JOURNAL OF ALGEBRA **210,** 385–418 (1998)
ARTICLE NO. JA987584

Isotropic Trialitarian Algebraic Groups

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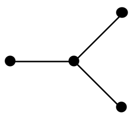
Received June 5, 1997

By modifying a construction from Knus *et al.*, we construct all isotropic algebraic groups of type ${}^3D_4$ and ${}^6D_4$ over an arbitrary field of characteristic $\neq 2$. We also provide a nice isomorphism criterion for such groups. The results of this paper extend the main results of Allison (using entirely different methods) to fields of nonzero characteristic and algebraic groups.

The four infinite families of absolutely simple affine algebraic groups (i.e., groups of type A_n, B_n, C_n, and D_n) over a field F of characteristic $\neq 2$ can be considered to be more or less well understood (because of the correspondence with algebras with involution—see [Wei60], [Mer93], or [Tit66]), except for the groups of type D_4. Such groups have Dynkin diagram



If F_s is a separable closure of F, then the Galois group, Γ, of F_s over F acts by graph automorphisms on the Dynkin diagram. Since the diagram has automorphism group isomorphic to $\mathscr{S}_3$, we have a homomorphism $\mathscr{G}(F_s/F) \to \mathscr{S}_3$. The group is said to be of type ${}^tD_4$ if the image of $\mathscr{G}(F_s/F)$ in $\mathscr{S}_3$ has order t.

Groups of type ${}^1D_4$ and ${}^2D_4$ may also be considered to be well understood, because of the correspondence mentioned previously (see [Bor91, 23.4], [Mer93], [MPW96, pp. 572–585], and [MPW98, Sect. 9]). So we will

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0021-8693/98 $25.00

focus our attention on the so-called *trialitarian* groups, i.e., groups of type ${}^3D_4$ and ${}^6D_4$.

The simplest kind of trialitarian group is a quasi-split group (i.e., one which contains a Borel subgroup defined over our ground field F). We will show in Application 4.3 that the best-known examples of trialitarian groups, the "Steinberg groups" [Ste59, p. 887, Sects. 10 and 11], are quasi-split and we will produce a Borel subgroup defined over the ground field in the type ${}^3D_4$ case.

The conclusion that the Steinberg groups are quasi-split is almost certainly known, although I have not been able to find it stated anywhere in the literature. The explicit description of a Borel subgroup seems to be new.

For any trialitarian group G defined over F there is a separable cubic field extension L determined up to F-algebra isomorphism, such that G is of type ${}^1D_4$ over the Galois closure of L over F. There is a central simple algebra over L of degree 8, also determined up to F-algebra isomorphism, called the *Allen invariant* of G. We will denote this invariant by $\mathscr{E}_F(G)$. (The terminology is due to Allison.) We will have more to say about the Allen invariant in Section 2.

The main results of this paper fully describe the next simplest kind of trialitarian groups, namely isotropic groups, in terms of their Allen invariants. Recall that an algebraic group is called *isotropic* over F if it contains a nontrivial F-split torus [Bor91, 20.1].

MAIN THEOREM 0.1. *Let F be a field of characteristic $\neq 2$ with a separable cubic field extension L.*

Invariant restriction: *If G is an anisotropic trialitarian group over F which is of type ${}^1D_4$ or ${}^2D_4$ over L then the Allen invariant $\mathscr{E}_F(G)$ is F-algebra isomorphic to $M_4(Q)$ for Q a (possibly split) quaternion algebra over L.*

Existence: *If Q is a quaternion algebra over L then the corestriction of Q down to F is trivial if and only if $M_4(Q)$ is the Allen invariant of an isotropic trialitarian group defined over F.*

Uniqueness: *Two isotropic trialitarian groups defined over F are centrally F-isogenous if and only if their Allen invariants are F-isomorphic.*

Our Main Theorem is proven in Lemma 2.4, Proposition 2.5, Application 4.7, and Proposition 5.6. The invariant restriction part is an easy consequence of general facts about central simple algebras and trialities (see Section 2). One direction of the existence part is also a standard fact about trialitarian groups, namely that if G is a group of type D_4 over F, then

$$\operatorname{cor}_{L/F}[\mathscr{E}_F(G)] \text{ is trivial for } L \text{ the center of } \mathscr{E}_F(G) \qquad (0.2)$$

(see, e.g., [All92, p. 216, Prop. 3.3] or [KMRT, 43.6]).

The existence and uniqueness parts were proven by Allison for Lie algebras over fields of characteristic were using structurable algebras (the existence part is [All92, p. 229, Lemma 6.8] and [All90, p. 7, Theorem 5.1] and the uniqueness part is [All90, p. 15, Theorem 8.1]). Seligman also demonstrated the existence part for Lie algebras over fields of characteristic 0 in [Sel88, Chap. 8, especially p. 172, Theorem 8.1].

An important point is that Allison and Seligman both worked with Lie algebras, whereas we will work with algebraic groups. This allows us to take a more characteristic free approach, and in particular their methods do not seem to generalize to fields of positive characteristic. To see how our Main Theorem corresponds with their results, one first observes that every Lie algebra $\mathfrak{g}$ of type D_4 is the Lie algebra of some algebraic group G and that the Allen invariant of $\mathfrak{g}$ as defined by Allen is precisely the Allen invariant of G. Moreover, one knows in general that if is G is isotropic then so is $\mathfrak{g}$ (i.e., $\mathfrak{g}$ contains a nontrivial abelian subalgebra whose image under $\mathrm{ad}_{\mathfrak{g}}$ is diagonalizable), and that the converse holds in characteristic 0. Thus our Main Theorem implies the aforementioned results of Allison and Seligman (in characteristic 0) and we also get that the Lie algebra version of the existence part of our Main Theorem holds in all characteristics (except perhaps characteristic 2).

In Section 6 we will apply our Main Theorem to get isotropy criteria for trialitarian groups over certain fields. This is useful because in general it is almost never possible to tell whether a given trialitarian group is isotropic or not. In particular, we show in Corollary 6.3 that over a finite extension of $\mathbb{F}_p(t)$ for p an odd prime (= a global field of odd characteristic) every trialitarian group is isotropic.

Please see [Dra83] for information about central simple algebras, [Dra83, Sect. 14] or [Lam73, Chap. III] for information about quaternion algebras, [Bor91] for information about algebraic groups, and [Sch85, Chap. 8] or [KMRT] for information about central simple algebras with involution.

For the rest of the paper, all of our fields will be assumed to have characteristic not 2. For a base field F, F_s will always denote a separable closure of F and Γ the absolute Galois group of F (i.e., the Galois group of F_s over F). If G is an affine algebraic group defined over F, then we write G^+ for its identity component. If K is an extension field of F, we write $\mathscr{G}(K/F)$ for the Galois group of K over F, $G(K)$ for the K-points of G, and $H^1(F, G)$ for the Galois cohomology set $H^1(\Gamma, G(F_s))$ [Ser94, I.5].

For any object A, we write $A^{\times 3}$ for $A \times A \times A$.

If A is an F-algebra and ρ is a ring automorphism of F, then as in [Dra83, p. 50, Definition 1] we write ${}^{\rho}A$ for the F-algebra which has the same underlying ring structure as A but has its F-action twisted by ρ. Specifically, if $\cdot$ denotes the twisted F-action and juxtaposition denotes the standard F-action, then we define

$$f \cdot a := \rho^{-1}(f)a \qquad \text{for } f \in F \text{ and } a \in A.$$

For q an invertible element of A we write $\operatorname{Int}(q)$ for the automorphism $a \mapsto qaq^{-1}$.

1. DEFINITION OF THE SPLIT OBJECT

Our approach to the construction of isotropic trialitarian groups will be via Galois descent from the split group over a Galois extension. Thus we will begin by describing the split simply connected group of type ${}^1D_4$, which for our purposes is best done in terms of the split Cayley algebra.

Over a fixed base field F, we define the split Cayley algebra, $\mathfrak{C}$, to be the F-vector space with basis $u_1, u_2, \ldots, u_8$ and multiplication given by the following table, where the entry in the table is $x \bigstar y$ and "$\cdot$" represents 0 for clarity of reading.

					y				
		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	u_1	$\cdot$	$\cdot$	$\cdot$	$-u_1$	$\cdot$	$-u_2$	u_3	$-u_4$
	u_2	$\cdot$	$\cdot$	u_1	$\cdot$	$-u_2$	$\cdot$	$-u_5$	$-u_6$
	u_3	$\cdot$	$-u_1$	$\cdot$	$\cdot$	$-u_3$	$-u_5$	$\cdot$	u_7
x	u_4	$\cdot$	$-u_2$	$-u_3$	u_5	$\cdot$	$\cdot$	$\cdot$	$-u_8$
	u_5	$-u_1$	$\cdot$	$\cdot$	$\cdot$	u_4	$-u_6$	$-u_7$	$\cdot$
	u_6	u_2	$\cdot$	$-u_4$	$-u_6$	$\cdot$	$\cdot$	$-u_8$	$\cdot$
	u_7	$-u_3$	$-u_4$	$\cdot$	$-u_7$	$\cdot$	u_8	$\cdot$	$\cdot$
	u_8	$-u_5$	u_6	$-u_7$	$\cdot$	$-u_8$	$\cdot$	$\cdot$	$\cdot$

There is an involution, π, on $\mathfrak{C}$ given by

$$\pi(u_i) = \begin{cases} u_5 & \text{if } i = 4, \\ u_4 & \text{if } i = 5, \\ -u_i & \text{otherwise}. \end{cases}$$

Remark 1.1. This multiplication is not the usual unital, alternative, nonassociative, and noncommutative multiplication defined for the split Cayley algebra (see [Sch66, Chap. III, Sect. 4] or [KMRT, Sect. 33.C] for a definition of the standard multiplication). If $\diamond$ denotes the usual multiplication, then the multiplication $\bigstar$ is given by

$$x \bigstar y = \pi(x) \diamond \pi(y).$$

This ($\bigstar$) multiplication is nonunital, noncommutative, and not even power-associative [KMRT, p. 464].

Remark 1.2. The basis we have chosen for our split Cayley algebra differs from those in the literature. (This is because our basis gives a nice description of a Borel subgroup of a quasi-split group of type ${}^3D_4$; see Application 4.3.) Our basis is a permutation of the basis in [KMRT, p. 556], which is itself the basis from [AF68, p. 483] with some vectors multiplied by a factor of 2 or $1/2$.

There is a symmetric bilinear norm

$$\mathfrak{n}\colon \mathfrak{C} \times \mathfrak{C} \to F$$

such that $\mathfrak{n}(x \bigstar y, z \bigstar y) = \mathfrak{n}(x, z)\mathfrak{n}(y, y)$ and the Gram matrix of $\mathfrak{n}$ with respect to our basis of $\mathfrak{C}$ is

$$S := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}. \tag{1.3}$$

It is standard that $\mathfrak{n}$ induces an involution $\sigma_{\mathfrak{n}}$ on $\operatorname{End}_F(\mathfrak{C})$ such that $\mathfrak{n}(fx, y) = \mathfrak{n}(x, \sigma(f)y)$ for all $f \in \operatorname{End}_F(\mathfrak{C})$, and that that involution is given by

$$\sigma_{\mathfrak{n}}(f) = Sf^tS$$

when we write f as a matrix with respect to the given basis and t denotes the transpose. We note in passing that $\sigma_{\mathfrak{n}}(\pi) = \pi \in \operatorname{End}_F(\mathfrak{C})$.

Our first algebraic group is the group of *similitudes* of $(\mathfrak{C}, \mathfrak{n})$, whose F-points are

$$\begin{aligned} GO(\mathfrak{C}, \mathfrak{n})(F) &= \left\{ f \in \operatorname{End}_F(\mathfrak{C}) \,\middle|\, \begin{array}{l} \text{for some } \lambda \in F^*, \ \mathfrak{n}(fx, fy) = \\ \lambda\mathfrak{n}(x, y) \text{ for all } x, y \in \mathfrak{C}, \end{array} \right\} \\ &= \{ f \in \operatorname{End}_F(\mathfrak{C}) \mid \sigma_{\mathfrak{n}}(f) f = \lambda I \text{ for some } \lambda \in F^* \}, \end{aligned}$$

where I denotes the identity matrix. If f is a similitude of $(\mathfrak{C}, \mathfrak{n})$ (i.e., $f \in GO(\mathfrak{C}, \mathfrak{n})(F)$), then λ is called the multiplier of f and is denoted $\mu(f)$. Observe that

$$(\det f)^2 = \det(\sigma(f) f) = \mu(f)^8 I,$$

so $\det f = \pm\mu(f)^4 I$. The similitude f is called proper if $\det f = \mu(f)^4 I$, and *improper* otherwise. It is known that the group of proper similitudes is precisely the identity component of $GO(\mathfrak{C}, \mathfrak{n})$, which we will denote by $GO^+(\mathfrak{C}, \mathfrak{n})$.

DEFINITION 1.4. A triple $(t_0, t_1, t_2) \in GO^+(\mathfrak{C}, \mathfrak{n})^{\times 3}$ is called *related* if

$$\mu(t_i)^{-1} t_i(x \bigstar y) = t_{i+2}(x) \bigstar t_{i+1}(y)$$

for all $x, y \in \mathfrak{C}$ and $i = 1, 2, 3$ where the subscripts are taken modulo 3.

Jacobson [Jac64b, p. 135, Definition 2] also defines a related triple, but he does so in terms of a trilinear form $(x, y, z) = \mathfrak{n}(\pi(x)\bigstar\pi(y), \pi(z))$. He calls a triple (t_0, t_1, t_2) related if for some $\lambda \in F^*$, $(t_0 x, t_1 y, t_2 z) = \lambda(x, y, z)$ for all $x, y, z \in \mathfrak{C}$. It is an easy check using basic properties of the trilinear form that a triple is related in our sense if and only if it is related in his sense with $\lambda = 1$.

If the formula in our definition holds for one value of i, then it holds for all values of i by [Jac64b, p. 135, Lemma 3] or [KMRT, 35.4]. Furthermore, $\mu(t_0)\mu(t_1)\mu(t_2) = 1$.

We make some simple observations about related triples.

Proposition 1.5. (1) *If (t_0, t_1, t_2) is a related triple, then so is (t_1, t_2, t_0) and $(\pi t_0 \pi, \pi t_2 \pi, \pi t_1 \pi)$.*

(2) *For $\lambda \in F^*$, the triple $(1, \lambda I, \lambda^{-1} I)$ is related, and no other triple $(1, t_1, t_2)$ is related.*

(3) *The set of related triples in $GO^+(\mathfrak{C}, \mathfrak{n})^{\times 3}$ forms a closed subgroup.*

Proof. (2) and the first part of (1) are clear from the definition. The second part of (1) is a straightforward calculation based on the observation that $\mu(\pi t_i \pi) = \mu(t_i)$.

For (3), we note that the set of related triples in $GO^+(\mathfrak{C}, \mathfrak{n})^{\times 3}$ is certainly Zariski-closed and that Jacobson's definition of a related triple makes it clear that it also contains 1 and is closed under multiplication and inversion. ∎

Given a reasonably nice similarity t_0, we can write down explicitly a related triple (t_0, t_1, t_2) (such a triple always exists by [Jac64b, p. 135, Lemma 4] or [KMRT, 35.4]). In particular, if t_0 is a monomial matrix (i.e., has precisely one nonzero entry in each row and column) or is "close enough" to such a matrix, one can find a t_1 and t_2 quite easily by directly applying the definition of a related triple. We do so and provide a few examples here which will be of use later.

Example 1.6. Write $\mathrm{diag}(d_1, d_2, \dots, d_8)$ for the diagonal 8×8 matrix whose (i, i)-entry is d_i. Then

$$t_0 = \mathrm{diag}\left(d_1, d_2, d_3, d_4, \frac{a}{d_4}, \frac{a}{d_3}, \frac{a}{d_2}, \frac{a}{d_1}\right)$$

is a similitude with multiplier a for d_i, $a \in F^*$.

If we take any $\lambda \in F^*$ and set

$$t_1 = \frac{1}{\lambda}\mathrm{diag}\left(\frac{d_2 d_3}{a d_4}, \frac{d_2}{a}, \frac{d_3}{a}, \frac{1}{d_4}, \frac{d_2 d_3}{a d_1}, \frac{d_2}{d_1 d_4}, \frac{d_3}{d_1 d_4}, \frac{1}{d_1}\right)$$

and

$$t_2 = \lambda \operatorname{diag}\left(\frac{d_1 d_4}{a}, \frac{d_1}{d_3}, \frac{d_1}{d_2}, 1, \frac{d_1 d_4}{d_2 d_3}, \frac{d_4}{d_3}, \frac{d_4}{d_2}, \frac{a}{d_2 d_3} \right),$$

then t_1 and t_2 are similtudes, t_1 has multiplier

$$\mu(t_1) = \frac{\lambda^2 d_1 d_4}{d_2 d_3},$$

and (t_0, t_1, t_2) is a related triple. Moreover, any triple with t_0 in the first position must be of this form.

To see this, suppose we have a related triple (t_0, t_1, t_2) with t_0 as given. Then we can apply the definition of a related triple with $i = 1$ to see that if $u_j \star v = 0$ for $v \in \mathfrak{C}$, then

$$0 = \mu(t_1)^{-1} t_1(u_j \star v) = t_0(u_j) \star t_2(v).$$

So $u_j \star t_2(v) = 0$. For example, if $v = u_1$, then $u_j \star t_2(u_1) = 0$ for $j = 1, 2, 3, 4$. This forces $t_2(u_1)$ to be a scalar multiple of u_1. Similar computations show that t_2 and t_1 preserve each of the linear subspaces spanned by the u_k's. Hence t_1 and t_2 are diagonal.

If $u_j \star u_k \neq 0$, then the definition of a related triple gives us equations relating the entries of t_0, t_1, and t_2. For example, $u_1 \star u_6 = -u_2$, so

$$-\mu(t_0)^{-1} t_0(u_2) = t_2(u_1) \star t_1(u_6)$$

and

$$\frac{d_2}{a} = (t_2)_{11} (t_1)_{66},$$

where $(t_2)_{11}$ denotes the $(1, 1)$-entry of the matrix t_2. Solving the 32 such equations simultaneously gives us that t_1 and t_2 are of the desired form. (It is a straightforward calculation using Mathematica or any similar package to verify that this and all the rest of the triples which we will assert are related are in fact related.)

One can see by similar sorts of arguments that if a triple (t_0, t_1, t_2) is related with some t_i a monomial matrix, then so are the other two.

EXAMPLE 1.7. (S, S, S) is a related triple for S as in (1.3).

EXAMPLE 1.8. We can create a map

$$P \colon \mathscr{S}_8 \to \operatorname{End}_F(\mathfrak{C})$$

by setting $P(g)$ to be the endomorphism which sends u_i to $u_{g(i)}$.

If we set

$$t = \operatorname{diag}(a, b, c, 1, 1, c, b, a)P((1\ 2)(3\ 6)(4\ 5)(7\ 8))$$

for $a, b, c \in \{1, -1\}$, then (t, t, t) is a related triple if and only if $abc = -1$.

Those similitudes of $(\mathfrak{C}, \mathfrak{n})$ with multipitier 1 are called isometries, and they form a closed subgroup $O(\mathfrak{C}, \mathfrak{n})$ of $GO(\mathfrak{C}, \mathfrak{n})$. The identity component $O^+(\mathfrak{C}, \mathfrak{n})$ of $O(\mathfrak{C}, \mathfrak{n})$ consists of the proper isometries, so

$$O^+(\mathfrak{C}, \mathfrak{n})(F) = \left\{ f \in \operatorname{End}_F(\mathfrak{C}) \mid \sigma_{\mathfrak{n}}(f)f = \det f = 1 \right\}.$$

This is the natural setting for our next example.

EXAMPLE 1.9. For $1 \leq i, j \leq 8$, define E_{ij} to be the matrix whose only nonzero entry is the (i, j)-entry, which is 1. For $i \neq j$, we can define a morphism of algebraic groups

$$U_{ij} \colon \mathbb{G}_a \to O^+(\mathfrak{C}, \mathfrak{n})$$

by

$$U_{ij}(r) = 1 + r\left(E_{ij} - E_{j^* i^*}\right),$$

where $i^* := 9 - i$. For example,

$$U_{12}(r) = \begin{pmatrix} 1 & r & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & -r \\ & & & & & & & 1 \end{pmatrix}.$$

Then $(U_{12}(r), U_{35}(-r), U_{34}(r))$, $(U_{23}(r), U_{23}(r), U_{23}(r))$, $(U_{21}(r), U_{53}(-r), U_{43}(r))$, and $(U_{32}(r), U_{32}(r), U_{32}(r))$ are related.

We define another algebraic group $\operatorname{Spin}(\mathfrak{C}, \mathfrak{n})$ whose F-points are given by

$$\operatorname{Spin}(\mathfrak{C}, \mathfrak{n})(F) = \left\{ \underline{t} \in O^+(\mathfrak{C}, \mathfrak{n})(F)^{\times 3} \,\middle|\, \begin{array}{l} \underline{t} = (t_0, t_1, t_2) \text{ is a} \\ \text{related triple} \end{array} \right\}. \tag{1.10}$$

This is an algebraic group by Proposition 1.5(3). It is known by [KMRT, 35.8] that this group is split simply connected of type ${}^1D_4$.

There is an obvious map $\varphi \colon \operatorname{Spin}(\mathfrak{C}, \mathfrak{n}) \to O^+(\mathfrak{C}, \mathfrak{n})$ given by $\varphi(t_0, t_1, t_2) = t_0$. Although this map is a surjection of algebraic groups, it is a surjection on the F-points only when F is quadratically closed. One can see this by taking any $r \in F^*$ and applying Example 1.6 with $d_1 = d_3 = d_4 = 1$ and $d_2 = r$. Then $t_0 \in O^+(\mathfrak{C}, \mathfrak{n})$ and if $\varphi(t_0, t_1, t_2) = t_0$ then t_1 would have multiplier 1, so $\lambda^2 = r$.

2. TRIALITIES

One knows that every absolutely simple adjoint nontrialitarian group of type D_n is of the form $PGO^+(A, \sigma)$ for A a central simple algebra of degree $2n$ and σ an orthogonal involution by [Tit66, pp. 56–57] or [Mer93, pp. 8–9]. This fact can be useful for proving things about such groups (see, [Mer96, p. 211, Theorem 3], [MPW96], [MPW98], and [MT95, Sect. 2.4]).

Thus it is reasonable to think that an analogous construction for trialitarian groups would also prove useful. Such a construction is given by something we call a *triality*, which is described in [KMRT, sect. 43.A] (where they are called "trialitarian algebras"). Since [KMRT] is not yet available to the general public, we will define what a triality is after a few preparatory definitions.

Cubic Étale Algebras

An *étale* F-algebra is a finite direct sum of finite separable field extensions of F (see also [Bou74, Chap. V, Sect. 6] or [KMRT, Sect. 18]). We will be particularly interested in those of dimension 3 or 2, which are called *cubic* and *quadratic*, respectively. It is known that étale F-algebras of dimension n are classified by the pointed set $H^1(F, \mathscr{S}_n)$, where the distinguished element corresponds to a direct sum of n copies of F. This distinguished element is called the *diagonal étale algebra*. Furthermore, to any n-dimensional étale algebra there is an associated quadratic étale algebra called the discriminant algebra (cf. [Wat87, p. 211]), given by the map $H^1(F, \mathscr{S}_n) \to H^1(F, \mathscr{S}_2)$ which is induced by the sign map $\mathscr{S}_n \to \boldsymbol{\mu}_2 \cong \mathscr{S}_2$. We denote the discriminant algebra of L over F by $\Delta_F(L)$.

Since the Galois action on $\mathscr{S}_n$ is trivial, one knows that $H^1(F, \mathscr{S}_n)$ is a quotient of $\mathrm{Hom}(\Gamma, \mathscr{S}_n)$ (for Γ the absolute Galois group of F) such that two maps $f, f' \in \mathrm{Hom}(\Gamma, \mathscr{S}_n)$ are equivalent if and only if they differ by an inner automorphism of $\mathscr{S}_n$. Thus anything equivalent to f has the same kernel, and if L is an n-dimensional étale F algebra and $(f) \in H^1(F, \mathscr{S}_n)$ corresponds to the isomorphism class of L, then $\ker f$ is an invariant associated to L. We say that L is of type t if $t = [\Gamma : \ker f]$. Also, we denote by L^c the subfield of F_s fixed by $\ker f$. It is clear for purely Galois cohomological reasons that if K is an extension of F, then $K \otimes_F L$ is diagonal over K if and only if K contains L^c.

We will mainly be interested in cubic étale algebras. Here is a table summarizing the possibilities, where L is a cubic étale algebra and Δ is any quadratic field extension of F.

L	Type L	L^c
$F \times F \times F$	1	F
$F \times \Delta$	2	Δ
Galois field extension	3	L
Non-Galois field extension	6	Normal closure of L/F

We point out that in the type 1 and 3 cases the discriminant algebra of L is just $F \times F$, in the type 2 case it is Δ, and in the type 3 and 6 cases it is precisely $F[x]/(x^2 - \delta)$ where δ is a representative of the standard discriminant for L as a field extension of F.

Central Simple Algebras with Involution

We need to generalize the idea of a central simple algebra over a field to algebras over an étale algebra. If $A = \oplus_{i=1}^n A_i$ is an F-algebra such that each of the A_i's is a central simple algebra of degree d with center $Z(A_i)$ a finite separable field extension of F, then we say that A is a central simple algebra of degree d over the étale F-algebra $\oplus_{i=1}^n Z(A_i)$. (In other words, A is an Azumaya—a.k.a. central separable—algebra of constant rank d over an étale algebra.) The A_i are called the *components* of A.

Suppose now that we have a central simple algebra E over an étale algebra L and an L-linear involution a on E. We say that σ is *orthogonal* if it is restricted to be an orthogonal involution on each component of E.

Trialities Defined [*KMRT*, *Sect*. 43.*A*]

"Triality" shows up in algebraic groups of type D_4 as an outer automorphism of order 3. This corresponds to a phenomenon for central simple algebras of degree 8 with orthogonal involution: if (A_i, σ_i) is such an algebra with center F for $i = 0, 1, 2$, and we are given an F-isomorphism

$$\psi_0 \colon \big(C_0(A_0, \sigma_0), \underline{\sigma_0}\big) \xrightarrow{\sim} (A_1, \sigma_1) \times (A_2, \sigma_2),$$

then there is a recipe for producing F-isomorphisms

$$\psi_i \colon \big(C_0(A_i, \sigma_i), \underline{\sigma_i}\big) \xrightarrow{\sim} (A_{i+1}, \sigma_{i+1}) \times (A_{i+2}, \sigma_{i+2})$$

for $i = 1, 2$ with subscripts taken modulo 3 [KMRT, 42.3]. (Here $(C_0(A_i, \sigma_i), \underline{\sigma_i})$ denotes the even Clifford algebra of (A_i, σ_i) endowed with the standard involution $\underline{\sigma_i}$; for more information, see [Jac64a] or [KMRT, Sect. 8], where $C_0(A_i, \sigma_i)$ is denoted by $C(A_i, \sigma_i)$.) Further, given ψ_i the same process produces ψ_{i+1} and ψ_{i+2}. Somehow this corresponds to a triality automorphism for groups of type ${}^1D_4$. The objects we call trialities encode similar information for all groups of type D_4.

DEFINITION 2.1. A *triality* over F is a 4-tuple (E, L, σ, α) where L is a cubic étale F-algebra and E is a central simple algebra over L of degree 8 with orthogonal involution σ. For ρ a generator of $\mathscr{G}(L \otimes_F \Delta_F(L)/\Delta_F(L))$, α is an L-algebra isomorphism

$$\alpha\colon \bigl(C_0(E, \sigma), \underline{\sigma}\bigr) \to {}^{\rho}\bigl((E, \sigma) \otimes_F \Delta_F(L)\bigr),$$

where $\underline{\sigma}$ is the canonical involution on $C_0(E, \sigma)$ induced by σ. (If L is of type 1 or 2, then $L \otimes_F \Delta_F(L) \cong_\Delta \Delta_F(L)^{\times 3}$ and in that case we take ρ to be a cyclic permutation of the components.) There is an added restriction on α as follows:

If we write $(\tilde{E}, \tilde{L}, \tilde{\sigma}, \tilde{\alpha})$ for $(E, L, \sigma, \alpha) \otimes_F L^c = (E \otimes_F L^c, L \otimes_F L^c, \sigma \otimes \mathrm{Id}, \alpha \otimes \mathrm{Id})$ and $\tilde{F}$ for L^c (so we have extended scalars to make L diagonal), then

$${}^{\rho}\bigl((\overline{E}, \tilde{\sigma}) \otimes_{\tilde{F}} \Delta_{\tilde{F}}(\tilde{L})\bigr) \cong_{\tilde{L}} {}^{\rho}(\tilde{E}, \tilde{\sigma}) \times {}^{\rho^2}(\tilde{E}, \tilde{\sigma}).$$

Since $\tilde{L}$ is of type 1, $(\tilde{E}, \tilde{\sigma}) \cong_{\tilde{F}} (A_0, \sigma_0) \times (A_1, \sigma_1) \times (A_2, \sigma_2)$ for (A_i, σ_i) central simple algebras of degree 8 over $\tilde{F}$ with orthogonal involution. Thus $\tilde{\alpha}$ restricts to isomorphisms

$$\tilde{\alpha}_i\colon \bigl(C_0(A_i, \sigma_i), \underline{\sigma_i}\bigr) \xrightarrow{\sim} (A_{i+1}, \sigma_{i+1}) \times (A_{i+2}, \sigma_{i+2}) \qquad \text{for } i = 0, 1, 2.$$

Our requirement on α is that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be the isomorphisms induced by $\tilde{\alpha}_0$ via the triality phenomenon given previously.

We say that a triality is of *type* t if L is of type t.

An isomorphism of trialities $\varphi\colon (E, L, \sigma, \alpha) \to (E', L', \sigma', \alpha')$ is an isomorphism $\varphi\colon (E, \sigma) \to (E', \sigma')$ of F-algebras with involution such that the following diagram commutes:

$$\begin{array}{ccc}
(C_0(E, \sigma), \underline{\sigma}) & \xrightarrow{\alpha} & {}^{\varphi}((E, \sigma) \otimes_F \Delta_F(L)) \\
{\scriptstyle C_0(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi \otimes \mathrm{Id}} \\
(C_0(E', \sigma'), \underline{\sigma}') & \xrightarrow{\alpha'} & {}^{\rho}((E', \sigma') \otimes_F \Delta_F(L'))
\end{array}$$

By [KMRT, Sect. 44] there is a split triality

$$T^d = \bigl(M_8(F)^{\times 3}, F^{\times 3}, \sigma_{\mathscr{H}}^{\times 3}, \alpha\bigr),$$

where $\sigma_{\mathscr{H}}$ denotes a hyperbolic involution on $M_8(F)$ (if you wish, $\sigma_{\mathscr{H}} = \mathrm{Int}(S) \circ t$ where t denotes the transpose), which has automorphism group isomorphic to $PGO^+(\mathfrak{C}, \mathfrak{n}) \rtimes \mathscr{S}_3$ [KMRT, 44.2], for $PGO^+(\mathfrak{C}, \mathfrak{n})$ the image of $GO^+(\mathfrak{C}, \mathfrak{n}) \subset GL(\mathfrak{C})$ in $PGL(\mathfrak{C})$. Over a separably closed field all trialities are isomorphic to T^d. In fact, for any triality T of type t, the group $\mathrm{Aut}^+(T)$ is absolutely simple adjoint of type ${}^tD_4$ and we denote it by $PGO^+(T)$. This provides an equivalence of categories between the category of trialities over F and the category of absolutely simple adjoint groups of type D_4 over F [KMRT, 44.8], where both categories have isomorphisms for morphisms.

Similarly, there is a canonically associated absolutely almost simple simply connected group of type ${}^tD_4$ which is the simply connected cover of $PGO^+(T)$. We denote it by $\mathrm{Spin}(T)$, and point out that $\mathrm{Spin}(T^d)$ is precisely the group $\mathrm{Spin}(\mathfrak{C}, \mathfrak{n})$ defined in the last section.

To illustrate the connection with the classical description of groups of type ${}^1D_4$ and ${}^2D_4$, let $T = (A \times B, L, \sigma \times \tau, \alpha)$ be a triality of type 1 or 2 so that the center of B is a quadratic étale F-algebra. Then (A, σ) is a central simple algebra of degree 8 with orthogonal involution σ, $\mathrm{Spin}(A, \sigma) \cong_F \mathrm{Spin}(T)$, and the definition of α forces that $(B, \tau) \cong_F (C_0(A, \sigma), \underline{\sigma})$. (For a definition of $\mathrm{Spin}(\mathrm{A}, \sigma)$, see [KMRT, p. 187] or [MPW96, p. 574].)

Invariants

Trialities provide the natural setting for several invariants of groups of type D_4. If we take such a group G then there is some corresponding triality $T = (E, L, \sigma, \alpha)$ determined up to F-isomorphism. Then L^c is the unique smallest field over which G becomes of inner type (i.e., of type ${}^1D_4$). We call L^c the *inner extension* of G. It is clear by the preceding description of groups of type ${}^1D_4$ and ${}^2D_4$ in terms of trialities that if $\mathrm{Spin}(T)$ is of type ${}^3D_4$ or ${}^6D_4$, then $\mathrm{Spin}(T) \times_F L \cong_L \mathrm{Spin}(E, \sigma)$.

The L-algebra E (which the F-isomorphism class of T determines up to F-algebra isomorphism) is the so-called *Allen invariant* of G over F, denoted by $\mathscr{E}_F(G)$. We say that the Allen invariant is trivial if all of its components are split.

Remark 2.2. The Allen invariant is in some sense precisely the Tits algebras associated to G (see [Tit71] for a defintion or [MPW96, Sect. 2] for an overview of Tits algebras). To see this, let $\tilde{G}$ be a simply connected cover of G and let L be a cubic étale F-algebra such that L^c is the inner extension for $\tilde{G}$. The cocenter C of $\tilde{G}$ (i.e., the dual of the center of $\tilde{G}$) is noncanonically isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ with a Galois action on the nontrivial elements which is the same as the Galois action permuting the connected components of $L \otimes_F L^c \cong (L^c)^{\times 3}$. Tits provides a map for those elements of C fixed by the Galois action to $\mathrm{Br}\, F$. Every nontrivial element $c \in C$ has a unique minimal field extension F_c over which the

Galois action fixes c. Tits' map then provides a central simple algebra A_c over F_c associated to c such that if c and c' are in the same Γ-orbit in C, then $A_c \cong_F A_{c'}$. Using this notation, we pick a set of representatives $1, c_1, \ldots, c_r$ for the orbits of Γ in C and observe that

$$\mathscr{E}_F(G) \cong_F \bigoplus_{i=1}^{r} A_{c_i}.$$

When we throw in the involution data, we get a previously studied invariant [Jac64b, Sect. 4], which I will call the *involution invariant*,

$$\mathscr{I}_F(G) \coloneqq (E, \sigma).$$

Observe that, for a field $K \supseteq F$,

$$\mathscr{I}_F(G) \otimes_F K \cong_K \mathscr{I}_K(G)$$

and that an analogous formula holds for the Allen invariant.

EXAMPLE 2.3. One knows by the description of the quasi-split groups of type ${}^1D_4$ [MPW96, p. 572] and ${}^2D_4$ [MPW98, Sect. 9] and standard scalar extension arguments that if G^q is a quasi-split trialitarian group with inner extension L^c, the involution invariant is $\mathscr{I}_F(G^q) \cong (M_8(L), \sigma)$ where σ is adjoint to $3\mathscr{H} \perp \langle 1, -\delta \rangle$ for $\delta F^{*2} = \operatorname{disc}_F L$ and $\mathscr{H}$ a hyperbolic plane.

From the standpoint of the theory of algebras with involution, one would like to say that two trialities over F are isomorphic if and only if their involution invariants are F-isomorphic. This would say that the existence of the α is what is important and not any particular choice of α. However, this does not hold in general as is mentioned in [All67, p. 256] (or see Example 2.6).

Examples of cases where the involution invariant classifies trialities:

- trialities of type 1 and 2 [Jac64b, pp. 143–144, Theorems 4 and 5]
- the reals or any field with no separable cubic field extensions (since every triality over such a field is of type 1 or 2)
- number fields (this is an easy consequence of Allison's injectivity theorem [All92, p. 235, 7.6])
- perfect fields F such that cd $F(\sqrt{-1}) \leq 1$ (by the Hasse principle for such fields; see [Sch96] or [Duc96])

In some cases, the Allen invariant alone classifies trialities:

- $\mathfrak{p}$-adic fields ($=$ fields complete with respect to a discrete valuation with finite residue field) [All67, p. 264]
- totally imaginary number fields (by Allison's injectivity theorem and the preceding fact about $\mathfrak{p}$-adic fields)

• global fields of characteristic p (= finite extensions of $\mathbb{Z}_p(t)$—this holds by observing that Allison's proof of his injectivity theorem [All92, p. 235, 7.6] goes over easily to the prime characteristic case)

A Caveat about Isotropy

In the nontrialitarian case, it is easy to see using the type of arguments in [Bor91, 23.4] that for (A, σ) a central simple algebra with orthogonal involution over F, $PGO^+(A, \sigma)$ is isotropic (i.e., contains a nontrivial F-split torus) if and only if (A, σ) is isotropic (i.e., there is an element $a \in A$ such that $\sigma(a)a = 0$). In general, there is no similar correspondence in the trialitarian case.

In particular, we point out the interesting phenomenon that one can have a trialitarian group over F with inner extension L^c such that the group is anisotropic over F, but is isotropic (or even split) over L. This is unexpected because L is a cubic extension, and generally one does not expect cubic extensions to make anisotropic groups of type D_n isotropic. An example where the group is anisotropic over F but splits over L is given in Example 2.6.

Just as one uses the Witt index to measure how isotropic a quadratic form is, there is an analogue for a central simple algebra E over a field with orthogonal involution σ. We say that the *Witt index* of (E, σ), denoted by $w(E, \sigma)$, is defined to be the maximum of $\dim_F I/\deg E$ for right ideals I of E such that $\sigma(I)I = 0$. It is clear that $w(E, \sigma) \geq 1$ if and only if σ is isotropic, and that in all cases $\operatorname{ind} E$ divides $w(E, \sigma)$. If $w(E, \sigma)$ is as large as possible given E (i.e., $w(E, \sigma) = (\deg E)/2$), then we say that σ is *hyperbolic*.

We call a triality *isotropic* if $\mathrm{Spin}(T)$ is isotropic.

LEMMA 2.4. *If $T = (E, L, \sigma, \alpha)$ is an isotropic triality of type* 3 *or* 6, *then $w(E, \sigma) \geq 2$.*

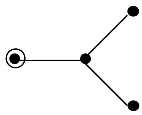
Proof. First, since $\mathrm{Spin}(T)$ is isotropic, so is $\mathrm{Spin}(T) \times_F L \cong \mathrm{Spin}(E, \sigma)$, so (E, σ) is isotropic and $w(E, \sigma) \geq 1$.

To eliminate the possibility that $w(E, \sigma) = 1$, we examine the Dynkin diagram of $\mathrm{Spin}(T)$. Since $\mathrm{Spin}(T)$ is isotropic, it contains a nontrivial maximal F-split torus S lying in a maximal torus S' defined over F. If Δ is a set of simple roots for $\mathrm{Spin}(T)$ with respect to S' (we will also use Δ to refer to the Dynkin diagram of $\mathrm{Spin}(T)$), then as in [Tit66, 2.1] we write Δ_0 for the subset of Δ of roots which vanish on S. Since S is nontrivial and Δ is a set of simple roots (so it forms a $\mathbb{Q}$-basis for $X(S') \otimes_{\mathbb{Z}} \mathbb{Q}$), $\Delta_0 \neq \Delta$.

Tits points out that $\Delta \setminus \Delta_0$ is stable under the Galois action and he circles those orbits in Δ which belong to $\Delta \setminus \Delta_0$ in the Dynkin diagram for

Spin(T) (which he calls the "F-index" [Tit66, 2.3]). Thus the center vertex of Δ must be circled or all of the outer vertices must be circled.

Now we have a Dynkin diagram over L (the "L-index"), and every vertex circled over F must necessarily be circled over L. However, if $w(E, \sigma) = 1$ then the L-index would be



by [Tit66, pp. 56–57], which is a contradiction. ∎

Proposition 2.5 (Invariant Restriction). *If a triality (E, L, σ, α) of type 3 or 6 has σ isotropic, then E has index at most 2. Further, if σ is hyperbolic, then A is split and L is of type 3.*

Proof. Since σ is isotropic, E cannot be a skew field, so E cannot have index 8.

If E has index 4, then since ind E divides $w(E, \sigma) \leq (\deg E)/2 = 4$, σ must be hyperbolic.

If σ is hyperbolic, then $C_0(E, \sigma) \cong_L E \times M_8(L)$ by [All68, Theorem 3], [Tit68, Prop. 8], or [MPW96, p. 585, Lemma 5.9]. Since α is an isomorphism $C_0(E, \sigma) \cong {}^{\rho}((E, \sigma) \otimes_F \Delta_F(L))$, $\Delta_F(L)$ is isomorphic to $F \times F$ and so L is of type 3. Since the two components of $C_0(E, \sigma)$ will be ring-isomorphic to E, E must be split. ∎

It follows from [All90, Theorems 5.1 and 6.3] that over a field of characteristic 0, any isotropic triality T of type 3 or 6 has Allen invariant $\mathscr{E}_F(\mathrm{Spin}(T))$ a central simple algebra of index at most 2. The preceding proposition (along with Lemma 2.4) shows that this holds over any field of characteristic not 2.

Example 2.6. In this example we will produce an anisotropic group over an appropriately chosen base field F with L of type 3 and with the same involution invariant as the associated quasi-split group (i.e., a split algebra with hyperbolic involution; see Example 2.3). Since involution invariants classify groups of type ${}^1D_4$ this group will be split over L.

Let F_0 be a field which supports a nonsplit central simple algebra A of degree 3. If we set $F = F_0(x)$ or $F = F_0((x))$ then we have a nonsplit central simple algebra $A_F := A \otimes_{F_0} F$ of degree 3 over F such that x is not a reduced norm from A_F. (This is [Jac68, p. 417, Lemma 1] for $F = F_0(x)$ and is easy to prove for $F = F_0(x)$.) Thus the first Tits construction (see [PR94] or [Jac68, Chap. IX] for this and other undefined Jordan algebra terms) provides a nonreduced Jordan algebra $J := J(A_F, x)$. Now

A_F contains a cubic Galois field extension L of F and L embeds in J. Then we write $\mathrm{Aut}(J/L)$ for the set of Jordan algebra automorphisms of J which fix L. This is known to be a simply connected group of type D_4 with inner extension L^c by [Jac60, p. 86], [KMRT, Sect. 38], or [Jac71, p. 31, Theorem 6].

There is some triality $T = (\mathrm{M}_8(L), L, \sigma, \alpha)$ with $\mathrm{Spin}(T) \cong_F \mathrm{Aut}(J/L)$. ($T$ has trivial Allen invariant by [All67, p. 258, Theorem I].) Moreover, σ is adjoint to the coordinate norm of J (as follows from the Springer construction [KMRT, Sect. 38], which is hyperbolic since J is a first Tits construction [PR84, p. 269, Theorem 4.7]. Thus $\mathscr{I}_F(\mathrm{Spin}(T)) \cong_F (M_8(L), \sigma_{\mathscr{H}})$ for $\sigma_{\mathscr{H}}$ a hyperbolic involution on $M_8(L)$.

However, $\mathrm{Aut}(J/L)$ injects into $\mathrm{Aut}(J)$, the group of Jordan algebra automorphisms of J. Since J is nonsplit and a first Tits construction, it has no nonzero nilpotent elements and so by [Tit66, p. 61], $\mathrm{Aut}(J)$ is anisotropic. Thus $\mathrm{Aut}(J/L) \cong \mathrm{Spin}(T)$ is anisotropic.

3. QUATERNION ALGEBRAS

It is clear from the invariant restriction part of the Main Theorem (which we proved in Proposition 2.5) that we should be interested in quaternion algebras over L cubic separable over F whose corestriction down to F is trivial. This turns out to be a strong condition. We collect in this section some useful facts about such algebras.

First, we describe such quaternion algebras fully. This result was proven by Allison for fields of characteristic 0 in [All92, p. 229, Lemma 6.8] and in full generality in [KMRT, 43.9]:

LEMMA 3.1 [KMRT, 43.9]. *Suppose that Q is a quaternion algebra over L a cubic étale F-algebra. Then $\mathrm{cor}_{L/F}[Q]$ is trivial if and only if there is some $a \in L^*$ such that $N_{L/F}(a) = 1$ and some $b \in F^*$ such that $Q \cong (a, b/L)$.*

Moreover, we can say that such a quaternion algebra is certain to be split in one very particular case.

LEMMA 3.2. *Suppose that Q is a quaternion algebra over L a separable cubic field extension of F such that the corestriction of Q down to F is trivial. If Q is split by L^c, it is split.*

Proof (Merkurjev). Let $\delta F^{*2} = \mathrm{disc}_{\mathrm{F}}\, L$, so that $L^c = L(\sqrt{\delta}) \neq L$.

Claim. We may assume that $Q \cong (a, \delta/L)$ for some $a \in L^*$ such that $N_{L/F}(a) = 1$. Since Q is split by $L(\sqrt{\delta})$, we can certainly write $Q \cong (b, \delta/L)$

for some $b \in L^*$. The projection formula [Tig87, Theorem 3.2] tells us that

$$\operatorname{cor}_{L/F}[Q] = \left[\frac{N_{L/F}(b), \delta}{F}\right]$$

and since this is split by hypothesis, $N_{L/F}(b) \in N_{F(\sqrt{\delta})/F}(F(\sqrt{\delta})^*)$. Setting $a \coloneqq b^3/N_{L/F}(b)$, we get that $N_{L/F}(a) = 1$ and that $(a, \delta/L) \cong_L (b, \delta/L) \cong_L Q$. This proves the claim.

Let ι be an element of order 2 of $\mathscr{G}(L^c/F)$ such that ι is the identity on L and let ρ be an element of order 3. Then

$$1 = a\rho(a)\rho^2(a) = a\rho(a)\rho^2(\iota a) = a(\rho(a)\iota(\rho(a))).$$

Thus a^{-1} (and hence a) is a norm from $L^c = L(\sqrt{\delta})$ down to L and so Q is split. ∎

This last lemma gives an elementary argument for the following:

COROLLARY 3.3 [All67, p. 263, Corollary]. Let G be a group of type ${}^6D_4$ over F with discriminant δF^{*2}. Then G has trivial Allen invariant over F if and only if it has trivial Allen invariant over $F(\sqrt{\delta})$.

Proof. We simply observe that

$$\mathscr{E}_{F(\sqrt{\delta})}(G) \cong \mathscr{E}_F(G) \otimes_F F(\sqrt{\delta}) \cong \mathscr{E}_F(G) \otimes_L L(\sqrt{\delta}).$$

So $\mathscr{E}_F(G)$ is Brauer-equivalent to a (possibly split) quaternion algebra, which (as always) has trivial corestriction down to F by (0.2). The lemma finishes the proof. ∎

4. THE DESCENT AND THE EXISTENCE PART

In this section we will produce the existence part of our Main Theorem. Our construction is a small modification of a descent argument from [KMRT, 43.11], which itself builds on work of Allen [All67, pp. 4–5] and Allen and Ferrar [AF68, Sect. 1].

The idea is to take L a separable cubic field extension of F and $b \in F^*$ such that $b \not\equiv \operatorname{disc}_F L$ or 1 mod F^{*2}, start with a split group of type D_4 over $P \coloneqq L^c(\sqrt{b})$, and descend to a group over F with inner extension L^c and nontrivial Allen invariant. Of course, doing the descent directly with the groups is not so easy, so we will instead work with the corresponding trialities.

First, we need a little notation. The field P is Galois over F with Galois group

$$G := \mathscr{G}(P/F) \cong \begin{cases} \mathscr{A}_3 \times \boldsymbol{\mu}_2 & \text{if } L \text{ is of type 3,} \\ \mathscr{S}_3 \times \boldsymbol{\mu}_2 & \text{if } L \text{ is of type 6.} \end{cases}$$

Set

$$\zeta := ((1\ 2\ 3), -1) \qquad \text{and} \qquad \iota := ((2\ 3), 1),$$

so that ζ permutes L and its conjugates in L^c while simultaneously flipping the sign of $\sqrt{b}$, and ι flips the sign of $\sqrt{\delta}$ for $L(\sqrt{\delta}) = L^c$ but fixes L and $\sqrt{b}$. (Of course, in the type 3 case one should simply ignore ι.) Note also that ζ is of order 6 and $\zeta\iota = \iota\zeta^5$.

For the purposes of descent, define F-vector space endomorphisms $\underline{\zeta}$ and $\underline{\pi}$ of $(\mathfrak{C} \otimes_F P)^{\times 3}$ by

$$\underline{\zeta}(x_0, x_1, x_2) := (\zeta x_1, \zeta x_2, \zeta x_0)$$

and

$$\underline{\pi}(x_0, x_1, x_2) := ((\pi \otimes \iota)x_0, (\pi \otimes \iota)x_2, (\pi \otimes \iota)x_1).$$

Note that $\underline{\zeta}\underline{\pi} = \underline{\pi}\underline{\zeta}^5$. Now we may state our descent result, which axiomatizes the argument in [KMRT, 43.11].

Descent Proposition 4.1. *Suppose L is a separable cubic field extension of F, $a \in L^*$ such that $N_{L/F}(a) = 1$, and $b \in F^*$ such that $b \not\equiv \operatorname{disc}_F L$ or 1 mod F^{*2}. Then if $\underline{t} = (t_0, t_1, t_2)$ is a related triple of similarities in $(\operatorname{End}_{L^c}(\mathfrak{C} \otimes_F L^c))^{\times 3}$ such that*

- $\mu(t_0) = a$,
- $\sigma_{\mathrm{n}}(t_i) = t_i$ *for* $i = 0, 1, 2$,
- $\zeta t_i = t_{i+2}\zeta$ *for all* i *(subscripts modulo 3)*

and if L is of type 6 we further require

- $(\pi \otimes \iota)t_i = t_{-i}(\pi \otimes \iota)$

then one gets a descent of the split triality T^d over $L^c(\sqrt{b})$ down to a triality over F with inner extension L^c and Allen invariant $M_4(a, b/L)$. The associated Spin group is the set of fixed points in $\operatorname{Spin}(T^d)$.

The descent is provided by $\langle \operatorname{Int}(\underline{t}\underline{\zeta})\rangle$ in the type 3 case and by $\langle \operatorname{Int}(\underline{t}\underline{\zeta}), \operatorname{Int}(\underline{\pi})\rangle$ in the type 6 case.

Proof. All of this proof besides the determination of the Allen invariant is paraphrased from [KMRT, 43.11]. (Actually, our descent differs slightly in the type 6 case.) We reproduce the material from there here for the convenience of the reader.

We do the "generic case" where L is of type 6, so that $P := L^c(\sqrt{b})$ is an $(\mathscr{S}_3 \times \boldsymbol{\mu}_2)$-Galois extension of F. The type 3 case is only easier.

The various conditions on $\underline{t}$ force $\underline{t}$ to commute with $\underline{\zeta}$ and $\underline{\pi}$ as F-vector space automorphisms of $(\mathfrak{C} \otimes_F P)^{\times 3}$. Since $\underline{\zeta}$ is ζ-semilinear and $\underline{\pi}$ is ι-semilinear, one checks that the group $\langle \operatorname{Int}(\underline{t}\underline{\zeta}), \operatorname{Int}(\underline{\pi}) \rangle$ provides a descent on the P-algebra with involution

$$\left((\operatorname{End}_F(\mathfrak{C}) \otimes_F P)^{\times 3}, \sigma_{\mathscr{H}}^{\times 3} \right).$$

Since $\underline{t}$ is a related triple, these automorphisms are actually automorphisms of the split triality over P; i.e., they respect the α in the definition of a triality.

Thus we need only check that the Allen invariant information is correct. We first examine the center. For $(p_0, p_1, p_2) \in P^{\times 3}$,

$$\operatorname{Int}(\underline{\pi})(p_0, p_1, p_2) = \left(\iota(p_0), \iota(p_2), \iota(p_1) \right)$$

and

$$\operatorname{Int}(\underline{t}\underline{\zeta})(p_0, p_1, p_2) = \left(\zeta(p_1), \zeta(p_2), \zeta(p_0) \right).$$

The first equation forces that $p_0 \in L^c$ and the second equation tells us that $p_i = \zeta p_{i+1}$ for all i. So we have that p_0 determines p_1 and p_2 and that p_0 is fixed by ζ^3. Thus $(P^{\times 3})^G \cong L$.

To see that the Allen invariant is the desired one, we need only descend down the biquadratic extension from P to L and look at the first component, for there the Allen invariant will be $E \times C_0(E, \sigma)$ if (E, σ) is the involution invariant of our descended group. Thus E is the L-subalgebra of $\operatorname{End}_F(\mathfrak{C}) \otimes_F P$ fixed by $\operatorname{Int}(t_0) \otimes \zeta^3$ and $\operatorname{Int}(\pi) \otimes \iota$.

We compute E using Galois cohomology. Let $G := \mathscr{G}(P/L) = \{1, \iota, \zeta^3, \iota\zeta^3\}$ and set $A := \operatorname{End}_F(\mathfrak{C}) \otimes_F L$. Then there is an exact sequence of P-points of algebraic groups

$$1 \to P^* \to GL_1(A)(P) \to \operatorname{Aut}(A)(P) \to 1,$$

which induces an exact sequence on cohomology

$$H^1(G, GL_1(A)(P)) \to H^1(G, \operatorname{Aut}(A)(P)) \xrightarrow{\partial} H^2(G, P^*).$$

Now $H^1(G, \operatorname{Aut}(A)(P))$ classifies central simple algebras of degree 8 over L which are split by P by [KMRT, Sect. 29.B] or [Ser94, III.1.3], and as is standard we can identify $H^2(G, P^*)$ with $\operatorname{Br}(P/L)$, the subgroup of $\operatorname{Br} L$

consisting of those central simple algebras which are split by P; cf. [Ser79, X.4, Corollary]. Moreover, the connecting homomorphism ∂ is just the map which sends a given central simple algebra to its Brauer class.

Since our descent is given by $\mathrm{Int}(t_0) \otimes \zeta^3$ and $\mathrm{Int}(\pi) \otimes \iota$, the isomorphism class of E is represented by the 1-cocycle η: $G \to \mathrm{Aut}(A)(P)$ given by (in the notation of [Ser94, I.5.1])

$$\eta_\iota = \mathrm{Int}(\pi), \qquad \eta_{\zeta^3} = \mathrm{Int}(t_0),$$

and

$$\eta_{\iota\zeta^3} = \eta_\iota {}^\iota\eta_{\zeta^3} = \mathrm{Int}\big(\pi\iota(t_0)\big) = \mathrm{Int}(t_0\pi).$$

Since the map $GL_1(A) \to \mathrm{Aut}(A)$ is given by $a \mapsto \mathrm{Int}(a)$, there is a set map γ: $G \to GL_1(A)(P)$ which "lifts" η and is given by

$$\gamma_\iota = \pi, \qquad \gamma_{\zeta^3} = t_0, \qquad \text{and} \qquad \gamma_{\iota\zeta^3} = t_0\pi.$$

Then the Brauer class of E corresponds to the image of η under the connecting homomorphism ∂, which by [Ser94, I.5.6] is the cohomology class of the 2-cocycle f given by

$$f_{s,t} := \gamma_s {}^s\gamma_t \gamma_{s,t}^{-1}.$$

Fix some square root of b and define ψ_b: $G \to \{0, 1\}$ by

$${}^s\sqrt{b} = (-1)^{\psi_b(s)}\sqrt{b}.$$

Since P is only a biquadratic extension of L, one can check all the possible $s, t \in G$ to see that η maps to the 2-cocycle f given by $f_{s,t} = a^{\psi_b(s)\psi_b(t)}$. (To see this, one makes use of the fact that $t_0^2 = \mu(t_0) = a$.) Then by [Spr59, pp. 250–251] or [GTW97, Lemma 3.5(2)], the Brauer class corresponding to f is $[a, b/L]$. Alternately, one can observe that for $H := \mathscr{G}(L(\sqrt{b})/L) = \{1, \zeta^3\}$, f is the inflation from H to G of the 2-cocycle g given by $g_{\zeta^3, \zeta^3} = a$. Since g corresponds to the class of the cyclic algebra $(a, L(\sqrt{b})/L, \zeta^3) \cong (a, b/L)$ in $\mathrm{Br}(L(\sqrt{b})/L)$ and the inflation map on cohomology corresponds to the inclusion $\mathrm{Br}(L(\sqrt{b})/L) \subseteq \mathrm{Br}(P/L)$ [Dra83, p. 97, Theorem 1], we see again that $[E] = [a, b/L]$. ∎

Our first application of this proposition will be to show that the Steinberg groups are quasi-split. Our proof will require a good criterion for being quasi-split over F in terms of the rank of a maximal F-split torus. The rank of such a torus in a group G over F is called the F-rank, and we denote it by $\mathrm{rank}_F\, G$ (cf. [Bor91, 21.1]).

Recall that if G is a semisimple group defined over F with a maximal F-split torus S lying in some maximal torus T defined over F, and Δ is a set of simple roots (or Dynkin diagram) for G relative to T, Tits [Tit66, 2.1] denotes by Δ_0 the subset of Δ consisting of roots which vanish on S. Moreover, every element of a given Γ-orbit of Δ has the same restriction to S [Tit66, 2.5.1] and Tits circles those orbits of Δ which do not belong to Δ_0.

LEMMA 4.2. *The notation is as in the preceding paragraph.*

(1) *The number of orbits circled in the Dynkin diagram of G is the F-rank of G* $(= \dim S)$.

(2) *All of the orbits in the Dynkin diagram of G are circled* (*i.e.*, Δ_0 *is empty*) *if and only if G is quasi-split.*

(3) *If G^q is the* (*unique*) *quasi-split group which is an inner form of G, then* $\operatorname{rank}_F G \leq \operatorname{rank}_F G^q$, *with equality if and only if G is quasi-split.*

Proof. (1) By [Tit66, 2.5.1] the set of simple roots of G with respect to S is in bijection with the set of Γ-orbits in $\Delta \setminus \Delta_0$, i.e., the circled orbits of Δ. The size of this set is the rank of the system of roots of G with respect to S, and by [BT65, p. 96, 5.3] that is the F-rank of G.

(2) This is [KR94, 1.9(iii)]. Or, the uncircled vertices of the Dynkin diagram of G form the Dynkin diagram for the semisimple anisotropic kernel of G. So all the vertices are circled iff G has trivial semisimple anisotropic kernel iff G is quasi-split [Tit66, 2.2].

(3) This is a straightforward consequence of (1) and (2) since the Γ-action on the Dynkin diagrams of G and G^q is the same by [BT87, 1.3] or [MPW96, p. 531, Prop. 1.10]. ∎

APPLICATION 4.3. *The trialitarian Steinberg groups are quasi-split, and so have trivial Allen invariant.*

Demonstration. The Steinberg groups are defined to be exactly those gotten by taking $\underline{t} = (1,1,1)$ in the statement of the Descent Proposition; see [Ste59, p. 887, Sects. 10 and 11] or [Jac64b, p. 140]. So we set $a = 1$ and pick some $b \in F^*$ such that $b \not\equiv \operatorname{disc}_F L$ or 1 mod F^{*2}. If there is no such b, we set $\Delta := F(\sqrt{\operatorname{disc}_F L})$, ρ to be a generator of $\mathscr{G}(L^c/\Delta)$, and ι a generator of $\mathscr{G}(\Delta/F)$ (possibly trivial). Then our descent is given by $\langle \operatorname{Int}(\underline{\rho})\rangle$ or $\langle \operatorname{Int}(\underline{\rho}), \operatorname{Int}(\underline{\pi})\rangle$ depending on whether L is of type 3 or 6 where we define

$$\underline{\rho}(x_0, x_1, x_2) := (\rho x_1, \rho x_2, \rho x_0).$$

One sees quickly that this provides a descent of the split triality over L^c to a triality over F which is essentially the same as the descent given in the proposition, and such that the descended triality corresponds to a Steinberg group.

That the Allen invariant of a Steinberg group is trivial will follow from the fact that it is quasi-split by Example 2.3.

In any case, let G be a Steinberg group. If one looks at the triple each of whose entries looks like

$$\operatorname{diag}\left(a, b, \frac{a}{b}, 1, 1, \frac{b}{a}, \frac{1}{b}, \frac{1}{a}\right) \qquad \text{for } a, b \in F^*,$$

then by Example 1.6 we see that it is a related triple. The set of such triples forms a rank 2 split torus in G which is also defined and split over F by the proposition. Since the Dynkin diagram of G has two orbits under the Galois action, the preceding lemma shows that G is quasi-split. However, we promised in the Introduction to produce the Borel subgroup explicitly if L is Galois over F (i.e., G is of type ${}^3D_4$).

We define a maximal torus T to be the connected component of the group

$$\left\{(t_0, t_1, t_2) \in \operatorname{Spin}(\mathfrak{C}, \mathfrak{n}) \mid t_0 \text{ is diagonal}\right\}.$$

Now $\varphi(T)$ (recall that φ is the surjection $\operatorname{Spin}(\mathfrak{C}, \mathfrak{n}) \to O^+(\mathfrak{C}, \mathfrak{n})$ which projects $\underline{t}$ onto the first factor) is a maximal torus in $O^+(\mathfrak{C}, \mathfrak{n})$. Its character group is generated by χ_1, χ_2, χ_3, and χ_4 where χ_i gives the (i,i)-entry of the matrix in $\varphi(T)$. There is a commonly made choice of simple roots relative to this torus, namely,

$$\alpha_1 := \chi_1 - \chi_2, \qquad \alpha_2 := \chi_2 - \chi_3, \qquad \alpha_3 := \chi_3 - \chi_4, \qquad \alpha_4 := \chi_3 + \chi_4.$$

The roots α_i lift via φ to characters of T, which we will also denote by α_i for $1 \le i \le 4$.

Since T is a maximal torus of $\operatorname{Spin}(\mathbb{C}, \mathfrak{n})$, we have uniquely determined connected one-dimensional unipotent subgroups which are the images of homomorphisms

$$U_{\alpha_i} \colon \mathbb{G}_a \to \operatorname{Spin}(\mathfrak{C}, \mathfrak{n})$$

such that $tU_{\alpha_i}(r)t^{-1} = U_{\alpha_i}(\alpha_i(t)r)$ for $1 \le i \le 4$ and all $t \in T$ [Bor91, 13.18]. Certainly such a root group would have to map under φ to a corresponding root group in $O^+(\mathfrak{C}, \mathfrak{n})$. Since we have chosen a set of simple roots with respect to our maximal torus in $O^+(\mathfrak{C}, \mathfrak{n})$, the root groups there are fully determined. Thus we may conclude that

$$\varphi\left(\operatorname{im} U_{\alpha_1}\right) = \operatorname{im} U_{12}, \qquad \varphi\left(\operatorname{im} U_{\alpha_2}\right) = \operatorname{im} U_{23},$$

$$\varphi\left(\operatorname{im} U_{\alpha_3}\right) = \operatorname{im} U_{34}, \qquad \varphi\left(\operatorname{im} U_{\alpha_4}\right) = \operatorname{im} U_{35}.$$

What are the images of U_{α_i}'s? We determine $\operatorname{im} U_{\alpha_2}$. Since $(U_{23}(r), U_{23}(r), U_{23}(r))$ is a related triple for all $r \in L$ by Example 1.9, and such triples form a connected, one-dimensional, unipotent subgroup of $\operatorname{Spin}(\mathfrak{C}, \mathfrak{n})$, we must have

$$\operatorname{im} U_{\alpha_2}(L) = \{(U_{23}(r), U_{23}(r), U_{23}(r)) \mid r \in L\}.$$

Similarly, one sees that

$$\operatorname{im} U_{\alpha_1}(L) = \{(U_{12}(r), U_{35}(-r), U_{34}(r)) \mid r \in L\},$$

$$\operatorname{im} U_{\alpha_3}(L) = \{(U_{34}(r), U_{12}(r), U_{35}(-r)) \mid r \in L\},$$

$$\operatorname{im} U_{\alpha_4}(L) = \{(U_{35}(-r), U_{35}(r), U_{12}(r)) \mid r \in L\}.$$

The choice of the α_i's as our set of simple roots corresponds to a choice of Borel subgroup B containing T, which is the subgroup generated by T and the $\operatorname{im} U_{\alpha_i}$'s by [Tit66, 1.6] or [Bor91, 14.18]. Again, using the corresponding facts about $O^+(\mathfrak{C}, \mathfrak{n})$, we see that $\varphi(B)$ is precisely the upper triangular matrices in $O^+(\mathfrak{C}, \mathfrak{n})$. In fact, B is the inverse image of the upper triangular matrices in $O^+(\mathfrak{C}, \mathfrak{n})$ by [Bor91, 22.6(i)], since φ is a central L-isogeny.

The description of the $\mathscr{A}_3$-action on T and on our root groups makes it clear that if (t_0, t_1, t_2) is a related triple with to upper triangular, then t_1 and t_2 are upper triangular as well. Since (t_1, t_2, t_0) and (t_2, t_0, t_1) are also related triples by Proposition 1.5(1), we see that a related triple (t_0, t_1, t_2) consists of upper triangular matrices if and only if one of the t_i is upper triangular. Consequently,

$$B(L) = \{(t_0, t_1, t_2) \in \operatorname{Spin}(\mathfrak{C}, \mathfrak{n})(L) \mid t_i \text{ is upper triangular for } i = 0, 1, 2\},$$

and we see that B is defined over F as desired. ∎

COROLLARY 4.4. *If L is a separable cubic field extension of F lying in J^d, the split 27-dimensional exceptional Jordan algebra over F, then* $\operatorname{Aut}(J^d/L)$, *the group of Jordan algebra automorphisms of J^d which fix L, is quasi-split of type D_4 with inner extension L^c.*

Proof. As stated in [Sod66, p. 150], the group described is exactly the Steinberg group of type D_4 with inner extension L^c, so the preceding application gives us the claim. ∎

This result is to be expected since one knows by [All67, p. 261, Corollary] that the group $\operatorname{Aut}(J^d/L)$ is determined up to F-isomorphism by the

F-isomorphism class of L and also that $\mathrm{Aut}(J^d)$, the group of Jordan algebra automorphisms of J^d, is itself split of type F_4. However, it does not seem to be in the literature anywhere.

It should also be pointed out that by [Jac60, p. 81] the special Jordan algebra defined by $M_3(F)$ embeds in J^d, so every cubic étale F algebra embeds in J^d, and thus every quasi-split group of type D_4 is of the form $\mathrm{Aut}(J^d/L)$.

Remark 4.5. We saw that the maximal torus T in Application 4.3 is defined over F, but it is clearly not F-split (it is of rank 4 and the maximal F-split torus has rank 2). The maximal anisotropic subtorus T_a of T [Bor91, 8.15] is $R_{L/F}(L^*)/F^*$, where $R_{L/F}(G)$ denotes the Weil restriction of scalars (a.k.a. the transfer) of the group G defined over L to a group defined over F.

APPLICATION 4.6. *If Q is a quaternion algebra over L a separable cubic field extension of F such that Q has trivial corestriction down to F, there is a group of type D_4 over F with Allen invariant $M_4(Q)$.*

Demonstration. This is the original descent from Allen and Ferrar [AF68] and [KMRT, 43.11] combined with Lemma 3.1. We recapitulate it here for the reader's convenience. (Note that Allen and Ferrar did not address the type 6 case, and our descent is slightly different from the one in [KMRT] in that case.)

If Q is split, then we can take the group to be the quasi-split group with inner extension L^c. Otherwise, use Lemma 3.1 to get an $a \in L^*$ and $b \in F^*$ such that $Q \cong_L (a, b/L)$ and $N_{L/F}(a) = 1$. Since Q is nonsplit, $b \not\equiv 1 \bmod F^*$ and Lemma 3.2 assures us that $b \not\equiv \mathrm{disc}_F L \bmod F^*$. Define

$$m_i(a) := \mathrm{diag}\big(1, \rho^i(a), \rho^i(a), \rho^{i+2}(a)^{-1}, \rho^{i+1}(a)^{-1}, 1, 1, \rho^i(a)\big).$$

(We will use this again in the next application.) If we set $t_i = m_i(a)S$ for S as in (1.3), then by Examples 1.6 and 1.7 and Proposition 1.5(3), $\underline{t} = (t_0, t_1, t_2)$ is a related triple. Straightforward calculations show that it satisfies the criteria of the Descent Proposition. ∎

This choice of t produces anisotropic groups as well as isotropic ones (see [AF68, pp. 482–483] for details), which makes it unsuitable for our purposes. Our goals are met by the next application.

APPLICATION 4.7 (Existence). *If Q is a quaternion algebra over L a separable cubic field extension of F such that Q has trivial corestriction down to F, then there is an isotropic group of type D_4 over F with Allen invariant $M_4(Q)$.*

Demonstration. As in the preceding application, we easily reduce to the case where we have an a and b satisfying the hypotheses of the Descent Proposition and $Q \cong_L (a, b/L)$.

Set

$$d = \operatorname{diag}(1, 1, -1, 1, 1, -1, 1, 1)$$

and

$$t_i = m_i(a)\, d P((1\ 2)(3\ 6)(4\ 5)(7\ 8)),$$

where $m_i(a)$ is as defined in the preceding application. Then $\underline{t} = (t_0, t_1, t_2)$ is a related triple by Examples 1.6 and 1.8 and Proposition 1.5(3). Once one observes that the permutation matrix is fixed by σ_{n} and commutes with d and that d commutes with $m_i(a)$, straightforward calculations show that $\underline{t}$ satisfies the hypotheses of the Descent Proposition. Furthermore, the descent given by the proposition fixes elementwise the split torus consisting of related triples (s, s, s) for s of the form

$$\operatorname{diag}(a, a, 1, 1, 1, 1, a^{-1}, a^{-1}) \qquad \text{with } a \in F^*.$$

Thus the descended group is isotropic. ∎

Remark 4.8. In the preceding descent, the torus T consisting of triples (t_0, t_1, t_2) such that every t_i is diagonal is defined over F and contains the rank 1 split torus which we produced. As in Remark 4.5, we point out that the maximal anisotropic subtorus T_a of the maximal torus T is precisely

$$R_{L/F}\left(L(\sqrt{b})^* / L^*\right)$$

for b as in the application.

5. THE UNIQUENESS PART

In this section we will prove the uniqueness part of the Main Theorem, i.e., that isotropic trialitarian groups are classified up to central F-isogeny by their Allen invariants.

One of the reasons this works is by a peculiar phenomenon involving central simple algebras of degree 4 with orthogonal involution, which is that if (A, σ) is such an algebra with $C_0(A, \sigma) \cong Q_1 \times Q_2$ for Q_i a quaternion algebra with unique symplectic involution γ_i for ii $= 1, 2$, then $(A, \sigma) \cong (Q_1, \gamma_1) \otimes (Q_2, \gamma_2)$ [KPS91, Theorem 5.2]. A version which includes the case where the center of $C_0(A, \sigma)$ is a field can be found in [KMRT].

LEMMA 5.1 [KMRT, 15.7]. *Suppose that* (A, σ) *and* (A', σ') *are two central simple algebras of degree* 4 *with orthogonal involutions such that* $C_0(A, \sigma)$ *and* $C_0(A', \sigma')$ *are isomorphic as* F*-algebras. Then* $(A, \sigma) \cong (A', \sigma')$.

We must necessarily work with central simple algebras of degree 8, so we need a way to get algebras of smaller degree.

Suppose that D is a skew field over F with orthogonal involution "$-$". A *hermitian form* h over $(D, -)$ is a biadditive map $h\colon V \times V \to D$ for some right D-vector space V such that

$$h(xd_1, yd_2) = \overline{d_1} h(x, y) d_2 \text{ for all } d_1, d_2 \in D,\ x, y \in V,$$

$$\text{and } \overline{h(x, y)} = h(y, x).$$

We require also that h is *nondegenerate*, i.e., that there is no nonzero $x \in V$ such that $h(x, y) = 0$ for all $y \in V$. Then h induces its adjoint involution σ_h on $\operatorname{End}_D(V)$ defined by

$$h(fx, y) = h(x, \sigma_h(f)y) \qquad \text{for all } f \in \operatorname{End}_D(V) \text{ and } x, y \in V.$$

We say that h is *associated* to (A, σ) if $(A, \sigma) \cong (\operatorname{End}_D(V), \sigma_h)$; given (A, σ) such an h always exists by [Sch85, p. 302].

If h is isotropic, i.e., $h \cong h' \perp \mathscr{H}$ for some hermitian form h' over $(D, -)$ and $\mathscr{H}$ the hyperbolic form on $(D, -)$ (cf. [Sch85, Sect. 7.7]), then we say that h' is obtained from h by splitting off a hyperbolic plane. By Witt cancellation [Sch85, 7.9.2], h' is determined up to isometry. Since $\mathscr{H} \cong \langle \lambda \rangle \mathscr{H}$ for any $\lambda \in F^*$, if h is associated to (A, σ) (in which case (A, σ) is isotropic) we get a well-determined central simple algebra with involution (B, τ) such that h' is associated to (B, τ). We say that one gets (B, τ) from (A, σ) by splitting off a hyperbolic plane. If (A, σ) is any central simple algebra with isotropic orthogonal involution, then one can always apply this construction to get an algebra of smaller degree with orthogonal involution.

DEFINITION 5.2. If (A, σ) and (B, τ) are central simple algebras with orthogonal involution such that one gets (B, τ) from (A, σ) by splitting off one or more hyperbolic planes, then we say that (A, σ) is a *hyperbolic extension* of (B, τ).

Certainly in that case A and B will be Brauer-equivalent. Better yet, we can say something about their even Clifford algebras.

LEMMA 5.3. *If* (A, σ) *is a hyperbolic extension of* (B, τ)*, then* $C_0(A, \sigma)$ *is* F*-algebra isomorphic to some size matrices over* $C_0(B, \tau)$.

Proof. Note that the centers of $C_0(A, \sigma)$ and $C_0(B, \tau)$ are F-isomorphic.

One knows that the conclusion holds if σ (and hence τ) has trivial discriminant (i.e., $C_0(A, \sigma)$ has center isomorphic to $F \times F$) by [MPW96, pp. 584–585, Lemmas 5.8 and 5.9] if A is nonsplit and by [Lam73, p. 121, (3.13)] otherwise. So we may suppose that both of the even Clifford algebras have center Z a quadratic field extension of F.

Since C_0 is compatible with scalar extension [Tit68, p. 32, Corollary 2]

$$C_0((A, \sigma) \otimes_F Z) \cong_Z C_0(A, \sigma) \otimes_F Z \cong_F C_0(A, \sigma) \times {}^{\iota}C_0(A, \sigma),$$

where ι is the nontrivial F-automorphism of Z. A similar formula holds for (B, τ). Thus by the trivial discriminant case $C_0(A, \sigma)$ is Z-isomorphic to some size matrices over $C_0(B, \tau)$ or ${}^{\iota}C_0(B, \tau) \cong_F C_0(B, \tau)$. Hence the conclusion. ∎

LEMMA 5.4. *Two central simple algebras over F of degree 8 with orthogonal involutions of Witt index ≥ 2 are isomorphic (as F-algebras with involution) if and only their even Clifford algebras are isomorphic (as F-algebras).*

Proof. Clearly we need only show "if."

Since the involutions in question have Witt index at least 2, there are two central simple algebras of degree 4 with orthogonal involutions (A, σ) and (A', σ') such that the original algebras in question are hyperbolic extensions of these. Then our hypothesis and the preceding lemma imply that $C_0(A, \sigma) \cong_F C_0(A', \sigma')$. By Lemma 5.1, $(A, \sigma) \cong_F (A', \sigma')$. ∎

What this means for us is that if we have two trialities (E, L, σ, α) and (E, L, σ', α') (so that they have the same Allen invariant E) of type 3 or 6 and with involution of Witt index at least 2, then we know that actually $(E, \sigma) \cong_L (E, \sigma')$, so that E determines σ. We will use this property to describe the involution σ explicitly.

For convenience of notation, set $\Delta := \Delta_F(L)$. If ι is the unique F-algebra automorphism of Δ and $(A, *)$ is a central simple algebra over Δ with orthogonal involution $*$, then since A and ${}^{\iota}A$ have the same underlying ring structure $*$ is also an involution on ${}^{\iota}A$. It is easy to see that this induces an involution on $\operatorname{cor}_{\Delta/F}(A)$ and we denote this algebra with involution by $\operatorname{cor}_{\Delta/F}(A, *)$.

PROPOSITION 5.5. *If (E, L, σ, α) is a triality of type 3 or 6 such that $w(E, \sigma) \geq 2$ (which occurs if the triality is isotropic), then $E \cong_L M_4(Q)$ for some (possibly split) quaternion algebra Q and (E, σ) is a hyperbolic extension of*

$$\operatorname{cor}_{(L \otimes \Delta)/L} {}^{\rho}((Q, \gamma) \otimes_F \Delta)$$

for γ the unique symplectic involution on Q and ρ an element of order 3 in $\mathcal{G}((L \otimes_F \Delta)/\Delta)$.

Proof. Certainly (E, σ) is a hyperbolic extension of $(M_2(Q), \theta)$ for some orthogonal involution θ. Since α gives us an isomorphism

$$C_0(E, \sigma) \cong_L {}^{\rho}(M_4(Q) \otimes_F \Delta),$$

Lemma 5.3 tells us that

$$C_0(M_2(Q), \theta) \cong_L {}^{\rho}(Q \otimes_F \Delta).$$

We set $(B, \tau) := \mathrm{cor}_{(L \otimes \Delta)/\Delta}\, {}^{\rho}((Q, \gamma) \otimes_F \Delta)$. We want to show that $(M_2(Q), \theta) \cong_L (B, \tau)$. We have by [KMRT, Sect. 15.B] that τ and θ have the same discriminant, meaning that the centers of $C_0(B, \tau)$ and $C_0(M_2(Q), \theta)$ can both be identified with Δ.

If L is of type 3 (i.e., $\Delta \cong_F F \times F$) and we set ρ to be induced by some nontrivial F-automorphism of L, then

$${}^{\rho}((Q, \gamma) \otimes_F \Delta) \cong_L ({}^{\rho}Q, \gamma) \times ({}^{\rho^2}Q, \gamma).$$

Thus

$$(B, \tau) \cong_L ({}^{\rho}Q, \gamma) \otimes_L ({}^{\rho^2}Q, \gamma),$$

and so by [Tao95, p. 202, Theorem 4.16] or [KMRT, 8.19],

$$C_0(B, \tau) \cong_L {}^{\rho}Q \times {}^{\rho^2}Q \cong_L C_0(M_2(Q), \theta).$$

By Lemma 5.1, $(B, \tau) \cong_L (M_2(Q), \theta)$.

If L is of type 6 (i.e., Δ is a quadratic field extension of F) then $L \otimes_F \Delta \cong_F L^c$ and we set ρ to be a nontrivial F-automorphism of L^c which restricts to the identity on Δ. Then

$$\begin{aligned} C_0(B, \tau) \times {}^{\iota}C_0(B, \tau) &\cong_{L^c} C_0(B, \tau) \otimes_F \Delta \\ &\cong_{L^c} C_0(M_2(Q), \theta) \otimes_F \Delta \qquad \text{by the type 3 case} \\ &\cong_{L^c} {}^{\rho}(Q \otimes_F \Delta) \times {}^{\rho^2}(Q \otimes_F \Delta). \end{aligned}$$

So $C_0(B, \tau) \cong_L {}^{\rho}(Q \otimes_F \Delta) \cong_L C_0(M_2(Q), \theta)$ and we are done. ∎

PROPOSITION 5.6 (Uniqueness). *Two isotropic algebraic groups of type ${}^3D_4$ or ${}^6D_4$ lie in the same central F-isogeny class if and only if their Allen invariants are the same.*

Proof. Clearly we only need to show "if".

By hypothesis, we have two isotropic groups, say G_1 and G_2, which are defined over F. We would like to show that they lie in the same central F-isogeny class. Since it is equivalent to show that their associated simply connected groups are F-isomorphic by [Tit66, 2.6.1], we proceed under the assumption that G_1 and G_2 are simply connected and show that they are isomorphic.

If G_1 and G_2 are quasi-split, then since their Allen invariants are isomorphic, they have the same inner extension and hence are inner forms of each other. But in that case we are done, as two quasi-split groups which are inner forms of each other are isomorphic by [BT87, 1.3] or [MPW96, Prop. 1.10]. Thus we can assume that at least one of our groups is not quasi-split.

Fix a maximal F-split torus S_i in G_i and denote by G_i' the derived group of $Z_{G_i}(S_i)$ (= the centralizer of S_i in G_i). Tits calls G_i' the semisimple anisotropic kernel of G_i [Tit66, 2.2].

The idea of the proof is to produce an F-isomorphism

$$\psi \colon G_1' \to G_2'$$

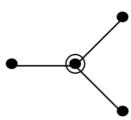
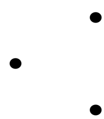
and an F_s-isomorphism

$$\phi \colon G_1 \to G_2$$

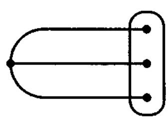
such that ϕ and ψ satisfy certain compatibility conditions. This will allow us to apply Tits' "Witt-type theorem" [Tit66, p. 43, 2.7.1] to get our conclusion.

Fix a cubic field extension L of F such that $L \cong Z(\mathscr{E}_F(G_i))$ for $i = 1, 2$. By the invariant restriction part of the Main Theorem and the fact that G_1 and G_2 have the same Allen invariant, there is some (possibly split) quaternion algebra Q over L such that $M_4(Q) \cong_F \mathscr{E}_F(G_i)$ for $i = 1, 2$.

Step 1: *We show that the semisimple anisotropic kernel G_i' of G_i is F-isomorphic to $R_{L/F}(SL_1(Q))$ for $i = 1, 2$.* For the moment suppose that G_i is not quasi-split. Then S_i has rank 1 by Lemma 4.2 since the Galois action on the Dynkin diagram of G_i has precisely two orbits. So we see that the Dynkin diagrams Δ_i and Δ_i' for G_i and G_i' respectively are

G_i   G_i'

where the Galois action on the uncircled vertices in each case is the same and transitive. (By [Tit66, p. 58], or observe that the only other possibility for the Dynkin diagram of G_i is



which cannot occur, as mentioned in [Tit66, 3.2.3] or see [Sel76, p. 45, 3.2.3] for more discussion. This can also be seen by using Jordan algebra techniques.) The Galois action tells us that G_i' is the transfer from L to F of a group of type A_1 over L [Tit66, 3.1.2].

To find out what group that is, we look at $G_i \times_F L$, where $G_i' \times_F L$ is of type $A_1^{\times 3}$. Since $G_i \times_F L \cong \mathrm{Spin}(\mathscr{I}_F(G_i))$, we will look at $O^+(\mathscr{I}_F(G_i))$ to see that one of the three components of $G_i' \times_F L$ is exactly $SL_1(Q)$. Specifically, we write $\mathscr{I}_F(G_i) \cong (M_4(Q), \sigma_i)$ and since σ_i is isotropic we can do a change of basis so that

$$\sigma_i = \mathrm{Int}\begin{pmatrix} & & 1 \\ & A & \\ -1 & & \end{pmatrix} \circ *$$

for $*$ the conjugate transpose on $M_4(Q)$ and some $A \in GL_2(Q)$. Then

$$O^+\big(M_4(Q), \sigma_i\big)(L) = \big\{m \in M_4(Q) \mid \sigma_i(m)m = \mathrm{Nrd}(m) = 1\big\}$$

and this group has a (maximal) L-split torus whose elements are of the form $\mathrm{diag}(\lambda, 1, 1, \lambda^{-1})$ for $\lambda \in L^*$. The corresponding semisimple anisotropic kernel consists of elements of the form

$$\begin{pmatrix} q & & \\ & B & \\ & & q \end{pmatrix}$$

for all $q \in SL_1(Q)$ and B in some subset of $GL_2(Q)$. Thus the semisimple anisotropic kernel of $O^+(\mathscr{I}_F(G_i))$ has one component isomorphic to $SL_1(Q)$. Since the map $\mathrm{Spin}(\mathscr{I}_F(G_i)) \to O^+(\mathscr{I}_F(G_i))$ is a central L-isogeny, it restricts to be a central L-isogeny on the semisimple anisotropic kernels by [Bor91, 22.6]. Since $SL_1(Q)$ is simply connected and some simple component of the anisotropic kernel of $\mathrm{Spin}(\mathscr{I}_F(G_i))$ maps onto it, that component of $\mathrm{Spin}(\mathscr{I}_F(G_i))$ must be isomorphic to $SL_1(Q)$. Thus

$$G_i' \cong_F R_{L/F}\big(SL_1(Q)\big).$$

Now we know that one of our groups, say G_1, is not quasi-split. So $G_1' \cong_F R_{L/F}(SL_1(Q))$, and since G_1' is anisotropic it is not quasi-split. As the transfer of a quasi-split group is quasi-split, $SL_1(Q)$ is not quasi-split and so Q is nonsplit. Thus G_2 has nontrivial Allen invariant since $\mathscr{E}_F(G_2) \cong_F \mathscr{E}_F(G_1) \cong_F M_4(Q)$. Hence G_2 is also not quasi-split, since quasi-split groups have trivial Allen invariant as mentioned in Example 2.3. Thus $G_2' \cong_F R_{L/F}(SL_1(Q)) \cong_F G_1'$ and we set ψ to be any F-isomorphism $G_1' \to G_2'$.

(Incidentally, the preceding shows that an isotropic trialitarian group with trivial Allen invariant is quasi-split.)

Step 2: *We produce a map* ϕ. Since G_1 and G_2 are both split of type D_4 over F_s, there is certainly an isomorphism $\phi\colon G_1 \to G_2$ defined over F_s. However, this ϕ need not satisfy our compatibility conditions so we will have to modify it.

Step 3: *We modify our* ϕ *to be compatible with* ψ. The condition required for the application of Tits' theorem is that the restriction of ϕ to the index of G_1' is induced by ψ. We first sort out what this means and then modify ϕ so that it is satisfied.

Taking T_1 to be a maximal torus of G_1 defined over F and containing S_1, and taking B_1 to be a Borel subgroup of G_1 containing T_1, we can set Δ_1 to be the induced set of simple roots of G_1 with respect to T_1. Then $T_1' := T_1 \cap G_1'$ is a maximal torus of G_1' [Tit66, 2.2] and $B_1' := B_1 \cap G_1'$ is a Borel subgroup of G_1' [Bor91, 21.13(i)]. We set $T_2' := \psi(T_1')$, $T_2 := \phi(T_1)$, $B_2' := \psi(B_1')$, and $B_2 := \phi(B_1)$. By [Bor91, 21.13(ii)], we can modify ϕ by an inner automorphism of G_2 if necessary so that $T_2' = T_2 \cap G_2'$ and $B_2' = B_2 \cap G_2'$. We set Δ_2 to be the set of simple roots of G_2 with respect to T_2 given by the Borel B_2. Clearly ψ induces a uniquely determined map $\Delta_1' \xrightarrow{\sim} \Delta_2'$ and ϕ induces a uniquely determined map $\Delta_1 \xrightarrow{\sim} \Delta_2$.

Tits points out in [Tit66, 2.2] that Δ_i' can be canonically identified as the set of roots in Δ_i which vanish on S_i. Since ϕ must map the central vertex of Δ_1 to the central vertex of Δ_2, ϕ restricts to a uniquely determined map $\Delta_1' \xrightarrow{\sim} \Delta_2'$. Our compatibility condition is that the map $\phi\psi^{-1} \in \mathrm{Aut}(\Delta_2')$ is the identity.

The map $\mathrm{Aut}(G_2)(F_s) \to \mathrm{Aut}(\Delta_2)$ is surjective by [MPW96, p. 530, Lemma 1.9] since G_2 is simply connected, and the map $\mathrm{Aut}(\Delta_2) \to \mathrm{Aut}(\Delta_2')$ is clearly surjective, so there is some $\tau \in \mathrm{Aut}(G_2)(F_s)$ such that τ maps to the inverse of $\phi\psi^{-1}$ in $\mathrm{Aut}(\Delta_2')$. The map $\tau\phi$ satisfies our compatibility criterion. ∎

6. COROLLARIES

In Section 2 we mentioned that there are some fields over which trialities are classified by their involution invariants (= algebraic groups of type D_4 are classified up to central isogeny by their involution invariants). For such fields our Main Theorem has a nice consequence.

COROLLARY 6.1. *If F is a field over which trialitarian groups are classified up to central F-isogeny by their involution invariants (e.g., a number field), then such a group G is isotropic if and only if* $w(\mathscr{I}_F(G)) \geq 2$.

Proof. The $\Rightarrow$ direction is Lemma 2.4. For the other direction, we know by the existence part of the Main Theorem there is an isotropic triality (E, L, σ, α) for $E = \mathscr{E}_F(G)$. By Lemma 5.4 we see that $(E, \sigma) \cong_F \mathscr{I}_F(G)$, so by hypothesis the trialities are isomorphic. ∎

If trialitarian groups are classified by their Allen invariants the result is even stronger.

COROLLARY 6.2. *If F is a field over which trialitarian groups are classified up to central F-isogeny by their Allen invariants, then such a group G is isotropic if and only if* $\operatorname{ind} \mathscr{E}_F(G) \leq 2$.

Proof. Since $\operatorname{ind} \mathscr{E}_F(G) \leq 2$, the existence part of the Main Theorem says that there is another group with the same Allen invariant which is isotropic. ∎

COROLLARY 6.3. *Over any finite extension of* $\mathbb{F}_p(t)$ *for p an odd prime* (*i.e., a global field of odd characteristic*) *or a totally imaginary number field, all trialitarian groups are isotropic.*

Proof. Let (E, L, σ, α) be a triality of type 3 or 6. Since E supports an orthogonal involution it has exponent 1 or 2 in the Brauer group [Dra83, p. 114, Theorem 1], and since exponent = index over any global field [Rei75, 32.19], $\operatorname{ind} E \leq 2$. As mentioned in Section 2, trialities over our field are classified by their Allen invariants, so we apply the preceding corollary. ∎

For number fields this last corollary is also a consequence of [All92, Corollary 9.4]. I do not know if this result is known for global fields of odd characteristic.

ACKNOWLEDGMENT

The material in this paper is part of my Ph.D. thesis at the University of California, San Diego, and I thank my advisor, Prof. Adrian Wadsworth, for his guidance and for suggesting the trialitarian groups as an object of study.

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