Young-Diagrammatic Methods for the Representation Theory of the Classical Groups of Type $B_n$, $C_n$, $D_n$

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INTRODUCTION

Since the appearance of the celebrated article "The Classical Groups" of Weyl [7], numerous influences have occurred in the representation theory of the classical groups. Especially it has become folklore that the facts in this field should be expressed only in terms of Young diagrams. The aim of this paper is one of the realizations of this principle.

A classical model of our approach is the results for $GL(n)$ which are known as Littlewood-Richardson coefficients and the Kostka coefficients. These two coefficients can be stated in the combinatorial manner using only Young diagrams independently of the rank $n$ of $GL(n)$. We generalize these results to the other classical groups over $\mathbb{C}$. Namely we give explicit formulas to decompose the tensor product of two irreducible representations into irreducible constituents. Also we have obtained an interesting relationship linking the irreducible characters of $Sp$ and $SO$, which we call "duality relation" between the representations of $Sp$ and $SO$.

As for explicit formulas for the multiplicities of weights in the form of polynomials in the rank $n$, we refer to [3].

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Our strategy is the following. It is known (see Sect. 1) that the irreducible representations of $SO(2n+1)$, $Sp(2n)$ ($\subset GL(2n)$) and $SO(2n)$ are, roughly speaking, parametrized concretely by the Young diagrams of depths at most $n$. In case of $GL(n)$, I. G. Macdonald defined the "universal character" $\chi_{GL}(\lambda)$ of $GL$ (for each Young diagram $\lambda$) in the ring $A$ of symmetric functions in countably many variables (we call this $A$ the Universal Character Ring). Motivated by these $\chi_{GL}(\lambda)$, we introduce the universal characters $\chi_{O}(\lambda)$, $\chi_{Sp}(\lambda)$ of the groups $SO$ and $Sp$ in the same ring $A$. Each of these $\{\chi_{GL}(\lambda)\}$, $\{\chi_{O}(\lambda)\}$, $\{\chi_{Sp}(\lambda)\}$ becomes a $\mathbb{Z}$-linear basis of $A$ and the transformation rules between these bases are given by the Character Interrelation Theorem (see Theorem 2.3.1). An interesting fact we have noticed in this paper is the existence of the ring homomorphisms $\pi_{SO(n)}$ and $\pi_{Sp(2n)}$ from $A$ to the character ring $R(SO(n))$ of $SO(n)$ and to the character ring $R(2n)$ of $Sp(2n)$ such that $\pi_{SO(n)}(\chi_{O}(\lambda))$ and $\pi_{Sp(2n)}(\chi_{Sp}(\lambda))$ are either zero or $\pm \{\text{irreducible character}\}$ and that these images can be calculated easily in a concrete manner using only Young diagrams (see Sect. 2.4). We call these homomorphisms the specialization homomorphisms.

Under these formulations we show the following.

(I) Duality between $Sp$ and $SO$. We prove that the well-known involutive ring automorphism $\omega$ of $A$ defined by $\omega(\chi_{GL}(\lambda)) = \chi_{GL}(\lambda')$ (see Sect. 1.4) transforms the $\chi_{O}(\lambda)$'s onto the $\chi_{Sp}(\lambda)$'s simultaneously, transposing Young diagrams, i.e., $\omega(\chi_{O}(\lambda)) = \chi_{Sp}(\lambda')$. This is what we call the duality between $Sp$ and $SO$. Since $\omega$ is an algebra automorphism, the decomposition rules of products of universal characters of $SO$ and $Sp$ totally coincide via the transpose of Young diagrams.

(II) Symmetry with respect to transpose. We prove that the involutive $\mathbb{Z}$-module homomorphism $i_{Sp}$ (resp. $i_{O}$) of $A$ defined by $i_{Sp}(\chi_{Sp}(\lambda)) = \chi_{Sp}(\lambda')$ (resp. $i_{O}(\chi_{O}(\lambda)) = \chi_{O}(\lambda')$) is an algebra automorphism. (See the last part of Sect. 2.1 and Theorem 2.3.2.) $i_{Sp}$ (resp. $i_{O}$) plays the same role for $\{\chi_{Sp}(\lambda)\}$ (resp. $\{\chi_{O}(\lambda)\}$) as $\omega$ does for $\{\chi_{GL}(\lambda)\}$.

(III) Expression in terms of fundamental characters. We give a formula expressing $\chi_{Sp(2n)}(\lambda)$ (resp. $\chi_{SO(n)}(\lambda)$) as a determinant of the characters of the fundamental representations of $Sp(2n)$ (resp. $SO(n)$) (see Theorem 2.3.3, Corollary 2.4.2).

(IV) Branching rules. If we apply the specialization homomorphism to the Character Interrelation Theorem, we have the "branching rule" describing how an irreducible representation of the general linear group decomposes into irreducible constituents when it is restricted to the symplectic group (resp. the orthogonal group) under the natural embedding (see Proposition 2.5.1).
(V) The generalization of the Littlewood–Richardson rules to $B_n$, $C_n$, $D_n$-case. We can decompose the products $\chi_{Sp}(\lambda) \cdot \chi_{Sp}(\mu)$ (resp. $\chi_O(\lambda) \cdot \chi_O(\mu)$) into sums of the $\chi_{Sp}(v)$'s (resp. the $\chi_O(v)$'s) if we combine the Character Interrelation Theorem with the Littlewood–Richardson rules. Thus applying the specialization homomorphism to the resulting formula, we get explicitly the decomposition of the tensor product of two given irreducible representations of each classical group into irreducible constituents (cf. Sect. 2.5 for explicit formulas.)

(VI) If $n$ is greater than depth($\lambda$) + depth($\mu$), then the decomposition rules of the product $\chi_{G_n}(\lambda) \chi_{G_n}(\mu)$ coincide completely for $G_n = Sp(2n)$, $SO(2n + 1)$, $SO(2n)$ under the parametrization of Section 1. Also they do not depend on the rank $n$ (see Corollary 2.5.3).

Finally let us mention the relation between this paper and the work of D. E. Littlewood and R. C. King. Littlewood had shown the key lemma of this paper as we state it in Section 1.5 of this paper (see [5]), and for classical groups of sufficiently large ranks, he had illustrated with examples the algorithm to calculate the decompositions of tensor products. Reading our first version of this paper, R. P. Stanley suggested to us the existence of King's papers [1, 2]. King has presented algorithms similar to ours concerning the specialization homomorphisms which are stated in Section 2.4, and he also has given similar results to ours. However it seems that these papers of King have several points to be clarified in the formulations and proofs. This is an additional motivation for us to write this paper in the present style.

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1. REVIEW OF THE REPRESENTATION THEORY OF THE CLASSICAL GROUPS AND CRUCIAL REMARKS

1.1. Preliminaries

Series of the Classical Groups

There are four series of the classical groups, that is,

Type $A_{n-1}$: \[ SL(n, \mathbb{C}) \ (\subset GL(n, \mathbb{C})) \],
Type $B_n$: \[ SO(2n + 1, \mathbb{C}) = \{ X \in SL(2n + 1, \mathbb{C}) ; XJ_O^tX - J_O \} \],
Type $C_n$: \[ Sp(2n, \mathbb{C}) = \{ X \in SL(2n, \mathbb{C}) ; XJ_{Sp}^tX = J_{Sp} \} \],
Type $D_n$: \[ SO(2n, \mathbb{C}) = \{ X \in SL(2n, \mathbb{C}) ; XJ_O^tX = J_O \} \],
where $J_O \in M(n, \mathbb{C})$ and $J_{sp} \in M(2n, \mathbb{C})$ are given by the matrices

$$J_O = \begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
1 & 0
\end{pmatrix}, \quad J_{sp} = \begin{pmatrix}
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
-1 & \cdots & 0
\end{pmatrix}$$

The Lie algebra $\mathfrak{g}$ of each classical group is given by

- Type $A_{n-1}$: $\mathfrak{sl}(n, \mathbb{C}) = \{X \in M(n, \mathbb{C}); \text{Tr} X = 0\}$,

- Type $B_n$: $\mathfrak{so}(2n+1, \mathbb{C}) = \{X \in \mathfrak{sl}(2n+1, \mathbb{C}); XJ_O + J_O^T X = 0\}$,

- Type $C_n$: $\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}); XJ_{sp} + J_{sp}^T X = 0\}$,

- Type $D_n$: $\mathfrak{so}(2n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}); XJ_O + J_O^T X = 0\}$.

In either case we can take as a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ the set of all diagonal matrices in $\mathfrak{g}$. Let $d_m$ be the linear subspace of $M(m, \mathbb{C})$ consisting of all diagonal matrices, and let $\varepsilon_i$ be the linear map $\varepsilon_i: d_m \to \mathbb{C}$, $\varepsilon_i(H) = h_i$, where $H$ is a diagonal matrix whose $(i, i)$-component is $h_i$. Then the root system $\Delta$ (subset of $\Delta^*$) of each Lie algebra is given by

- Type $A_{n-1}$: $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i < j \leq n}$,

- Type $B_n$: $\Delta = \{\pm (\varepsilon_i, \varepsilon_j)\}_{1 \leq i < j \leq n} \cup \{\pm \varepsilon_i\}_{1 \leq i \leq n}$,

- Type $C_n$: $\Delta = \{\pm (\varepsilon_i, \varepsilon_j)\}_{1 \leq i < j \leq n} \cup \{\pm 2\varepsilon_i\}_{1 \leq i \leq n}$,

- Type $D_n$: $\Delta = \{\pm (\varepsilon_i, \varepsilon_j)\}_{1 \leq i < j \leq n}$,

where for each Lie algebra $\mathfrak{g}$, the above $\varepsilon_i$ is considered as the linear map restricted to $\mathfrak{h}$, i.e., $\varepsilon_i: \mathfrak{h} \to \mathbb{C}$. Let $\mathfrak{h}_\mathbb{R}^*$ be the real part of $\mathfrak{h}^*$ and let $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ (subset of $\Delta$) be a simple root system of $\Delta$, then $\Pi$ is a base of $\mathfrak{h}_\mathbb{R}^*$. For example we can take $\Pi$ as follows:

- Type $A_{n-1}$: $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, ..., \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\}$,

- Type $B_n$: $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, ..., \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n\}$,

- Type $C_n$: $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, ..., \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n\}$,

- Type $D_n$: $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, ..., \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$. 
Then for each classical group the weight lattice \( P \) and the set of dominant integral weights \( P_+ \) are given by

Type \( A_{n-1} (G = GL(n, \mathbb{C})) \): \( P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2 + \cdots + \mathbb{Z} \varepsilon_n \),

\[
P_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_n \},
\]

\((G = SL(n, \mathbb{C}))\): \( P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2 + \cdots + \mathbb{Z} \varepsilon_n / \langle \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n \rangle \),

\[
P_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_n \},
\]

Type \( B_n (G = Sp(2n + 1, \mathbb{C})) \): \( P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2 + \cdots + \mathbb{Z} \varepsilon_n \),

\[
P_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_n \geq 0 \},
\]

Type \( C_n (G = SP(2n, \mathbb{C})) \): \( P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2 + \cdots + \mathbb{Z} \varepsilon_n \),

\[
P_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_n \geq 0 \},
\]

Type \( D_n (G = SO(2n, \mathbb{C})) \): \( P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2 + \cdots + \mathbb{Z} \varepsilon_n \),

\[
P_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_{n-1} \geq \lvert f_n \rvert \geq 0 \}.
\]

Now we refer to the following theorem.

**Theorem 1.1.1** (E. Cartan–H. Weyl). Let \( G \) be a connected, complex semi-simple Lie group. Then the equivalence classes of the irreducible representations of \( G \) correspond bijectively to the elements of \( P_+ \), i.e., the dominant integral weights of \( G \). The correspondence is afforded by taking the highest weight of each irreducible representation of \( G \).

Therefore for each \( \omega = f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \), let \( \chi_G(\omega) \) or \( \chi_G(f_1, f_2, \ldots, f_n) \) denote the character of the irreducible representation corresponding to \( \omega \).

In general any (finite or infinite) sequence

\[
\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots)
\]

of nonnegative integers in decreasing order \((\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots)\) and containing only finitely many nonzero terms is called a *partition*. For a partition \( \lambda \), the *depth* of \( \lambda \) (denoted by \( d(\lambda) \)) is defined to be the number of nonzero terms in \( \lambda \), that is, \( \lambda_{d(\lambda)} \neq 0 \) and \( \lambda_{d(\lambda)+1} = 0 \). Also the *size* of \( \lambda \) (denoted by \( \lvert \lambda \rvert \)) is defined to be the sum of all terms in \( \lambda \), i.e., \( \lvert \lambda \rvert = \lambda_1 + \lambda_2 + \cdots + \lambda_n + \cdots \).

From the above theorem, the irreducible representations of \( G = Sp(2n, \mathbb{C}) \) are parametrized bijectively by the partitions whose depths are less than or equal to \( n \).
In order to deal with the representations of \(SO(n, \mathbb{C})\) uniformly, whether \(n\) is even or odd, we prepare the following consideration. First we shall consider the case \(G = SO(2n, \mathbb{C})\), i.e., type \(D\). Let \(\sigma\) be the involutive outer automorphism of \(SO(2n, \mathbb{C})\), namely \(\sigma\) is induced from the automorphism of the Dynkin diagram of type \(D\) given in Fig. 1.

Let \((\rho_\omega, V_\omega)\) be the irreducible representation of \(SO(2n)\) corresponding to \(\omega \in P_+\), and let \(\chi^o_\omega(\omega)\) denote the character of the representation \(\rho_\omega \circ \sigma: G \to GL(V_\omega)\). Then for \(P, \exists \omega = f_1 e_1 + f_2 e_2 + \cdots + f_n e_n\) it holds that

\[
\chi^o_{SO(2n)}(f_1 e_1 + f_2 e_2 + \cdots + f_n e_n) = \chi_{SO(2n)}(f_1 e_1 + \cdots + f_{n-1} e_{n-1} - f_n e_n).
\]

We shall denote the character ring of \(SO(2n, \mathbb{C})\) by \(R(SO(2n))\) and define the element \(\chi_{O(2n)}(f_1, f_2, \ldots, f_n)\) by

\[
\chi_{O(2n)}(f_1, f_2, \ldots, f_n) = \begin{cases} 
\chi_{SO(2n)}(f_1, f_2, \ldots, f_n) & \text{if } f_n = 0, \\
\chi_{SO(2n)}(f_1, f_2, \ldots, f_n) + \chi^o_{SO(2n)}(f_1, f_2, \ldots, f_n) & \text{if } f_n > 0.
\end{cases}
\]

Remark. \(\chi_{O(2n)}\) is not a symbol denoting an irreducible representation of \(O(2n)\).

As is well known (see Weyl [7]) \(\chi_{O(2n)}(f_1, f_2, \ldots, f_n)\) is the character of the representation of \(SO(2n)\) which is obtained by restricting an irreducible representation of \(O(2n)\) to \(SO(2n)\). Similarly, in the case \(G = SO(2n + 1)\), i.e., type \(B\), we shall define \(\chi_{O(2n+1)}(f_1, f_2, \ldots, f_n)\) by

\[
\chi_{O(2n+1)}(f_1, f_2, \ldots, f_n) = \chi_{SO(2n+1)}(f_1, f_2, \ldots, f_n).
\]

Namely \(\chi_{O(n)}(\lambda)\) is the character of the representation which is obtained by restricting an irreducible representation of \(O(n)\) to \(SO(n)\). Let \(R(O(n))\) be the subring of \(R(SO(n))\) spanned by \(\chi_{O(n)}(\lambda)\), where \(\lambda\) runs over all the partitions whose depths are less than or equal to the rank of \(SO(n)\). In other words, \(R(O(2n)) = R(SO(2n))^\sigma\) and \(R(O(2n+1)) = R(SO(2n+1))\). We shall "parametrize" representations of \(SO(n)\) using these \(\chi_{O(n)}(\lambda)\), that is, for a fixed \(n \in \mathbb{N}\), the parametrization is given by

\[\text{Figure 1}\]
partitions whose depths are less than or equal to the rank of \( SO(n) \)

\[
\begin{align*}
\{ \text{partitions whose depths are less than or equal to the rank of } SO(n) \} & \overset{\text{inj}}{\longrightarrow} R(O(n)) \subset R(SO(n)) \\
\bigcup \lambda & \longrightarrow \chi_{O(n)}(\lambda)
\end{align*}
\]

It should be noticed that in the case of \( SO(2n+1) \) the above correspondence gives a complete parametrization of the irreducible representations of \( SO(2n+1) \), but in the case of \( SO(2n) \), for a partition \( \lambda \) of depth \( n \), \( \chi_{O(2n)}(\lambda) \) is no longer an irreducible character of \( SO(2n) \) and the above correspondence cannot give a complete parametrization of the irreducible characters of \( SO(2n) \).

Finally we shall consider the case \( G = GL(n, \mathbb{C}) \) or \( G = SL(n, \mathbb{C}) \). As for the reductive group \( G = GL(n, \mathbb{C}) \), it is well known that the irreducible representations of \( G \) are parametrized by \( P^+ \) completely (see [7, p. 201, Theorem 7.5C]). Moreover the equivalence classes of irreducible polynomial-representations of \( G \) correspond bijectively to the subset \( P^+ \) of \( P \), where \( P^+ \) is given by

\[
P^+_+ = \{ f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P_+ ; f_1 \geq f_2 \geq \cdots \geq f_n \geq 0 \}
\]

(see Theorem 1.3.1). Any polynomial-representation of \( G \) can be decomposed into a direct sum of irreducible polynomial-representations since \( G \) is a reductive group. The highest weight of the linear character "determinant" of \( GL(n, \mathbb{C}) \) is \( \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n \), hence if we denote the representation corresponding to \( \omega = f_1 \varepsilon_1 + f_2 \varepsilon_2 + \cdots + f_n \varepsilon_n \in P_+ \) by \( (\rho_\omega, V_\omega) \), then the highest weight of \( (\rho_\omega \otimes (\det)^s, V_\omega) \) is afforded by \( (f_1 + s) \varepsilon_1 + \cdots + (f_n + s) \varepsilon_n \), and the weights of \( \rho_\omega \) added to \( s(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) \) amount exactly to the weights of \( \rho_\omega \otimes (\det)^s \). Moreover the multiplicity of a weight \( \eta \) in \( (\rho_\omega, V_\omega) \) is equal to that of the weight \( \eta + s(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) \) in \( (\rho_\omega \otimes (\det)^s, V_\omega) \). Hence when we consider the representation theory of \( GL(n, \mathbb{C}) \), we have only to consider the polynomial-representation of \( GL(n, \mathbb{C}) \). The irreducible polynomial-representations of \( GL(n) \) are parametrized by the partitions whose depths are less than or equal to \( n \), that is,

\[
\begin{align*}
\{ \text{partitions whose depths are less than or equal to } n \} & \longrightarrow R_+(GL(n)) \\
\bigcup \lambda & \longrightarrow \chi_{GL(n)}(\lambda)
\end{align*}
\]

where \( R_+(GL(n)) \) is the subring of \( R(GL(n)) \) spanned by the characters of the polynomial-representations.
1.2. The Structure of the Character Rings

1.2.1. The Case of $G = GL(n, \mathbb{C})$

The characters of the rational representations of $GL(n)$ are considered as functions defined on a maximal torus

$$T = \{ \text{diag}(t_1, t_2, \ldots, t_n) \mid t_i \neq 0 \ (1 \leq i \leq n) \}.$$  

Because of the reductivity of $G$, $\bigcup_{g \in G} gT g^{-1}$ is dense in $G$, hence a character is uniquely determined by the values on $T$. Therefore we can consider $R(GL(n))$ as a subring of $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ and from now on we proceed with our argument under the above embedding $R(GL(n)) \subset \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Since the values of a character are invariant under the conjugation of the elements of $G$, a character is an invariant polynomial of $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ under the action of the Weyl group $N_G(T)/T \cong S_n$, where $S_n$ acts on $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ naturally by the permutations of the variables $t_i$. Let $p_i$ be the character of the $i$th symmetric power of the natural representation $(\rho_{e_i}, V_{e_i})$ of $GL(n)$, and let $e_i$ be the character of $i$-th alternating power of $(\rho_{e_i}, V_{e_i})$. That is, $p_i$ is the character of the representation of $GL(n)$ on the $i$th symmetric power $S_i(V_{e_i})$ of $V_{e_i}$, and $e_i$ is the character of the representation of $GL(n)$ on the $i$th alternating power $\Lambda^i V_{e_i}$ of $V_{e_i}$. The $p_i$ and $e_i$ are both irreducible characters and the generating functions for the $p_i$ and for the $e_i$ are given respectively by

$$\prod_{i=1}^n (1 + t_i x) = \sum_{i=0}^n e_i x^i,$$

$$\prod_{i=1}^n (1 - t_i x)^{-1} = \sum_{i=0}^\infty p_i x^i,$$

where $x$ is an indeterminate. Now we shall consider only the polynomial-representations. Then by using these generators of the ring of symmetric functions, one has

**Proposition 1.2.1.** $R_+(GL(n)) = \mathbb{Z}[t_1, t_2, \ldots, t_n]^{S_n} = \mathbb{Z}[p_1, p_2, \ldots, p_n]$ (polynomial ring) = $\mathbb{Z}[e_1, e_2, \ldots, e_n]$ (polynomial ring).

1.2.2. The Case of the Other Classical Groups

A maximal torus $T$ of each classical group $G$ is given by

- Type $B_n$: $SO(2n + 1) \supset T = \{ \text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \}$,
- Type $C_n$: $Sp(2n) \supset T = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \}$,
- Type $D_n$: $SO(2n) \supset T = \{ \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \}$. 
As in the case of $GL(n)$, each character ring is considered as a subring of $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}]$. Also the Weyl group $N_G(T)/T$ of each classical group is considered as a subgroup of the automorphism group of the ring $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}]$ and is denoted by $W(B_n)$, $W(C_n)$, and $W(D_n)$, respectively, according to the type of $G$. It is known that $W(B_n)$ and $W(C_n)$ are isomorphic to each other and is generated by the permutations of the variables $t_1, t_2, ..., t_n$ and the transpositions of $t_i$ and $t_i^{-1}$, when considered as a subgroup of $\text{Aut}(\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}])$. We shall denote this group by $W_n$. Its structure is

$$W_n \cong W(B_n) \cong W(C_n) \cong \mathbb{Z}_2^2 \times \mathfrak{S}_n.$$  

In the case of type $D_n$, i.e., $G = SO(2n)$, we shall study $R(O(n)) (= R(SO(n))$ instead of $R(SO(n))$. We fix an involutive outer automorphism $\sigma$ of $SO(2n)$ which is given by

$$\sigma: SO(2n) \ni x \mapsto \sigma(x) = \sigma_0 x \sigma_0^{-1} \in SO(2n),$$

where $\sigma_0$ is the element of $O(2n) = \{ X \in GL(2n); XJ_0X = J_0 \}$ given by

$$\sigma_0 = \begin{pmatrix}
1 & \cdots & 0 & & & & \\
0 & 1 & \cdots & & & & \\
\vdots & & & & & & \ddots & \\
0 & \cdots & & & & & 1
\end{pmatrix}$$

Then $\sigma$ makes the maximal torus $T$ of $SO(2n)$ stable and the action of $\sigma$ on $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}]$ is given by

$$\sigma(t_1) = t_1 \ (1 \leq i \leq n - 1), \quad \sigma(t_n) = t_n^{-1}.$$  

Hence the subgroup of $\text{Aut}(\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}])$ which is generated by $\sigma$ and $W(D_n)$ coincides with $W_n$.

**Lemma 1.2.2.** $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}]^{W_n} = \mathbb{Z}[c_1, c_2, ..., c_n]^{\mathfrak{S}_n}$, where $c_i = t_i + t_i^{-1}$ and $\mathfrak{S}_n$ acts on $\mathbb{Z}[c_1, c_2, ..., c_n]$ by the permutations of the variables $c_i$.

**Proof.** Let $\varepsilon_i$ be the element of $W_n$ whose action on $\mathbb{Z}[t_1, t_1^{-1}, ..., t_n^{-1}]$ is given by $\varepsilon_i(t_j) = t_j$ if $i \neq j$, $\varepsilon_i(t_i) = t_i^{-1}$. Then it is sufficient to prove that
\[ Z[t_1, t_1^{-1}, \ldots, t_n^{-1}] < e_1, e_2, \ldots, e_n > = Z[c_1, c_2, \ldots, c_n] \]. We shall use the induction on \( n \). If \( n = 1 \), \( \{ t_1^k + t_1^{-k}; k = 0, 1, 2, \ldots, n, \ldots \} \) is a \( Z \)-base of \( Z[t_1, t_1^{-1}] < e_1 > \) and it is clear that \( t_1^k + t_1^{-k} \) can be expressed as a polynomial in \( t_1 + t_1^{-1} \) with integral coefficients. Suppose that the claim holds for \( n - 1 \). Let \( g(t_1, t_2, \ldots, t_n) \) be an element of \( Z[t_1, t_1^{-1}, \ldots, t_n^{-1}] < e_1, \ldots, e_n > \). Since \( g \) is \( e_n \)-invariant, if we expand \( g(t_1, t_2, \ldots, t_n) \) with respect to \( t_n \), we have

\[
g(t_1, t_2, \ldots, t_n) = g_0(t_1, \ldots, t_{n-1}) + \sum_{1 \leq j \leq s} g_j(t_1, \ldots, t_{n-1})(t_n^j + t_n^{-j}).
\]

Therefore due to the induction hypothesis and the case \( n = 1 \), the claim holds for \( n \).

By Proposition 1.2.1, \( Z[c_1, c_2, \ldots, c_n] \) is a polynomial ring whose generator system is given by \( \{ c_i(c) \}^{n}_{i=1} \) or \( \{ p_i(c) \}^{n}_{i=1} \), where the \( e_i(c) \) and \( p_i(c) \) are the polynomials in \( c_i \)'s defined by the generating functions with the indeterminate \( x \),

\[
\prod_{i=1}^{n} (1 + c_i x) = \sum_{i=0}^{n} e_i(c) x^i,
\]

\[
\prod_{i=1}^{n} (1 - c_i x)^{-1} = \sum_{i=0}^{\infty} p_i(c) x^i.
\]

Let us prove now \( Z[t_1, t_1^{-1}, \ldots, t_n^{-1}]^{W_n} = R(O(2n + 1)) = R(Sp(2n)) = R(O(2n)) \). Since we know that \( Z[t_1, t_1^{-1}, \ldots, t_n^{-1}] \) includes the character rings, it is sufficient to prove that there exists a generator system of \( Z[t_1, t_1^{-1}, \ldots, t_n^{-1}]^{W_n} \) consisting of elements of each character ring. From now on, as common notations for all classical groups, we denote the character of the \( i \)th alternating power of the natural representation \( (\rho_{e_1}, V_{e_1}) \) by \( e_i \) and the character of the \( i \)th symmetric power of \( (\rho_{e_1}, V_{e_1}) \) by \( p_i \). Also for convention, we put \( p_i = e_i = 0 \) for any negative integer \( i \) and for brevity we write \( V \) for \( V_{e_1} \).

First, we shall consider the type \( B_n \). Let us state the following proposition together with the proof for the sake of convenience of the readers.

**Proposition 1.2.3.** In the case of \( G = SO(2n + 1) \) we have

1. \( e_i \) is an irreducible character and it holds that

\[
e_i = e_{2n+1-i} = \chi_{SO(2n+1)}(1') \).

2. If \( i \geq 2 \), the character \( p_i \) is not irreducible and if we put \( p_i^\circ = p_i - p_{i-2} \), then \( p_i^\circ \) is an irreducible character of \( G = SO(2n + 1) \) and \( p_i^\circ = \chi_{SO(2n+1)}(1) \).
Proof. (1) Let \( f_i = (1, 0, \ldots, 0), f_2 = (0, 1, 0, \ldots, 0), \ldots, f_n = (0, 0, \ldots, 1) \) be the natural base of \( V \) (see Sect. 1.1). Since the maximal torus consists of diagonal matrices with respect to this base, the weight vectors of \( \Lambda^i V \) are given by the form \( f_{s_1} \wedge f_{s_2} \wedge \cdots \wedge f_{s_n} \), where \( 1 \leq s_1 < \cdots < s_i \leq n \), and therefore the highest weight of \( \Lambda^i V \) is given by \( \varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n \) and \( \Lambda^i V \supset \chi_{SO(2n+1)}(i) \). (From now on we identify the representation with its character.) Applying Weyl's degree formula to \( \chi_{SO(2n+1)}(i) \), we can verify easily \( \deg(\chi_{SO(2n+1)}(i)) = \deg(\varepsilon_i) \), hence we have \( e_i = \chi_{SO(2n+1)}(i) \). In fact \( e_i = e_{2n+1-i} \) follows easily from the facts that (1) there exists a \( G \)-equivariant isomorphism \( \rho: \Lambda^i V \to \text{Hom}(A^{2n+1-i} V, \mathbb{C}) \), where for \( A^{2n+1-i} V \supset v \) and \( A^{2n+1-i} V \supset w, \rho(v)(w) \) is defined by \( v \wedge w = \rho(v)(w) \cdot f_1 \wedge f_2 \wedge \cdots \wedge f_n \), and that (2) each representation of \( G \) is self-contragredient (since the Weyl group of \( G \) contains \(-1\)).

(2) Let \( Q_{J_0} \) be the quadratic form corresponding to the symmetric matrix \( J_0 \), i.e., \( Q_{J_0} = f_1 f_{2n+1} + f_2 f_{2n+2} + \cdots + f_n^2 + 1 \). Then we have \( S_i(V) \supset Q_{J_0} S_{i-2} (V) \) and \( Q_{J_0} S_{i-2} (V) \) is \( G \)-invariant, since \( Q_{J_0} \) is also \( G \)-invariant. Since \( f_i^2 \) is the highest weight vector of \( S_i(V) \), we have \( S_i(V) \supset \chi_{SO(2n+1)}(i) \) and \( Q_{J_0} S_{i-2} (V) \supset \chi_{SO(2n+1)}(i) \). By checking the degree of the representation \( \chi_{SO(2n+1)}(i) \) using Weyl's degree formula, we have \( \deg(p_i) = \deg(\chi_{SO(2n+1)}(i)) + \deg(p_{i-2}) \), hence we have \( p_i = \chi_{SO(2n+1)}(i) \).

Remark. In other words \( p_i \) is the character of the \( i \)th homogeneous part of \( S(V)/\langle Q_{J_0} \rangle \) where \( \langle Q_{J_0} \rangle \) is the homogeneous ideal of \( S(V) \) generated by \( Q_{J_0} \).

Let us consider now the type \( D_n \) before the type \( C_n \).

**Proposition 1.2.4.** In the case of \( G = SO(2n, \mathbb{C}) \) we have

1. If \( i \neq n, e_i \) is an irreducible character and \( e_i = e_{2n-i} = \chi_{SO(2n)}(i) \).
2. \( e_n = \chi_{SO(2n)}(1^n) + \chi_{SO(2n)}(1^n) = \chi_{O(2n)}(1^n) \).
3. If \( i \geq 2 \), the character \( p_i \) is not irreducible and if we put \( p_i = p_i - p_{i-2} \), then \( p_i \) is an irreducible character of \( SO(2n, \mathbb{C}) \) and \( p_i = \chi_{SO(2n)}(i) \).

**Proof.** The proof of (1) and (3) are the same as in the type \( B_n \) and (2) is also clear.

Finally we consider the type \( C_n \).

**Proposition 1.2.5.** In the case of \( G = Sp(2n, \mathbb{C}) \) we have

1. If \( i \geq 2 \), the character \( e_i \) is not irreducible and if we put \( e_i = e_i - e_{i-2} \) (\( i = 0, 1, \ldots \)), then we have \( e_i = -e_{2n+2-i} \), especially \( e_n = 0 \). Moreover if \( 0 \leq i \leq n, e_i \) is an irreducible character and \( e_i = \chi_{Sp(2n)}(1^n) \).

2. \( p_i \) is an irreducible character and \( p_i = \chi_{Sp(2n)}(i) \).
Proof. To prove (2) we have only to check the degrees of the representations \( p_i \) and \( \chi_{\text{Sp}(2n)}(i) \) and we leave it to the reader. We shall prove (1). Let \( f_1 = \langle (1, 0, \ldots) \rangle \), \( f_2 = \langle (0, 1, 0, \ldots) \rangle \), \( f_{2n} = \langle (0, 0, \ldots, 1) \rangle \) be the natural base of \( V \) (i.e., the base which gives the matrix expression of \( \text{Sp}(2n) \) in Sect. 1.1) and let \( \rho: A^i V \to \text{Hom}(A^{2n-i} V, \mathbb{C}) \) be the \( G \)-equivariant isomorphism defined by

\[
\rho(v)(w) = \sum_{i=1}^{2n} f_1 \wedge f_2 \wedge \cdots \wedge f_{2n} = v \wedge w.
\]

Since each representation of \( G = \text{Sp}(2n) \) is self-contragredient, it follows that \( e_i = e_{2n-i} \). From the definition of \( e_k^o \) we have, for any \( k \), \( e_k = e_k^o + e_{k-2}^o + \cdots + e_{2[k/2]}^o \). Hence rewriting the relation \( e_i = e_{2n-i}^o \) \((i = 0, 1, \ldots)\) using the \( e_i^o \), we obtain \( e_i^o = e_{2n+2-i}^o \). Finally we must show that \( e_i^o = \chi_{\text{Sp}(2n)}(1) \) \((0 \leq i \leq n) \). Let \( F \) be the 2-form defined by the skew symmetric matrix \( J_{2n} \), i.e., \( F = \sum f_1 \wedge f_2 + f_3 \wedge f_4 + \cdots + f_{n-1} \wedge f_n + f_{n+1} \wedge f_{n+2} \). By definition \( F \) is invariant under the action of \( G = \text{Sp}(2n, \mathbb{C}) \). The crucial point of the proof is in the following lemma.

**Lemma.** For \( 0 \leq k \leq n-2 \), \( \tilde{F}: A^k V \to A^{k+2} V \), where \( \tilde{F}: v \in V \to v \wedge F \in A^{k+2} V \), is a \( G \)-equivariant injective homomorphism and the \( G \)-invariant complementary space of \( \text{Im}(\tilde{F}) \) in \( A^{k+2} V \) is irreducible and its character is given by \( e_{k+2}^o = \chi_{\text{Sp}(2n)}(1, 2, \ldots) \).

**Proof of the Lemma.** It is clear that \( \tilde{F} \) is \( G \)-equivariant and also if we consider the highest weight of \( A^{k+2} V \), we have \( A^{k+2} V = \chi_{\text{Sp}(2n)}(1, 2, \ldots) \) and \( \text{deg} \chi_{\text{Sp}(2n)}(1, 2, \ldots) \). According to Weyl's degree formula, we have \( \dim A^{k+2} V = \deg \chi_{\text{Sp}(2n)}(1, 2, \ldots) + \dim A^k V \). Therefore it is sufficient to show that \( \tilde{F} \) is injective. We shall use the induction on \( k \).

If \( k = 0 \), calculating the degree of \( \chi_{\text{Sp}(2n)}(1, 2) \) we have \( A^2 V = \langle F \rangle + \chi_{\text{Sp}(2n)}(1, 2) \). Assume that the claim is true until \( k - 1 \). Then by the induction hypothesis, we have

\[
A^k V = e_k^o + e_{k-2}^o \wedge F + e_{k-4}^o \wedge F \wedge F + \cdots + e_{k-2[k/2]}^o \wedge F \wedge F \wedge \cdots \wedge F,
\]

where \( e_i^o \) is regarded as a subspace of \( A^i V \). Therefore we have

\[
(A^k V) \wedge F = e_k^o \wedge F + e_{k-2}^o \wedge F \wedge F + \cdots + e_{k-2[k/2]}^o \wedge F \wedge F \wedge \cdots \wedge F.
\]

\( e_{k-2i}^o \wedge F \wedge F \wedge \cdots \wedge F \) is stable under the action of \( G = \text{Sp}(2n) \) and if \( e_{k-2i}^o \wedge F \wedge \cdots \wedge F = 0 \), then \( e_{k-2i}^o \wedge F \wedge F \wedge \cdots \wedge F \) is isomorphic to \( e_{k-2i}^o \). Since \( e_{k-2i}^o \) is irreducible. Since the highest weight vector of \( e_{k-2i}^o \) is \( f_1 \wedge f_2 \wedge \cdots \wedge f_{k-2i} \), it is sufficient to show that

\[
A^{k+2} V \ni f_1 \wedge f_2 \wedge \cdots \wedge f_{k-2i} \wedge F \wedge F \wedge \cdots \wedge F \neq 0.
\]
But this is clear, hence the linear map $\bar{F}: \Lambda^kV \to \Lambda^{k+2}V$ for $k \leq n-2$ is injective. The proof of the lemma is completed.

For $k \leq n$, $e_k^\circ = \chi_{Sp(2n)}(1^k)$ follows immediately from the lemma.

Due to the above preparations, we have

**Proposition 1.2.6.** (1) Type $B_n$ ($G = SO(2n + 1)$)

$$R(O(2n + 1)) = \mathbb{Z}[e_1, e_2, \ldots, e_n] = \mathbb{Z}[p_1^\circ, p_2^\circ, \ldots, p_n^\circ]$$

(both are polynomial rings).

(2) Type $C_n$ ($G = Sp(2n)$)

$$R(Sp(2n)) = \mathbb{Z}[e_1^\circ, e_2^\circ, \ldots, e_n^\circ] = \mathbb{Z}[p_1^\circ, p_2^\circ, \ldots, p_n^\circ]$$

(both are polynomial rings).

(3) Type $D_n$ ($G = SO(2n)$)

$$R(O(2n)) = R(SO(2n)) = \mathbb{Z}[e_1, e_2, \ldots, e_n] = \mathbb{Z}[p_1^\circ, p_2^\circ, \ldots, p_n^\circ]$$

(both are polynomial rings).

**Proof.** There are two generator systems $\{e_i(c)\}_{i=1}^n$, $\{p_i(c)\}_{i=1}^n$ of the polynomial ring $R = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]^{W_n}$, which are defined by the generating functions in the indeterminate $x$:

$$\prod_{i=1}^n (1 + c_ix) = \sum_{i=0}^n e_i(c) x^i, \quad \prod_{i=1}^n (1 - c_ix)^{-1} = \sum_{i=0}^\infty p_i(c) x^i,$$

where $c_i = t_i + t_i^{-1}$, $i = 1, 2, \ldots, n$. Then for each classical group $G$, one can easily verify:

$$e_i = e_i(c) + \text{polynomial in the } e_j(c) \text{ where } 1 \leq j \leq i - 1,$$

$$p_i = p_i(c) + \text{polynomial in the } p_j(c) \text{ where } 1 \leq j \leq i - 1.$$

This implies that $e_k(c)$ can be expressed as a polynomial in $\{e_j\}_{j=1}^k$. Hence $\{e_i\}_{i=1}^n$ is also a generator system of $R$, i.e.,

$$R(Sp(2n)) = R(O(2n + 1)) = R(O(2n)) = R.$$

For the other systems $\{p_j\}$, $\{e_i^\circ\}$, and $\{p_i^\circ\}$, the argument is the same, so we omit the proof.
1.3. The Characters of the Classical Groups

In this section we review some fundamental theorems which give the formulas expressing the characters of the classical groups as polynomials in the symmetric characters \( p_i \). We refer to [7, Chap. VII, p. 201, Theorem 7.5B; p. 203, Theorem 7.6B; p. 219, Theorem 7.8E; p. 228, Theorem 7.9A].

**Notation.** As before, \( p_i \) denotes the character of the \( i \)th symmetric power of the natural representation of each classical group and for convention we put \( p_0 = 1 \) and \( p_i = 0 \) for \( i < 0 \). For a fixed nonnegative integer \( n \) and a partition \( \lambda \) satisfying \( d(\lambda) \leq n \), we regard \( \lambda \) as an \( n \)-dimensional vector \((\lambda_1, \lambda_2, ..., \lambda_n)\) and we define \( \lambda^{*{(n)}} \) as the following vector of \( \mathbb{Z}^n \),

\[
\lambda^{*{(n)}} = (\lambda_1 - 1, \lambda_2 - 2, ..., \lambda_n - (n - 1))
\]

In each of the following theorems, \( n \) denotes the rank of \( G \) and for abbreviation we write \( \lambda^* \) for \( \lambda^{*{(n)}} \).

**Theorem 1.3.1.** In the case of \( G = GL(n) \), the character \( \chi_{GL(n)}(\lambda) \) of \( GL(n) \) corresponding to the partition \( \lambda \) has the following several expressions.

1. In terms of \( t_1, t_2, ..., t_n \),

\[
\chi_{GL(n)}(\lambda)(t_1, t_2, ..., t_n) = \frac{\prod_{i=1}^{n} t_i^{\lambda_i} + (n-1) \prod_{i=1}^{n} t_i^{2\lambda_i + (n-2)} \cdots \prod_{i=1}^{n} t_i^{n\lambda_i}}{\prod_{i=1}^{n-1} t_i^{n_1} t_i^{2\lambda_i + (n-2)} \cdots t_i^{n\lambda_i}}
\]

2. In terms of the \( p_i \)'s:

\[
\chi_{GL(n)}(\lambda) = \left| p_{\lambda^*}, p_{\lambda^* + (1^n)}, p_{\lambda^* + 2(1^n)}, ..., p_{\lambda^* + (n-1)(1^n)} \right|
\]

\[
= \left| p_{\lambda_1}, p_{\lambda_1 + 1}, p_{\lambda_1 + 2}, ..., p_{\lambda_1 + (n-1)} \right|
\]

\[
\left| p_{\lambda_2 - 1}, p_{\lambda_2}, p_{\lambda_2 + 1}, p_{\lambda_2 + (n-2)} \right|
\]

\[
\vdots
\]

\[
\left| p_{\lambda_n - (n-1)}, p_{\lambda_n - (n-2)}, p_{\lambda_n - (n-3)}, ..., p_{\lambda_n} \right|
\]
where \((1^n) = (1, 1, \ldots, 1) \in \mathbb{Z}^n\) and \(\lambda^* + i(1^n)\) denotes the sum in \(\mathbb{Z}^n\). Furthermore for \(\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n\), we use the convention that \(p_\mu\) denotes the column vector which is obtained by transposing \((p_{\mu_1}, p_{\mu_2}, \ldots, p_{\mu_n})\) and for \(i \in \mathbb{Z}\), \(t^i\) denotes the column vector which is obtained by transposing \((t^i_1, t^i_2, \ldots, t^i_n)\).

We state the theorem for the types \(B_n\) and \(D_n\) simultaneously.

**Theorem 1.3.2.** In the case of \(G = SO(2n + 1)\) and \(SO(2n)\) each character \(\chi_{SO(2n+1)}(\lambda)\) and \(\chi_{SO(2n)}(\lambda)\) corresponding to the partition \(\lambda = (\lambda_1, \ldots, \lambda_n)\) can be expressed by the same formula in the symmetric characters \(p_i\) of each group as follows:

\[
\chi_{SO(2n+1)}(\lambda) = |p_{\lambda^*} - p_{\lambda^* - 2(1^n)}, p_{\lambda^* + (1^n)} - p_{\lambda^* - 3(1^n)}, \ldots, p_{\lambda^* + (n-1)(1^n)} - p_{\lambda^* -(n+1)(1^n)}|,
\]

\[
\chi_{SO(2n)}(\lambda) = |p_{\lambda^*} - p_{\lambda^* - 2(1^n)}, p_{\lambda^* + (1^n)} - p_{\lambda^* - 3(1^n)}, \ldots, p_{\lambda^* + (n-1)(1^n)} - p_{\lambda^* -(n+1)(1^n)}|.
\]

**Theorem 1.3.3.** In the case of \(G = Sp(2n)\) the character \(\chi_{Sp(2n)}(\lambda)\) can be expressed by a polynomial in \(p_i\)'s as follows:

\[
\chi_{Sp(2n)}(\lambda) = |p_{\lambda^*}, p_{\lambda^* + (1^n)} + p_{\lambda^* - (1^n)}, \ldots, p_{\lambda^* + (n-1)(1^n)} + p_{\lambda^* -(n-1)(1^n)}|.
\]

The next remark is crucial in Section 2 to define \(\chi_{Sp}(\lambda)\) and \(\chi_{SO}(\lambda)\) in \(A\).

**Remark 1.3.4.** In the above three theorems, if we note the form of each determinant, we see that each determinant is equal to its minor consisting of the first \(k\) rows and the first \(k\) columns, where \(k\) is the depth of \(\lambda\). That is, for example, in the case of \(G = GL(n)\) and \(d(\lambda) = k\) it follows that

\[
\chi_{GL(n)}(\lambda) = \begin{pmatrix}
p_{\lambda_1} & p_{\lambda_1 + 1} & \cdots & p_{\lambda_1 + (k-1)} \\
p_{\lambda_2 - 1} & p_{\lambda_2} & \cdots & p_{\lambda_2 + (k-2)} \\
\vdots & \vdots & \ddots & \vdots \\
p_{\lambda_k - (k-1)} & p_{\lambda_k - (k-2)} & \cdots & p_{\lambda_k}
\end{pmatrix}.
\]

In other words, if \(n \geq d(\lambda)\), then the expression above shows that \(\chi_{GL(n)}(\lambda)\) can be expressed as the polynomial in \(p_i\)'s which depends only on the partition \(\lambda\) and is independent of \(n\). The same is true for the other classical groups, hence it is sufficient to consider the determinant of size \(k\), where \(k\) is the depth of \(\lambda\) and the polynomials of \(p_i\)'s are independent of \(n\).
1.4. The Universal Character Ring, Young Diagrams, and Schur Functions

In this section we review various notions needed later, following Mac- 
donald's book [6]. As for the proofs and details see Chap. 1 of [6].

The definition of the Universal Character Ring

Let $A_n = \mathbb{Z}[t_1, t_2, \ldots, t_n]$ be the graded algebra consisting 
of the symmetric polynomials in $n$ variables and let $\tilde{\rho}_{m,n}: \mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{Z}[t_1, \ldots, t_n]$ be the homomorphism of graded algebras defined by $\tilde{\rho}_{m,n}(t_i) = t_i$ if $1 \leq i \leq n$ and $\tilde{\rho}_{m,n}(t_i) = 0$ if $n < i$. $\tilde{\rho}_{m,n}$ induces a 
homomorphism $\rho_{m,n}: A_m \to A_n$. Then $(A_n, \rho_{m,n})$ becomes a projective 
system and the projective limit of this system in the category of graded 
algebras is denoted by $A$, i.e., $A = \lim_{n \to \infty} A_n$. We call $A$ the Universal Character 
Ring. By definition $A$ is also a graded algebra: $A = \sum_{k \geq 0} A^k$ where $A^k = \lim_{n \to \infty} A^n_k$ ($A^n_k$ is the homogeneous part of degree $k$ of $A_n$.) Note that $A$ can be considered as the ring consisting of symmetric functions in 
countably many variables $t_1, t_2, \ldots, t_n, \ldots$. Let $\pi_n: A \to A_n$ be the natural projection.

In Section 1.2 we defined $p_i$ in each $A_n = R_+(GL(n))$. In order to avoid 
confusion, let us denote $p_i \in A_n$ by $p_i(GL(n))$. $p_i(GL(n))$ is the sum of all 
monomials with coefficient 1 in $t_1, \ldots, t_n$ of degree $i$. Since $\rho_{m,n}(p_i(GL(m))) = p_i(GL(n))$, whenever $m \geq n$, we can define a $p_i \in A$ such that $\pi_n(p_i) = p_i(GL(n))$ for 
all $n$. On the other hand, we also defined $e_i$ in each $A_n$ such that $n \geq i$ (we 
denote this $e_i$ by $e_i(GL(n))$). $e_i(GL(n))$ is the $i$th elementary symmetric 
polynomial in $t_1, \ldots, t_n$. Again since $\rho_{m,n}(e_i(GL(m))) = e_i,GL(n)$ whenever $m \geq n \geq i$, we can define $e_i \in A$ by $\pi_n(e_i) = e_i(GL(n))$ for all $n \geq i$. Then the 
generating functions of $p_i$ and $e_i$ are given by

$$E(x) = \prod_{i=1}^{\infty} (1 + t_i x) = \sum_{i=0}^{\infty} e_i x^i \quad \text{and} \quad P(x) = \prod_{i=1}^{\infty} (1 - t_i x)^{-1} = \sum_{i=0}^{\infty} p_i x^i,$$

respectively. ($p_0 = e_0 = 1.$)

**Proposition 1.4.1** (cf. [6, Chap. 1, p. 13 (2.4), p. 14 (2.8)]).

$A = \mathbb{Z}[e_1, e_2, \ldots, e_n, \ldots] = \mathbb{Z}[p_1, p_2, \ldots, p_n, \ldots]$ 

(both are polynomial rings).

Due to Proposition 1.4.1, we can define an algebra homorphism 
$\omega: A \to A$ by $\omega(p_i) = e_i$, $i = 1, 2, \ldots, n, \ldots$.

Moreover since $P(x) E(-x) = 1$, $\{p_i\}_{i=0}^{\infty}$ and $\{e_i\}_{i=0}^{\infty}$ are related to each 
other by

$$\sum_{r=0}^{n} (-1)^r e_r p_{n-r} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \quad (1.4)$$
Since (1.4) is invariant under the simultaneous transposition of the \( p_i \) and the \( e_i \) \((i=0, 1, 2, \ldots)\), solving Eqs. (1.4) recursively we have the expressions of the same type as follows:

\[
p_n = f_n(e_1, e_2, \ldots, e_n), \quad e_n = f_n(p_1, p_2, \ldots, p_n),
\]

where \( f_n \) is a polynomial in \( n \) variables. Therefore if we apply \( \omega \) to the above equalities, we have \( \omega(e_n) = \omega(f_n(p_1, \ldots, p_n)) = f_n(e_1, \ldots, e_n) = p_n \). Consequently \( \omega^2 = 1 \) and \( \omega \) becomes an involutive automorphism of the graded algebra \( \mathcal{A} \).

Let us recall the notion of the Young diagrams. To each partition corresponds a Young diagram (see [6, p. 1–2]). This correspondence is illustrated as follows.

**Example.** For the partition \( \lambda = (5, 3, 2) \), the corresponding Young diagram is illustrated in Fig. 2.

From now on we identify a partition with the corresponding Young diagram and also denote the Young diagram corresponding to \( \lambda \) by the same symbol \( \lambda \). For a Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), the transpose \( '\lambda = (\lambda'_1, \lambda'_2, \ldots, \lambda'_s) \) is defined by \( \lambda'_i = \# \{k; \lambda_k \geq i\} \), where \( \# \) denotes the cardinality of the set. In terms of Young diagrams, \( '\lambda \) is the diagram obtained by reflecting \( \lambda \) with respect to the main diagonal. For example, if \( \lambda = (5, 3, 2) \), then \( '\lambda = (3, 3, 2, 1, 1) \) (see Fig. 3). As is well known,
\{\chi_{GL(n)}(\lambda)\}_{\lambda: \text{partition}, d(\lambda) \leq n} \text{ is a } \mathbb{Z}\text{-base of } A_n = R_+(GL(n)). \text{ As stated above, for } m \geq n \text{ we have}

\begin{align*}
\rho_{m,n}(p_{i,GL(m)}) &= \rho_{i,GL(n)}, \\
\pi_n(p_i) &= p_{i,GL(n)} \in R_+(GI(n)).
\end{align*}

Therefore from Remark 1.3.4 it follows that for \(m \geq n \geq d(\lambda)\),

\[\rho_{m,n}(\chi_{GL(n)}(\lambda)) = \chi_{GL(n)}(\lambda).\]

According to Theorem 1.3.1(1), if \(n = d(\lambda)\), i.e., \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) and \(\lambda_n > 0\), \(\chi_{GL(n)}(\lambda)\) is divisible by \((t_1, t_2, \ldots, t_n)^{\lambda_n}\). Consequently if \(d(\lambda) > k\), \(\rho_{n,k}(\chi_{GL(n)}(\lambda)) = 0\). Hence the \(\chi_{GL(n)}(\lambda)\)'s form a projective system and we may define \(\chi_{GL}(\lambda) \in A\), where \(\pi_n(\chi_{GL}(\lambda)) = \chi_{GL(n)}(\lambda)\) if \(n \geq d(\lambda)\) and \(\pi_n(\chi_{GL}(\lambda)) = 0\) if \(n < d(\lambda)\). \(\chi_{GL}(\lambda)\) is called a Schur function. (The symbol \(s_\lambda\) is also used for \(\chi_{GL}(\lambda)\)). Clearly \(\{\chi_{GL}(\lambda)\}_{\lambda: \text{partition}}\) becomes a \(\mathbb{Z}\)-linear base of \(A\).

For a partition \(\lambda\) of depth \(k\), let us regard \(\lambda\) as a vector in \(\mathbb{Z}^k\). Then \(\lambda^* = (\lambda_1, \lambda_2 - 1, \ldots, \lambda_k - (k - 1))\). From now on \(\lambda^*\) is an abbreviation of \(\lambda^*^{(k)}\). Then from Remark 1.3.4 we have

**Proposition 1.4.2.** In the algebra \(A\) it holds that

\[\chi_{GL}(\lambda) = |p_{\lambda^*}, p_{\lambda^* + (1^1)}, p_{\lambda^* + 2(1^1)}, \ldots, p_{\lambda^* + (k-1)(1^1)}|.
\]

Moreover \(\chi_{GL}(\lambda)\) can be expressed as a polynomial in \(\{e_i\}_{i=1}^\infty\) as follows:

**Proposition 1.4.3.** Let \(\lambda = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)\) be the transposed Young diagram of \(\lambda\) and \(d(\lambda') = m\). Then in the algebra \(A\) one has

\[\chi_{GL}(\lambda) = |e_{(\lambda'_1)^\ast}, e_{(\lambda'_2)^\ast + (1^{m_1})}, \ldots, e_{(\lambda'_m)^\ast + (m-1)(1^{m_1})}|.
\]

From Propositions 1.4.2 and 1.4.3 we have

\[\omega(\chi_{GL}(\lambda)) = |e_{\lambda^*}, e_{\lambda^* + (1^1)}, \ldots, e_{\lambda^* + (k-1)(1^1)}| = \chi_{GL}(\lambda^*).\]

Namely, \(\omega \in \text{Aut}(A)\) acts on the set \(\{\chi_{GL}(\lambda)\}\) as the permutation transposing the Young diagrams.

The structure constants \(\{LR_{\mu\nu}^\lambda\}_{\lambda,\mu,\nu: \text{partitions}}\) of the algebra \(A\) with respect to the base \(\{\chi_{GL}(\lambda)\}\) are called the Littlewood–Richardson coefficients. Thus, \(LR_{\mu\nu}^\lambda\) are defined by

\[\chi_{GL}(\mu) \chi_{GL}(\nu) = \sum_{\lambda: \text{partition}} LR_{\mu\nu}^\lambda \chi_{GL}(\lambda), \quad LR_{\mu\nu}^\lambda \in \mathbb{Z}_{\geq 0}.
\]
LR\*\mu has the following combinatorial description. If partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) satisfy the condition that \( \lambda_i \geq \mu_i \) for all \( i \geq 1 \), we say that the Young diagram \( \lambda \) contains the Young diagram \( \mu \) and write \( \lambda \triangleright \mu \). If \( \lambda \triangleright \mu \), put \( \mu \) on \( \lambda \) with the same top-left corner and remove \( \mu \) out of \( \lambda \). Then the resulting diagram is called a skew diagram and is denoted by \( \lambda - \mu \). For example, if \( \lambda = (5, 3, 2) \) and \( \mu = (1, 1) \) the skew diagram \( \lambda - \mu \) is the diagram consisting of the white squares in Fig. 4. If we take the empty set as \( \mu \), then \( \lambda \) itself can be considered as a skew diagram. The skew diagram such that each square of which is filled with a positive integer is called a numbered skew diagram. Moreover numbered skew diagrams which satisfy the following condition (*) are called Young tableaux.

\[ (*) \text{ The numbers written in the squares must increase strictly downwards along each column and increase (admitting equality) along each row from left to right.} \]

That is, (*) means that the numbers written in the skew diagram satisfies the inequalities in Fig. 5.

For a Young tableau \( T \), the original skew diagram obtained by erasing the numbers written in the squares of \( T \) is called the shape of \( T \).

**Example.**

(1) \[
\begin{array}{cc}
1 & 1 \\
2 & 2 \\
\end{array}
\]

is a Young tableau

\[
\begin{array}{c}
1 \\
3 \\
\end{array}
\]

but

(2) \[
\begin{array}{cc}
1 & 1 \\
2 & 2 \\
\end{array}
\]

is not,

\[
\begin{array}{c}
1 \\
3 \\
\end{array}
\]

since in the second column

\[
\begin{array}{c}
1 \\
1 \\
3 \\
\end{array}
\]

does not increase strictly.

\[ \lambda - \mu = \]

\[
\begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}
\]

**Figure 4**
Let $T$ be a Young tableau and let $v_i$ be the times of occurrences of the number $i$ in the tableau $T$. Then we get a sequence $(v_1, v_2, ..., v_n, ...)$ and call this sequence the \textit{weight} of $T$. In the above example (1) the weight of $T$ is $(3, 4, 1, 0, ..., 0, ...)$. From a Young tableau $T$ we derive a "word" or sequence $w(T)$ by reading the numbers in $T$ from right to left in successive rows, starting with the top row. In the case of example (1) the word of $T$ is given by

$$w(T) = 2, 2, 1, 1, 2, 2, 3, 1.$$

**Definition.** The word $w(T) = a_1, a_2, ..., a_n$ is said to be a \textit{lattice permutation} if for any $r$ ($1 < r < n$) and any $i \in \mathbb{N}$, the number of times of occurrences of the number $i$ in the sequence $a_1, a_2, ..., a_r$ is not less than that of the number $i + 1$.

**Theorem 1.4.4** (Littlewood–Richardson's rule). Let $\lambda, \mu, \nu$ be partitions. Then $LR_{\mu\nu}^\lambda$ is given by

$$LR_{\mu\nu}^\lambda = \# \{ T; \ T \text{ is a Young tableau whose shape is } \lambda - \mu \text{ and whose weight is } \nu = (v_1, v_2, ...) \text{ and such that } w(T) \text{ is a lattice permutation} \}.$$ 

By the definition of $LR_{\mu\nu}^\lambda$, it is clear that $LR_{\mu\nu}^\lambda = LR_{\nu\mu}^\lambda$ and consequently from Theorem 1.4.4, $LR_{\mu\nu}^\lambda = 0$ unless $\lambda \geq \mu$ and $\lambda \geq \nu$. $LR_{\mu\nu}^\lambda$ depends only on the pair of skew diagrams $(\lambda - \mu, \nu)$ or $(\lambda - \nu, \mu)$. If we apply $\omega \in \text{Aut}(\mathcal{A})$ to $\chi_{GL}(\mu) \chi_{GL}(\nu) = \sum \chi_{GL}(\lambda)$, then we get $LR_{\mu'\nu'}^\lambda = LR_{\mu\nu}^\lambda$, namely if we take the simultaneous transpose of $\lambda, \mu$, and $\nu$, the Littlewood–Richardson coefficient $LR_{\mu\nu}^\lambda$ does not change. Moreover for partitions $\mu, \nu$ whose depths are not greater than $n$, if we apply $\pi_n$ to $\chi_{GL}(\mu) \chi_{GL}(\nu) = \sum \chi_{GL}(\lambda)$, we obtain

$$\chi_{GL(n)}(\mu) \chi_{GL(n)}(\nu) = \sum_{d(\lambda) \leq n} LR_{\mu\nu}^\lambda \chi_{GL(n)}(\lambda).$$

This is the decomposition of the tensor product of two irreducible representations of $GL(n)$ into irreducible constituents.
1.5. D. E. Littlewood's Lemma

In this section, we state Littlewood's lemmas, i.e., Lemmas 15.1 and 1.52 and Proposition 15.3. Since they play the crucial roles in this paper and we would like to clarify the situation in which the equalities hold, we give the proof of Lemma 1.5.1 for the sake of completeness. As for the proof of Lemma 1.5.2, see [4].

Let \( R = \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}] \) be the Laurent polynomial ring in variables \( t_1, t_2, \ldots, t_n \) and let \( R[[z_1, z_2, \ldots, z_n]] \) be the formal power series ring of \( z_1, z_2, \ldots, z_n \) over the coefficient ring \( R \). In this section, all the equalities are considered in the algebra \( R[[z_1, z_2, \ldots, z_n]] \). Moreover we consider the character functions \( \chi_{GL(n)}(\lambda)(t_1, t_2, \ldots, t_n) \), \( \chi_{O(2n+1)}(\lambda)(t_1, t_2, \ldots, t_n) \), \( \chi_{Sp(2n)}(\lambda)(t_1, t_2, \ldots, t_n) \) and \( \chi_{O(2n)}(\lambda)(t_1, t_2, \ldots, t_n) \) simply as polynomials in \( t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1} \). In the following lemmas, as the variables of \( \chi_{GL(n)}(\lambda) \) we use \( z_1, z_2, \ldots, z_n \) instead of \( t_1, t_2, \ldots, t_n \).

**Lemma 15.1 (Littlewood).** In the algebra \( R[[z_1, z_2, \ldots, z_n]] \) the following equalities hold:

\[
\begin{align*}
(1) & \quad \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{O(2n+1)}(\lambda)(t_1, t_2, \ldots, t_n) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n) \\
& \quad \cdot \prod_{1 \leq i < j \leq n} (1 - z_i z_j) \\
& \quad = \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{GL(2n)}(\lambda)(t_1, t_2, \ldots, t_n, t_n^{-1}) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n). \\
(2) & \quad \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{O(2n+1)}(\lambda)(t_1, t_2, \ldots, t_n) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n) \\
& \quad \cdot \prod_{1 \leq i < j \leq n} (1 - z_i z_j) \\
& \quad = \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{GL(2n+1)}(\lambda)(t_1, t_2, \ldots, t_n, 1, t_n^{-1}) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n). \\
(3) & \quad \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{O(2n)}(\lambda)(t_1, t_2, \ldots, t_n) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n) \\
& \quad \cdot \prod_{1 \leq i < j \leq n} (1 - z_i z_j) \\
& \quad = \sum_{\lambda, \text{partition} \atop \lambda(\lambda) \leq n} \chi_{GL(2n)}(\lambda)(t_1, t_2, \ldots, t_n, t_n^{-1}) \chi_{GL(n)}(\lambda)(z_1, z_2, \ldots, z_n). 
\end{align*}
\]

**Proof.** (1) We put

\[
|z_1^{n-1}, z_1^n + z_1^{n-2}, \ldots, z_1^{2(n-1)} + 1| = \det \begin{pmatrix}
z_1^{n-1} & z_1^n & z_1^{n-2} & \cdots & z_1^{2(n-1)} + 1 \\
z_2^{n-1} & z_2^n & z_2^{n-2} & \cdots & z_2^{2(n-1)} + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
z_n^{n-1} & z_n^n & z_n^{n-2} & \cdots & z_n^{2(n-1)} + 1
\end{pmatrix}
\]

Let us introduce the polynomial \( \phi_{Sp}(z) = \prod_{i=1}^n (1 - t_i z)(1 - t_i^{-1} z) \) corresponding to the characteristic polynomial of the element \( t = \)
diag(t₁, ..., tₙ, tₙ⁻¹, ..., t₁⁻¹) of Sp(2n). Then in the algebra \( R[[z]] \), \( \phi_{Sp}(z)^{-1} \) is given by

\[
\phi_{Sp}(z)^{-1} = \sum_{f \geq 0} p_f(t) z^f,
\]
where \( p_f(t) \) is the symmetric character of degree \( f \) of Sp(2n). Therefore if we put

\[
\Phi = \frac{\left| z^{n-1}, z^n, z^{n-2}, ..., z^{2(n-1)} + 1 \right|}{\phi_{Sp}(z_1) \phi_{Sp}(z_2) \cdots \phi_{Sp}(z_n)},
\]
then we have

\[
= \sum_{f_t \geq 0} p_{f_t}(t) p_{f_2}(t) \cdots p_{f_n}(t) z^{f_1} z^{f_2} \cdots z^{f_n} \left| z^{n-1}, z^n, z^{n-2}, ..., z^{2(n-1)} + 1 \right|.
\]
Since

\[
\left| z^{n-1}, z^n, z^{n-2}, ..., z^{2(n-1)} + 1 \right| = (z_1 z_2 \cdots z_n)^{n-1} \left| 1, z + z^{-1}, ..., z^{n-1} + z^{-(n-1)} \right|
\]
we have

\[
\Phi = \sum_{f_t \geq 0} p_{f_1}(t) p_{f_2}(t) \cdots p_{f_n}(t)
\times \sum_{\sigma \in \mathfrak{S}_n} \left\{ \text{sgn}(\sigma)(z_1^{\sigma(n-1)} + z_1^{-\sigma(n-1)}) \cdots (z_1^{\sigma(0)} + z_1^{-\sigma(0)}) \right\}
\times z_1^{f_1} z_2^{f_2} \cdots z_n^{f_n}.
\]
Expanding the above expression with respect to variables \( z \), we have (for brevity we omit the variables \( t \) of \( p \)):

\[
\Phi = \sum_{l_i \geq 1} \left( \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) p_{l_1} \cdots p_{l_n} \right) z_1^{l_1-1} z_2^{l_2-1} \cdots z_n^{l_n-1},
\]
where the coefficient of \( z_1^{l_1-1} z_2^{l_2-1} \cdots z_n^{l_n-1} \) in the above runs over all permutations \( \sigma \) on the set \( \{0, 1, 2, ..., n-1\} \) and all \( i_1 \geq 0, i_2 \geq 0, ..., i_n \geq 0 \) satisfying \( l_1 = i_1 + n + \sigma(0), \ l_2 = i_2 + n + \sigma(1), ..., \ l_n = i_n + n + \sigma(n-1) \) (the choices of \(+, -\) are arbitrary).

Hence the coefficient of \( z_1^{l_1-1} \cdots z_n^{l_n-1} \) equals

\[
\sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) p_{l_1 - n + \sigma(0)} p_{l_2 - n + \sigma(1)} \cdots p_{l_n - n + \sigma(n-1)}
= |p_{l - n(1^n)}, p_{l - (n-1)(1^n)} + p_{l - (n+1)(1^n)}, ..., p_{l - (1^n)} + p_{l - (2n-1)(1^n)}|,
\]
where \( l = (l_1, l_2, ..., l_n) \in \mathbb{Z}^n \) and \( l_i \geq 1 \). Clearly if \( l_1, l_2, ..., l_n \) are not distinct from each other, the coefficient of \( z_1^{l_1-1} \cdots z_n^{l_n-1} \) is reduced to 0. In the above determinant, permuting the rows if necessary, we may assume that \( l_1 > l_2 > \cdots > l_n \geq 1 \) (up to sign), and collecting together the monomials in
the $z_i$'s having the same determinant in the $p_i$'s as their coefficients (up to sign), we have

$$
\Phi = \sum |p_{l-n}(1^n)|, \ p_{l-(n-1)}(1^n)+p_{l-(n+1)}(1^n), \ldots, p_{l-(2n-1)}(1^n)|
\times |z^{l-1}_1, z^{l-1}_2, \ldots, z^{l-1}_n|
$$

where the sum runs over $l = (l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$ ($l_1 > l_2 > \cdots > l_n$).

If we put $l_1 = f_1 + n, l_2 = f_2 + n - 1, \ldots, l_n = f_n + 1$, we obtain

$$
\Phi = \sum |p_{\lambda^*}, p_{\lambda^* + (1^n)}, p_{\lambda^* + (n-1)}(1^n)|
\times |z^{f_1 + (n-1)}_1, z^{f_2 + (n-2)}_2, \ldots, z^{f_n}|
$$

where the sum runs over $\lambda = (f_1, f_2, \ldots, f_n)$ ($f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$).

On the other hand, applying the elementary transformations of columns to the determinant $|1, z+z^{-1}, \ldots, z^{n-1}+z^{-(n-1)}|$, we see that

$$
|1, z+z^{-1}, \ldots, z^{n-1}+z^{-(n-1)}| = |1, z+z^{-1}, (z+z^{-1})^2, \ldots, (z+z^{-1})^{n-1}|
$$

where

$$
|1, z+z^{-1}, (z+z^{-1})^2, \ldots, (z+z^{-1})^{n-1}|
= \det \begin{pmatrix}
1 & z_1+z_1^{-1} & (z_1+z_1^{-1})^2 & \cdots & (z_1+z_1^{-1})^{n-1} \\
1 & z_2+z_2^{-1} & (z_2+z_2^{-1})^2 & \cdots & (z_2+z_2^{-1})^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & z_n+z_n^{-1} & (z_n+z_n^{-1})^2 & \cdots & (z_n+z_n^{-1})^{n-1}
\end{pmatrix}.
$$

The right-hand side of the above equality is the Vandermonde determinant, and therefore equal to

$$
\prod_{i > j} (z_i+z_i^{-1}-z_j-z_j^{-1}) = \prod_{i > j} \{z_i^{-1}z_j^{-1}(z_i-z_j)(1-z_i z_j)\}.
$$

Finally we have

$$
|z^{n-1}, z^n+z^{n-2}, \ldots, z^{2(n-1)}+1| = |z^{n-1}, z^{n-2}, \ldots, 1| \prod_{1 \leq i < j \leq n} (1-z_i z_j).
$$

According to Theorems 1.3.3 and 1.3.1(1), we get

$$
\frac{\prod_{1 \leq i < j \leq n} (1-z_i z_j)}{\phi_{Sp}(z_1) \phi_{Sp}(z_2) \cdots \phi_{Sp}(z_n)} = \sum_{d(\lambda) \leq n} \chi_{Sp(2n)}(\lambda)(t) \chi_{GL(n)}(\lambda)(z). \quad (1.5.1)
$$

Let $\phi_{GL(2n)}(z) = \prod_{i=1}^{2n} (1-t_i z)$ be the polynomial corresponding to the characteristic polynomial of the element $t = \text{diag}(t_1, t_2, \ldots, t_{2n})$ of $GL(2n)$. As
is well known (see [7, Chap. VII, p. 202, Lemma 7.6A; p. 210, (7.10)]), the following equality holds in the algebra $\mathbb{C}[t_1, t_2, \ldots, t_{2n}][[ z_1, z_2, \ldots, z_{2n}]]$:

$$\frac{1}{\phi_{GL(2n)}(z_1) \cdots \phi_{GL(2n)}(z_{2n})} = \sum_{d(\lambda) \leq 2n} \chi_{GL(2n)}(\lambda)(t) \chi_{GL(2n)}(\lambda)(z).$$

In fact, a similar calculation starting with

$$\prod_{i=1}^{2n} z_i$$

leads us to this equality.

In the above equality, if we put $t_1 = t, t_2 = t_1, \ldots, t_{2n} = t_1$, and $z_1 = z+2 = \cdots = z_{2n} = 0$, we have an algebra homomorphism $\mathbb{C}[t_1, \ldots, t_{2n}][[ z_1, z_2, \ldots, z_{2n}]] \to R[[ z_1, z_2, \ldots, z_{2n}]]$. Since $\phi_{GL(2n)}(0) = 1$ and

$$\rho_{2n,n}(\chi_{GL(2n)}(\lambda)) = \begin{cases} \chi_{GL(n)}(\lambda) & \text{if } d(\lambda) \leq n, \\ 0 & \text{if } d(\lambda) > n, \end{cases}$$

we have

$$\frac{1}{\phi_{Sp}(z_1) \cdots \phi_{Sp}(z_n)} = \sum_{d(\lambda) \leq n} \chi_{GL(2n)}(\lambda)(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_{2n}^{-1}) \chi_{GL(n)}(\lambda)(z).$$

Comparing this with Eq. (1.5.1), we obtain (1).

(2) Let $\phi_{SO(2n+1)}(z) = (1-z) \prod_{i=1}^{n} \{(1-t_i z)(1-t_i^{-1} z)\}$ be the polynomial corresponding to the characteristic polynomial of the element $t = \text{diag}(t_1, \ldots, t_n, 1, t_1^{-1}, \ldots, t_n^{-1})$ of $SO(2n+1)$. The same calculation as in the proof of (1) gives the result that

$$\frac{1}{\phi_{SO(2n+1)}(z_1) \cdots \phi_{SO(2n+1)}(z_n)}$$

Further we have

$$|z^{n-1} - z^n + 1 - z^{2n}|$$

$$= z_1^n z_2^n \cdots z_n^n (-1)^n |z - z^{-1}, z^2 - z^{-2}, \ldots, z^n - z^{-n}|$$

$$= z_1^n \cdots z_n^n (-1)^n \prod (z_i - z_i^{-1}) |1, z + z^{-1}, \ldots, (z + z^{-1})^{n-1}|$$

$$= \prod_{1 \leq i < j \leq n} (1 - z_i z_j) \times \prod_{1 \leq i < j \leq n} (z_i - z_j).$$
According to Theorem 1.3.2, we get
\[
\prod_{1 \leq i < j \leq n} (1 - z_i z_j) \cdot \phi_{SO(2n+1)}(z_1) \cdots \phi_{SO(2n+1)}(z_n) = \sum_{d(\lambda) \leq n} \chi_{GL(2n+1)}(\lambda)(t) \chi_{GL(n)}(\lambda)(z).
\] (1.5.2)

On the other hand, for the polynomial \( \phi_{GL(2n+1)}(z) \), the following equality holds in \( \mathbb{C}[t_1, t_2, \ldots, t_{2n+1}][[z_1, z_2, \ldots, z_{2n+1}]] \) as before,
\[
1 \cdot \phi_{GL(2n+1)}(z_1) \cdots \phi_{GL(2n+1)}(z_{2n+1}) = \sum_{d(\lambda) \leq 2n+1} \chi_{GL(2n+1)}(\lambda)(t) \chi_{GL(2n+1)}(\lambda)(z).
\]

In this equality, if we put \( t_{2n+1} = t_1^{-1}, t_{2n} = t_2^{-1}, \ldots, t_{n+2} = t_n^{-1}, t_{n+1} = 1 \) and \( z_{n+1} = z_{n+2} = \cdots = z_{2n+1} = 0 \) and compare this with the formula (1.5.2), we obtain the formula (2).

The proof of (3) is almost the same as that of (2). The only different point is that we must use \( \phi_{SO(2n)}(z) = \prod_{i=1}^{n} (1 - t_i z)(1 - t_i^{-1} z) \) instead of \( \phi_{SO(2n+1)}(z) \). So we omit it. 

We shall consider the rational functions in \( z \) appearing on the left-hand side of Littlewood's Lemma 1.5.1. They are
\[
(1) \prod_{1 \leq i < j \leq n} (1 - z_i z_j)^{-1}, \quad (2) \prod_{1 \leq i < j \leq n} (1 - z_i z_j), \quad (3) \prod_{1 \leq i < j \leq n} (1 - z_i z_j)^{-1}, \quad (4) \prod_{1 \leq i < j \leq n} (1 - z_i z_j).
\]

The rational functions (1), (2), (3), and (4) are all \( \mathfrak{S}_n \)-invariant (where \( \mathfrak{S}_n \) acts by the permutations of variables \( \{z_i\}_{i=1}^{n} \)). Therefore if we embed the rational functions (1), (2), (3), and (4) into the formal power series ring \( \mathbb{C}[[z_1, z_2, \ldots, z_n]] \), they can be expressed as linear combinations (finite or infinite) of \( \chi_{GL(n)}(\lambda)(z) \)'s. Thus, if we know the expansion of functions (1), (2), (3), and (4) into sums of \( \chi_{GL(n)}(\lambda)(z) \)'s, then Littlewood's lemma gives us the formulas linking the characters of \( GL(n) \) and the characters of the other classical groups. We must prepare a few notations first.

For a partition \( \kappa = (k_1, k_2, \ldots, k_n) \), \( 2\kappa \) denotes the even partition \( 2\kappa = (2k_1, 2k_2, \ldots, 2k_n) \). For a distinct partition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \) \((\alpha_1 > \alpha_2 > \cdots > \alpha_s \geq 1)\), \( I(\alpha) \) denotes the partition \( I(\alpha) = (\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_s - 1 | \alpha_1, \alpha_2, \ldots, \alpha_s) \), using the Frobenius notation. The Frobenius notation
\((\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_r)\) expresses the Young diagram \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) whose diagonal consists of \(r\) squares and the \(\alpha_i, \beta_i\ (1 \leq i \leq r)\) and the \(\tilde{\lambda}_i (1 \leq i \leq n)\) are combined with the relations

\[ \alpha_i = \lambda_i - i, \quad \beta_i = \tilde{\lambda}_i - i, \quad 1 \leq i \leq r, \]

where we put \(\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_r)\). In terms of Young diagrams, \((\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \ldots, \beta_r)\) is the diagram illustrated in Fig. 6a. For example, \(\Gamma(3, 2)\) is the one in Fig. 6b.

The following fundamental Lemma 1.5.2(1)–(4) was also found by Littlewood. As for the proof of these formulas under the setting of modern terminology, see Macdonald [6, p. 45] and also [4].

**Lemma 1.5.2 (Littlewood).**

1. \[
\frac{1}{\prod_{1 \leq i < j \leq n} (1 - z_i z_j)} = \sum_{f=0}^{\infty} \sum_{d(\kappa) \leq n, \vert \kappa \vert = f} \chi_{GL(n)}(\kappa(2\kappa))(z),
\]

2. \[
\frac{1}{\prod_{1 \leq i < j \leq n} (1 - z_i z_j)} = \sum_{f=0}^{\infty} \sum_{d(\kappa) \leq n, \vert \kappa \vert = f} \chi_{GL(n)}(2\kappa)(z),
\]

3. \[
\prod_{1 \leq i < j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \sum_{d(\kappa) \leq n, \vert \kappa \vert = f} (-1)^{|\kappa|} \chi_{GL(n)}(\Gamma(\kappa))(z),
\]

4. \[
\prod_{1 \leq i < j \leq n} (1 - z_i z_j) = \sum_{f=0}^{\infty} \sum_{d(\kappa) \leq n, \vert \kappa \vert = f} (-1)^{|\kappa|} \chi_{GL(n)}(\Gamma(\kappa))(z).
\]
Now let us combine the above Lemma 15.2 with Lemma 1.51. According to Section 1.4, the product \( \chi_{GL(n)}(\mu) \chi_{GL(n)}(v) \) is decomposed into the following sum:

\[
\chi_{GL(n)}(\mu)(z_1, \ldots, z_n) \chi_{GL(n)}(v)(z_1, \ldots, z_n) = \sum_{\lambda \vdash \mu, \nu} LR_{\mu \nu}^\chi \chi_{GL(n)}(\lambda)(z_1, \ldots, z_n).
\]

Moreover if we note that \( LR_{\mu \nu}^\chi = 0 \) unless \( \lambda \supset \mu \) and \( \lambda \supset \nu \), we obtain the following proposition.

**PROPOSITION 1.53 (Littlewood).** Let \( \lambda \) be a partition whose depth is less than or equal to \( n \). Then we have

\begin{align*}
(1) \quad & \chi_{GL(2n)}(\lambda)(t) \downarrow_{Sp(2n)} GL(2n) = \sum_{\kappa \text{; partition}} LR_{\mu \nu}^\chi \chi_{Sp(2n)}(\mu)(t), \\
(2) \quad & \chi_{Sp(2n)}(\lambda)(t) = \sum_{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \neq (\lambda_1, \lambda_2, \ldots, \lambda_s)} (-1)^{s-1} LR_{\mu \lambda}^\chi \chi_{GL(2n)}(\mu)(t) \downarrow_{Sp(2n)} GL(2n), \\
(3) \quad & \chi_{O(2n+1)}(\lambda)(t) \downarrow_{SO(2n+1)} GL(2n+1) = \sum_{\kappa \text{; partition}} LR_{\mu \nu}^\chi \chi_{O(2n+1)}(\mu)(t), \\
(4) \quad & \chi_{O(2n)}(\lambda)(t) \downarrow_{SO(2n)} GL(2n) = \sum_{\kappa \text{; partition}} LR_{\mu \nu}^\chi \chi_{O(2n)}(\mu)(t), \\
(5) \quad & \chi_{GL(2n)}(\lambda)(t) \downarrow_{SO(2n)} GL(2n) = \sum_{\kappa \text{; partition}} LR_{\mu \nu}^\chi \chi_{GL(2n)}(\mu)(t), \\
(6) \quad & \chi_{O(2n)}(\lambda)(t) = \sum_{\kappa \text{; partition}} (-1)^{s-1} LR_{\mu \nu}^\chi \chi_{O(2n)}(\mu)(t) \downarrow_{SO(2n)} GL(2n)'
\end{align*}

where \( \downarrow_{Sp(2n)} GL(2n) \) in (1) and (2) denotes the restriction of the representation of \( GL(2n) \) to \( Sp(2n) \) under the embedding \( Sp(2n) \subset GL(2n) \) of Section 1.1. In other words let \( r_{Sp} \) denote the homomorphism \( r_{Sp}: R_+(GL(2n)) \to R(Sp(2n)) \) or

\[
\mathbb{Z}[t_1, t_2, \ldots, t_{2n}] \to \mathbb{Z}[t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}] W^.
\]

defined by \( r_{Sp}(t_i) = t_i \) for \( 1 \leq i \leq n \) and \( r_{Sp}(t_{n+i}) = t_n^{-1} \) for \( 1 \leq i \leq n \). Then \( \downarrow_{Sp(2n)} GL(2n) \) is given by \( \chi_{GL(2n)}(\lambda)(t) \downarrow_{Sp(2n)} GL(2n) = r_{Sp}(\chi_{GL(2n)}(\lambda)(t)) \) if we recall the
maximal torus of each group given in Section 1.2. The other notations $GL(2n+1)$ and $SO(2n)$ convey similar meanings.

2. ON THE RELATIONS BETWEEN THE CHARACTERS
   OF THE CLASSICAL GROUPS

In this section, first we establish a duality linking the characters of symplectic groups and those of orthogonal groups. Next we give the "restriction rules" describing how an irreducible representation of the general linear group decomposes when it is restricted to the symplectic group or to the orthogonal group. Finally we give a formula to decompose the tensor product of two irreducible representations of a classical group $G$ into irreducible constituents. We use the same notations as in Section 1.

2.1. Some More Bases and Generator Systems of
   the Universal Character Ring

We have defined a $\mathbb{Z}$-base of $A$, namely $\{\chi_{GL}(\lambda)\}$, in Section 1.4. Here we shall define two more bases $\{\chi_{Sp}(\lambda)\}$, $\{\chi_{O}(\lambda)\}$ ($\lambda$: partitions), on the basis of Theorems 1.3.2 and 1.3.3 in Section 1.3. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of depth $k$, as in Section 1.4, we denote by $\lambda^*$ the element of $\mathbb{Z}^k$ defined by $\lambda^* = \lambda^{* (k)} = (\lambda_1, \lambda_2 - 1, \ldots, \lambda_k - (k - 1))$. (See the definition of $\chi_{CI}(\lambda)$ in Section 1.4 and Remark 1.3.4.) We shall define the elements $\chi_{Sp}(\lambda)$, $\chi_{O}(\lambda)$ of $A$ as follows.

DEFINITION 2.1.1.

$$\chi_{Sp}(\lambda) = |p_{\lambda^*} - p_{\lambda^* - 2(\ell^1)}, p_{\lambda^* + (\ell^1)}, p_{\lambda^* + (k - 1)(\ell^1)}|,$$

$$\chi_{O}(\lambda) = |p_{\lambda^*} - p_{\lambda^* - 2(\ell^1)}, p_{\lambda^* + (\ell^1)}, p_{\lambda^* + (k - 1)(\ell^1)} - p_{\lambda^* - (k + 1)(\ell^1)}|.$$

Then we have

PROPOSITION 2.1.2. Both $\{\chi_{Sp}(\lambda)\}_{\lambda: \text{partition}}$ and $\{\chi_{O}(\lambda)\}_{\lambda: \text{partition}}$ are $\mathbb{Z}$-bases of $A$.

Proof. If we recall the definition of $\chi_{GL}(\lambda)$, we see that the degrees of
\(\chi_{S_p}(\lambda), \chi_{O}(\lambda), \) and \(\chi_{GL}(\lambda)\) in \(A\) are all equal to \(|\lambda|\) and therefore \(\chi_{S_p}(\lambda)\) and \(\chi_{O}(\lambda)\) can be written as follows.

\[
\chi_{S_p}(\lambda) = \chi_{GL}(\lambda) + \sum_{|\lambda| > |\mu|} c_{\mu \lambda} \chi_{GL}(\mu) \quad (c_{\mu \lambda} \in \mathbb{Z}), \tag{2.1.1}
\]

\[
\chi_{O}(\lambda) = \chi_{GL}(\lambda) + \sum_{|\lambda| > |\mu|} b_{\mu \lambda} \chi_{GL}(\mu) \quad (b_{\mu \lambda} \in \mathbb{Z}). \tag{2.1.2}
\]

We arrange the partitions in the nondecreasing order of their sizes. (As for the partitions of the same size, we can arrange them arbitrarily.) According to the formula (2.1.1), the (infinite size) matrix \((c_{\mu \lambda})_{\mu, \lambda}\) partitions is upper triangular. Also, its diagonal components \(c_{\lambda \lambda}\) are all equal to 1. Therefore the inverse matrix of \((c_{\mu \lambda})\) exists and \(\chi_{GL}(\lambda)\) can be expressed by a linear combination of \(\chi_{S_p}(\mu)\)'s. Hence \(\{\chi_{S_p}(\lambda)\}\) is also a \(\mathbb{Z}\)-basis of \(A\). Similarly, by means of the formula (2.1.2), we can show that \(\{\chi_{O}(\lambda)\}\) is a \(\mathbb{Z}\)-basis of \(A\).

In the algebra \(A\) we shall define two more generator systems \(\{p_i^0\}_{i=1}^{\infty}\) and \(\{e_i^0\}_{i=1}^{\infty}\), other than \(\{p_i\}\) and \(\{e_i\}\), based on Propositions 1.2.3, 1.2.4, and 1.2.5.

**Definition.** We put \(p_i^0 := p_i - p_{i-2}\) and \(e_i^0 := e_i - e_{i-2}\) for \(i \in \mathbb{Z}\).

In the above definition we should note that if \(i < 0\), we have put \(e_i = p_i = 0\). Then clearly \(A = \mathbb{Z}[p_1^0, p_2^0, ..., p_n^0, ...] \) (polynomial ring) \(- \mathbb{Z}[e_1^0, e_2^0, ..., e_n^0, ...] \) (polynomial ring) holds. The generating functions of the \(p_i^0\) and \(e_i^0\) are given by

\[
P(x) = \frac{1 - x^2}{\prod_{i=1}^{\infty} (1 - t_i x)} = \sum_{i=0}^{\infty} p_i^0 x^i,
\]

and

\[
E(x) = (1 - x^2) \prod_{i=1}^{\infty} (1 + t_i x) = \sum_{i=0}^{\infty} e_i^0 x^i,
\]

respectively. If we recall the generating functions of \(\{p_i\}\) and \(\{e_i\}\):

\[
P(x) = \prod_{i=1}^{\infty} (1 - t_i x)^{-1} \quad \text{and} \quad E(x) = \prod_{i=1}^{\infty} (1 + t_i x),
\]

we have \(P(x) E(-x) = P(x) E^2(-x) = 1 - x^2\). Therefore we have the following relations.
Since the equalities (2.1.3) are symmetric with respect to the \( e_r \) and the \( p_r^o \), solving the system of equations (2.1.3) recursively, we see that there exists a polynomial \( f_n \) in \( n \) variables for each \( n \) such that both \( e_n = f_n(p_1^o, p_2^o, \ldots, p_n^o) \) and \( p_n^o = f_n(e_1, e_2, \ldots, e_n) \) hold. Let \( \iota_O \) be the endomorphism of \( A \) defined by \( \iota_O(p_r^o) = e_r, r = 1, 2, \ldots \), then \( \iota_O(e_r) = \iota_O(f_n(p_1^o, p_2^o, \ldots, p_n^o)) = f_n(e_1, e_2, \ldots, e_n) = p_r^o \). Therefore \( \iota_O^2 = \text{id} \), and \( \iota_O \) becomes an involutive automorphism of \( A \) (not as a graded algebra but only as an algebra).

Also let \( \iota_{Sp} \) be the endomorphism of \( A \) defined by \( \iota_{Sp}(e_r^o) = p_r \). The similar argument shows that \( \iota_{Sp}^2 = \text{id} \) and \( \iota_{Sp} \) becomes an involutive automorphism of \( A \).

**Remark.** The reason why we note \( p_i^o \) and \( e_i^o \) in the case of \( Sp \) and \( p_r^o \) and \( e_r \) in the case of \( SO \) is that we take into consideration the irreducible decompositions of symmetric and alternating characters of each group (Propositions 1.2.3–1.2.5).

Later we shall show that \( \iota_O \) and \( \iota_{Sp} \) are the operators corresponding to transposing Young diagrams when we parametrize the representations of \( SO \) and \( Sp \) respectively by Young diagrams as in Section 1.1. Moreover \( \omega, \iota_O, \) and \( \iota_{Sp} \) have the following relation:

\[
\omega \iota_O = \iota_{Sp} \omega.
\]

### 2.2. The Specialization Homomorphisms

In this section we shall define a homomorphism from \( A \) to the character ring of each classical group \( G \). In the case of \( G = GL(n) \), we have already defined the homomorphism \( \pi_n : A \rightarrow A_n = R_+(GL(n)) \), that is, the natural projection from \( A \) to \( A_n \).

In the case of \( G = Sp(2n) \), let \( \pi_{Sp(2n)} : A \rightarrow R(Sp(2n)) \) be the homomorphism defined by

\[
\pi_{Sp(2n)} = r_{Sp} \circ \pi_{2n} : A \xrightarrow{\pi_{2n}} A_{2n} = R_+(GL(2n)) \xrightarrow{r_{Sp} \text{ restr. map}} R(Sp(2n)).
\]
In the case of $G = \text{SO}(n)$, let $\pi_{O(n)}: A \to R(O(n)) (\subset R(\text{SO}(n)))$ be the homomorphism defined by

$$
\pi_{O(n)} = r_{O} \circ \pi_{n}: A \xrightarrow{\pi_{n}\text{ proj. map}} A_{n} = R_{+}(GL(n)) \xrightarrow{r_{O}\text{ restr. map}} R(O(n)).
$$

(Since $GL(n) \supset O(n) \supset SO(n)$, the characters of $SO(n)$ which are obtained by restricting the representations of $GL(n)$ to $SO(n)$ belong to $R(O(n))$. See Sect. 1.1. These three algebra homomorphisms $\pi_{n}, \pi_{Sp(2n)},$ and $\pi_{O(n)}$ are called the specialization homomorphisms.

Then Theorems 1.3.2 and 1.3.3 in Section 1.3 can be rewritten as follows.

**Proposition 2.2.1.** For a partition $\lambda$ of $d(\lambda) \leq n$ we have

1. $\pi_{Sp(2n)}(\chi_{Sp}(\lambda)) = \chi_{Sp(2n)}(\lambda)(t),$
2. $\pi_{O(2n)}(\lambda) = \chi_{O(2n)}(\lambda)(t) = \chi_{SO(2n+1)}(\lambda)(t),$
3. $\pi_{O(2n)}(\lambda) = \chi_{O(2n)}(\lambda)(t).$

Especially due to Propositions 1.2.3, 1.2.4, and 1.2.5, the images of the generators of $A$ under $\pi_{Sp(2n)}$ and $\pi_{O(n)}$ are given by

- $\pi_{O(n)}(p_{i}^{c}) = \pi_{O(n)}(p_{i} - p_{i-2}) = p_{i}^{c}(t) \in R(O(n))$,  
- $\pi_{O(n)}(e_{i}) = e_{i}(t) \in R(O(n))$,  
- $\pi_{Sp(2n)}(p_{i}) = p_{i}(t) \in R(\text{Sp}(2n))$,  
- $\pi_{Sp(2n)}(e_{i}^{c}) = \pi_{Sp(2n)}(e_{i} - e_{i-2}) = e_{i}^{c}(t) \in R(\text{Sp}(2n)).$

We shall determine the kernel of each specialization homomorphism.

**Proposition 2.2.2.** Let $I_{GL(n)} = \langle e_{n+i} \rangle_{i \geq 0}$, $I_{Sp(2n)} = \langle e_{i}^{c} + e_{2n+2-i} \rangle_{i \in \mathbb{Z}}$ and $I_{O(n)} = \langle e_{n-i} - e_{i} \rangle_{i \in \mathbb{Z}}$ be the ideals of $A$. Then we have the following exact sequences.

1. $0 \longrightarrow I_{GL(n)} \longrightarrow A \xrightarrow{\pi_{n}} R_{+}(GL(n)) \longrightarrow 0$ (exact)
2. $0 \longrightarrow I_{Sp(2n)} \longrightarrow A \xrightarrow{\pi_{Sp(2n)}} R(\text{Sp}(2n)) \longrightarrow 0$ (exact)
3. $0 \longrightarrow I_{O(n)} \longrightarrow A \xrightarrow{\pi_{O(n)}} R(O(n)) \longrightarrow 0$ (exact)

**Proof.** (1) The surjectivity of $\pi_{n}$ follows immediately from Proposition 2.2.1(1). We shall prove the exactness at $A$. It is clear that $\text{Ker} \pi_{n}$ includes $I_{GL(n)}$. Therefore a homomorphism $\tilde{\pi}_{n}: A/I_{GL(n)} \to R_{+}(GL(n))$ is induced by
\[ \pi_n. \] It is sufficient to prove that \( \pi_n \) is injective. But this is clear, since \( R_+(GL(n)) = \mathbb{Z}[e_1, e_2, ..., e_n] \) is a polynomial ring.

Proofs of (2) and (3) are the same as above, so we omit them.  

Although it is against the order of the statements in the Introduction, we shall stop describing the specialization homomorphisms for the moment and resume it later in Section 2.4, since we have to use the Character Interrelation Theorem 2.3.1 to describe the specialization images \( \pi_{Sp(2n)}(\chi_{Sp}(\lambda)) \), \( \pi_{O(2n)}(\chi_{O}(\lambda)) \) and \( \pi_{O(2n+1)}(\chi_{O}(\lambda)) \) for a partition \( \lambda \) such that \( d(\lambda) > n. \)

### 2.3. The Character Interrelation Theorem and a Duality

In Section 2.1, we have defined two \( \mathbb{Z} \)-bases of \( \Lambda \), that is \( \{ \chi_O(\lambda) \} \) and \( \{ \chi_{Sp}(\lambda) \} \). In this section, first we are describing the transition matrices from the basis \( \{ \chi_{GL}(\lambda) \} \) to the bases \( \{ \chi_O(\lambda) \} \) and \( \{ \chi_{Sp}(\lambda) \} \), as well as their inverse matrices. Namely let \( (c_{\mu, \lambda})_{\mu \geq \lambda} \) and \( (b_{\mu, \lambda})_{\mu \geq \lambda} \) be the transition matrices from the basis \( \{ \chi_{GL}(\lambda) \} \) to the bases \( \{ \chi_O(\lambda) \} \) and \( \{ \chi_{Sp}(\lambda) \} \), respectively, i.e.,

\[
\chi_{Sp}(\lambda) = \sum_{\mu} c_{\mu, \lambda} \chi_{GL}(\mu), \quad \chi_O(\lambda) = \sum_{\mu} b_{\mu, \lambda} \chi_{GL}(\mu).
\]

Let \( (c'_{\mu, \lambda})_{\mu \geq \lambda} \) and \( (b'_{\mu, \lambda})_{\mu \geq \lambda} \) be their inverse transition matrices, i.e.,

\[
\chi_{GL}(\lambda) = \sum_{\mu} c'_{\mu, \lambda} \chi_{Sp}(\mu), \quad \chi_{GL}(\lambda) = \sum_{\mu} b'_{\mu, \lambda} \chi_O(\mu).
\]

We have already known that for any partition \( \lambda \),

\[
c_{\lambda, \lambda} = b_{\lambda, \lambda} = c'_{\lambda, \lambda} = b'_{\lambda, \lambda} = 1
\]

and that if \( |\mu| \geq |\lambda| \) and \( \mu \neq \lambda \),

\[
c_{\mu, \lambda} = b_{\mu, \lambda} = c'_{\mu, \lambda} = b'_{\mu, \lambda} = 0.
\]

We shall determine the remaining coefficients \( c_{\mu, \lambda}, b_{\mu, \lambda}, c'_{\mu, \lambda}, \) and \( b'_{\mu, \lambda} \) explicitly.

**Theorem 2.3.1 (The Character Interrelation Theorem).** In the algebra \( \Lambda \) the following equalities hold for any partition \( \lambda \):

*Restriction rules*

1. \( \chi_{GL}(\lambda) = \sum_{\mu} \sum_{\kappa} LR_{(2\kappa), \mu}(\lambda) \chi_{Sp}(\mu) \),
2. \( \chi_{GL}(\lambda) = \sum_{\mu} \sum_{\kappa} LR_{2\kappa, \mu}(\lambda) \chi_O(\mu) \).
(Construction rules)

(3) $\chi_{Sp}(\lambda) = \sum_\mu \sum_\alpha (-1)^{|\alpha|} LR_{\tau(\alpha),\mu}^\lambda \chi_{GL}(\mu)$,

(4) $\chi_{O}(\lambda) = \sum_\mu \sum_\alpha (-1)^{|\alpha|} LR_{\tau(\alpha),\mu}^\lambda \chi_{GL}(\mu)$,

where in the above sums $\mu$ and $\kappa$ run over all partitions and $\alpha$ runs over all distinct partitions, i.e., $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i), \alpha_1 > \alpha_2 > \cdots > \alpha_i > 0$.

**Proof.** (1) For a partition $\lambda$ we fix a positive integer $n$ such that $n \geq |\lambda|$ and apply $\pi_{Sp(2n)}$ to the formula

$$\chi_{GL}(\lambda) = \chi_{Sp}(\lambda) + \sum_\mu c'_\mu \chi_{Sp}(\mu).$$

According to Proposition 2.2.1, we have

$$\chi_{GL(2n)}(\lambda) \downarrow_{Sp(2n)}^{GL(2n)} = \chi_{Sp(2n)}(\lambda) + \sum_{|\mu| < |\lambda|} \sum_\mu c'_\mu \chi_{Sp(2n)}(\lambda).$$

Since the $\chi_{Sp(2n)}(\lambda)$ ($d(\lambda) \leq n$) are linearly independent, comparing this with the formula (1) in Proposition 1.5.3, we have $c'_\mu = \sum_\kappa LR_{\tau(\lambda),\kappa}^\mu$.

(3) As in (1) let $n$ be a positive integer satisfying $n \geq |\lambda|$. If we apply $\pi_{Sp(2n)}$ to the formula

$$\chi_{Sp}(\lambda) = \chi_{GL}(\lambda) + \sum_{|\mu| < |\lambda|} c_{\mu, \lambda} \chi_{GL}(\mu),$$

we have

$$\chi_{Sp(2n)}(\lambda) = \chi_{GL(2n)}(\lambda) \downarrow_{Sp(2n)}^{GL(2n)} + \sum_{|\mu| < |\lambda|} c_{\mu, \lambda} \chi_{GL(2n)}(\mu) \downarrow_{Sp(2n)}^{GL(2n)}.$$ (2.3.1)

According to Proposition 1.5.3(1) it follows that

$$\chi_{GL(2n)}(\lambda) \downarrow_{Sp(2n)}^{GL(2n)} = \chi_{Sp(2n)}(\lambda) + \{\text{linear combination of the } \chi_{Sp(2n)}(\mu) \text{ with } |\mu| < |\lambda|\}. $$

Therefore the $\chi_{GL(2n)}(\lambda) \downarrow_{Sp(2n)}^{GL(2n)}$ ($d(\lambda) \leq n$) are linearly independent in $R(\text{Sp}(2n))$. Comparing the formula (2.3.1) with the formula (2) in Proposition 1.5.3, we obtain

$$c_{\mu, \lambda} = \sum_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i), \alpha_1 > \alpha_2 > \cdots > \alpha_i > 0} (-1)^{|\alpha|} LR_{\tau(\alpha),\mu}^\lambda.$$

The proofs of (2) and (4) are the same as above.
Now we are stating a few results following immediately from Theorem 2.3.1.

**Theorem 2.3.2 (Duality of the characters of $Sp$ and $SO$).**

$$\omega(\chi_{Sp}(\lambda)) = \chi_{O}(\lambda).$$

*Remark.* We use the word "duality" here because this theorem implies the following contents.

1. $b_{\mu}^\lambda = c_{\mu}^{\lambda}$, $b_{\mu}^\lambda = c_{\mu}^{\lambda}$.
2. The two kinds of structure constants of $A$, that is, the $c_{\mu}^{\lambda}$ (with respect to the base $\{\chi_{Sp}(\lambda)\}_{\lambda}$) and the $b_{\mu}^\lambda$ (with respect to the base $\{\chi_{O}(\lambda)\}_{\lambda}$) are combined by the following relations:

$$c_{\mu}^{\lambda} = b_{\mu}^{\lambda},$$

since $\omega$ is an algebra automorphism.

*Proof.* If we apply the automorphism $\omega$ to the formula (3) in the Character Interrelation Theorem, we have

$$\omega(\chi_{Sp}(\lambda)) = \sum_{\mu} \sum_{\lambda} (-1)^{|\lambda|} LR_{I, (\lambda), \mu}^{\lambda} \chi_{GL}(\lambda)$$

$$= \sum_{\mu} \sum_{\lambda} (-1)^{|\lambda|} LR_{I, (\lambda), \mu}^{\lambda} \chi_{GL}(\mu).$$

If we rewrite the last expression using $LR_{I, (\lambda), \mu}^{\lambda} = LR_{I, (\lambda), \mu}^{\lambda}$, we have

$$\omega(\chi_{Sp}(\lambda)) = \sum_{\mu} \sum_{\lambda} (-1)^{|\lambda|} LR_{I, (\lambda), \mu}^{\lambda} \chi_{GL}(\mu).$$

Comparing this with the construction rule (4), we have

$$\omega(\chi_{Sp}(\lambda)) = \chi_{O}(\lambda).$$

Now we are expressing $\chi_{Sp}(\lambda)$ and $\chi_{O}(\lambda)$ by polynomials in the generator systems other than $\{p_i\}_{i=1}^{\infty}$. These expressions are very useful for the description of the specialization homomorphisms.

**Theorem 2.3.3.** Let $\lambda$ be a partition of depth $k$ and let $l$ be the depth of $\lambda$. Then we have

$$(1) \quad \chi_{Sp}(\lambda) = \prod_{i} p_{\lambda^{+}(1)} p_{\lambda^{+}(1^{k})} + p_{\lambda^{+}-1^{k}} \cdots p_{\lambda^{+}(k-1)(1^{k})} + p_{\lambda^{+}-(k-1)(1^{k})}$$
\begin{equation}
\chi_O(\lambda) = \left| \begin{array}{c}
-p_{2^*} - p_{2^* - 2(1^k)} - p_{2^* + (1^k)} - p_{2^* - 3(1^k)} \\
\cdots \\
-p_{2^* + (k - 1)(1^k)} - p_{2^* - (k + 1)(1^k)} \\
\end{array} \right|
\end{equation}

\begin{enumerate}
\item[(2)] \begin{equation}
\chi_O(\lambda) = \left| \begin{array}{c}
P_{2^*} + p_{2^* + (1^k)} + p_{2^* - (1^k)} \\
\cdots \\
P_{2^* + (k - 1)(1^k)} + p_{2^* - (k - 1)(1^k)} \\
\end{array} \right|
\end{equation}
\item[(4)] \begin{equation}
\chi_O(\lambda) = \left| \begin{array}{c}
K_{2^*}, \ e_{2^*} + (1^k) + e_{2^* - (1^k)} \\
\cdots \\
K_{2^* + (k - 1)(1^k)} + e_{2^* - (k - 1)(1^k)} \\
\end{array} \right|
\end{equation}
\end{enumerate}

\textbf{Remark.} The reason why we use the \( p_i \) and \( e_i^o \) as the main variables and the \( e_i \) as supplementary variables in the case of \( Sp \), while we use the \( p_i \) and \( e_i \) as the main variables and the \( p_i \) as supplementary variables in the case of \( SO \) is that we take into consideration the irreducibilities of \( p_i \) and \( e_i \) of each group.

\textbf{Proof.} The formulas (1) and (4) are exactly the definitions of \( \chi_{Sp}(\lambda) \) and \( \chi_O(\lambda) \). If we apply \( \omega \) to the formula (1), in view of the duality theorem we have

\begin{equation}
\chi_O(\lambda') = \omega(\chi_{Sp}(\lambda)) = \left| \begin{array}{c}
K_{2^*}, \ e_{2^*} + (1^k) + e_{2^* - (1^k)} \\
\cdots \\
K_{2^* + (k - 1)(1^k)} + e_{2^* - (k - 1)(1^k)} \\
\end{array} \right|
\end{equation}

If we replace \( \lambda \) by \( \lambda' \) in the above formula, we obtain the formula (6). Similarly if we apply \( \omega \) to the formula (4) and replace \( \lambda \) by \( \lambda' \), we have the formula (2). Each column vector appearing in the determinant of the formula (4) can be expressed as follows.

\begin{align}
P_{2^* + j(1^k)} - p_{2^* - (j + 2)(1^k)} \\
= \sum_{i=0}^{j} (p_{2^* + (j - 2i)(1^k)} - p_{2^* + (j - 2(i + 1))(1^k)}) \\
= \sum_{i=0}^{j} p_{2^* + (j - 2i)(1^k)} \\
= p_{2^* + j(1^k)} + p_{2^* + (j - 2)(1^k)} + \cdots + p_{2^* - j(1^k)}.
\end{align}

(2.3.2)

We replace each column vector by the formula (2.3.2). And, using elementary transformations, we start from the last column and subtract suc-
cessively the \((k - 2)\)th column from the \(k\)th column, the \((k - 3)\)th column from the \((k - 1)\)th column, and so on. Then we have the formula (5). The formula (3) follows from the formula (2) through similar arguments.

Let \(\{c_{\mu \nu}^\lambda\}\) and \(\{b_{\mu \nu}^\lambda\}\) be the structure constants of \(A\) with respect to the bases \(\{\chi_{Sp}(\lambda)\}\) and \(\{\chi_{O}(\lambda)\}\), respectively. That is,

\[
\chi_{Sp}(\mu) \cdot \chi_{Sp}(\nu) = \sum_{\lambda} c_{\mu \nu}^\lambda \chi_{Sp}(\lambda),
\]

\[
\chi_{O}(\mu) \cdot \chi_{O}(\nu) = \sum_{\lambda} b_{\mu \nu}^\lambda \chi_{O}(\lambda).
\]

Then the next theorem follows immediately from the above proposition.

**Theorem 2.3.4.** (1) \(\chi_{Sp}(\chi_{Sp}(\lambda)) = \chi_{Sp}(\lambda)\). Therefore \(c_{\mu \nu}^\lambda = c_{\mu \nu}^{\lambda'}\).

(2) \(\chi_{O}(\chi_{O}(\lambda)) = \chi_{O}(\lambda)\). Therefore \(b_{\mu \nu}^\lambda = b_{\mu \nu}^{\lambda'}\).

(3) The two kinds of structure constants with respect to the bases \(\{\chi_{Sp}(\lambda)\}\) and \(\{\chi_{O}(\lambda)\}\) totally coincide. Namely for all \(\lambda, \mu, \nu,
\]
\[
c_{\mu \nu}^\lambda = b_{\mu \nu}^\lambda.
\]

**Remark.** (1) and (2) mean that in \(A\) the two operations (the tensor product and the transpose of Young diagrams) are commutative as in the case of \(\{\chi_{GL}(\lambda)\}\).

**Proof.** If we apply \(i_{Sp}\) to the formula (1) in the above theorem, we have

\[
i_{Sp}(\chi_{Sp}(\lambda)) = [e_{\lambda^* (l)} + e_{\lambda^* -(l)} + \ldots + e_{\lambda^* +(k-1)(l)} + e_{\lambda^* -(k-1)(l)}].
\]

Comparing this with the formula (3) in the above theorem, we have

\[
i_{Sp}(\chi_{Sp}(\lambda)) = \chi_{Sp}(\lambda).
\]

Moreover if we apply \(i_{Sp}\) to \(\chi_{Sp}(\mu) \cdot \chi_{Sp}(\nu) = \sum_{\lambda} c_{\mu \nu}^\lambda \chi_{Sp}(\lambda)\), we have \(c_{\mu \nu}^\lambda = c_{\mu \nu}^{\lambda'}\).

The proof of (2) is the same as above.

We apply \(i_{Sp}\) to \(\chi_{Sp}(\mu) \cdot \chi_{Sp}(\nu) = \sum_{\lambda} c_{\mu \nu}^\lambda \chi_{Sp}(\lambda)\). Since \(i_{Sp}(\chi_{Sp}(\lambda)) = \chi_{O}(\lambda)\), (3) follows.

2.4. The Specialization Homomorphism (Continued)

This section is a continuation of Section 2.2. As we announced in the Introduction, we shall determine here what \(\pi_{O(n + 1)}(\chi_{O}(\lambda))\), \(\pi_{Sp(2n)}(\chi_{Sp}(\lambda))\), and \(\pi_{O(2n)}(\chi_{O}(\lambda))\) actually are, especially if \(d(\lambda) > n\). Our arguments are based on the expressions (3) and (6) in Theorem 2.3.3.

Let us recall the results in Proposition 2.2.2: \(I_{Sp(2n)} = \text{Ker } \pi_{Sp(2n)} = \langle e_{2n+2-i} + e_{i} \rangle_{i \in \mathbb{Z}}\) and \(I_{O(n)} = \text{Ker } \pi_{O(n)} = \langle e_{n-i} - e_{i} \rangle_{i \in \mathbb{Z}}\).

Fix an arbitrary partition \(\lambda\), and put \(k = d(\lambda)\). Also put \(\lambda' = (\lambda'_1, \ldots, \lambda'_t)\).
Case. \( G = \text{SO}(2n + 1) \) (type B).

Let us recall the formula (6) in Theorem 2.3.3:

\[
\chi_{O}(\lambda) = |e_{(\lambda)}^{*}, e_{(\lambda)}^{*} + (1) + e_{(\lambda)}^{*} - (1), \ldots, e_{(\lambda)}^{*} + (l - 1)(1) + e_{(\lambda)}^{*} - (l - 1)(1)|. \tag{6}
\]

We shall rewrite its image under \( \pi_{O(2n + 1)} \) in terms of \( e_{1}, e_{2}, \ldots, e_{n} \) using the fundamental relations: \( \text{Ker } \pi_{O(2n + 1)} = I_{O(2n + 1)} = \langle e_{2n + 1} - e_{i}, i \in \mathbb{Z} \rangle \).

If \( d(\lambda) \leq n \), that is \( \lambda'_{i} \leq n \), then we have

\[
\pi_{O(2n + 1)}(\chi_{O}(\lambda)) = \chi_{O(2n + 1)}(\lambda) = \chi_{\text{SO}(2n + 1)}(\lambda)
\]

Thus we can express \( \chi_{\text{SO}(2n + 1)}(\lambda) \) in terms of \( e_{1}, e_{2}, \ldots, e_{n} \) if we replace the \( e_{n + i} (i \geq 1) \) in the last determinant by the \( e_{n + 1} - i \).

Now let us consider the case where \( d(\lambda) > n \), that is, \( \lambda'_{i} > n \).

The row vectors in the expression (6) have the form

\[
e_{j} = (e_{j}, e_{j+1} + e_{j-1}, \ldots, e_{j+(l-1)} + e_{j-(l-1)})
\]

so that we have

\[
\chi_{O}(\lambda) = \begin{vmatrix}
    e_{\lambda_{1}} \\
    e_{\lambda_{2} - 1} \\
    \vdots \\
    e_{\lambda_{l} - (l-1)}
\end{vmatrix}.
\]

Using the fundamental relations, we have

\[
e_{j} = e_{2n + 1 - j} \mod I_{O(2n + 1)},
\]

since \( e_{j} = e_{2n + 1 - j} \) and \( e_{j+s} + e_{j-s} = e_{2n + 1 - j-s} + e_{2n + 1 - j+s} \mod I_{O(2n + 1)} \).

Hence, we may replace the row vectors \( e_{\lambda_{i} - (i-1)} \) satisfying \( \lambda'_{i} - (i-1) > n \) by \( e_{2n + 1 - \lambda'_{i} + i - 1} \). Thus we have

\[
\chi_{O}(\lambda) = \begin{vmatrix}
    e_{k_{1}} \\
    e_{k_{2} - 1} \\
    \vdots \\
    e_{k_{l} - (l-1)}
\end{vmatrix} \mod I_{O(2n + 1)}, \tag{2.4.1}
\]

where the \( k_{i} \) are determined by the following rules:

- if \( n \geq \lambda'_{i} - (i-1) \) then \( k_{i} = \lambda'_{i} \),
- if \( n < \lambda'_{i} - (i-1) \) then \( k_{i} - (i-1) = 2n + 1 - \lambda'_{i} + (i-1) \),

i.e.,

\[
\lambda'_{i} - (n + \frac{1}{2} + (i-1)) = (n + \frac{1}{2} + (i-1)) - k_{i}.
\]
Let us describe how to get the indices \((k_1, k_2, \ldots, k_r) \in \mathbb{Z}^r\) in terms of Young diagrams. Fold back the \(i\)-th column at the depth \(n + \frac{i}{2} + (i - 1)\) and remove the overlapping squares. Then the length of the remaining column is \(k_i\) (which may be negative).

**Example 1.** \(\lambda = (8, 6, 6, 3)\), \(n = 3\). (See Fig. 7.)

We shall reorder the rows in (2.4.1) according to their indices. Put \(t_i = k_j - (i - 1)\) \((1 \leq i \leq l)\). Since \(n + \frac{i}{2} + (i - 1) \geq k_j\), we have \(n \geq t_i\). If \(t_i = t_j\) for some \(i, j\) \((i \neq j)\) then the determinant (2.4.1) is clearly equal to zero. So hereafter assume \(t_i \neq t_j\) for all \(i, j\) \((i \neq j)\). Arranging the \(t_i\) in the decreasing order, we have \(n \geq t_{i_1} > t_{i_2} > \cdots > t_{i_l}\). Putting \(\mu'_i = t_{i_1}, \mu'_2 = t_{i_2} + 1, \ldots, \mu'_i = t_{i_i} + (l - 1)\), we have \(n \geq \mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_i\) and

\[
\chi_0(\lambda) \equiv \text{sgn} \begin{pmatrix}
1, 2, \ldots, l \\
i_1, i_2, \ldots, i_l
\end{pmatrix}
\left| e_{\mu_i} \right| e_{\mu_i - 1} \cdots e_{\mu_i - (l - 1)} \mod I_{O(2n + 1)}.
\]

If \(\mu'_i < 0\) then the determinant on the right-hand side is clearly equal to zero, and otherwise it is equal to \(\chi_0(\mu)\), where \(\mu = (\mu'_1, \mu'_2, \ldots, \mu'_i)\). Since \(n \geq \mu'_i\) we have \(d(\mu) \leq n\). Hence we have

\[
\chi_0(\lambda) \equiv \text{sgn} \begin{pmatrix}
1, 2, \ldots, l \\
i_1, i_2, \ldots, i_l
\end{pmatrix} \chi_0(\mu) \mod I_{O(2n + 1)}.
\]

In Example 1, \(t_1 = 1, t_2 = 2, t_3 = 3,\) and \(t_4 = 0\). Hence \(t_3 > t_2 > t_4 > t_1\) and \(\mu'_1 = 3, \mu'_2 = 3, \mu'_3 = 2,\) and \(\mu'_4 = 2\). In other words, \(\mu = (4, 4, 2)\). The signature is computed from the inversion number: \((-1)^{1+3} = 1\). Therefore,

\[
\pi_{O(2n + 1)}(\chi_0(4^3, 3^1, 1^2)) = \pi_{O(2n + 1)}(\chi_0(4^2, 2)) = \chi_{SO(2n + 1)}(4, 4, 2).
\]
Case. $G = S_p(2n)$ (type C).

Let us recall the formula (3) in Theorem 2.3.3:

$$
\chi_{Sp}(\lambda) = |e_{(1)}^{(1)} + e_{(1)}^{(1)} + (1) + e_{(1)}^{(1)} - (1) + \ldots + e_{(1)}^{(1)} + e_{(1)}^{(1)} - (1)|.
$$

(3)

This time, we shall describe its image under $\pi_{Sp(2n)}$ in terms of $e_1^0, e_2^0, \ldots, e_n^0$ using the fundamental relations: $\ker I_{Sp(2n)} = \langle e_{i}^0 + e_{2n+2-i}^0, i \in \mathbb{Z} \rangle$.

If $d(\lambda) \leq n$, we have

$$
\pi_{Sp(2n)}(\chi_{Sp}(\lambda)) = \chi_{Sp(2n)}(\lambda)
$$

$$
= |e_{(1)}^{(1)} + e_{(1)}^{(1)} + (1) + e_{(1)}^{(1)} - (1) + \ldots + e_{(1)}^{(1)} + e_{(1)}^{(1)} - (1)|.
$$

In the last determinant we replace $e_{n+1}^0$ by 0 and $e_{n+i}^0 (i \geq 2)$ by $-e_{n+2-i}^0$. Thus we obtain the expression of $\chi_{Sp(2n)}(\lambda)$ as a polynomial in $e_1^0, e_2^0, \ldots, e_n^0$, i.e., the characters of the fundamental representations of $Sp(2n)$.

Next we assume $d(\lambda) > n$, in other words, $\lambda'_i > n$.

Each row in the expression (3) has the form

$$
e_j = (e_j^0, e_{j+1}^0 + e_{j-1}^0, \ldots, e_{j+(t-1)}^0 + e_{j-(t-1)}^0).
$$

So we have

$$
\chi_{Sp}(\lambda) = \begin{vmatrix}
e_{\lambda_{1z}}^0 \\
e_{\lambda_{i-1}^0} \\
\vdots \\
e_{\lambda_{(t-1)}^0}
\end{vmatrix}
$$

By the fundamental relations we have $e_j^0 \equiv -e_{2n+2-j}^0 \mod I_{Sp(2n)}$. Especially, $e_{n+1}^0 \equiv 0 \mod I_{Sp(2n)}$. This shows that, if there exists an $i$ for which $\lambda'_i - (i-1) = n + 1$, then $\pi_{Sp(2n)}(\chi_{Sp}(\lambda)) = 0$. So hereafter we assume $\lambda'_i - (i-1) \neq n + 1$ for all $i (1 \leq i \leq n)$. For every $i$ satisfying $\lambda'_i - (i-1) > n$, we replace $e_{\lambda'_i - (i-1)}^0$ by $-e_{2n+2-\lambda'_i+i-1}^0$. Then we have

$$
\chi_{Sp}(\lambda) \equiv (-1)^{\lambda_{i-1}} \begin{vmatrix}
e_{k_1}^0 \\
e_{k_2}^0 - 1 \\
\vdots \\
e_{k_{t-1}}^0 - (t-1)
\end{vmatrix} \mod I_{Sp(2n)},
$$

where the $k_i$ are determined as follows: if $\lambda'_i - (i-1) \leq n$ then $k_i = \lambda'_i$, while if $\lambda'_i - (i-1) > n + 1$ then $k_i = (i-1) = 2n + 2 - \lambda'_i + i - 1$, in other words,
\[ \lambda'_i - (n + 1 + (i - 1)) = n + 1 + (i - 1) - k_i. \] 
\( s \) denotes the number of \( i \)'s satisfying \( \lambda'_i - (i - 1) > n \).

In terms of Young diagrams, the indices \((k_1, k_2, \ldots, k_i) \in \mathbb{Z}^i\) are obtained as follows: Fold back the \( i \)th column at the depth \( n + 1 + (i - 1) \) and remove the overlapping squares. Then \( k_i \) is the length of the remaining column (possibly negative).

**Example 2.** \( \lambda' = (8, 6, 4, 1) \), \( n = 3 \). (See Fig. 8.)

The only remaining thing to do is to reorder the rows in the nonincreasing order. Put \( t_i = k_i - (i - 1) \) \((1 \leq i \leq l)\). Since \( n + 1 - (i - 1) \neq \lambda_i \), we have \( n + 1 + (i - 1) > k_i \). Hence \( n \geq t_i \). The rest goes in the same way as type B. For Example 2, \( t_1 = 0, t_2 = 3, t_3 = 2, \) and \( t_4 = -2 \), therefore \( t_2 > t_3 > t_1 > t_4 \). So \( \mu_1 = 3, \mu_2 = 3, \mu_3 = 2, \) and \( \mu_4 = 1 \). Hence \( \mu = (4, 3, 2) \), and

\[
\pi_{Sp(2n)}(\chi_{Sp}(4, 3^3, 2^2, 1^2)) = (-1)^2 + 2 \pi_{Sp(2n)}(\chi_{Sp}(4, 3, 2))
= \chi_{Sp(2n)}(4, 3, 2).
\]

**Case.** \( G = SO(2n) \) (type D).

The procedure is exactly the same as that for type B, except one point. That is, the \( i \)th column must be folded back at the depth \( n + (i - 1) \). We simply illustrate the procedure with an example here.

**Example 3.** \( \lambda' = (7, 6, 4, 1) \), \( n = 3 \). (See Fig. 9.)

Since \( t_1 = -1, t_2 = 1, t_3 = 2, \) and \( t_4 = -2 \), we have \( t_3 > t_2 > t_4 > t_1 \), therefore \( \mu'_1 = 2, \mu'_2 = 2, \mu'_3 = 1, \) and \( \mu'_4 = 1 \). Hence \( \mu = (4, 2) \) and

\[
\pi_{O(2n)}(\chi_{O}(4, 3^3, 2^2, 1)) = (-1)^4 \chi_{O(6)}(4, 2)
= \chi_{SO(6)}(4, 2).
\]

We assemble the above results into

**Figure 8**
Proposition 2.4.1 (Description of specialization homomorphisms). Let $\lambda$ be a partition.

(i) $\pi_{O(2n+1)}(\chi_\lambda(\lambda)) = 0$ or $\pm \chi_{O(2n+1)}(\mu_1)$, where $\mu_1$ is obtained from $\lambda$ by the procedure for type B.

(ii) $\pi_{Sp(2n)}(\chi_{Sp}(\lambda)) = 0$ or $\pm \chi_{Sp(2n)}(\mu_2)$, where $\mu_2$ is obtained from $\lambda$ by the procedure for type C.

(iii) $\pi_{O(2n)}(\chi_\lambda(\lambda)) = 0$ or $\pm \chi_{O(2n)}(\mu_3)$, where $\mu_3$ is obtained from $\lambda$ by the procedure for type D.

Corollary 2.4.2. If $d(\lambda) \leq n$, then we obtain the formula to express an irreducible character of each classical group by a determinant of the characters of its fundamental representations.

2.5. Representations of the Classical Groups

In this section we bring the results holding in $A$ to the character ring of each classical group through the specialization homomorphism. We give restriction rules from $GL$ to $SO$ and $Sp$ and decomposition rules of tensor products. Let us recall the situations. In Section 1.1 we have the natural embeddings $SO(2n+1) \subset GL(2n+1)$, $Sp(2n) \subset GL(2n)$, $SO(2n) \subset GL(2n)$. Then the restriction rules in the Character Interrelation Theorem imply that

Proposition 2.5.1 (The restriction rules from $GL$ to $SO$ and $Sp$).

\[
\chi_{GL(2n+1)}(\lambda) \bigg|_{SO(2n+1)} = \sum_{\mu} \sum_{\kappa} LR_{2\kappa,\mu}^2 \pi_{O(2n+1)}(\chi_\lambda(\mu)),
\]
Especially if \( d(\lambda) \leq n \), the restriction rules are independent of the rank \( n \) of the classical groups.

Furthermore we would like to calculate the decomposition of tensor products \( \chi_{Sp(2n)}(\mu) \cdot \chi_{Sp(2n)}(v) \), \( \chi_{O(2n+1)}(\mu) \cdot \chi_{O(2n+1)}(v) \), \( \chi_{O(2n)}(\mu) \cdot \chi_{O(2n)}(v) \), where \( \mu, v \) are partitions of depth \( \leq n \). For that purpose we have only to calculate the structure constants \( b_{\mu\nu}^{\lambda} \) and \( c_{\mu\nu}^{\lambda} \) of the algebra \( A \). Since Theorem 2.3.4(3) states that \( b_{\mu\nu}^{\lambda} = c_{\mu\nu}^{\lambda} \) for all \( \lambda, \mu, v \), it is sufficient to calculate \( c_{\mu\nu}^{\lambda} \).

**Proposition 2.5.2.**

\[
\sum_{\xi, \eta, \beta, \kappa, x, x'} (-1)^{|\xi| + |\eta|} L_{R_{\lambda}(x), \xi}^{\mu} L_{R_{\lambda}(x'), \eta}^{\nu} L_{R_{\lambda}(x), \kappa}^{\beta} L_{R_{\lambda}(x'), \kappa}^{\beta}
\]

Moreover for a partition \( \lambda \) satisfying \( |\lambda| = |\mu| + |v| \), we have \( c_{\mu\nu}^{\lambda} = L_{R_{\lambda}(\lambda)}^{\mu\nu} \).

**Proof.** Let us recall the construction rules in the Character Interrelation Theorem:

\[
\chi_{Sp}(\mu) = \sum_{\xi} \sum_{x} (-1)^{|\xi|} L_{R_{\lambda}(x), \xi}^{\mu} \chi_{GL}(\xi),
\]

\[
\chi_{Sp}(v) = \sum_{\eta} \sum_{x'} (-1)^{|\eta|} L_{R_{\lambda}(x'), \eta}^{\nu} \chi_{GL}(\eta).
\]

Therefore we have

\[
\chi_{Sp}(\mu) \cdot \chi_{Sp}(v) = \sum_{\xi, \eta} \sum_{x, x'} (-1)^{|\xi| + |\eta|} L_{R_{\lambda}(x), \xi}^{\mu} L_{R_{\lambda}(x'), \eta}^{\nu} \chi_{GL}(\xi) \cdot \chi_{GL}(\eta)
\]

\[
= \sum_{\xi, \eta} \sum_{x, x'} (-1)^{|\xi| + |\eta|} L_{R_{\lambda}(x), \xi}^{\mu} L_{R_{\lambda}(x'), \eta}^{\nu} L_{R_{\lambda}(x), \xi}^{\beta} L_{R_{\lambda}(x'), \eta}^{\beta}
\]

\[
= \sum_{\xi, \eta} \sum_{x, x'} (-1)^{|\xi| + |\eta|} L_{R_{\lambda}(x), \xi}^{\mu} L_{R_{\lambda}(x'), \eta}^{\nu} L_{R_{\lambda}(x), \xi}^{\beta} L_{R_{\lambda}(x'), \eta}^{\beta} \chi_{Sp}(\lambda).
\]

Thus we obtain the former part of the proposition. 

\( |\lambda| = |\mu| + |v| \) can occur only in the case \( |\Gamma(x)| = |\Gamma(x')| = |\Gamma(2\kappa)| = 0 \), hence \( \mu = \xi, v = \eta, \beta = \lambda \), and we have \( c_{\mu\nu}^{\lambda} = L_{R_{\lambda}(\lambda)}^{\mu\nu} \).
Corollary 2.5.3. If two partitions $\mu$, $\nu$ satisfy the condition $d(\mu) + d(\nu) \leq n$, we have

$$\chi_{SO(2n+1)}(\mu) \cdot \chi_{SO(2n+1)}(\nu) = \sum_{\lambda} c_{\mu\nu}^{\lambda} \chi_{SO(2n+1)}(\lambda),$$

$$\chi_{Sp(2n)}(\mu) \cdot \chi_{Sp(2n)}(\nu) = \sum_{\lambda} c_{\mu\nu}^{\lambda} \chi_{Sp(2n)}(\lambda),$$

$$\chi_{O(2n)}(\mu) \cdot \chi_{O(2n)}(\nu) = \sum_{\lambda} c_{\mu\nu}^{\lambda} \chi_{O(2n)}(\lambda).$$

Namely if $d(\mu) + d(\nu) \leq n$, then all the decomposition rules of tensor products totally coincide for $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$. (However, in the case of $SO(2n)$, the reader should note that the character $\chi_{O(2n)}(\lambda)$ of $SO(2n)$ is not always irreducible. See Sect. 1.1.)

Proof. In the proof of the above proposition it can be easily verified that nonzero coefficients $c_{\mu\nu}^{\lambda}$ appear only for the partitions $\lambda$ of depth $\leq n$. If we apply the specialization homomorphisms to the formula in the above proposition, we obtain the corollary.

Example. We denote $\chi_{GL}(\lambda)$, $\chi_{Sp}(\lambda)$, and $\chi_{O}(\lambda) \in A$ simply by $\lambda_{GL}$, $\lambda_{Sp}$, and $\lambda_{O}$ respectively. For example $\chi_{Sp}(2, 2) = (2, 2)_{Sp}$, etc. We shall calculate the decomposition of the tensor product $(2, 2)_{Sp} \times (2)_{Sp}$. According to the construction rules, we have $(2, 2)_{Sp} = (2, 2)_{GL} - (1, 1)_{GL}$, and $(2)_{Sp} = (2)_{GL}$. By virtue of Littlewood–Richardson’s rules of $GL$, we have

$$(2, 2)_{GL} \times (2)_{GL} = (4, 2)_{GL} + (3, 2, 1)_{GL} + (2, 2, 2)_{GL}$$

and

$$(1, 1)_{GL} \times (2)_{GL} = (3, 1)_{GL} + (2, 1, 1)_{GL}.$$ 

Hence

$$(2, 2)_{Sp} \times (2)_{Sp} = (4, 2)_{GL} + (3, 2, 1)_{GL} + (2, 2, 2)_{GL}$$

$$- (3, 1)_{GL} - (2, 1, 1)_{GL}. \quad (2.5.1)$$

On the other hand, by the restriction rules we have

$$(4, 2)_{GL} = (4, 2)_{Sp} + (3, 1)_{Sp} + (2)_{Sp},$$

$$(3, 2, 1)_{GL} = (3, 2, 1)_{Sp} + (2, 2)_{Sp} + (2, 1, 1)_{Sp} + (3, 1)_{Sp} + (1, 1)_{Sp} + (2)_{Sp},$$

$$(2, 2, 2)_{GL} = (2, 2, 2)_{Sp} + (2, 1, 1)_{Sp} + (2)_{Sp},$$

$$(3, 1)_{GL} = (3, 1)_{Sp} + (2)_{Sp},$$

$$(2, 1, 1)_{GL} = (2, 1, 1)_{Sp} + (2)_{Sp} + (1, 1)_{Sp}.$$
Carrying out these substitutions in the formula (2.5.1), finally we obtain

\[(2, 2)_{Sp} \times (2)_{Sp} = (4, 2)_{Sp} + (3, 2, 1)_{Sp} + (2, 2, 2)_{Sp} + (3, 1)_{Sp} + (2, 2)_{Sp} + (2, 1, 1)_{Sp} + (2)_{Sp}.\]

Moreover Theorem 2.3.4(3) asserts that the same formula holds for \(SO\), that is,

\[(2, 2)_{O} \times (2)_{O} = (4, 2)_{O} + (3, 2, 1)_{O} + (2, 2, 2)_{O} + (3, 1)_{O} + (2, 2)_{O} + (2, 1, 1)_{O} + (2)_{O}.\]

Applying the specialization homomorphisms, these formulas hold without any change for \(SO(2n + 1)\), \(Sp(2n)\), and \(SO(2n)\) if \(n > 3\). If \(n = 2\), these formulas change into the following:

\[(2, 2)_{Sp(4)} \times (2)_{Sp(4)} = (4, 2)_{Sp(4)} + (3, 1)_{Sp(4)} + (2, 2)_{Sp(4)} + (2)_{Sp(4)} + (2, 2)_{Sp(4)} + (2, 1, 1)_{Sp(4)} + (2)_{Sp(4)}.\]

and

\[(2, 2)_{SO(5)} \times (2)_{SO(5)} = (4, 2)_{SO(5)} + (3, 2)_{SO(5)} + (3, 1)_{SO(5)} + (2, 2)_{SO(5)} + (2, 1)_{SO(5)} + (2)_{SO(5)},\]

and

\[(2, 2)_{O(4)} \times (2)_{O(4)} = (4, 2)_{O(4)} + (3, 1)_{O(4)} + (2)_{O(4)} + (2)_{O(4)} + (2, 2)_{O(4)} + (2, 1, 1)_{O(4)} + (2)_{O(4)}.\]

Some Other Examples of Decomposition of Tensor Products

Since the decomposition of tensor products totally coincides for \(Sp\) and \(SO\), we are stating only the case of \(Sp\). Of course the same formulas hold for \(SO\).

(1) Tensor product with the natural representation \((1)_{Sp}\):

\[\lambda_{Sp} \times (1)_{Sp} = \sum \lambda'_{Sp},\]

where the above sum runs over all the Young diagrams \(\lambda'\) obtained by either adding one square to \(\lambda\) or erasing one square from \(\lambda\).

This follows immediately from the fact that the weight decomposition of \((1)_{Sp(2n)}\) is given by
\( (1)_{\text{Sp}(2n)} = t_1 + t_2 + \cdots + t_n + t_1^{-1} + t_2^{-1} + \cdots + t_n^{-1} \in R(\text{Sp}(2n)). \)

(2) Tensor product of two symmetric characters:

\[
(r)_{\text{Sp}} \times (s)_{\text{Sp}} = \sum_{j=0}^{s} \sum_{i=0}^{j} (r + s - i, j - i)_{\text{Sp}},
\]

where \( r, s \in \mathbb{N} \) and \( r \geq s \).

This follows from the facts that \( (1^0) (r)_{\text{Sp}} = (r)_{\text{GL}} \) and \( (s)_{\text{Sp}} = (s)_{\text{GL}} \), and \( (2^0) (r)_{\text{GL}} \times (s)_{\text{GL}} = \sum_{j=0}^{s} (r + s - j, j)_{\text{GL}} \), and \( (3^0) \) the Young diagrams \( \lambda(2\kappa) \) whose depths are at most 2 are nothing but those given by the partitions \( (i, i), i \in \mathbb{Z}_+ \).

Especially applying \( \pi_{\text{Sp}(2)} \) to the above formula we obtain the Clebsch–Gordan formula for \( \text{Sp}(2, \mathbb{C}) \cong \text{SL}(2, \mathbb{C}) \), since the partitions of depth 2 reduce to zero under \( \pi_{\text{Sp}(2)} \), that is

\[
(r)_{\text{Sp}(2)} \times (s)_{\text{Sp}(2)} = \sum_{j=0}^{s} (r + s - 2j)_{\text{Sp}(2)}.
\]

(3) Tensor product of the character of a fundamental representation and a symmetric character:

\[
(1^r)_{\text{Sp}} \times (s)_{\text{Sp}} = (s + 1, 1^r - 1)_{\text{Sp}} + (s, 1^r)_{\text{Sp}} + (s - 1, 1^r)_{\text{Sp}} + (s, 1^r - 2)_{\text{Sp}},
\]

where \( r, s \in \mathbb{N} \) and \( r > 1 \), and \( s > 1 \). (If \( r = 2 \), we should regard \( 1^0 = \phi \) on the right hand of the above formula.)

This can be proved by induction on \( r \).

(4) Tensor product of two fundamental representations:

\[
(1^r)_{\text{Sp}} \times (1^s)_{\text{Sp}} = \sum_{j=0}^{r} \sum_{i=0}^{j} (2^j, 1^r + s - 2j)_{\text{Sp}},
\]

where, \( r, s \in \mathbb{Z}_+ \) and \( r \geq s \).

This is obtained by applying \( \iota_{\text{Sp}} \) to the formula (2).

Remark. In fact combining (1), (2), (3), (4) (especially (1)) with Remark to Theorem 2.3.4, we can obtain the decomposition formulas of tensor products for two Young diagrams of small sizes.

**Proposition 2.5.3.** (1) The trivial representation of \( \text{Sp}(2n) \) occurs in \( \chi_{\text{SL}(2n)}(\lambda) \downarrow_{\text{Sp}(2n)}^{\text{SL}(2n)} \) if and only if \( \lambda \) is the transpose of an even partition, i.e., \( \lambda = ^t(2\kappa) \).

(2) The trivial representation of \( \text{SO}(n) \) occurs in \( \chi_{\text{SL}(n)}(\lambda) \downarrow_{\text{SO}(n)}^{\text{SL}(n)} \) if and only if either \( \lambda \) or \( \lambda + (1^n) \) is an even partition, i.e., \( \lambda = 2\kappa \) or \( \lambda + (1^n) = 2\kappa \).
Remark. Since \((SL(2n), Sp(2n)), (SL(n), SO(n))\) are symmetric pairs, the multiplicity of the trivial representation is at most one in either case.

Proof. Instead of \(SL\), we may start with \(GL\). According to Section 2.4, for Young diagram \(\lambda\) of \(2n \geq d(\lambda) > n\), \(\pm \pi_{SP(2;2n)}(\chi_{SP(\lambda)})\) cannot be the trivial representation of \(Sp(2n)\). Therefore the restriction rules imply (1) in Proposition 2.5.3.

In the case of \(SO(n)\), for a partition \(\lambda\) of \(n \geq d(\lambda) > [n/2]\), \(\pm \pi_{SO(n)}(\chi_{SO(\lambda)})\) becomes the trivial representation of \(SO(n)\) if and only if \(\lambda = (1^n)\), hence (2) follows.

REFERENCES