtain

$$J_3 = \|\varphi\|_\infty \int_\Omega \frac{|f(x,u_n) - f(x,u_0)|}{|x|^a}\,dx \to 0 \quad \text{as } n \to \infty.$$

Thus (4.2) holds, and consequently

$$\left\langle I'(u_n), u_n\right\rangle = \left\langle I'(u_0), u_0\right\rangle + \|w_n\|^N - \int_\Omega \frac{f(x,u_n)}{|x|^a} w_n \,dx + o(1).$$

This implies,

$$\|w_n\|^N = \int_\Omega \frac{f(x,u_n)}{|x|^a} w_n \,\mathrm{d}x + o(1).$$

Since (u_n) is bounded in $W_0^{1,N}(\Omega)$ and $f(x,s)$ has subcritical growth, we can choose $q>1$ sufficiently close to 1 and $\alpha>0$ sufficiently small such that $\alpha q\|u_n\|^{N/(N-1)}/\alpha_N + qa/N < 1$. Then

$$\int_\Omega \left(\frac{|f(x,u_n)|}{|x|^a}\right)^q \mathrm{d}x \leqslant C\int_\Omega \frac{e^{q\alpha\|u_n\|^{N/(N-1)}|\frac{u_n}{\|u_n\|}|^{N/(N-1)}}}{|x|^{aq}}\,\mathrm{d}x \leqslant C.$$

Thus,

$$\int_\Omega \frac{f(x,u_n)}{|x|^a} w_n\,\mathrm{d}x \leqslant C\|w_n\|_{q'} \to 0.$$

Consequently $\|w_n\| \to 0$ and the result follows. □

In view of Lemmas 2.2 and 2.3 we can apply the mountain-pass theorem to obtain the following result:

Proposition 4.1. *There exists $\eta_1>0$ such that if $\|h\|_* \leqslant \eta_1$, then the functional I has a critical point u_M at the minimax level*

$$c_M = \inf_{g\in\Gamma}\max_{t\in[0,1]} I\big(g(t)\big),$$

where

$$\Gamma = \big\{g\in C\big([0,1], W_0^{1,N}(\Omega)\big):\ g(0)=0 \text{ and } g(1)=e\big\}.$$

Proposition 4.2. *For each $h\in W^{-1,N'}$ with $h\not\equiv 0$, Eq.* (1.2) *has a local minimum solution u_0 with $I(u_0)=c_0<0$, where c_0 is defined in* (4.1)*.*

Proof. Let ρ_h be as in Lemma 2.2. Since $\overline{B}_{\rho_h}$ is convex and a complete metric space with the metric given by the norm of $W_0^{1,N}(\Omega)$, and I is of class C^1 and bounded below on $\overline{B}_{\rho_h}$, by Ekeland's variational principle there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n)\to c_0 = \inf_{\|u\|\leqslant\rho_h} I(u) < 0 \quad \text{and} \quad \big\|I'(u_n)\big\|_* \to 0,$$

and the proof follows by Lemma 4.1. □

Proof of Theorem 1.2. The proof follows from Propositions 4.1 and 4.2. □

4.2. Critical case

In order to get a more precise information about the minimax level obtained by the mountain-pass theorem, it was crucial in our argument to consider the following sequence, which it was introduced in [12]: For $n\in\mathbb{N}$ set $\delta_n = \frac{2\log n}{n}$, and let

$$y_n(t) = \begin{cases} \frac{t}{n^{1/N}}(1-\delta_n)^{(N-1)/N} & \text{if } 0\leqslant t\leqslant n,\\ \frac{N-1}{(n(1-\delta_n))^{1/N}}\log\frac{A_n+1}{A_n+e^{-(t-n)/(N-1)}} + (n(1-\delta_n)^{(N-1)/N}) & \text{if } n\leqslant t, \end{cases}$$

where A_n is defined as follows

$$A_n = \frac{1}{n^2}\frac{1}{e^{1+1/2+\cdots+1/(N-1)}} + \begin{cases} O(1/n^4) & \text{if } N=2,\\ O(\log^2(n)/n^3) & \text{if } N\geqslant 3. \end{cases}$$

The sequence of function (y_n) satisfies the following proprieties:

- $(y_n)\subset C([0,+\infty))$, piecewise differentiable, with $y_n(0)=0$ and $y_n'(t)\geqslant 0$;
- $\int_0^{+\infty}|y_n'(t)|^N\,\mathrm{d}t = 1$;
- $\lim_{n\to+\infty}\int_0^{+\infty} e^{y_n^{N/(N-1)}(t)-t}\,\mathrm{d}t = 1+e^{1+1/2+\cdots 1/(N-1)}$.

(See more details about this sequence in [12].)

Now, $y_n(t) = N^{(N-1)/N}\omega_{N-1}^{1/N} V_n(e^{-t/N})$, with $|x|^N = e^{-t}$, define a function $V_n(x) = V_n(|x|)$ on $\overline{B_1(0)}$, which is non-negative and radially symmetric. Moreover,

$$\int_{B_1(0)} \left|\nabla V_n(x)\right|^N \mathrm{d}x = \int_0^{+\infty} \left|y_n'(t)\right|^N \mathrm{d}t = 1.$$

Let $\beta = \frac{N-a}{N}$, then V_n define another function non-negative and radially symmetric $\widetilde{M_n}$ as follows:

$$V_n(\rho) = \beta^{(N-1)/N} \widetilde{M_n}\big(\rho^{1/\beta}\big), \quad \text{for } \rho \in [0,1].$$

Notice that

$$\int_0^1 \left|V_n'(\rho)\right|^N \rho^{N-1}\,\mathrm{d}\rho = \int_0^1 \left|\widetilde{M_n}'(\rho)\right|^N \rho^{N-1}\,\mathrm{d}\rho.$$

Thus, $\|V_n\| = \|\widetilde{M_n}\| = 1$.

For the next lemma, let us consider the following sequence $M_n(x,r) = \widetilde{M}_n(x/r)$. Notice that $M_n(x/r) \in W_0^{1,N}(\Omega)$, $\operatorname{supp}(M_n(x,r)) = \overline{B_r(0)}$ and $\|M_n(\cdot,r)\| = 1$.

Lemma 4.2. *Assume (f_2), (f_3) and (f_4^+). Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_\Omega \frac{F(x,tM_n)}{|x|^a}\,\mathrm{d}x\right\} < \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Suppose, for the sake of contradiction, that for all $n \in \mathbb{N}$, we have

$$\max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_\Omega \frac{F(x,tM_n)}{|x|^a}\,\mathrm{d}x\right\} \geqslant \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

In view of Lemma 2.2 and Lemma 2.3, for all $n \in \mathbb{N}$, there exists $t_n > 0$ such that

$$\frac{t_n^N}{N} - \int_\Omega \frac{F(x,t_nM_n)}{|x|^a}\,\mathrm{d}x = \max_{t \geqslant 0}\left\{\frac{t^N}{N} - \int_\Omega \frac{F(x,tM_n)}{|x|^a}\,\mathrm{d}x\right\}.$$

Up to a subsequence, we have

$$t_n^N \to \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.4}$$

Indeed, since

$$\frac{t_n^N}{N} - \int_\Omega \frac{F(x,t_nM_n)}{|x|^a}\,\mathrm{d}x \geqslant \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

and $F(x,t_nM_n) \geqslant 0$ in Ω, we have

$$t_n^N \geqslant \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.5}$$

Also at $t = t_n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^N}{N} - \int_\Omega \frac{F(x,tM_n)}{|x|^a}\,\mathrm{d}x\right)\bigg|_{t=t_n} = 0.$$

Thus,

$$t_n^N = \int_\Omega \frac{f(x,t_nM_n)}{|x|^a} t_nM_n\,\mathrm{d}x \geqslant \int_{B_{\frac{r}{n}}(0)} \frac{f(x,t_nM_n)}{|x|^a} t_nM_n\,\mathrm{d}x. \tag{4.6}$$

By (f_4^+), given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$uf(x,u) \geqslant (\beta_0 - \epsilon)e^{\alpha_0 |u|^{N/(N-1)}} \quad \text{for all } u \geqslant R_\epsilon. \tag{4.7}$$

Since $t_n M_n \geqslant R_\epsilon$ in $B_{\frac{r}{n}}(0)$, for n sufficiently large, using (4.6) and (4.7), we obtain

$$\begin{aligned} t_n^N &\geqslant (\beta_0 - \epsilon) \int_{B_{\frac{r}{n}}(0)} \frac{e^{\alpha_0 |t_n M_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x \\ &= (\beta_0 - \epsilon)\left(\frac{r}{n}\right)^{N-a} \int_{B_1(0)} \frac{e^{\alpha_0 |t_n \widetilde{M}_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x \\ &= (\beta_0 - \epsilon)\omega_{N-1}\left(\frac{r}{n}\right)^{N-a} \int_0^1 e^{\alpha_0 |t_n \widetilde{M}_n(\rho)|^{N/(N-1)}} \rho^{N-1-a}\,\mathrm{d}\rho. \end{aligned}$$

By performing the change of variable $\rho = \tau^{1/\beta}$, we get

$$t_n^N \geqslant (\beta_0 - \epsilon)\omega_{N-1}\frac{N}{N-a}\left(\frac{r}{n}\right)^{N-a} \int_0^1 e^{\alpha_0 \frac{N}{N-a} |t_n V_n(\tau)|^{N/(N-1)}} \tau^{N-1}\,\mathrm{d}\tau.$$

Also setting $\tau = e^{-t/N}$, we obtain

$$t_n^N \geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a}\left(\frac{r}{n}\right)^{N-a} \int_0^{+\infty} e^{\frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} |t_n y_n(t)|^{N/(N-1)}} e^{-t}\,\mathrm{d}t. \tag{4.8}$$

Thus,

$$\begin{aligned} t_n^N &\geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a}\left(\frac{r}{n}\right)^{N-a} \int_n^{+\infty} e^{\frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} t_n^{N/(N-1)}(n - 2\log n)} e^{-t}\,\mathrm{d}t \\ &= (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a} r^{N-a} e^{\frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} t_n^{N/(N-1)}(n - 2\log n) - (N-a)\log n - n}, \end{aligned}$$

and hence

$$1 \geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a} r^{N-a} e^{\frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} t_n^{N/(N-1)}(n - 2\log n) - (N-a)\log n - n - \log t_n^N}, \tag{4.9}$$

which implies that (t_n) is bounded, otherwise we have

$$t_n^{N/(N-1)} n\left[\frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)}\left(1 - \frac{2\log n}{t_n^{N/(N-1)} n}\right) - \frac{(N-a)\log n + n}{t_n^{N/(N-1)} n} - \frac{\log t_n^N}{t_n^{N/(N-1)} n}\right] \to +\infty,$$

which is a contradiction with (4.9).

Next, assuming that (4.4) does not hold and using (4.5), there exists $\delta > 0$ such that, for n sufficiently large,

$$t_n^{N/(N-1)} \geqslant \delta + \frac{N-a}{N}\frac{\alpha_N}{\alpha_0}.$$

By (4.8),

$$t_n^N \geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a}\left(\frac{r}{n}\right)^{N-a} \int_n^{+\infty} e^{(\delta \frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} + 1)(n - 2\log n)} e^{-t}\,\mathrm{d}t,$$

thus,

$$t_n^N \geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a} r^{N-a} e^{\delta \frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} n - (\delta \frac{\alpha_0}{\alpha_N}\frac{N}{(N-a)} + 1) 2\log n - (N-a)\log n}, \tag{4.10}$$

which implies that $t_n \to +\infty$. Thus, (4.4) holds.

Now, consider

$$A_n = \{x \in B_r(0)\colon\ t_n M_n \geqslant R_\epsilon\} \quad \text{and} \quad B_n = B_r(0) \setminus A_n.$$

By (4.6), we have

$$t_n^N \geqslant (\beta_0 - \epsilon)\Bigg[\int\limits_{B_r(0)} \frac{e^{(\alpha_0 |t_n M_n|^{N/(N-1)})}}{|x|^a}\,\mathrm{d}x - \int\limits_{B_n} \frac{e^{(\alpha_0 |t_n M_n|^{N/(N-1)})}}{|x|^a}\,\mathrm{d}x\Bigg] + \int\limits_{B_n} \frac{f(x, t_n M_n)}{|x|^a} t_n M_n\,\mathrm{d}x. \tag{4.11}$$

Notice that $M_n(x) \to 0$, almost everywhere in $B_r(0)$, and the characteristic functions $\chi_{B_n}(x) \to 1$ almost everywhere in $B_r(0)$ and $t_n M_n(x) \leqslant R_\epsilon$ in B_n. Therefore, the Lebesgue dominated convergence theorem implies

$$\int\limits_{B_n} \frac{f(x, t_n M_n)}{|x|^a} t_n M_n\,\mathrm{d}x \to 0$$

and

$$\int\limits_{B_n} \frac{e^{(\alpha_0 |t_n M_n|^{N/(N-1)})}}{|x|^a}\,\mathrm{d}x \to \frac{\omega_{N-1}}{N-a} r^{N-a}.$$

Notice that

$$\int\limits_{B_r(0)} \frac{e^{\alpha_0 |t_n M_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x = r^{N-a} \int\limits_{B_1(0)} \frac{e^{\alpha_0 |t_n \widetilde{M}_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x$$
$$= \omega_{N-1} r^{N-a} \int_0^1 e^{\alpha_0 |t_n \widetilde{M}_n(\rho)|^{N/(N-1)}} \rho^{N-1-a}\,\mathrm{d}\rho.$$

Changing variables in the integral above, $\rho = \tau^{1/\beta}$, we get

$$\int\limits_{B_r(0)} \frac{e^{\alpha_0 |t_n M_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x = \omega_{N-1} \frac{N}{N-a} r^{N-a} \int_0^1 e^{\alpha_0 \frac{N}{N-a} |t_n V_n(\tau)|^{N/(N-1)}} \tau^{N-1}\,\mathrm{d}\tau,$$

and setting $\tau = e^{-t/N}$, we obtain

$$\int\limits_{B_r(0)} \frac{e^{\alpha_0 |t_n M_n|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x = \frac{\omega_{N-1}}{N-a} r^{N-a} \int_0^{+\infty} e^{\frac{\alpha_0}{\alpha_N} \frac{N}{(N-a)} |t_n y_n(t)|^{N/(N-1)}} e^{-t}\,\mathrm{d}t$$
$$\geqslant \frac{\omega_{N-1}}{N-a} r^{N-a} \int_0^{+\infty} e^{y_n^{N/(N-1)}(t) - t}\,\mathrm{d}t.$$

Passing to the limit in (4.11),

$$\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geqslant (\beta_0 - \epsilon)\left(\frac{\omega_{N-1}}{N-a} r^{N-a}\big(1 + e^{1+1/2+\cdots+1/(N-1)}\big) - \frac{\omega_{N-1}}{N-a} r^{N-a}\right),$$

which implies that

$$\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \geqslant (\beta_0 - \epsilon)\frac{\omega_{N-1}}{N-a} r^{N-a} e^{1+1/2+\cdots+1/(N-1)}.$$

Thus,

$$\beta_0 \leqslant \frac{N-a}{r^{N-a} e^{1+1/2+\cdots+1/(N-1)}} \left(\frac{N-a}{\alpha_0}\right)^{N-1},$$

which is a contradiction with the assumption (f_4^+). □

Corollary 4.3. *Under the conditions* $(f_2)-(f_4^+)$, *if* $\|h\|_*$ *is sufficiently small, it holds*

$$\max_{t\geqslant 0}\left\{\frac{t^N}{N}-\int_\Omega \frac{F(x,tM_n)}{|x|^a}\,dx - t\int_\Omega hM_n\,dx\right\}<\frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Notice that $\|hM_n\|_1\leqslant\|h\|_*$. Thus, taking $\|h\|_*$ sufficiently small and using Lemma 4.2 the result follows. □

In order to obtain convergence results, we need to improve the estimate of Lemma 2.2.

Corollary 4.4. *Under the hypotheses* $(f_2)-(f_4^+)$, *there exists* $\delta_2>0$ *such that for all* $h\in W^{-1,N'}$ *with* $0<\|h\|_*<\delta_2$, *there exists* $u\in W_0^{1,N}(\Omega)$ *with compact support verifying*

$$I(tu)<c_0+\frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},\quad \textit{for all } t\geqslant 0.$$

Proof. It is possible to raise the infimum c_0 by reducing $\|h\|_*$. By Lemma 2.2, $\rho_h\to 0$ as $\|h\|_*\to 0$. Consequently, c_0 increases as $\|h\|_*$ decreases and $c_0\to 0$ as $\|h\|_*\to 0$. Thus, there exists $\delta_2>0$ such that if $0<\|h\|_*<\delta_2$ then, by Corollary 4.3, we have

$$\max_{t\geqslant 0} I(tM_n)<c_0+\frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Taking $u=M_n\in W_0^{1,N}(\Omega)$, the result is proved. □

Lemma 4.5. *If* (u_n) *is a Palais–Smale sequence for* I *at any level such that*

$$\liminf_{n\to\infty}\|u_n\|^N<\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \tag{4.12}$$

then (u_n) *possesses a subsequence which converges strongly in* $W_0^{1,N}(\Omega)$ *to a weak solution* u_0 *of* (1.2).

Proof. From Lemma 3.2 and Corollary 3.3, up to a subsequence, we may assume that

$$\begin{aligned}
&u_n\rightharpoonup u_0\quad \text{in } W_0^{1,N}(\Omega),\\
&u_n\to u_0\quad \text{in } L^q(\Omega) \text{ for all } q\in[1,\infty),\\
&u_n(x)\to u_0(x)\quad \text{almost everywhere in } \Omega,
\end{aligned}$$

where u_0 is a weak solution of (1.2).

Assertion 3. $u_n\to u_0$ strongly in $W_0^{1,N}(\Omega)$.

Indeed, writing $u_n=u_0+w_n$, it follows that $w_n\rightharpoonup 0$ in $W_0^{1,N}(\Omega)$. Thus $w_n\to 0$ in $L^q(\Omega)$ for all $q\in[1,\infty)$. By the Brezis–Lieb lemma (see [7]), we get

$$\|u_n\|^N=\|u_0\|^N+\|w_n\|^N+o(1). \tag{4.13}$$

Using similar argument as in the proof of (4.2), we have

$$\int_\Omega \frac{f(x,u_n)}{|x|^a}u_0\,dx\to\int_\Omega \frac{f(x,u_0)}{|x|^a}u_0\,dx\quad \text{as } n\to\infty. \tag{4.14}$$

By (4.13) and (4.14), we can write

$$\langle I'(u_n),u_n\rangle=\langle I'(u_0),u_0\rangle+\|w_n\|^N-\int_\Omega \frac{f(x,u_n)}{|x|^a}w_n\,dx+o(1),$$

that is,

$$\|w_n\|^N=\int_\Omega \frac{f(x,u_n)}{|x|^a}w_n\,dx+o(1). \tag{4.15}$$

Since

$$\|u_n\|^N < \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

for n sufficiently large, we can choose $q > 1$ sufficiently close to 1 such that

$$\alpha_0 q \|u_n\|^{N/(N-1)}/\alpha_N + qa/N < 1.$$

Then,

$$\int_\Omega \left(\frac{|f(x,u_n)|}{|x|^a}\right)^q \mathrm{d}x \leqslant C \int_\Omega \frac{e^{q\alpha_0\|u_n\|^{N/(N-1)}|\frac{u_n}{\|u_n\|}|^{N/(N-1)}}}{|x|^a}\,\mathrm{d}x \leqslant C,$$

which implies that

$$\int_\Omega \frac{f(x,u_n)}{|x|^a} w_n\,\mathrm{d}x \leqslant C\|w_n\|_{q'} \to 0.$$

Consequently $\|w_n\| \to 0$ and the result follows. □

Next, we will prove the existence of a local minimum solution.

Lemma 4.6. *For each $h \in W^{-1,N'}$ with $0 < \|h\|_* < \delta_1$, Eq.* (1.2) *has a local minimum solution u_0 with $I(u_0) = c_0 < 0$, where c_0 is defined in* (4.1)*.*

Proof. Let ρ_h be as in Lemma 2.2. We can choose $\|h\|_*$ sufficiently small such that

$$\rho_h < \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}.$$

Since $\overline{B}_{\rho_h}$ is convex and a complete metric space with the metric given by the norm of $W_0^{1,N}(\Omega)$, and I is of class C^1 and bounded below on $\overline{B}_{\rho_h}$, by Ekeland's variational principle there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n) \to c_0 = \inf_{\|u\|\leqslant\rho_h} I(u) \quad \text{and} \quad \left\|I'(u_n)\right\|_* \to 0.$$

Observing that

$$\|u_n\|^N \leqslant \rho_h^N < \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

by Lemma 4.5, there exists a subsequence of (u_n) which converges strongly to a weak solution u_0 of (1.2). Therefore, $I(u_0) = c_0 < 0$. □

Lemma 4.7. *Under the assumptions $(f_2)-(f_4^+)$,* (1.2) *has a mountain-pass type solution u_M, provided that $\|h\|_* < \delta_1$.*

Proof. By Lemmas 2.2 and 2.3, we have that I has a mountain-pass geometry. Thus, using the mountain-pass theorem without the Palais–Smale condition (see [10]), there exists a sequence (u_n) in $W_0^{1,N}(\Omega)$ satisfying

$$I(u_n) \to c_M > 0 \quad \text{and} \quad \left\|I'(u_n)\right\|_* \to 0,$$

where c_M is the mountain-pass level. Now, by Lemma 3.2 and Corollary 3.3, the sequence (u_n) converges weakly to a weak solution u_M of (1.2). □

Remark 4.1. By Corollary 4.4, we can conclude that

$$0 < c_M < c_0 + \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

For the critical case we obtain a Palais–Smale sequence converge strongly for level c_0, but for level c_M we obtain only that the Palais–Smale sequence converge weakly, then to prove that these two solutions are distinct we shall use the following result due to Adimurthi and Sandeep [2] (see also [19] for a nonsingular case):

Lemma 4.8. *Let $\{u_k\colon \|u_k\|=1\}$ be a sequence in $W_0^{1,N}(\Omega)$ converging weakly to a non-zero function u. Then, for every $p<(1-\|u\|^N)^{-1/(N-1)}$ and $a\in[0,N)$*

$$\sup_k \int_\Omega \frac{e^{p\alpha_N \frac{N-a}{N}|u_k|^{N/(N-1)}}}{|x|^a}\,dx<\infty.$$

We also will use the following convergence result:

Lemma 4.9. *Assume that $f(x,s)$ satisfies (f_2) and has critical growth at both $+\infty$ and $-\infty$. If $(u_n)\subseteq W_0^{1,N}(\Omega)$ is a Palais–Smale sequence for I and u_0 is its weak limit then, up to a subsequence,*

$$\frac{F(x,u_n)}{|x|^a}\to\frac{F(x,u_0)}{|x|^a}\quad \text{in } L^1(\Omega).$$

Proof. As a consequence of Lemma 3.2, we get

$$\frac{f(x,u_n)}{|x|^a}\to\frac{f(x,u_0)}{|x|^a}\quad \text{in } L^1(\Omega).$$

Thus, there exists $g\in L^1(\Omega)$ such that $\frac{|f(x,u_n)|}{|x|^a}\leqslant g$ almost everywhere in Ω. From (f_2) we can conclude that

$$\left|F(x,u_n)\right|\leqslant \sup_{(x,u_n)\in\Omega\times[-R,R]}\left|F(x,u_n)\right|+M_0 f(x,u_n)\quad \text{almost everywhere in } \Omega,$$

thus, by generalized Lebesgue dominated convergence theorem

$$\frac{F(x,u_n)}{|x|^a}\to\frac{F(x,u_0)}{|x|^a}\quad \text{in } L^1(\Omega). \qquad \square$$

Proposition 4.3. *If $\delta_2>0$ is small enough, then the solutions of* (1.2) *obtained in Lemma* 4.6 *and Lemma* 4.7 *are distinct.*

Proof. Let (u_n) be the minimizing sequence and let (v_n) be the mountain pass sequence, so that

$$\begin{aligned}
&u_n\rightharpoonup u_0\quad \text{in } W_0^{1,N}(\Omega)\quad \text{and}\quad v_n\rightharpoonup u_M\quad \text{in } W_0^{1,N}(\Omega),\\
&I(u_n)\to c_0<0\quad \text{and}\quad I(v_n)\to c_M>0,\\
&\left\langle I'(u_n),u_n\right\rangle\to 0\quad \text{and}\quad \left\langle I'(v_n),v_n\right\rangle\to 0.
\end{aligned}\tag{4.16}$$

Suppose that $u_0=u_M$. Then from Lemma 4.9

$$I(u_n)=\frac{1}{N}\|u_n\|^N-\int_\Omega \frac{F(x,u_0)}{|x|^a}\,dx-\int_\Omega h(x)u_0\,dx+o(1)=c_0$$

and

$$I(v_n)=\frac{1}{N}\|v_n\|^N-\int_\Omega \frac{F(x,u_0)}{|x|^a}\,dx-\int_\Omega h(x)u_0\,dx+o(1)=c_M$$

and subtracting one from the other, we have

$$\|u_n\|^N-\|v_n\|^N\to N(c_0-c_M)<0\quad \text{as } n\to\infty.\tag{4.17}$$

Since (u_n) and (v_n) are both Palais–Smale sequences

$$\begin{aligned}
&\left\langle I'(u_n),u_n\right\rangle=\|u_n\|^N-\int_\Omega \frac{f(x,u_n)}{|x|^a}u_n\,dx-\int_\Omega h(x)u_n\,dx\to 0,\\
&\left\langle I'(v_n),v_n\right\rangle=\|v_n\|^N-\int_\Omega \frac{f(x,v_n)}{|x|^a}v_n\,dx-\int_\Omega h(x)v_n\,dx\to 0,
\end{aligned}$$

to give

$$\left(\|u_n\|^N - \|v_n\|^N\right) - \int_\Omega \left[\frac{f(x,u_n)}{|x|^a}u_n - \frac{f(x,u_n)}{|x|^a}v_n + \frac{f(x,u_n)}{|x|^a}v_n - \frac{f(x,v_n)}{|x|^a}v_n\right]\mathrm{d}x$$
$$- \int_\Omega \left[h(u_n-u_0) - h(v_n-u_0)\right]\mathrm{d}x \to 0 \quad \text{as } n\to\infty. \tag{4.18}$$

Notice that the last term in (4.18) tends to zero, because $h \in W^{-1,N'}$, $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup u_0$ weakly in $W_0^{1,N}(\Omega)$.

The second term in (4.18) may be written as:

$$\int_\Omega \frac{f(x,u_n)}{|x|^a}(u_n - v_n)\mathrm{d}x - \int_\Omega \frac{f(x,u_n)-f(x,v_n)}{|x|^a}v_n\,\mathrm{d}x.$$

Notice that

$$\int_\Omega \frac{f(x,u_n)}{|x|^a}(u_n - v_n)\,\mathrm{d}x \to 0 \quad \text{as } n\to\infty.$$

Indeed, we have derived that for $\|h\|_*$ in the range $(0,\delta_1)$, the minimizing sequence (u_n) must satisfy

$$\|u_n\| < \left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{(N-1)/N}, \tag{4.19}$$

thus using Lemma 4.5 we can conclude that $u_n \to u_0$ strongly in $W_0^{1,N}(\Omega)$ and since I is continuous we have $I(u_n) \to I(u_o) < 0$. Notice that if $v_n \to u_M$ strongly in $W_0^{1,N}(\Omega)$ we have $I(v_n) \to I(u_M) > 0$. Therefore, $u_0 \neq u_M$.

Next, we assume that $v_n \rightharpoonup u_0$ weakly in $W_0^{1,N}(\Omega)$ but $v_n \not\to u_0$ strongly in $W_0^{1,N}(\Omega)$. Let $v_n = u_0 + w_n$, so $w_n \rightharpoonup 0$ and $\lim_{n\to\infty}\|w_n\| > 0$.

Using (4.19), we can choose $q > 1$ sufficiently close to 1 such that

$$q\alpha_0\|u_n\|^{N/(N-1)}/\alpha_N + aq/N < 1.$$

Thus

$$\int_\Omega \left(\frac{|f(x,u_n)|}{|x|^a}\right)^q \mathrm{d}x \leqslant C\int_\Omega \frac{e^{(q\alpha_0\|u_n\|^{N/(N-1)}|\frac{u_n}{\|u_n\|}|^{N/(N-1)})}}{|x|^{aq}}\,\mathrm{d}x \leqslant C, \tag{4.20}$$

which together with the Hölder inequality implies that

$$\int_\Omega \frac{f(x,u_n)}{|x|^a}(u_n - v_n)\,\mathrm{d}x \leqslant C\|u_n - v_n\|_{q'} \to 0 \quad \text{as } n \to +\infty.$$

It remains to show that

$$\int_\Omega \frac{f(x,u_n)-f(x,v_n)}{|x|^a}v_n\,\mathrm{d}x \to 0 \quad \text{as } n\to\infty. \tag{4.21}$$

Now, (4.21) may be expressed as

$$\int_\Omega \frac{f(x,u_n)-f(x,v_n)}{|x|^a}u_0\,\mathrm{d}x + \int_\Omega \frac{f(x,u_n)-f(x,v_n)}{|x|^a}w_n\,\mathrm{d}x.$$

Using the same argument as in the proof of (4.2) in Lemma 4.1, we have that the first term vanishes. Considering the second of these terms,

$$\int_\Omega \frac{f(x,u_n)-f(x,v_n)}{|x|^a}w_n\,\mathrm{d}x = \int_\Omega \frac{f(x,u_n)}{|x|^a}w_n\,\mathrm{d}x - \int_\Omega \frac{f(x,v_n)}{|x|^a}w_n\,\mathrm{d}x.$$

Using (4.20), the Hölder inequality and Sobolev embedding, we get

$$\int_\Omega \frac{f(x,u_n)}{|x|^a}w_n\,\mathrm{d}x \leqslant \left(\int_\Omega \frac{e^{(q\alpha_0\|u_n\|^{N/(N-1)}|\frac{u_n}{\|u_n\|}|^{N/(N-1)})}}{|x|^{aq}}\,\mathrm{d}x\right)^{1/q}\|w_n\|_{q'} \leqslant C\|w_n\|_{q'} \to 0. \tag{4.22}$$

We are now left with only the term $\int_\Omega \frac{f(x,v_n)}{|x|^a}w_n\,\mathrm{d}x$.

By Corollary 4.4, taking δ_2 is sufficiently small, we conclude that

$$0 < c_M < c_0 + \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Consequently, for large n

$$\begin{aligned} c_M - c_0 = I(v_n) - I(u_n) + o(1) &= \frac{1}{N}\|v_n\|^N - \frac{1}{N}\|u_n\|^N + o(1) \\ &= \frac{1}{N}\|v_n\|^N - \frac{1}{N}\|u_0\|^N + o(1) \\ &< \frac{1}{N}\left(\frac{N-a}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \end{aligned}$$

Thus, there exists $s > 1$ sufficiently close to 1 such that for large n,

$$\|v_n\|^N - \|u_0\|^N < \left(\frac{N-as}{N}\frac{\alpha_N}{s\alpha_0}\right)^{N-1}.$$

As a direct implication,

$$s\alpha_0\|v_n\|^{N/(N-1)} < \alpha_N\frac{N-as}{N}\left(1 - \left\|\frac{u_0}{\|v_n\|}\right\|\right)^{-1/(N-1)}. \tag{4.23}$$

Define $U_n = \frac{v_n}{\|v_n\|}$. Thus $\|U_n\| = 1$, $U_n \rightharpoonup U_0 = \frac{u_0}{\lim\|v_n\|}$ and $\|U_0\| < 1$. Now,

$$\int_\Omega \frac{f(x,v_n)}{|x|^a} w_n \, dx \leqslant C\left(\int_\Omega \frac{e^{(s\alpha_0\|v_n\|^{N/(N-1)}|\frac{v_n}{\|v_n\|}|^{N/(N-1)})}}{|x|^{as}} \, dx\right)^{1/s} \|w_n\|_{s'}. \tag{4.24}$$

By Lemma 4.8 and using the information that $\|w_n\|_{s'} \to 0$, it follows that

$$\int_\Omega \frac{f(x,v_n)}{|x|^a} w_n \, dx \to 0.$$

Hence expression (4.18) gives that $\|u_n\|^N - \|v_n\|^N \to 0$. But this contradicts (4.17), and thus $u_0 \not\equiv u_M$ and the solutions are distinct. $\square$

Now, the proof of Theorems 1.4 and 1.5 follows directly from Lemmas 4.6, 4.7 and Proposition 4.3.

4.3. Proof of Theorems 1.3 and 1.6

In order to prove Theorems 1.3 and 1.6 in the case $h(x) \geqslant 0$, we redefine $f(x,s)$ as

$$\widetilde{f}(x,s) = \begin{cases} f(x,s), & \text{if } (x,s) \in \Omega \times [0,+\infty), \\ 0, & \text{if } (x,s) \in \Omega \times (-\infty, 0]. \end{cases}$$

Thus, in the subcritical case (f_1) holds for $s \geqslant s_1$ and in the critical case (f_2) holds for $s \geqslant R$. Notice that hypotheses (f_1) and (f_2) was required to help verify the Palais–Smale condition and Lemma 3.2, which is valid also for this modified nonlinearity.

The proof is a consequence of the following result.

Corollary 4.10. *If $h(x) \geqslant 0$ almost everywhere in Ω, then the weak solutions of* (1.2) *are nonnegative.*

Proof. Let $u \in W_0^{1,N}(\Omega)$ be a weak solution of (1.2). Setting $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and taking $v = u^-$ as a testing function in $\langle I'(u), v\rangle = 0$, we obtain

$$\|u^-\|^N = -\int_\Omega h(x)u^- \, dx \leqslant 0,$$

because $f(x,u(x))u^-(x) = 0$ in Ω. Consequently, $u = u^+ \geqslant 0$. $\square$

Now, in the case $h(x) \leqslant 0$, in order to prove Theorems 1.3 and 1.6, we redefine $f(x,s)$ as

$$\widetilde{f}(x,s) = \begin{cases} -f(x,-s), & \text{if } (x,s) \in \Omega \times (-\infty, 0), \\ f(x,s), & \text{if } (x,s) \in \Omega \times [0,+\infty). \end{cases}$$

In this case, the proof of Theorems 1.3 and 1.6 is given in the following corollary:

Corollary 4.11. *Suppose that* (f_4^-) *holds and* $h(x) \leqslant 0$ *almost everywhere in* Ω*. Then there exist at least two nonpositive weak solutions of* (1.2)*.*

Proof. Consider the functional defined by

$$\widetilde{I}(u) = \frac{1}{N}\|u\|^N - \int_\Omega \frac{\widetilde{F}(x,u)}{|x|^a}\,dx - \int_\Omega \bigl(-h(x)\bigr)u\,dx,$$

where $\widetilde{F}$ is the primitive of $\widetilde{f}$. Notice that $\widetilde{f}$ satisfies the same hypotheses of f. Since $-h(x) \geqslant 0$ almost everywhere in Ω, by Corollary 4.10, $\widetilde{I}(u)$ has two nonnegative and nontrivial critical points. Let $\widetilde{u}$ be one such critical point, that is

$$\int_\Omega |\nabla \widetilde{u}|^{N-2}\nabla\widetilde{u}\nabla v\,dx - \int_\Omega \frac{\widetilde{f}(x,\widetilde{u})}{|x|^a} v\,dx + \int_\Omega h(x)v\,dx = 0, \quad \forall v \in W_0^{1,N}(\Omega). \tag{4.25}$$

Recalling the construction of $\widetilde{f}$, we have that $\widetilde{f}(x,\widetilde{u}) = -f(x,-\widetilde{u})$ and replacing v by $-v$ in (4.25), we obtain

$$\int_\Omega |\nabla(-\widetilde{u})|^{N-2}\nabla(-\widetilde{u})\nabla v\,dx - \int_\Omega \frac{f(x,-\widetilde{u})}{|x|^a} v\,dx - \int_\Omega h(x)v\,dx = 0, \quad \forall v \in W_0^{1,N}(\Omega),$$

which implies that $-\widetilde{u}$ is a nonpositive weak solution of (1.2). □

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