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Global weak solutions and blow-up structure for the Degasperis–Procesi equation

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Abstract

In this paper we study several qualitative properties of the Degasperis–Procesi equation. We first established the precise blow-up rate and then determine the blow-up set of blow-up strong solutions to this equation for a large class of initial data. We finally prove the existence and uniqueness of global weak solutions to the equation provided the initial data satisfies appropriate conditions.

Keywords: Degasperis–Procesi equation; Global weak solutions; Blow-up rate; Blow-up set

1. Introduction

Recently, Degasperis and Procesi [21] studied the following family of third order dispersive conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.1}$$

where α, c_0, c_1, c_2, and c_3 are real constants and indices denote partial derivatives. In [21] the authors found that there are only three equations that satisfy the asymptotic integrability

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condition within this family: the KdV equation, the Camassa–Holm equation and the Degasperis–Procesi equation.

If $\alpha = c_2 = c_3 = 0$, then Eq. (1.1) becomes the well-known KdV equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. In this model $u(t,x)$ represents the wave's height above a flat bottom, x is proportional to distance in the direction of propagation and t is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [22,40]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand (for example, see [30,45]). It is shown that the KdV equation is globally well-posed for $u_0 \in L^2(\mathbb{R})$, cf. [45]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave remains bounded but its slope becomes unbounded in finite time [48]).

For $c_1 = -\frac{3}{2}c_3/\alpha^2$ and $c_2 = c_3/2$, Eq. (1.1) becomes the Camassa–Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom. Again $u(t,x)$ stands for the fluid velocity at time t in the spatial x direction and c_0 is a nonnegative parameter related to the critical shallow water speed [3,23,28]. The Camassa–Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [17,19]. It has a bi-Hamiltonian structure [25,32] and is completely integrable [3,8]. Its solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$, see [4]. The orbital stability of the peaked solitons is proved in [16], and that of the smooth solitons in [18]. The explicit interaction of the peaked solitons is given in [1].

The Cauchy problem of the Camassa–Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [10,33,44] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. More interestingly, it has global strong solutions [7,10] and also finite time blow-up solutions [7,10,11, 14,34]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2,12,15,47,49,50]. It is also known that if u is the solution of the Camassa–Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, then we have the following a priori estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2}\|u(t,\cdot)\|_{H^1(\mathbb{R})} \leqslant \sqrt{2}\|u_0(\cdot)\|_{H^1(\mathbb{R})}$$

for all $t > 0$. The advantage of the Camassa–Holm equation in comparison with the KdV equation lies in the fact that the Camassa–Holm equation has peaked solitons and models wave breaking [4].

If $c_1 = -2c_3/\alpha^2$ and $c_2 = c_3$ in Eq. (1.1), then, after rescaling, shifting the dependent variable, and applying a Galilean boost [20], we find the Degasperis–Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

Degasperis, Holm and Hone [20] proved the formal integrability of Eq. (1.2) by constructing a Lax pair. They also showed that Eq. (1.2) has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and that it admits exact peakon solutions which are analogous to the Camassa–Holm peakons.

The Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm shallow water equation. Dullin, Gottwald and Holm [24] showed that the Degasperis–Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. Lundmark and Szmigielski [37] presented an inverse scattering approach for computing n-peakon solutions to Eq. (1.2). Vakhnenko and Parkes [46] investigated traveling wave solutions of Eq. (1.2) and Holm and Staley [27] studied stability of solitons and peakons numerically.

After the Degasperis–Procesi equation (1.2) was derived, many papers were devoted to its study, cf. [5,26,31,35,36,39,51–54] and the citations therein. For example, Yin proved local well-posedness of Eq. (1.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, on the line [51] and on the circle [52]. In these two papers the precise blow-up scenario and a blow-up result were derived. The global existence of strong solutions and global weak solutions to Eq. (1.2) are also investigated in [53, 54]. Recently, Lenells [31] classified all weak traveling wave solutions. Matsuno [39] studied multisoliton solutions and their peakon limits. Analogous to the case of Camassa–Holm equation [9], Henry [26] and Mustafa [42] showed that smooth solutions to Eq. (1.2) have infinite speed of propagation. Coclite and Karlsen [5] also obtained global existence results for entropy solutions in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$.

Despite the similarities to the Camassa–Holm equation, it should be emphasized that these two equations are truly different. One of the important features of Eq. (1.2) is that it has not only peakon solitons [20], i.e. solutions at the form $u(t,x) = ce^{-|x-ct|}$, $c > 0$, but also shock peakons [6,36] which are given by

$$u(t,x) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \quad k > 0.$$

It is shown in [36] that the above shock-peakon solutions can be formally obtained by substituting $(x,t) \mapsto (\epsilon x, \epsilon t)$ to Eq. (1.2) and letting $\epsilon \to 0$ so that it becomes the "derivative Burger's equation" $(u_t + uu_x)_{xx} = 0$, from which shock waves form.

On the other hand, the isospectral problem for Eq. (1.2) has the third-order equation in the Lax pair

$$\psi_x - \psi_{xxx} - \lambda y \psi = 0,$$

cf. [20]. While the isospectral problem for the Camassa–Holm equation is the second-order equation

$$\psi_{xx} - \frac{1}{4}\psi - \lambda y \psi = 0,$$

cf. [3] (in both cases $y = u - u_{xx}$). Another indication of the fact that there is no simple transformation of Eq. (1.2) into the Camassa–Holm equation is the entirely different form of conservation laws for that two equations [3,20]. Furthermore, the Camassa–Holm equation is a re-expression of geodesic flow on the diffeomorphism group [13] or on the Bott–Virasoro group [41]. Up to now, no geometric derivation of the Degasperis–Procesi equation is available.

The following three conservation laws of the Degasperis–Procesi equation are very useful for our analysis:

$$E_1(u) = \int_{\mathbb{R}} y\,dx, \qquad E_2(u) = \int_{\mathbb{R}} yv\,dx, \qquad E_3(u) = \int_{\mathbb{R}} u^3\,dx.$$

Here we set $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$. The corresponding conservation laws of the Camassa–Holm equation are the following:

$$F_1(u)=\int_{\mathbb{R}} y\,dx, \qquad F_2(u)=\int_{\mathbb{R}} \left(u^2+u_x^2\right)dx, \qquad F_3(u)=\int_{\mathbb{R}} \left(u^3+uu_x^2\right)dx.$$

It turns out that the conservation laws of the Degasperis–Procesi equation are much weaker than those of the Camassa–Holm equation. Although the bi-Hamiltonian structure of Eq. (1.2) provides an infinite number of conservation laws [20], the conservation laws $E_i(u)$ cannot guarantee the boundedness of the slope of wave, and there is no way to find conservation laws controlling the H^1-norm, which plays important role in studying the Camassa–Holm equation.

It [35] is shown that the first blow-up must occur as wave breaking and shock waves possibly appear afterwards. Note that Eq. (1.2) admits peaked solitons which are global weak solutions [20]. Global weak solutions to Eq. (1.2) have recently been discussed in [54]. The goal of this paper is to present a new result for global weak solutions in $H^1(\mathbb{R})$ and to study the blow-up structure for the Degasperis–Procesi equation. We hope that our results shed some light on important physical phenomena of Eq. (1.2) such as wave breaking and shock waves. Our methods not only rely on a new conservation law and a very useful a priori estimate of the L^∞-norm of the strong solutions to Eq. (1.2), but also on an approximation procedure used first for the solutions to the Camassa–Holm equation [15,47], a partial integration result in Bochner spaces, and Helly's theorem.

The remainder of the paper is organized as follows. In Section 2, we recall the local well-posedness of the Cauchy problem of Eq. (1.2) with initial data $u_0\in H^s(\mathbb{R})$, $s>\frac{3}{2}$, and the precise blow-up scenario of strong solutions to Eq. (1.2) from [51,54]. We also recall several results and definitions on strong solutions and weak solutions to Eq. (1.2). Based on a new conservation law and a useful a priori estimate of the L^∞-norm of the strong solutions to Eq. (1.2), we will investigate in Section 3 the blow-up rate and the blow-up set of the strong blow-up solutions to Eq. (1.2). In the last section, we show the existence and uniqueness of global weak solutions to Eq. (1.2), provided the initial data satisfy appropriate conditions.

Notation. In the following, we denote by $*$ the spatial convolution. We write $\hat{f}$ for the Fourier transform of f. We also use $(\cdot|\cdot)$ to represent the standard inner product in $L^2(\mathbb{R})$. For $1\leqslant p\leqslant\infty$, the norm in the Lebesgue space L^p will be denoted by $\|\cdot\|_{L^p}$, while $\|\cdot\|_s$, $s\geqslant 0$, will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{R})$. Given a Banach space X, we denote its norm by $\|\cdot\|_X$. The duality paring between $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$ is denoted by $\langle\cdot,\cdot\rangle$. The space of all Radon measures on $\mathbb{R}$ with bounded total variation is denoted by $M(\mathbb{R})$ and $M^+(\mathbb{R})$ is the subset of all positive measures. Finally, we write $BV(\mathbb{R})$ for the space of functions with bounded variation and $\mathbb{V}(f)$ for the total variation of $f\in BV(\mathbb{R})$.

2. Preliminaries

In this section, we recall the local well-posedness result, some results on blow-up and global existence of strong solutions, the definitions and some properties of strong and weak solutions to (1.2) from [35,51,54]. A partial integration result for Bochner spaces from [38] and several approximation results from [15] are also presented.

With $y:=u-u_{xx}$, Eq. (1.2) takes the form:

$$\begin{cases} y_t+uy_x+3u_xy=0, & t>0,\ x\in\mathbb{R},\\ y(0,x)=u_0(x)-u_{0,xx}(x), & x\in\mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$. Using this identity, we can rewrite Eq. (2.1) as a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x + \partial_x p * (\frac{3}{2}u^2) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The local well-posedness of the Cauchy problem of Eq. (2.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, can be obtained by applying the Kato's theorem [29,51]. In fact, we have the following well-posedness result.

Lemma 2.1. [51] *Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique solution u to Eq.* (2.2) (*or to Eq.* (1.2))*, such that*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0)$: $H^s(\mathbb{R}) \to C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous and the maximal time of existence $T > 0$ is independent of s.

By using the local well-posedness in Lemma 2.1 and the energy method, one can get the following precise blow-up scenario of strong solutions to Eq. (2.2).

Lemma 2.2. [51] *Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, blow-up of the solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t \uparrow T} \Big\{ \inf_{x \in \mathbb{R}} \big[u_x(t,x)\big] \Big\} = -\infty.$$

Consider now the following differential equation:

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following result on q which is crucial in the proof of global existence and blow-up solutions.

Lemma 2.3. [54] *Let $u_0 \in H^s(\mathbb{R})$, $s \geqslant 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq.* (2.2)*. Then Eq.* (2.3) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x\big(s, q(s,x)\big)\, ds \right) > 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Furthermore, setting $y := u - u_{xx}$, we have

$$y\big(t, q(t,x)\big) q_x^3(t,x) = y_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Using the above lemmas, we can prove the following blow-up and global existence results.

Lemma 2.4. [35] *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and $y_0(x) := u_0(x) - u_{0,xx}(x)$. Assume that $y_0(x)$ changes sign and that there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \geqslant 0 & \text{if } x \leqslant x_0, \\ y_0(x) \leqslant 0 & \text{if } x \geqslant x_0. \end{cases}$$

Then the corresponding solution to Eq. (2.2) *blows up in finite time.*

Lemma 2.5. [35] *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and that there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \leqslant 0 & \text{if } x \leqslant x_0, \\ y_0(x) \geqslant 0 & \text{if } x \geqslant x_0. \end{cases}$$

Then Eq. (2.2) *has a unique global strong solution*

$$u = u(\cdot, u_0) \in C\big([0,\infty); H^s(\mathbb{R})\big) \cap C^1\big([0,\infty); H^{s-1}(\mathbb{R})\big).$$

Moreover, $E_2(u) = \int_{\mathbb{R}} yv\,dx$ is a conservation law, where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$, and for all $t \in \mathbb{R}_+$ we have

(i) $u_x(t,\cdot) \geqslant -|u(t,\cdot)|$ *on* $\mathbb{R}$,
(ii) $\|u\|_1^2 \leqslant 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2 \|u_0\|_{L^\infty} t + \|u_0\|_1^2$.

Note that Eq. (1.2) has the soliton interaction property of solitary waves with corners at their peaks, discovered in [20,21]. Obviously, such solutions are not strong solutions to Eq. (2.1) in the sense of Lemma 2.1. In order to provide a mathematical framework for the study of peaked solitons and their interaction, we shall introduce a suitable notion of weak solutions to Eq. (2.2).

Observe that, setting

$$F(u) = \left(\frac{u^2}{2} + p * \left(\frac{3}{2}u^2\right)\right),$$

Eq. (2.2) can be rewritten as the conservation law

$$u_t + F(u)_x = 0, \qquad u(0,x) = u_0, \quad t > 0,\ x \in \mathbb{R}.$$

The following notion was introduced in [12], see also [54].

Definition 2.1. Let $u_0 \in H^1(\mathbb{R})$ and $u \in L^\infty_{\mathrm{loc}}([0,T); H^1(\mathbb{R}))$ satisfies the following identity:

$$\int_0^T \int_{\mathbb{R}} \big(u\psi_t + F(u)\psi_x\big)\,dx\,dt + \int_{\mathbb{R}} u_0(x)\psi(0,x)\,dx = 0$$

for all $\psi \in C_c^\infty([0,T)\times\mathbb{R})$. Let $C_c^\infty([0,T)\times\mathbb{R})$ denote the space of all functions on $[0,T)\times\mathbb{R}$, which may be obtained as the restriction to $[0,T)\times\mathbb{R}$ of a smooth function on $\mathbb{R}^2$ with compact support contained in $(-T,T)\times\mathbb{R}$, then u is called a weak solution to Eq. (2.2). If u is a weak solution on $[0, T)$ for every $T > 0$, then it is called global weak solution to Eq. (2.2) (or to Eq. (1.2)).

The following result proved in [51] clarifies the relation between strong and weak solutions.

Proposition 2.1. [51]

(i) *Every strong solution is a weak solution.*
(ii) *If u is a weak solution and $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ with $s > \frac{3}{2}$, then it is a strong solution.*
(iii) *All nontrivial traveling waves of Eq.* (1.2) *are not strong solutions.*
(iv) *The peaked solitons are global weak solutions of Eq.* (1.2)*.*

Next, we recall a partial integration result for Bochner spaces.

Lemma 2.6. [38] *Let $T > 0$. If*

$$f, g \in L^2\big((0, T); H^1(\mathbb{R})\big) \quad \textit{and} \quad \frac{df}{dt}, \frac{dg}{dt} \in L^2\big((0, T); H^{-1}(\mathbb{R})\big),$$

then f, g are a.e. equal to a function continuous from $[0, T]$ into $L^2(\mathbb{R})$ and

$$\big\langle f(t), g(t)\big\rangle - \big\langle f(s), g(s)\big\rangle = \int_s^t \left\langle \frac{df(\tau)}{d\tau}, g(\tau)\right\rangle d\tau + \int_s^t \left\langle \frac{dg(\tau)}{d\tau}, f(\tau)\right\rangle d\tau$$

for all $s, t \in [0, T]$.

Throughout this paper, let $\{\rho_n\}_{n\geqslant 1}$ denote the mollifiers

$$\rho_n(x) := \left(\int_{\mathbb{R}} \rho(\xi)\, d\xi\right)^{-1} n\rho(nx), \quad x \in \mathbb{R},\ n \geqslant 1,$$

where $\rho \in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x) := \begin{cases} e^{\frac{1}{x^2-1}}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| \geqslant 1. \end{cases}$$

Then we have the following auxiliary result.

Lemma 2.7. [15]

(a) *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded.*
  (i) *If $\mu \in M(\mathbb{R})$, then*

$$\lim_{n\to\infty} \big[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)\big] = 0 \quad \textit{in } L^1(\mathbb{R}).$$

(ii) *If* $g \in L^\infty(\mathbb{R})$, *then*

$$\lim_{n\to\infty}\big[\rho_n * (fg) - (\rho_n * f)(\rho_n * g)\big] = 0 \quad \text{in } L^\infty(\mathbb{R}).$$

(b) *Assume that* $u(t,\cdot) \in W^{1,1}(\mathbb{R})$ *is uniformly bounded in* $W^{1,1}(\mathbb{R})$ *for all* $t \in \mathbb{R}_+$. *Then for a.e.* $t \in \mathbb{R}_+$

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * u|\,dx = \int_{\mathbb{R}} (\rho_n * u_t)\,\mathrm{sgn}(\rho_n * u)\,dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * u_x|\,dx = \int_{\mathbb{R}} (\rho_n * u_{xt})\,\mathrm{sgn}(\rho_n * u_x)\,dx.$$

3. Blow-up rate and blow-up set

In this section, we derive a conservation law for strong solutions to Eq. (2.2). Using this conservation law, we then obtain an a priori estimate of the L^∞-norm of strong solutions. This enables us to investigate the blow-up rate and the blow-up set of strong blow-up solutions to Eq. (2.1).

Lemma 3.1. *If* $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, *then as long as the solution* $u(t,x)$ *given by Lemma* 2.1 *exists, we have*

$$\int_{\mathbb{R}} y(t,x)v(t,x)\,dx = \int_{\mathbb{R}} y_0(x)v_0(x)\,dx,$$

where $y(t,x) = u(t,x) - u_{xx}(t,x)$ *and* $v(t,x) = (4-\partial_x^2)^{-1}u$. *Moreover, we have*

$$\big\|u(t)\big\|_{L^2}^2 \leqslant 4\|u_0\|_{L^2}^2.$$

Proof. Applying Lemma 2.1 and a simple density argument, it is clear that we may consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with initial data $u_0 \in H^3(\mathbb{R})$ such that $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$, which is guaranteed by Lemma 2.1. Applying the operator $(1-\partial_x^2)$ on both sides of Eq. (2.2), we have

$$y_t + \big(1-\partial_x^2\big)\partial_x\left(\frac{u^2}{2}\right) + \partial_x\left(\frac{3u^2}{2}\right) = 0.$$

Multiplying the above equation by $v(t,x)$ and integrating by parts with respect to x on $\mathbb{R}$, in view of $4v - v_{xx} = u$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} v y_t\,dx &= -\frac{1}{2}\int_{\mathbb{R}} v\big(1-\partial_x^2\big)\partial_x u^2\,dx - \frac{3}{2}\int_{\mathbb{R}} v\partial_x u^2\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}} v_x\big(1-\partial_x^2\big)u^2\,dx + \frac{3}{2}\int_{\mathbb{R}} v_x u^2\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}} v_x\partial_x^2 u^2\,dx + 2\int_{\mathbb{R}} v_x u^2\,dx = 2\int_{\mathbb{R}} v_x u^2\,dx - \frac{1}{2}\int_{\mathbb{R}} v_{xxx}u^2\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}} u^2\partial_x\big(4-\partial_x^2\big)v\,dx = \frac{1}{2}\int_{\mathbb{R}} u^2 u_x\,dx = 0.\end{aligned}$$

Note that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} yv\,dx = \frac{1}{2}\int_{\mathbb{R}} y_t v\,dx + \frac{1}{2}\int_{\mathbb{R}} y v_t\,dx = \int_{\mathbb{R}} y_t v\,dx.$$

Combining the above two relations, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} yv\,dx = \int_{\mathbb{R}} v y_t\,dx = 0.$$

Consequently, this implies the desired conserved quantity. In view of the above conservation law, it then follows that

$$\begin{aligned}\big\|u(t)\big\|_{L^2}^2 = \big\|\hat{u}(t)\big\|_{L^2}^2 &\leqslant 4\int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}\big|\hat{u}(t,\xi)\big|^2\,d\xi = 4\big(\hat{y}(t)|\hat{v}(t)\big)\\ &= 4\big(y(t)|v(t)\big) = 4(y_0|v_0) = 4(\hat{y}_0|\hat{v}_0)\\ &\leqslant 4\int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}\big|\hat{u}_0(\xi)\big|^2\,d\xi \leqslant 4\|\hat{u}_0\|_{L^2}^2 = 4\|u_0\|_{L^2}^2.\end{aligned}$$

This completes the proof of Lemma 3.1. □

The following important estimate is a consequence of Lemma 3.1.

Lemma 3.2. *Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to the Eq.* (2.2) *guaranteed by Lemma* 2.1. *Then we have*

$$\big\|u(t,x)\big\|_{L^\infty} \leqslant 3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}, \quad \forall t \in [0,T].$$

Proof. Applying Lemma 2.1 and a simple density argument, it suffices to consider the case $s=3$ to prove the above result. Let $T>0$ be the maximal time of existence of the solution u to Eq. (2.2) with the initial data $u_0\in H^3(\mathbb{R})$. By (2.2), we have

$$u_t+uu_x=-\partial_x p*\left(\frac{3}{2}u^2\right)=-3p*(uu_x). \tag{3.1}$$

Note that

$$\begin{aligned}-3p*(uu_x)&=-\frac{3}{2}\int_{-\infty}^{+\infty}e^{-|x-\eta|}uu_\eta\,d\eta=-\frac{3}{2}\int_{-\infty}^{x}e^{-x+\eta}uu_\eta\,d\eta-\frac{3}{2}\int_{x}^{+\infty}e^{x-\eta}uu_\eta\,d\eta\\&=\frac{3}{4}\int_{-\infty}^{x}e^{-|x-\eta|}u^2\,d\eta-\frac{3}{4}\int_{x}^{+\infty}e^{-|x-\eta|}u^2\,d\eta.\end{aligned}$$

By (2.3), we have

$$\frac{du(t,q(t,x))}{dt}=u_t\big(t,q(t,x)\big)+u_x\big(t,q(t,x)\big)\frac{dq(t,x)}{dt}=(u_t+uu_x)\big(t,q(t,x)\big).$$

It then follows from (3.1) that

$$-\frac{3}{4}\int_{q(t,x)}^{+\infty}e^{-|q(t,x)-\eta|}u^2\,d\eta\leqslant\frac{du(t,q(t,x))}{dt}\leqslant\frac{3}{4}\int_{-\infty}^{q(t,x)}e^{-|q(t,x)-\eta|}u^2\,d\eta.$$

It thus transpires that

$$\left|\frac{du(t,q(t,x))}{dt}\right|\leqslant\frac{3}{4}\int_{-\infty}^{+\infty}e^{-|q(t,x)-\eta|}u^2\,d\eta\leqslant\frac{3}{4}\int_{-\infty}^{+\infty}u^2(t,\eta)\,d\eta.$$

In view of Lemma 3.1, we have

$$-3\|u_0\|_{L^2}^2\leqslant\frac{du(t,q(t,x))}{dt}\leqslant3\|u_0\|_{L^2}^2.$$

Integrating the above inequality with respect to $t<T$ on $[0,t]$ yields

$$-3\|u_0\|_{L^2}^2t+u_0(x)\leqslant u\big(t,q(t,x)\big)\leqslant3\|u_0\|_{L^2}^2t+u_0(x).$$

Thus,

$$\big|u\big(t,q(t,x)\big)\big|\leqslant\big\|u\big(t,q(t,x)\big)\big\|_{L^\infty}\leqslant3\|u_0\|_{L^2}^2t+\|u_0\|_{L^\infty}. \tag{3.2}$$

We use the Sobolev embedding to ensure the uniform boundedness of $u_x(s,\eta)$ for $(s,\eta)\in[0,t]\times\mathbb{R}$ with $t\in[0,T)$. In view of Lemma 2.3, we get for every $t\in[0,T)$ a constant $C(t)>0$ such that

$$e^{-C(t)}\leqslant q_x(t,x)\leqslant e^{C(t)},\quad x\in\mathbb{R}.$$

We deduce from the above equation that the function $q(t,\cdot)$ is strictly increasing on $\mathbb{R}$ with $\lim_{x\to\pm\infty}q(t,x)=\pm\infty$ as long as $t\in[0,T)$. Thus, by (3.2) we can obtain

$$\big\|u(t,x)\big\|_{L^\infty}=\big\|u\big(t,q(t,x)\big)\big\|_{L^\infty}\leqslant 3\|u_0\|_{L^2}^2t+\|u_0\|_{L^\infty}. \tag{3.3}$$

This completes the proof of the lemma. □

We are now concerned with the rate of the blow-up of the slope of blow-up solutions to Eq. (2.2).

First, we recall the following useful lemma about the evolution of the minimum point of functions of two variables.

Lemma 3.3. [11] *Let $T>0$ and $v\in C^1([0,T);H^2(\mathbb{R}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{R}$ with*

$$m(t):=\inf_{x\in\mathbb{R}}\big[v_x(t,x)\big]=v_x\big(t,\xi(t)\big).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm}{dt}=v_{tx}\big(t,\xi(t)\big)\quad a.e.\ on\ (0,T).$$

Theorem 3.1. *Let $T<\infty$ be the blow-up time of the corresponding solution u to Eq.* (2.2) *with initial data $u_0\in H^s(\mathbb{R})$, $s>\frac{3}{2}$. Then we have*

$$\lim_{t\to T}\Big(\inf_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}(T-t)\Big)=-1$$

while the solution u remains bounded.

Proof. Again we may assume $s=3$ to prove the above theorem. Differentiating (3.1) with respect to x, we find

$$u_{tx}+uu_{xx}+u_x^2=-\partial_x^2\bigg(p*\frac{3}{2}u^2\bigg)=\frac{3}{2}u^2-\bigg(p*\frac{3}{2}u^2\bigg). \tag{3.4}$$

By Lemma 2.2, we know that

$$\liminf_{t\to T}m(t)=-\infty, \tag{3.5}$$

where $m(t) := \inf_{x\in\mathbb{R}}\{u_x(t,x)\}$ for $t \in [0,T)$. Obviously, one can check that the function m is locally Lipschitz. Moreover, by Lemma 3.3 we have

$$m(t) = u_x\big(t,\xi(t)\big), \quad t \in [0,T).$$

Note that $u_{xx}(t,\xi(t)) = 0$ for a.e. $t \in (0,T)$. Then, from (3.4) we deduce that

$$\frac{d}{dt}m(t) = -m^2(t) + \frac{3}{2}u^2(t,\xi) - \frac{3}{2}p * \big(u^2\big)(t,\xi). \tag{3.6}$$

By Young's inequality, in view of Lemma 3.1, we have

$$\big\|p * u^2(t,\cdot)\big\|_{L^\infty} \leqslant \|p\|_{L^\infty}\big\|u^2\big\|_{L^1} \leqslant \frac{1}{2}\big\|u(t,\cdot)\big\|_{L^2}^2 \leqslant 2\|u_0\|_{L^2}^2.$$

Now Lemma 3.2 and the above equation imply that

$$\begin{aligned}
&\left|\frac{3}{2}u^2\big(t,\xi(t)\big) - \frac{3}{2}p * u^2\big(t,\xi(t)\big)\right| \\
&\quad\leqslant \frac{3}{2}\Big(\big\|u(t,\cdot)\big\|_{L^\infty}^2 + \big\|p * u^2(t,\cdot)\big\|_{L^\infty}\Big) \\
&\quad\leqslant \frac{3}{2}\Big(\big(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\big)^2 + 2\|u_0\|_{L^2}^2\Big) \\
&\quad\leqslant \frac{3}{2}\Big(\big(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}\big)^2 + 2\|u_0\|_{L^2}^2\Big), \quad t \in [0,T].
\end{aligned} \tag{3.7}$$

Set

$$K(T) = \frac{3}{2}\Big(\big(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}\big)^2 + 2\|u_0\|_{L^2}^2\Big).$$

Combining (3.6) with (3.7), we deduce that

$$\frac{d}{dt}m(t) \leqslant -m^2(t) + K(T) \quad \text{for a.e. } t \in [0,T). \tag{3.8}$$

Choose now $\varepsilon \in (0,1)$. Using (3.5), we can find $t_0 \in [0,T)$ such that

$$m(t_0) < -\sqrt{K(T) + \frac{K(T)}{\varepsilon}}.$$

Since m is locally Lipschitz, it follows that m is absolutely continuous. By (3.8) and the absolute continuity of m, we deduce that if

$$m(t_0) < -\sqrt{K(T) + \frac{K(T)}{\varepsilon}},$$

then m is decreasing on $[t_0, T)$ and

$$m(t) < -\sqrt{K(T) + \frac{K(T)}{\varepsilon}} < -\sqrt{\frac{K(T)}{\varepsilon}}, \quad t \in [t_0, T).$$

Thus, in view of (3.5), we know that $\lim_{t\to T} m(t) = -\infty$.

Again, by (3.6) and (3.7), we obtain that

$$-m^2(t) - K(T) \leqslant \frac{d}{dt} m(t) \leqslant -m^2(t) + K(T) \quad \text{for a.e. } t \in (t_0, T).$$

Note that m is locally Lipschitz and less than $m(t_0) < 0$ on (t_0, T). From the above inequality, we obtain

$$1 - \varepsilon \leqslant \frac{d}{dt}\left(\frac{1}{m(t)}\right) \leqslant 1 + \varepsilon.$$

Integrating the above relation on (t, T) with $t \in [t_0, T)$ and noticing that $\lim_{t\to T} m(t) = -\infty$, we get

$$(1 - \varepsilon)(T - t) \leqslant -\frac{1}{m(t)} \leqslant (1 + \varepsilon)(T - t).$$

Since $\varepsilon \in (0, 1)$ is arbitrary, in view of the definition of $m(t)$, the above inequality implies the desired result of the theorem. □

Remark 3.1. Note that the blow-up rate of breaking waves to the Camassa–Holm equation is -2, see [7]. Theorem 3.1 shows that the blow-up rate of breaking waves to the Camassa–Holm is twice as much as that of the Degasperis–Procesi equation.

In the case of breaking waves corresponding to initial profiles satisfying the assumptions of Lemma 2.4, we have

Theorem 3.2. *Let $T < \infty$ be the blow-up time of the solution corresponding to some initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, such that the associated potential $y_0 = u_0 - u_{0,xx}$ satisfies $y_0(x) \geqslant 0$ on $(-\infty, x_0]$ and $y_0(x) \leqslant 0$ on $[x_0, \infty)$ for some points $x_0 \in \mathbb{R}$ and y_0 does not have a constant sign. Then*

$$\lim_{t\to T}\Big(\inf_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}(T - t)\Big) = -1 \quad \textit{and} \quad \lim_{t\to T}\Big(\sup_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}(T - t)\Big) = 0,$$

while the solution remains uniformly bounded.

Proof. As before we may assume $s = 3$. Let $T < \infty$ be the maximal time of existence of the solution u to Eq. (2.2) with initial data $u_0 \in H^3(\mathbb{R})$ which is guaranteed by Lemma 2.4. From Lemma 3.2, we see that the solution u is uniformly bounded, i.e.

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \big|u(t,x)\big| \leqslant 3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}. \tag{3.9}$$

By Theorem 3.1, we know that

$$\lim_{t\to T}\Big(\inf_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}(T-t)\Big)=-1.$$

Note that

$$u(t,x)=\frac{e^{-x}}{2}\int_{-\infty}^{x}e^{\eta}y(t,\eta)\,d\eta+\frac{e^{x}}{2}\int_{x}^{\infty}e^{-\eta}y(t,\eta)\,d\eta \tag{3.10}$$

and

$$u_x(t,x)=-\frac{e^{-x}}{2}\int_{-\infty}^{x}e^{\eta}y(\eta)\,d\eta+\frac{e^{x}}{2}\int_{x}^{\infty}e^{-\eta}y(\eta)\,d\eta. \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$u(t,x)+u_x(t,x)=e^{x}\int_{x}^{\infty}e^{-\eta}y(t,\eta)\,d\eta,$$

$$u(t,x)-u_x(t,x)=e^{-x}\int_{-\infty}^{x}e^{\eta}y(t,\eta)\,d\eta. \tag{3.12}$$

Note that the function $q(t,x)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t,x)>0$ with respect to time t. We infer from the assumptions of the theorem and Lemma 2.3 that for $t\in[0,T)$

$$\begin{cases} y(t,x)\geqslant 0 & \text{if } x\leqslant q(t,x_0),\\ y(t,x)\geqslant 0 & \text{if } x\geqslant q(t,x_0),\end{cases} \tag{3.13}$$

and $y(t,q(t,x_0))=0$, $t\in[0,T)$.

By (3.12) and (3.13), we obtain for $t\in[0,T)$,

$$\begin{cases} u_x(t,x)\leqslant u(t,x) & \text{if } x\leqslant q(t,x_0),\\ u_x(t,x)\leqslant -u(t,x) & \text{if } x\geqslant q(t,x_0).\end{cases} \tag{3.14}$$

Therefore, $u_x(t,\cdot)\leqslant|u(t,\cdot)|$ on $\mathbb{R}$ for all $t\in[0,T)$. By (3.9), we have

$$\sup_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}\leqslant 3\|u_0\|_{L^2}^2T+\|u_0\|_{L^\infty},\quad \forall t\in[0,T].$$

The above inequality implies

$$\lim_{t\to T}\Big(\sup_{x\in\mathbb{R}}\big\{u_x(t,x)\big\}(T-t)\Big)=0.$$

This completes the proof of the theorem. □

Let us now present some information about the blow-up set of a breaking wave for (2.2) with a large class of initial data.

Theorem 3.3. *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and $u_0 \not\equiv 0$ is odd such that the associated potential $y_0 := u_0 - u_{0,xx}$ is nonnegative on $\mathbb{R}_-$. Then the solution to Eq.* (2.2) *with initial data u_0 blows up in finite time only at zero point.*

Proof. As we mentioned before, here we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.2) with initial data $u_0 \in H^3(\mathbb{R})$.

Since $u_0 \not\equiv 0$ is odd and $y_0 := u_0 - u_{0,xx}$ is nonnegative on $\mathbb{R}_-$, it follows that y_0 is odd and nonpositive on $\mathbb{R}_+$. Note that Eq. (1.2) is invariant under the transformation $(u, x) \to (-u, -x)$. If u_0 is odd, then $t \in [0, T)$, $u(t, \cdot)$ is also odd. Therefore, by the continuity of u and u_{xx} with respect to x, we obtain

$$u(t, 0) = u_{xx}(t, 0) = 0, \quad t \in [0, T). \tag{3.15}$$

Set $h(t) = u_x(t, 0)$, $t \in [0, T)$. By (3.4) and (3.15), we obtain

$$\frac{dh}{dt}(t) = -h^2(t) - p * \left(\frac{3}{2}u^2\right)(t, 0), \quad t \in [0, T). \tag{3.16}$$

Due to $-p * (\frac{3}{2}u^2)(t, 0) \leqslant 0$ and $-h^2(t) \leqslant 0$, it follows from (3.16) that

$$\frac{dh}{dt}(t) \leqslant -h^2(t), \quad t \in [0, T), \tag{3.17}$$

and

$$\frac{dh}{dt}(t) \leqslant -\int_{\mathbb{R}} p(x)\left(\frac{3}{2}u^2(t, x)\right) dx, \quad t \in [0, T). \tag{3.18}$$

If there exists some $t' \in (0, T)$ such that

$$\int_{\mathbb{R}} p(x)\left(\frac{3}{2}u^2(t', x)\right) dx = 0,$$

then we have $u(t', x) \equiv 0$. Using the uniqueness of strong solution guaranteed by Lemma 2.1, we obtain $u_0(x) \equiv 0$. This contradicts the assumption $u_0(x) \not\equiv 0$. Thus, in view of the positivity of p and u^2, by (3.18) we have $dh/dt(t) < 0$, i.e., $h(t)$ is strictly decreasing on $[0, T)$. Since $h(0) \leqslant 0$, which is guaranteed by the assumptions of the theorem, it follows that there exists some $t_0 \in (0, T)$ such that $h(t_0) < 0$. Solving inequality (3.17), we can obtain

$$0 > \frac{1}{h(t)} \geqslant \frac{1}{h(t_0)} + t - t_0, \quad t \in [t_0, T).$$

Consequently,

$$T < t_0 - \frac{1}{h(t_0)} \quad \text{and} \quad \lim_{t\to T} h(t) = \lim_{t\to T} u_x(t,0) = -\infty.$$

Next, we give a precise description of the blow-up mechanism. As noted in (3.9), there is a uniform bounded on $u(t,x)$ for $(t,x) \in [0,T) \times \mathbb{R}$. We will see below that for any $x \neq 0$, the slope $u_x(t,x)$ remains bounded on $[0,T)$ while $u_x(t,0) \to -\infty$ as $t \to T$, that is, the wave breaks in finite time exact at zero and nowhere else.

Since y_0 is odd, in view of Lemma 2.3, we see that $y(t,x) = u(t,x) - u_{xx}(t,x)$ is also odd. Then we have for $(t,x) \in [0,T) \times \mathbb{R}_+$:

$$\begin{aligned} u(t,x) = p * y(t,x) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|} y(t,\eta)\,d\eta \\ &= \sinh(x)\int_x^{\infty} e^{-\eta} y(t,\eta)\,d\eta + e^{-x}\int_0^x \sinh(\eta) y(t,\eta)\,d\eta \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} u_x(t,x) &= \partial_x\left[\frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|} y(t,\eta)\,d\eta\right] \\ &= \cosh(x)\int_x^{\infty} e^{-\eta} y(t,\eta)\,d\eta - e^{-x}\int_0^x \sinh(\eta) y(t,\eta)\,d\eta. \end{aligned} \tag{3.20}$$

Note that $y(t,\eta) \leqslant 0$ for $\eta \geqslant 0$. By (3.19) and Lemma 3.2, we obtain for all $(t,x) \in [0,T) \times \mathbb{R}_+$:

$$\left|\sinh(x)\int_x^{\infty} e^{-\eta} y(t,\eta)\,d\eta\right| \leqslant \left|u(t,x)\right| \leqslant 3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}$$

and

$$\left|e^{-x}\int_0^x \sinh(\eta) y(t,\eta)\,d\eta\right| \leqslant \left|u(t,x)\right| \leqslant 3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}.$$

Using the above two estimates in (3.20), we obtain

$$\left|u_x(t,x)\right| \leqslant \left(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}\right)\left(1 + \frac{\cosh(x)}{\sinh(x)}\right), \quad t \in [0,T),\ x > 0.$$

The above inequality shows that $|u_x(t,x)| = |u_x(t,-x)|$ is uniformly bounded on $t \in [0,T)$, $x \geqslant \delta$ for $\delta > 0$ arbitrarily small. This completes the proof of the theorem. □

4. Global weak solutions

In this section, we will show that there exists a unique global weak solution to Eq. (2.1), provided the initial data u_0 satisfies a certain sign condition.

We first prove the following useful lemma.

Lemma 4.1. *Assume that $u_0 \in H^s(\mathbb{R})$, $s \geqslant 3$, and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \leqslant 0 & \text{if } x \leqslant x_0, \\ y_0(x) \geqslant 0 & \text{if } x \geqslant x_0. \end{cases}$$

Let u be the corresponding solution to (2.1) and set $y(t,x) = u(t,x) - u_{xx}(t,x)$. Then

(i) $\|u_x(t,\cdot)\|_{L^\infty} \leqslant \frac{1}{2}\|y(t,\cdot)\|_{L^1}$,
(ii) $\|u(t,\cdot)\|_{L^1} \leqslant \|y(t,\cdot)\|_{L^1}$,
(iii) $\|u_x(t,\cdot)\|_{L^1} \leqslant \|y(t,\cdot)\|_{L^1}$.

Moreover, if $y_0 \in L^1(\mathbb{R})$, then we have that $y \in C^1(\mathbb{R}_+; L^1(\mathbb{R}))$ and

$$\big\|y(t,\cdot)\big\|_{L^1} \leqslant e^{3t^2\|u_0(x)\|_{L^2}^2 + 2t\|u_0(x)\|_{L^\infty}} \|y_0\|_{L^1}.$$

Proof. Since $y(t,x) = u(t,x) - u_{xx}(t,x)$, it follows that $u = p * y$ and $u_x = p_x * y$. Note that $\|p_x\|_{L^\infty} = \frac{1}{2}$, $\|p\|_{L^1} = 1$, and $\|p_x\|_{L^1} = 1$. Applying Young's inequality, one can easily obtain inequalities (i)–(iii).

By (2.1), we have that $y_t = -y_x u - 3yu_x$. Since $y \in C(\mathbb{R}_+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}))$, it follows that $y_t \in C(\mathbb{R}_+; L^1(\mathbb{R}))$. Note that $y_0 \in L^1(\mathbb{R})$ and $y \in C^1(\mathbb{R}_+; L^2(\mathbb{R}))$. Then one can easily deduce that $y \in C^1(\mathbb{R}_+; L^1(\mathbb{R}))$.

Since $y(x_0) = 0$, it follows from Lemma 2.3 that $y(t, q(t,x_0)) = 0$. Using this relation and $y_t = -(yu)_x - 2yu_x$, in view of Lemmas 3.2 and 2.5, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} y^+\,dx &= \frac{d}{dt}\int_{q(t,x_0)}^{\infty} y\,dx = -2\int_{q(t,x_0)}^{\infty} yu_x\,dx \\ &\leqslant 2\sup_{x\in\mathbb{R}}\{-u_x\}\int_{\mathbb{R}} y^+\,dx \leqslant 2|u|\int_{\mathbb{R}} y^+\,dx \\ &\leqslant \big(6\|u_0(x)\|_{L^2}^2 t + 2\|u_0(x)\|_{L^\infty}\big)\int_{\mathbb{R}} y^+\,dx. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\int_{\mathbb{R}} y^+(t,x)\,dx \leqslant e^{3t^2\|u_0(x)\|_{L^2}^2 + 2t\|u_0(x)\|_{L^\infty}} \int_{\mathbb{R}} y_0^+\,dx.$$

Repeating the above proof, one can obtain a same estimate for y^-. This completes the proof of the lemma. □

Let us now present the existence and uniqueness result for global weak solutions to (2.2).

Theorem 4.1. *Assume that* $u_0 \in H^1(\mathbb{R})$ *and that* $y_0 := (u_0 - u_{0,xx}) \in M(\mathbb{R})$. *Further assume that there is a* $x_0 \in \mathbb{R}$ *such that* $\operatorname{supp} y_0^- \subset (-\infty, x_0)$ *and* $\operatorname{supp} y_0^+ \subset (x_0, \infty)$. *Then Eq.* (2.2) *has a unique weak solution*

$$u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; H^1(\mathbb{R})\big)$$

with initial data $u(0) = u_0$ *and*

$$y(t,\cdot) = \big(u(t,\cdot) - u_{xx}(t,\cdot)\big) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; M(\mathbb{R})\big).$$

Moreover, $E_1(u)$ *and* $E_2(u)$ *are two conservation laws.*

Proof. Assume that $u_0 \in H^1(\mathbb{R})$ and that $y_0 := u_0 - u_{0,xx} \in M(\mathbb{R})$. Then the relation $u_0 = p * y_0$ holds true. Thus we have

$$\begin{aligned}
\|u_0\|_{L^1(\mathbb{R})} = \|p * y_0\|_{L^1(\mathbb{R})} &= \sup_{\|f\|_{L^\infty(\mathbb{R})}\leqslant 1} \int_{\mathbb{R}} f(x)(p * y_0)(x)\,dx \\
&= \sup_{\|f\|_{L^\infty(\mathbb{R})}\leqslant 1} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} p(x-\xi)\,dy_0(\xi)\,dx \\
&= \sup_{\|f\|_{L^\infty(\mathbb{R})}\leqslant 1} \int_{\mathbb{R}} (p * f)(\xi)\,dy_0(\xi) \\
&= \sup_{\|f\|_{L^\infty(\mathbb{R})}\leqslant 1} \|p * f\|_{L^\infty(\mathbb{R})} \|y_0\|_{M(\mathbb{R})} \\
&\leqslant \sup_{\|f\|_{L^\infty(\mathbb{R})}\leqslant 1} \|p\|_{L^1(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R})} \|y_0\|_{M(\mathbb{R})} = \|y_0\|_{M(\mathbb{R})}.
\end{aligned} \tag{4.1}$$

We first prove that there exists a u with initial data u_0, which belongs to $H^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R}))$, satisfying Eq. (2.2) in the sense of distributions.

Set

$$y_0^n = l\left(-\frac{1}{n}\right)\big(\rho_n * y_0^+\big) - l\left(\frac{1}{n}\right)\big(\rho_n * y_0^-\big),$$

where $l(r)$ denotes right translation by $r \in \mathbb{R}$, $l(r)f(x) = f(x+r)$. By the definition of ρ_n and the assumptions of the theorem, we have

$$\operatorname{supp}\big(\rho_n * y_0^-\big) \subset \left(-\infty, x_0 + \frac{1}{n}\right] \quad \text{and} \quad \operatorname{supp}\big(\rho_n * y_0^+\big) \subset \left[x_0 - \frac{1}{n}, \infty\right).$$

Thus, it follows that

$$\begin{cases} y_0^n \leqslant 0 & \text{if } x \leqslant x_0, \\ y_0^n \geqslant 0 & \text{if } x \geqslant x_0. \end{cases} \tag{4.2}$$

Let us define $u_0^n := (1-\partial_x^2)^{-1} y_0^n = p * y_0^n \in H^\infty(\mathbb{R})$ for $n \geqslant 1$. By Theorem 2.5 and (4.2), we obtain a global strong solution

$$u^n = u^n\big(\cdot, u_0^n\big) \in C\big([0,\infty); H^s(\mathbb{R})\big) \cap C^1\big([0,\infty); H^{s-1}(\mathbb{R})\big),$$

for every $s > \frac{3}{2}$ and for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Note that

$$p * \big(y_0^{n\pm}\big) \in H^1(\mathbb{R}) \quad \text{and} \quad \left\| l\left(\mp\frac{1}{n}\right)\rho_n \right\|_{L^1} = 1.$$

Since $\operatorname{supp}(l(\mp\frac{1}{n})\rho_n) \to 0$, as $n \to \infty$, it follows that

$$\begin{aligned} u_0^n &= p * \left[l\left(-\frac{1}{n}\right)\big(\rho_n * y_0^+\big) - l\left(\frac{1}{n}\right)\big(\rho_n * y_0^-\big) \right] \\ &= l\left(-\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^+\big)\big) - l\left(\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^-\big)\big) \\ &\to p * y_0^+ - p * y_0^- = u_0 \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to \infty. \end{aligned} \tag{4.3}$$

Using Young's inequality, in view of (4.1), we have for all $n \geqslant 1$

$$\begin{aligned} \big\|u_0^n\big\|_{L^1} &= \big\|p * y_0^n\big\|_{L^1} \leqslant \big\|y_0^n\big\|_{L^1} \\ &= \left\| l\left(-\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^+\big)\big) - l\left(\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^-\big)\big) \right\|_{L^1} \\ &= \left\| l\left(-\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^+\big)\big) \right\|_{L^1} + \left\| l\left(\frac{1}{n}\right)\big(\rho_n * \big(p * y_0^-\big)\big) \right\|_{L^1} \\ &\leqslant \left\| l\left(-\frac{1}{n}\right)\rho_n \right\|_{L^1} \big\|p * y_0^+\big\|_{L^1} + \left\| l\left(\frac{1}{n}\right)\rho_n \right\|_{L^1} \big\|p * y_0^-\big\|_{L^1} \\ &\leqslant \|p\|_{L^1} \big\|y_0^+\big\|_M + \|p\|_{L^1} \big\|y_0^-\big\|_M \\ &\leqslant \big\|y_0^+\big\|_M + \big\|y_0^-\big\|_M = \|y_0\|_M. \end{aligned} \tag{4.4}$$

Similarly, one can also obtain the estimates

$$\begin{aligned} \big\|u_0^n\big\|_{L^2} &\leqslant \|p\|_{L^2} \big\|y_0^+\big\|_M + \|p\|_{L^2} \big\|y_0^-\big\|_M = \|p\|_{L^2} \|y_0\|_M, \\ \big\|u_0^n\big\|_{L^\infty} &\leqslant \|p\|_{L^\infty} \big\|y_0^+\big\|_M + \|p\|_{L^\infty} \big\|y_0^-\big\|_M = \frac{1}{2}\|y_0\|_M, \\ \big\|u_0^n\big\|_1 &\leqslant \|p\|_1 \big\|y_0^+\big\|_M + \|p\|_1 \big\|y_0^-\big\|_M = \|p\|_1 \|y_0\|_M. \end{aligned} \tag{4.5}$$

Note that for $n \geqslant 1$ and $t \geqslant 0$

$$\big[u^n(t,x)\big]^2 = \int_{-\infty}^{x} 2u^n(t,\xi)u_x^n(t,\xi)\,d\xi \leqslant \int_{\mathbb{R}} \big(\big[u^n(t,\xi)\big]^2 + \big[u_x^n(t,\xi)\big]^2\big)\,d\xi = \big\|u^n(t,\cdot)\big\|_1^2.$$

In view of Lemma 2.5(i)–(ii) and (4.5), we obtain that

$$\begin{aligned}\left\|u^n(t,\cdot)\right\|_{L^\infty}^2 &\leqslant \left\|u^n(t,\cdot)\right\|_1^2 \leqslant 6\left\|u_0^n\right\|_{L^2}^4 t^2 + 4\left\|u_0^n\right\|_{L^2}^2\left\|u_0^n\right\|_{L^\infty} t + \left\|u_0^n\right\|_1^2 \\ &\leqslant 6\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 2\|p\|_{L^2}^2\|y_0\|_M^3 t + \|p\|_1^2\|y_0\|_M^2. \end{aligned} \tag{4.6}$$

The above inequality implies that

$$\begin{aligned}\left\|u^n(t)u_x^n(t)\right\|_{L^2} &\leqslant \left\|u^n(t)\right\|_{L^\infty}\left\|u_x^n(t)\right\|_{L^2} \leqslant \left\|u^n(t)\right\|_1^2 \\ &\leqslant 6\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 2\|p\|_{L^2}^2\|y_0\|_M^3 t + \|p\|_1^2\|y_0\|_M^2 \end{aligned} \tag{4.7}$$

for all $t \geqslant 0$ and $n \geqslant 1$. By Young's inequality and (4.6), we get

$$\begin{aligned}&\left\|\partial_x p * \left(\frac{3}{2}\left[u^n(t)\right]^2\right)\right\|_{L^2} \\ &\quad\leqslant \frac{3}{2}\|p_x\|_{L^2}\left\|\left[u^n(t)\right]^2\right\|_{L^1} \leqslant \frac{3}{2}\|p\|_1\left\|u^n(t)\right\|_1^2 \\ &\quad\leqslant 9\|p\|_1\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 3\|p\|_1\|p\|_{L^2}^2\|y_0\|_M^3 t + \frac{3}{2}\|p\|_1^3\|y_0\|_M^2 \end{aligned} \tag{4.8}$$

for all $t \geqslant 0$ and $n \geqslant 1$.

Using (4.7), (4.8) and Eq. (2.2), we find that

$$\begin{aligned}&\left\|\frac{d}{dt}u^n(t,\cdot)\right\|_{L^2} = \left\|u_t^n(t,\cdot)\right\|_{L^2} \\ &\quad\leqslant \left(1+\frac{3}{2}\|p\|_1\right)\left(6\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 2\|p\|_{L^2}^2\|y_0\|_M^3 t + \|p\|_1^2\|y_0\|_M^2\right) \end{aligned} \tag{4.9}$$

for all $t \geqslant 0$ and $n \geqslant 1$. For fixed $T > 0$, by (4.6) and (4.9), we have

$$\int_0^T\int_{\mathbb{R}}\left(\left[u^n(t,x)\right]^2 + \left[u_x^n(t,x)\right]^2 + \left[u_t^n(t,x)\right]^2\right)dx\,dt \leqslant K, \tag{4.10}$$

where K is a positive constant depending only on $\|p\|_1$, $\|p\|_{L^2}$, $\|y_0\|_M$ and T. Therefore the sequence $\{u^n\}_{n\geqslant 1}$ is uniformly bounded in the space $H^1((0,T)\times\mathbb{R})$. Thus we can extract a subsequence such that

$$u^{n_k} \rightharpoonup u \quad \text{weakly in } H^1\big((0,T)\times\mathbb{R}\big) \quad \text{for } n_k \to \infty, \tag{4.11}$$

and

$$u^{n_k} \to u \quad \text{a.e. on } (0,T)\times\mathbb{R} \quad \text{for } n_k \to \infty, \tag{4.12}$$

for some $u \in H^1((0,T)\times\mathbb{R})$. Given $t \in (0,T)$, it follows from Lemma 4.1 and (4.4) that $u_x^{n_k}(t,\cdot) \in BV(\mathbb{R})$ with

$$\begin{aligned}\mathbb{V}\big[u_x^{n_k}(t,\cdot)\big] &= \big\|u_{xx}^{n_k}(t,\cdot)\big\|_{L^1} \leqslant \big\|u^{n_k}(t,\cdot)\big\|_{L^1} + \big\|y^{n_k}(t,\cdot)\big\|_{L^1} \\ &\leqslant 2\big\|y^{n_k}\big\|_{L^1} \leqslant 2e^{3t^2\|u_0^n\|_{L^2}^2 + 2t\|u_0^n\|_{L^\infty}} \big\|y_0^n\big\|_{L^1} \\ &\leqslant 2e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M.\end{aligned}$$

and

$$\big\|u_x^{n_k}(t,\cdot)\big\|_{L^\infty} \leqslant \frac{1}{2}\big\|y^{n_k}(t,\cdot)\big\|_{L^1} \leqslant \frac{1}{2}e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M.$$

Applying Helly's theorem [43], we find a subsequence, again denoted by $\{u_x^{n_k}(t,\cdot)\}$, which converges at every point to some function $v(t,\cdot)$ of finite variation with

$$\mathbb{V}\big(v(t,\cdot)\big) \leqslant 2e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M.$$

The limit (4.12) implies that $u_x^{n_k}(t,\cdot) \to u_x(t,\cdot)$ in $D'(\mathbb{R})$ for almost all $t \in (0,T)$. Thus it follows that $v(t,\cdot) = u_x(t,\cdot)$ for a.e. $t \in (0,T)$. Therefore, we have

$$u_x^{n_k}(t,\cdot) \to u_x(t,\cdot) \quad \text{a.e. on } (0,T)\times\mathbb{R} \quad \text{for } n_k \to \infty \tag{4.13}$$

and

$$\mathbb{V}\big[u_x(t,\cdot)\big] = \big\|u_{xx}(t,\cdot)\big\|_{M(\mathbb{R})} \leqslant 2e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M \tag{4.14}$$

for a.e. $t \in (0,T)$. By (4.6), we have

$$\begin{aligned}\left\|\frac{3}{2}\big[u^n(t)\big]^2\right\|_{L^2} &\leqslant \frac{3}{2}\big\|u^n(t)\big\|_{L^\infty}\big\|u^n(t)\big\|_{L^2} \leqslant \frac{3}{2}\big\|u^n\big\|_1^2 \\ &\leqslant 9\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 3\|p\|_{L^2}^2\|y_0\|_M^3 t + \frac{3}{2}\|p\|_1^2\|y_0\|_M^2.\end{aligned}$$

Consequently, given $t \in (0,T)$, the sequence $\{\frac{3}{2}[u^n(t)]^2\}_{n\geqslant 1}$ is uniformly bounded in $L^2(\mathbb{R})$. Therefore, there is a subsequence $\{\frac{3}{2}[u^{n_k}(t)]^2\}_{n_k\geqslant 1}$ which converges weakly in $L^2(\mathbb{R})$. By (4.12) we deduce that the weak $L^2(\mathbb{R})$-limit is $\frac{3}{2}[u(t,\cdot)]^2$. Note that $p_x \in L^2(\mathbb{R})$. Thus we conclude that

$$\partial_x p * \left(\frac{3}{2}\big[u^{n_k}(t)\big]^2\right) \to \partial_x p * \left(\frac{3}{2}u^2\right) \quad \text{for } n_k \to \infty. \tag{4.15}$$

By (4.12), (4.13) and (4.15), we see that u satisfies Eq. (2.1) in $D'((0,T)\times T)$.

Fix $T > 0$. Then the sequences $u_t^{n_k}(t,\cdot)$ and $u^{n_k}(t,\cdot)$ are uniformly bounded in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$, respectively, for all $t \in [0,T)$ and $k \in N$. Thus, the family $t \mapsto u^{n_k}(t,\cdot) \in H^1(\mathbb{R})$ is weakly equicontinuous on $[0,T]$. An application of the Arzela–Ascoli theorem yields that $\{u^{n_k}\}$ has a subsequence, denoted again $\{u^{n_k}\}$, which converges weakly in $H^1(\mathbb{R})$, uniformly in $t \in [0,T)$. The limit function is u. The above arguments are true for any $T > 0$. This implies that u is locally weakly continuous from $[0,\infty)$ into $H^1(\mathbb{R})$, i.e.

$$u \in C_{\mathrm{loc}}^w\big(\mathbb{R}_+; H^1(\mathbb{R})\big).$$

Since $u^{n_k}(t,\cdot) \rightharpoonup u(t,\cdot)$ weakly in $H^1(\mathbb{R})$, it follows from (4.6) that

$$\begin{aligned}\|u(t,\cdot)\|_{L^\infty} &\leqslant \|u(t,\cdot)\|_1 \leqslant \liminf_{n_k\to\infty} \|u^{n_k}(t,\cdot)\|_1 \\ &\leqslant 6\|p\|_{L^2}^4\|y_0\|_M^4 t^2 + 2\|p\|_{L^2}^2\|y_0\|_M^3 t + \|p\|_1^2\|y_0\|_M^2\end{aligned}$$

for a.e. $t \in \mathbb{R}_+$. The above inequality implies that

$$u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}) \cap L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; H^1(\mathbb{R})\big). \tag{4.16}$$

By Lemma 4.1 and (4.14), we have for $t \in \mathbb{R}_+$ that

$$\begin{aligned}\|u_x(t,\cdot)\|_{L^\infty} &= \lim_{n\to\infty}\|u_x^n(t,\cdot)\|_{L^\infty} \leqslant \liminf_{n\to\infty}\frac{1}{2}\|y^n(t,\cdot)\|_{L^1} \\ &\leqslant \frac{1}{2}e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M.\end{aligned}$$

The above inequality shows that $u_x \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$. Thus, in view of (4.16), we obtain

$$u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}) \cap L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; H^1(\mathbb{R})\big). \tag{4.17}$$

Next, we prove that $(u(t,\cdot) - u_{xx}(t,\cdot)) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; M(\mathbb{R}))$ and that $E_1(u)$ is a conservation law.

Since u solves (2.2) in the distributional sense, we have that

$$\rho_n * u_t + \rho_n * (uu_x) + \rho_n * \partial_x p * \left(\frac{3}{2}u^2\right) = 0 \tag{4.18}$$

for a.e. $t \in \mathbb{R}_+$. Integrating the above equation with respect to x on $\mathbb{R}$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_n * u\,dx + \int_{\mathbb{R}} \rho_n * (uu_x)\,dx + \int_{\mathbb{R}} \rho_n * \partial_x p * \left(\frac{3}{2}u^2\right)dx = 0.$$

Integration by parts yields further

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_n * u\,dx = 0, \quad t \in \mathbb{R}_+,\ n \geqslant 1.$$

Applying Lemma 2.6, we get

$$\int_{\mathbb{R}} \rho_n * u(t,\cdot)\,dx = \int_{\mathbb{R}} \rho_n * u_0\,dx.$$

Note that

$$\lim_{n\to\infty}\|\rho_n * u(t,\cdot) - u(t,\cdot)\|_{L^1(\mathbb{R})} = \lim_{n\to\infty}\|\rho_n * u_0 - u_0\|_{L^1(\mathbb{R})} = 0.$$

It follows that for a.e. $t \in \mathbb{R}_+$

$$\int_{\mathbb{R}} u(t,\cdot)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} \rho_n * u(t,\cdot)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} \rho_n * u_0\,dx = \int_{\mathbb{R}} u_0\,dx.$$

This proves that $E_1(u(t)) = \int_{\mathbb{R}} u\,dx$ is a conservation law.

Note that $L^1(\mathbb{R}) \subset (L^\infty(\mathbb{R}))^* \subset (C_0(\mathbb{R}))^* = M(\mathbb{R})$. By (4.1), (4.14) and the conservation law $E_1(u)$, we get for a.e. $t \in \mathbb{R}_+$ that

$$\begin{aligned} \big\|u(t,\cdot) - u_{xx}(t,\cdot)\big\|_{M(\mathbb{R})} &\leqslant \big\|u(t,\cdot)\big\|_{L^1(\mathbb{R})} + \big\|u_{xx}(t,\cdot)\big\|_{M(\mathbb{R})} \\ &\leqslant \|u_0\|_{L^1(\mathbb{R})} + \big\|u_{xx}(t,\cdot)\big\|_{M(\mathbb{R})} \\ &\leqslant \|y_0\|_{M(\mathbb{R})} + 2e^{3t^2\|p\|_{L^2}^2\|y_0\|_M^2 + t\|y_0\|_M}\|y_0\|_M. \end{aligned}$$

The above inequality shows that

$$\big(u(t,\cdot) - u_{xx}(t,\cdot)\big) \in L^\infty_{\mathrm{loc}}\big(\mathbb{R}_+; M(\mathbb{R})\big).$$

Next, we prove that $E_2(u)$ is a conservation law.

By (4.18) and the relation $u = p * y$, we have that

$$\frac{d}{dt}\rho_n * y + \big(1 - \partial_x^2\big)\rho_n * (uu_x) + \rho_n * (3uu_x) = 0$$

for a.e. $t \in \mathbb{R}$ and $n \geqslant 1$. Multiplying the above equation by $v = (4 - \partial_{xx})^{-1}u$ and integrating by parts with respect to x on $\mathbb{R}$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}} v\frac{d}{dt}\rho_n * y\,dx &= -\int_{\mathbb{R}} v\big(1 - \partial_x^2\big)\rho_n * (uu_x)\,dx - \int_{\mathbb{R}} v\rho_n * (3uu_x)\,dx \\ &= \int_{\mathbb{R}} v\partial_x^2\rho_n * (uu_x)\,dx - 4\int_{\mathbb{R}} v\rho_n * (uu_x)\,dx \\ &= \int_{\mathbb{R}} v_{xx}\rho_n * (uu_x)\,dx - 4\int_{\mathbb{R}} v\rho_n * (uu_x)\,dx \end{aligned} \tag{4.19}$$

for a.e. $t \in \mathbb{R}$ and $n \geqslant 1$. By (4.19) and the relation $v_{xx} = 4v - u$, we have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} v\rho_n * y\,dx &= \int_{\mathbb{R}} v\,\frac{d}{dt}\rho_n * y\,dx = \int_{\mathbb{R}} (4v - u)\rho_n * (uu_x)\,dx - 4\int_{\mathbb{R}} v\rho_n * (uu_x)\,dx \\ &= -\int_{\mathbb{R}} u\rho_n * (uu_x)\,dx = \frac{1}{2}\int_{\mathbb{R}} u_x\rho_n * \big(u^2\big)\,dx. \\ &\to \frac{1}{2}\int_{\mathbb{R}} u_x u^2\,dx = 0, \quad \text{as } n \to \infty. \end{aligned} \tag{4.20}$$

Given $t \in \mathbb{R}_+$ and $n \geqslant 1$. Define

$$E^n(t) := \int_{\mathbb{R}} v\rho_n * y\,dx \quad \text{and} \quad H_n(t) := \int_{\mathbb{R}} u_x\rho_n * u^2\,dx.$$

By (4.20), we get

$$\frac{d}{dt}E^n(t) = H_n(t) \quad \text{and} \quad \lim_{n\to\infty} H_n(t) = 0 \tag{4.21}$$

for a.e. $t \in \mathbb{R}_+$ and $n \geqslant 1$. Thus, in view of Lemma 2.6, we have

$$E^n(t) - E^n(0) = \int_0^t H_n(s)\,ds, \quad t \in \mathbb{R}_+,\ n \geqslant 1. \tag{4.22}$$

Note that $u_x \in L^\infty([0,T) \times \mathbb{R})$ and $u \in L^\infty([0,T); H^1(\mathbb{R}))$, for all $T > 0$. Using Young's inequality and Hölder's inequality, we can obtain that there exists a $K(T) > 0$ such that

$$\big|H_n(t)\big| \leqslant K(T), \quad t \in [0,T),\ n \geqslant 1.$$

In view of (4.21), (4.22), an application of Lebesgue's dominated convergence theorem yields that for fixed $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty}\big(E^n(t) - E^n(0)\big) = 0.$$

Let $t \in \mathbb{R}_+$ be given. By (4.12), we find that

$$E_2\big(u(t)\big) = \lim_{n\to\infty} E^n(t) = \lim_{n\to\infty} E^n(0) = E_2(u_0).$$

This proves that $E_2(u(t)) = \int_{\mathbb{R}} yv\,dx$ is a conservation law.

Finally, we prove the uniqueness of the global weak solution.

Assume that $u, v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R}))$ be two global weak solutions of (2.2) with initial data u_0 such that $(u(t,\cdot) - u_{xx}(t,\cdot))$ and $(v(t,\cdot) - v_{xx}(t,\cdot))$ belong to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; M(\mathbb{R}))$.

Fix $T > 0$ and set

$$N(T) := \sup_{t\in[0,T]}\big\{\big\|u(t,\cdot) - u_{xx}(t,\cdot)\big\|_M + \big\|v(t,\cdot) - v_{xx}(t,\cdot)\big\|_M\big\}.$$

By assumption, we have that $N(T) < \infty$. Thus, given $(t,x) \in [0,T] \times \mathbb{R}$, we have that

$$\begin{aligned}\big|u(t,x)\big| &= \big|\big[p * (u - u_{xx})\big](t,x)\big| \\ &\leqslant \|p\|_{L^\infty}\big\|u(t,\cdot) - u_{xx}(t,\cdot)\big\|_M \leqslant \frac{N(T)}{2}\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\left|u_x(t,x)\right| &= \left|\left[p_x*(u-u_{xx})\right](t,x)\right| \\ &\leqslant \|p_x\|_{L^\infty}\left\|(u-u_{xx})(t,\cdot)\right\|_M \leqslant \frac{N(T)}{2}.\end{aligned} \tag{4.24}$$

Similarly, we may obtain

$$\left|v(t,x)\right| \leqslant \frac{N(T)}{2}, \qquad \left|v_x(t,x)\right| \leqslant \frac{N(T)}{2}, \qquad (t,x)\in[0,T]\times\mathbb{R}. \tag{4.25}$$

In view of (4.1), we can also get that

$$\begin{aligned}&\left\|u(t,\cdot)\right\|_{L^1(\mathbb{R})} = \left\|\left[p*(u-u_{xx})\right](t,\cdot)\right\|_{L^1(\mathbb{R})} \leqslant N(T),\\ &\left\|u_x(t,\cdot)\right\|_{L^1} = \left\|\left[p_x*(u-u_{xx})\right](t,\cdot)\right\|_{L^1} \leqslant \|p_x\|_{L^1} N(t) = N(T),\\ &\left\|v(t,\cdot)\right\|_{L^1} \leqslant N(T), \quad \text{and} \quad \left\|v_x(t,\cdot)\right\|_{L^1} \leqslant N(T), \qquad t\in[0,T].\end{aligned} \tag{4.26}$$

Let us set

$$w(t,x) = u(t,x) - v(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}.$$

Convoluting Eq. (2.2) for u and v with ρ_n and using Lemma 2.7, we obtain for a.e. $t\in[0,T]$ and all $n\geqslant 1$ that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w|\,dx &= \int_{\mathbb{R}} (\rho_n * w_t)\,\mathrm{sgn}(\rho_n * w)\,dx \\ &= -\int_{\mathbb{R}} \rho_n * (w u_x)\,\mathrm{sgn}(\rho_n * w)\,dx - \int_{\mathbb{R}} \rho_n * (v w_x)\,\mathrm{sgn}(\rho_n * w)\,dx \\ &\quad - \frac{3}{2}\int_{\mathbb{R}} \left(\rho_n * p_x * \left[w(u+v)\right]\right)\mathrm{sgn}(\rho_n * w)\,dx.\end{aligned} \tag{4.27}$$

Using (4.23)–(4.26), Young's inequality and Lemma 2.7 and following the procedure described in [15, pp. 56–57], we can deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w|\,dx = C(T)\int_{\mathbb{R}} |\rho_n * w|\,dx + C(T)\int_{\mathbb{R}} |\rho_n * w_x|\,dx + R_n(t) \tag{4.28}$$

for a.e. $t\in[0,T]$ and all $n\geqslant 1$, where $C(T)$ is a generic constant depending on $N(T)$. Moreover, $R_n(t)$ satisfies

$$\begin{cases}\lim_{n\to\infty} R_n(t) = 0,\\ |R_n(t)| \leqslant C(T), \quad n\geqslant 1,\ t\in[0,T].\end{cases} \tag{4.29}$$

In the following $C(T)$ stands for a positive constant depending on $N(T)$ and the $H^1(\mathbb{R})$-norms of $u(0)$ and $v(0)$.

Similarly, convoluting Eq. (2.2) for u and v with $\rho_{n,x}$ and using Lemma 2.7, we obtain for a.e. $t \in [0,T]$ and all $n \geqslant 1$ that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w_x|\,dx &= \int_{\mathbb{R}} (\rho_n * w_{xt})\,\mathrm{sgn}(\rho_{n,x} * w)\,dx \\
&= -\int_{\mathbb{R}} \rho_n * \big(w_x(u_x + v_x)\big)\,\mathrm{sgn}(\rho_{n,x} * w)\,dx \\
&\quad - \int_{\mathbb{R}} \rho_n * (w v_{xx})\,\mathrm{sgn}(\rho_{n,x} * w)\,dx \\
&\quad - \int_{\mathbb{R}} \rho_n * (u w_{xx})\,\mathrm{sgn}(\rho_{n,x} * w)\,dx \\
&\quad - \frac{3}{2}\int_{\mathbb{R}} \rho_n * p_{xx} * \big(u^2 - v^2\big)\,\mathrm{sgn}(\rho_{n,x} * w)\,dx. \qquad (4.30)
\end{aligned}$$

Using (4.23)–(4.26), Young's inequality, Lemma 2.7 and the identity $\partial_x^2(p * f) = p * f - f$ and following the arguments given in [15, pp. 57–59], we can deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w_x|\,dx = C(T)\int_{\mathbb{R}} |\rho_n * w|\,dx + C(T)\int_{\mathbb{R}} |\rho_n * w_x|\,dx + R_n(t) \qquad (4.31)$$

for a.e. $t \in [0,T]$ and all $n \geqslant 1$.

Summing (4.28) and (4.31) and using Gronwall's inequality, we infer that

$$\begin{aligned}
\int_{\mathbb{R}} \big(|\rho_n * w| + |\rho_n * w_x|\big)(t,x)\,dx &\leqslant \int_0^t e^{2C(T)(t-s)} R_n(s)\,ds \\
&\quad + e^{2C(T)t}\int_{\mathbb{R}} \big(|\rho_n * w| + |\rho_n * w_x|\big)(0,x)\,dx
\end{aligned}$$

for all $t \in [0,T]$ and $n \geqslant 1$. Note that $w = u - v \in W^{1,1}(\mathbb{R})$. In view of (4.29), an application of Lebesgue's dominated convergence theorem yields that for all $t \in [0,T]$,

$$\int_{\mathbb{R}} \big(|w| + |w_x|\big)(t,x)\,dx \leqslant e^{2C(T)t}\int_{\mathbb{R}} \big(|w| + |w_x|\big)(0,x)\,dx.$$

Since $w(0) = w_x(0) = 0$, it follows from the above inequality that $u(t,x) = v(t,x)$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$. Recalling that T was chosen arbitrarily, the proof is complete. □

Example. Let

$$u_0(x) = c_1 e^{-|x-x_1|} + c_2 e^{-|x-x_2|}, \quad x \in \mathbb{R},$$

with $c_1 < 0$, $c_2 > 0$ and $x_1 < x_2$. One can easily check

$$y_0 = u_0 - u_{0,xx} = 2c_1\delta(x - x_1) + 2c_2\delta(x - x_2).$$

By Theorem 4.1, Eq. (2.2) has a unique global weak solution u with the initial data u_0. It has the explicit form [20]:

$$u(t,x) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

for some $p_1, p_2, q_1, q_2 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. Actually, u is the sum of a peakon and an antipeakon, the antipeakon moves off to the left and the peakon moves off to the right so that no collision occurs. Observe that Theorems 4.1 and 4.5 in [54] cannot be used in the present case.

Remark 4.1. Note that Theorem 4.1 improves considerably the previous results [54, Theorems 4.1 and 4.5]. They are special cases of Theorem 4.1 with $x_0 = -\infty$ and $x_0 = \infty$, respectively.

Remark 4.2. Note that the Degasperis–Procesi equation has shock waves. In order to investigate shock waves of the Degasperis–Procesi equation, Coclite and Karlsen [5] presented recently another notion of entropy weak solution in $L^\infty(\mathbb{R}_+, L^1(\mathbb{R}) \cap BV(\mathbb{R}))$ and in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}) \cap L^4(\mathbb{R}))$ for the Degasperis–Procesi equation. They prove global existence results for entropy solutions to the equation in $L^\infty(\mathbb{R}_+, L^1(\mathbb{R}) \cap BV(\mathbb{R}))$ and in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}) \cap L^4(\mathbb{R}))$.

Their notion of weak solution is much weaker than ours and is designed to study shock waves to the Degasperis–Procesi equation. However, the above example and [54, Examples 1, 2] show that our framework of weak solutions is quite suitable for the study of global peakon solutions to the Degasperis–Procesi equation.

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