A Well-Balanced Flux-Vector Splitting Scheme Designed for Hyperbolic Systems of Conservation Laws with Source Terms

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Abstract—We propose a way to construct robust numerical schemes for the computations of numerical solutions of one- and two-dimensional hyperbolic systems of balance laws. In order to reduce the computational cost, we selected the family of flux vector splitting schemes. We reformulate the source terms as nonconservative products and treat them directly in the definition of the numerical fluxes by means of generalized jump relations. This is applied to a 1D shallow water system with topography and to a 2D simplified model of two-phase flows with damping effects. Numerical results and comparisons with a classical centered discretizations scheme are supplied. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We consider numerical approximations of strictly hyperbolic systems of conservation laws with source terms (also called balance laws) which can be written in one space dimension under the following form:

\[ U_t + F(U)_x = G(U), \quad x \in \mathbb{R}, \quad t > 0 \]  

(1)

together with an initial data which is assumed to have bounded and moreover small total variation. The smooth functions \( F \) and \( G \) are called, respectively, the fluxes and the source terms. The mathematical theory for this class of problems has been carried out in, e.g., [1].

We are mainly interested in the derivation of an efficient numerical process providing a reliable approximation of the entropy solution of (1). There are several classical ways to discretize...
problems of this type. The first one is the fractional step method (see, e.g., [2]) in which one solves at each time step the homogeneous system rendering the convection effects, and then the ordinary differential equations associated with the source terms. This approach is, for example, well suited for the stiff systems one gets out of the relaxation schemes [3-5]. Another way is sometimes referred as the method of lines. It consists in making first a semidiscretization in space of (1) by means of a numerical flux function and a space averaging of the sources, then to solve numerically this differential system. One can also construct directly a Godunov [6] scheme for (1) considering generalized Riemann problems as elementary building blocks. We refer, for example, to [7] for details on this last method. In some cases (for example, in an industrial context), one may use these marching schemes iterated up to convergence in time in order to simulate numerically a stationary equilibrium curve which is expected to be a discretization of the solution of the differential algebraic system \( F(\bar{U})_x = G(\bar{U}) \) associated with some appropriate boundary conditions (see [1] for precise results in this direction and [8] for some numerical remarks). It is a common feature shared by most of these numerical schemes that their ability to reproduce efficiently these steady states curves remains not clear. It turns out that in some delicate situations, this goal can be reached only using a very fine discretization of the space domain, which leads to an expensive computational cost. As an example, one can consider the Euler system for an atmosphere with gravity [9]. Another interesting example is presented in [10] in a totally different context.

Consequently, a new kind of numerical processing of the source terms has been recently proposed in [11] for one-dimensional scalar equations for which the theory of Kruzkov [12] is available. This class of schemes is endowed with the so-called well-balanced property (see Definition 1 in the present paper): this means that convenient discretizations of any steady state curve are invariant under the action of these algorithms. Their main core is the use of some modified homogeneous Riemann problems to derive a Godunov type scheme. The action of the source term is taken into account by some boundary conditions by means of which one can easily ensure the well-balanced property. Moreover, this class of schemes remain stable under the homogeneous CFL restriction, and they converge towards the unique entropy solution at the usual rate [11,13,14].

A first extension to systems of balance laws has been proposed in [15]. The main idea was to convert any problem of form (1) into an augmented one involving a steady variable denoted by \( a \) whose space derivative is equal to one. This way, any source term can be rewritten like \( G(U) \cdot a, \) and this leads to take into account a nonconservative product directly in the numerical fluxes of a Godunov type scheme. To avoid the use of an intricate and computationally expensive nonconservative Riemann solver, a well-balanced Roe type linearized scheme giving satisfactory numerical results has been presented in [15] (see also [16-19]). But its main drawback is the lack of robustness in the stiff cases. Consequently, we introduce here another approach relying on the nonlinear flux vector splitting ideas [20] for which it is possible to ensure both the stability of the scheme independently of the size of the source terms and the well-balanced property. Unfortunately, it does not seem possible up to now to prove any rigorous convergence result for systems as it has been done in the scalar case. Consequently, we will rely mainly on the numerical experiments displayed at the end of this paper to validate our approach.

This work is organized as follows. In Section 2, we summarize some main results concerning well-balanced schemes for scalar problems. We show how to go from the Riemann solvers with boundary conditions to the nonconservative reformulation of the source terms. In Section 3, we propose a framework to construct well-balanced schemes starting from classical flux vector splitting ideas. We detail the generalized jump relations coming out of the new nonconservative terms and show how to handle them in the numerical scheme. Section 4 is devoted to the numerical simulation of a two-dimensional problem modeling a very simplified two-phase flow in a duct. We provide also an appendix to recall some basic results of the theory of nonconservative products [21-23].

This work is part of the author’s Ph.D. Thesis [24].
2. THE WELL-BALANCED SCHEMES FOR
ONE-DIMENSIONAL BALANCE LAWS

2.1. The Case of the Scalar Balance Law

We first consider the Cauchy problem for the following equation:

\[ w + f(u)_x = g(u), \quad \text{with } (x, t) \in \mathbb{R} \times \mathbb{R}_+, \]
\[ g(0) = 0 \quad \text{and} \quad \exists U \in \mathbb{R}, \quad \text{such that } |u| \geq U \Rightarrow u.g(u) \leq 0, \quad \text{(2)} \]
\[ u(x, 0) = u_0 \in BV(\mathbb{R}), \]

where \( f \) and \( g \) are smooth functions and \( BV(\mathbb{R}) \) denotes the space of bounded variation functions \([25]\). In \([12]\), Kruzkov showed the existence and uniqueness of the entropy solution in the space \( L^\infty(0, T; BV(\mathbb{R})) \) for \( (2) \). At the numerical level, the most commonly used schemes for computational approximations were based on splitting techniques solving iteratively the convection step and the ordinary differential equation associated to the right-hand side. Unfortunately, these approaches are not totally satisfactory as it has been quoted in, e.g., \([2,11,26]\). We now briefly recall the Godunov-type scheme proposed in \([14]\). We introduce a computational grid with uniform time-step and mesh size denoted, respectively, \( \Delta t \) and \( \Delta x \). Following standard notations, we denote by \( u^n_j \) the approximation of the local average of \( u(x, n\Delta t) \) on a spatial mesh element. The scheme consists in solving in each computational cell \( [(j - 1/2)\Delta x, (j + 1/2)\Delta x] \times [n\Delta t, (n + 1)\Delta t] \) a Riemann problem for the following equation (cf. \([11]\)):

\[ \frac{\partial u}{\partial t} + f(u)_x - g(u)u_x = 0, \]

where \( a(x) = x \). The solution of this problem is composed of two waves: one is associated with the homogeneous equation \( (i.e., g(v) \equiv 0) \), the other is a steady contact discontinuity modeling the action of the nonconservative term. We introduce now a function \( \phi \) such that

\[ \phi'(v) = \frac{f'(v)}{g(v)}. \]

If we assume that \( f' \geq \alpha > 0 \) in the domain of interest in order to avoid any problem with nonlinear resonant situations, we are led to solve in each cell the following initial-boundary value problem:

\[ \frac{\partial v}{\partial t} + f(v)_x = 0, \quad \text{for } x \in \left[ \left( j - \frac{1}{2} \right) \Delta x, \left( j + \frac{1}{2} \right) \Delta x \right], \quad t \in [n\Delta t, (n + 1)\Delta t] \]
\[ v(x, n\Delta t) = u^n_j; \quad v \left( \left( j - \frac{1}{2} \right) \Delta x, t \right) = u^n_{j-1/2}, \quad \text{(3)} \]

where \( u^n_{j-1/2} \) is defined by

\[ \phi \left( u^n_{j-1/2} \right) - \phi \left( u^n_{j-1} \right) = \Delta x, \quad \text{if } g \left( u^n_{j-1} \right) \neq 0, \]
\[ u^n_{j-1/2} = u^n_{j-1}, \quad \text{if } g \left( u^n_{j-1} \right) = 0. \]

The unusual jump relation on \( \phi(u) \) comes from the fact that the local averages of \( v \) present at each interface a discontinuity of size \( \Delta x \). One notices that the function \( \phi \) exhibits vertical asymptotic lines as \( u \) approaches a zero of \( g \). This means in particular that it is always possible to perform a jump of size \( \Delta x \) along the curve \( u \mapsto \phi(u) \) in this area; of course, the change for \( u \) will be very small.

The solution of \( (3) \) is given by a \( L^1 \)-contraction semigroup \([27]\): \( v(., t) = S(t)v(., 0) \), and consequently, we can define an approximation of \( u(., t) \) by the following process:

\[ u^n_{\Delta x}(., t) = S(t - n\Delta t) \left[ P^{\Delta x} \circ S(\Delta t) \right]^n P^{\Delta x}(u_0), \quad \text{(4)} \]

where \( n \) denotes the integer part of \( t/\Delta t \) and \( P^{\Delta x} \) the projector onto piecewise-constant functions

\[ P^{\Delta x}(u_0) = \left( \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_0(x) \frac{dx}{\Delta x} \right)_{j \in \mathbb{Z}}. \]
THEOREM 1. (See [13,14].) If $u_0 \in L^1 \cap BV(\mathbb{R})$ is such that $g(u_0)$ has a compact support, then scheme (3),(4) is stable in $L^\infty(0,T;BV(\mathbb{R})), T > 0$ under the homogeneous CFL condition

$$\sup_{j,n} |f'(u^n_j)| \Delta t \leq \Delta x.$$ (5)

Moreover, the following error estimate holds:

$$\forall t \in [0,T], \quad \left\| u^{\Delta x}(.t) - u(.t) \right\|_{L^1(\mathbb{R})} \leq e^{NT} \cdot O \left( TV(u_0) \sqrt{\Delta x} + \Delta x \right),$$ (6)

where $u$ is the unique entropy solution of (2) and $N = \max\{g'(|\xi|), |\xi| \leq \max\{\|u\|_{L^\infty}, \|u^{\Delta x}\|_{L^\infty}\}\}$.

We can state now a precise definition of this particular class of numerical schemes.

DEFINITION 1. Let $\bar{u} \in BV(\mathbb{R})$ be any steady-state solution of (2). A numerical scheme is said to be well-balanced (WB) if it leaves invariant $PAX(\bar{u})$ for any value of $\Delta x > 0$ under a given CFL condition.

We remark that the WB property obviously holds for the scalar scheme (3),(4). In fact, at least for any smooth steady state solution of (2), the differential equation $f(\bar{u})_x = g(\bar{u})$ holds with an appropriate initial datum at infinity. This means in particular that for all $j \in \mathbb{Z}$, we have $u^-_{j-1/2} = u^+_{j}$ and then $u^+_{j+1/2} = u^+_{j}$ under the homogeneous CFL restriction (5). Definition 1 admits clearly a straightforward extension to the case of systems.

2.2. A Nonconservative Version of the Scalar Well-Balanced Scheme

Shortly, we present the way we propose to make a link between the preceding ideas and the nonconservative formalism of [22]. Let us consider the following problem:

$$u_t + f(u)_x - g(u)a^{\Delta x} = 0$$

$$u_0 \in BV(\mathbb{R})$$

$$a^{\Delta x}(x) = jAx, \text{ for } x \in \left( j \frac{1}{2} \Delta x, (j + \frac{1}{2}) \Delta x \right).$$ (7)

Seeking a regularization of the ambiguous product $g(u)a^{\Delta x}$, one may define it as a Borel measure built on a locally Lipschitzian paths family $\Phi$ and denoted by $[g(u)a^{\Delta x}]_\Phi$. (In the Appendix, we recall some basic results about the theory of nonconservative products.) The path $\Phi$ is, therefore, written for a $BV$ function with two different kinds of discontinuities: the first one comes from the convection term and verifies classical Rankine-Hugoniot relations, but the second one is really a nonconservative one and renders locally the action of the source term. Consequently, we introduce the augmented problem

$$u_t + f(u)_x - [g(u)a^{\Delta x}]_\Phi = 0,$$

$$u_0^{\Delta x} = 0.$$ (8)

We denote $V = (u, a^{\Delta x})$ the unknowns vector. System (8) is nonconservative, nonstrictly hyperbolic with eigenvalues 0 and $f'(u)$. The field induced by 0 is obviously linearly degenerate: it gives a stationary contact discontinuity which is called the standing wave in [28]. According to [22], we can write generalized jump relations for (8) where the notation $[.]$ denotes the jump of a quantity across a line of discontinuity

$$\sigma[a^{\Delta x}] = [g(u)a^{\Delta x}]_\Phi.$$

Defining two functions $\psi(u) = \int^u \frac{dw}{g(w)}$ and $\phi(u) = \int^u \frac{f'(w)}{g(w)} dw$, we are able to write classical jump relations

$$\sigma[a^{\Delta x}] = 0,$$

$$\sigma[\psi(u)] = [\phi(u)] - [a^{\Delta x}].$$ (9)

Consequently, $\Phi$ has to be chosen in such a way that (9) and (10) admit the same steady contact discontinuities. An answer is given by the following result.
THEOREM 2. (See [15].) Suppose \( a^{\Delta x} \in BV_{\text{loc}}(\mathbb{R}) \) and \( u \mapsto (g(u)/f'(u)) \) is locally Lipschitz. For given \( u_L, a_L^{\Delta x}, a_R^{\Delta x} \), we introduce the following regularizations defined on the interval \( x \in [0, \Delta x] \):

\[
\bar{a}(x) = a_L^{\Delta x} + \left( a_R^{\Delta x} - a_L^{\Delta x} \right) \frac{x}{\Delta x},
\]

\[
\bar{u}(x) \text{ solution of } f(\bar{u})_x = g(\bar{u}) \bar{a}_x; \bar{u}(0) = u_L.
\]

Then, if \( u_R = \bar{u}(\Delta x) \), the family of paths \( \Phi \)

\[
[0,1] \ni s \mapsto \Phi \left( s; \left( \frac{u_L}{a_L^{\Delta x}} \right), \left( \frac{u_R}{a_R^{\Delta x}} \right) \right) = \left( \bar{u}(s \Delta x) \right)
\]

satisfies the requirements of Definition 2, in the Appendix. Moreover, every pair \( (V_L, V_R) \) such that \( u_R = \bar{u}(\Delta x) \) satisfying the generalized relations (9) satisfies also Rankine-Hugoniot conditions (10).

Consider a pair \( (V_L, V_R) \) such that \( a_R^{\Delta x} - a_L^{\Delta x} = \Delta x \) and \( \bar{u}(\Delta x) = u_R \). By definition, we have \( f(\bar{u})_x = g(\bar{u}) \bar{a}_x = g(\bar{u})[a^{\Delta x}] / \Delta x \). Assuming that \( g(u_L) \neq 0 \), we get

\[
\frac{f'(\bar{u})}{g(\bar{u})} \bar{a}_x = \phi(\bar{u})_x = \frac{[a^{\Delta x}]}{\Delta x}.
\]

An integration between \( x = 0 \) and \( x = \Delta x \) leads to \( [\phi(u)] = [a^{\Delta x}] \) which appears in both (3) and (10). If \( g(u_L) = 0 \), then thanks to the Lipschitz regularity of \( g/f' \), we just deduce that \( [u] = 0 \) (see (3)).

The regularity condition on \( g/f' \) means in particular that resonant regimes are excluded from this framework. We refer to [28] for a study of resonance in the context of balance laws. At least under the hypothesis \( f'(\bar{u}) \neq 0 \), the Riemann problem for (8) can be solved uniquely by wave curves intersection in the phase plane. According to (12), one has to join the states \( V_L = (u_L, a_L^{\Delta x}) \) and \( V_R = (u_R, a_R^{\Delta x}) \) thanks to a medium state \( V_M = (u_M, a_L^{\Delta x}) \) lying on the integral curve of \( f(\bar{u})_x = g(\bar{u}) \bar{a}_x \) coming from \( V_L \).

2.3. An Extension to Strictly Hyperbolic Systems of Balance Laws

We go one step further considering the following system where \( F \) and \( G \) are smooth \( C^1 \) functions:

\[
U_t + F(U)_x = G(U) \text{ with } (x,t) \in \mathbb{R} \times \mathbb{R}^+_t,
\]

\[
U(x,0) = U_0 \in [BV(\mathbb{R})]^N. \quad (13)
\]

We propose to regularize the nonconservative terms \( G(U) a_x \) by an integral curve of the steady-state system \( F(U)_x = G(U) \). The same way, we obtain a similar nonstrictly hyperbolic system operating on the augmented unknowns vector \( V = (U, a^{\Delta x}) \)

\[
U_t + F(U)_x = [G(U)a^{\Delta x}]]_x = 0, \quad a^{\Delta x}_t = 0. \quad (14)
\]

The regularizing family of paths is now

\[
\Phi \left( s, \frac{U_L}{a_L^{\Delta x}}, \frac{U_R}{a_R^{\Delta x}} \right) = \left( \tilde{U}(s \Delta x) \right),
\]

with \( \tilde{U} \) the solution of \( F(\tilde{U})_x = G(\tilde{U}) \tilde{a}_x; \tilde{U}(0) = U_L \)

\[
\text{and } a(x) = a_L^{\Delta x} + \left( a_R^{\Delta x} - a_L^{\Delta x} \right) \frac{x}{\Delta x}.
\]

At this level, a Godunov-type scheme for (13) relying on Riemann problems of type (14) is totally determined by the regularization (15). The major drawback for this approach is the
complexity of the nonlinear algebraic system one has to solve to derive the numerical fluxes at each interface. One possible way to circumvent this difficulty is to introduce linearized approximate Riemann solvers as building blocks for the numerical scheme. This matches the approach proposed in [15]. However, in this case one loses part of the robustness inherent to the scalar scheme (3), (4). Consequently, we present in the next section the derivation of an efficient robust and well-balanced numerical scheme for which one avoids intricate and computationally expensive calculations.

3. A WELL-BALANCED FLUX-SPLITTING NUMERICAL SCHEME

3.1. Introduction of Nonconservative Riemann Problems

We are now interested in the derivation of a robust well-balanced scheme to compute approximate solutions of the Cauchy problem for the following strictly hyperbolic system:

\[ U_t + F(U)_x = G(U), \quad \text{with } (x, t) \in \mathbb{R} \times \mathbb{R}^+, \]
\[ U(x, 0) = U_0 \in [BV(\mathbb{R})]^N. \]

As in the scalar case, we introduce a uniform discretization in space and time by means of the parameters \( \Delta x \) and \( \Delta t \) denoting, respectively, the cells width and the time step. We assume that the initial data for the numerical scheme is obtained by taking the local averages of \( U_0 \), which means

\[ (U^0_j)_{j \in \mathbb{Z}} = \mu_{\Delta x}(U_0) \]

and we want to derive a numerical scheme able to generate at each time \( t^n = n\Delta t \) a piecewise constant approximation \( U^{\Delta x} \) of \( U \) solution of problem (16). It is possible to derive a flux splitting assuming only smoothness for the function \( U \rightarrow F(U) \) (cf. [29]), that is,

\[ \forall U \in \mathbb{R}^N, \quad F(U) = F^+(U) + F^-(U), \quad \pm \Lambda(dF^\pm(U)) \subset \mathbb{R}^+ \]

where \( \Lambda(dF^\pm(U)) = \{ \lambda^\pm(U(i)) \}_{i=1,...,N} \) denotes the set of eigenvalues of each Jacobian matrix \( dF^\pm(U) \). In the homogeneous case for which \( G(U) \equiv 0 \), the flux-splitting approach can be introduced by taking in each mesh cell the average of the solutions of these two Riemann problems:

\[ U^+_t + F^+(U^+)_x = 0, \]
\[ (x, t) \in \left( j - \frac{1}{2} \right) \Delta x, \left( j + \frac{1}{2} \right) \Delta x \times [n\Delta t, (n + 1)\Delta t] \]
\[ U^+(x, n\Delta t) = \begin{cases} U^n_{j-1}, & \text{for } x \leq \left( j - \frac{1}{2} \right) \Delta x, \\ U^n_j, & \text{for } x > \left( j - \frac{1}{2} \right) \Delta x, \end{cases} \]

and

\[ U^-_t + F^-(U^-)_x = 0, \]
\[ (x, t) \in \left( j - \frac{1}{2} \right) \Delta x, \left( j + \frac{1}{2} \right) \Delta x \times [n\Delta t, (n + 1)\Delta t] \]
\[ U^-(x, n\Delta t) = \begin{cases} U^n_{j+1}, & \text{for } x < \left( j + \frac{1}{2} \right) \Delta x, \\ U^n_j, & \text{for } x \geq \left( j + \frac{1}{2} \right) \Delta x. \end{cases} \]
Thanks to the sign assumption for the eigenvalues of the Jacobians $dF^\pm(U)$, their solutions have a very simple structure and a numerical scheme can be easily deduced. We define

$$u_{jn+l} = \frac{1}{j-1} \left( \int_{x=(j-1)Ax}^{x=(j)Ax} U^+(x, (n+1)\Delta t) \, dx + \int_{x=(j)Ax}^{x=(j+1/2)Ax} U^-(x, (n+1)\Delta t) \, dx \right)$$

and under the CFL condition

$$\sup_{i,j,n} \left| \frac{\Delta x}{\Delta t} \right| \leq \frac{1}{2},$$

we get the following numerical scheme:

$$u_{jn+l} = u^n - 2 \left( F^+ (U^n) - F^+ (U_{j-1}^n) - F^- (U_{j+1}^n) - F^- (U_j^n) \right).$$

In order to extend to inhomogeneous problems, we are about to follow the same type of ideas. The main change will occur at the level of the elementary Riemann problems. More precisely, going back to the original system (16), we first introduce as in the former section the augmented unknowns vector $V = (U, a^{Ax})$ where $a^{Ax}$ still denotes the piecewise constant function written in (7). We are about to concentrate the effects of the source term at the borders of the space cells as in the scalar scheme; that means that the elementary Riemann problems we have to consider now are of the type

$$U^+_t + F^+ (U^+) \cdot x = \left[ G^+ (U^+) a^{Ax} \right] \Phi = 0,$$

$$(x, t) \in \left[ \left( j - \frac{1}{2} \right) Ax, j Ax \right] \times [n \Delta t, (n+1)\Delta t]$$

and

$$U^-_t + F^- (U^-) \cdot x = \left[ G^- (U^-) a^{Ax} \right] \Phi = 0,$$

$$(x, t) \in \left[ j Ax, \left( j + \frac{1}{2} \right) Ax \right] \times [n \Delta t, (n+1)\Delta t]$$

where the functions $G^\pm$ are to be determined in a convenient way. Of course, the family of paths $\Phi$ is still the one introduced in (15).

**Lemma 1.** Assume that $F^\pm \in C^1(\mathbb{R}^N)$ and $U \cdot dF(U)^{-1}G(U)$ is locally Lipschitz in $\Omega \subset \mathbb{R}^N$, there exists a unique couple of locally Lipschitz functions $G^\pm$ such that for any $\Delta x > 0$ :

$$\forall U \in \mathbb{R}^N, \quad G^+(U) + G^-(U) = G(U),$$

$$[F(U)] = [G(U)a^{Ax}] \Phi \text{ implies } [F^+(U)] = [G^+(U)a^{Ax}] \Phi,$$

$$[F(U)] = [G(U)a^{Ax}] \Phi \text{ implies } [F^-(U)] = [G^-(U)a^{Ax}] \Phi.$$

They are given by

$$\forall U \in \Omega, \quad G^\pm(U) = -dF^\pm(U) \cdot dF(U)^{-1}G(U).$$

**Proof.** We consider the stationary generalized jump relation (see (50) in the Appendix) between $V_L = (U_L, a^{Ax}_L)$ and $V_R = (U_R, a^{Ax}_R)$ for systems (14) and (15):

$$0 = F(U_R) - F(U_L) - [G(U)a^{Ax}] \Phi = \int_{0}^{1} \left( \begin{array}{cc} dF^- & -G^- \\ 0 & -G^+ \end{array} \right) (\Phi(s; V_L, V_R)) \cdot \frac{\partial \Phi}{\partial s}(s; V_L, V_R) \, ds,$$
where
\[ \frac{\partial \Phi}{\partial s}(s; V_s, V_h) = \frac{\partial}{\partial s} \left( \frac{\bar{U}(s \Delta x)}{\bar{a}(s \Delta x)} \right) \in \mathbb{R}^{N+1}. \]
Since \( \frac{\partial a}{\partial x} = a^2 \frac{\partial^2}{\partial x^2} = \Delta x \), \( U_l = \bar{U}(0) \), and \( U_R = \bar{U}(\Delta x) \), this leads to
\[ F\left( \bar{U} \right)(\Delta x) - F\left( \bar{U} \right)(0) = \int_0^{\Delta x} G\left( \bar{U} \right)(s \Delta x) \Delta x \, ds = \int_0^{\Delta x} G\left( \bar{U} \right)(x) \, dx, \]
and then to
\[ \int_0^{\Delta x} F\left( \bar{U} \right)_{\Delta x} - G\left( \bar{U} \right) \, dx = 0. \] (20)
Now we consider one of the splitted problems (18) \( U + F^+(U)_{x} - [G^+(U)a^2]_{\Phi} = 0 \). The steady jump relations read: \( [F^+(U)] = [G^+(U)a^2]_{\Phi} \). So, using the definition of \( \Phi \), the same way we derive
\[ F^+\left( \bar{U}^+ \right)(\Delta x) - F^+\left( \bar{U}^+ \right)(0) = \int_0^{\Delta x} G^+\left( \bar{U}^+ \right)(x) \, dx \]
and then
\[ \int_0^{\Delta x} F^+\left( \bar{U}^+ \right)_{\Delta x} - G^+\left( \bar{U}^+ \right) \, dx = 0. \] (21)
The WB property will be ensured if (21) preserves the microscopic profiles \( x \mapsto \bar{U}(x) \) given by (20) for any value of \( \Delta x > 0 \). Therefore, we can derive these two expressions with respect to the parameter \( \Delta x \) and the only choice for \( G^+ \) according to the underlying differential equations is given by \( G^+(U) = (dF^+ \cdot dF^{-1} \cdot G)(U) \). The same way we define \( G^+(U) = (dF^+ \cdot dF^{-1} \cdot G)(U) \) and the first property of (19) is obvious considering the requirements on \( F^\pm \).

Taking once again into account the sign requirements on the eigenvalues of \( dF^\pm(U) \), we see that the Riemann problems (18) boil down to
\[ U^+_t + F^+(U^+)_x = 0, \]
\[ (x,t) \in \left[ \left( j - \frac{1}{2} \right) \Delta x, j \Delta x \right] \times [n \Delta t, (n + 1) \Delta t] \]
\[ U^+(x,n \Delta t) = \begin{cases} \frac{U^+_{j-1/2}}{2}, & \text{for } x \leq \left( j - \frac{1}{2} \right) \Delta x, \\ U^+_{j+1/2}, & \text{for } x > \left( j - \frac{1}{2} \right) \Delta x, \end{cases} \]
and
\[ F^+\left( U^+_{j-1/2} \right) - F^+\left( U^+_{j+1/2} \right) = \left[ G^+(U)a^2 \right]_{\Phi}. \]
\[ \int_0^{\Delta x} F^+\left( \bar{U}^+ \right)_{\Delta x} - G^+\left( \bar{U}^+ \right) \, dx = 0. \]
These nonconservative problems are nonstrictly hyperbolic: they have a solution in the class of Lax if we assume no interference between the genuinely nonlinear fields and the steady nonconservative discontinuity [30,31]. At this point, we are in position to carry out the last steps exactly.
the same way as in the homogeneous case. The resulting (formally) first-order numerical scheme (see Figure 1) reads

$$U_{j+1}^{n+1} = U_{j}^{n} - \frac{\Delta t}{\Delta x} \left\{ F^{+} \left( U_{j}^{n} \right) - F^{+} \left( U_{j-1/2}^{n} \right) + F^{-} \left( U_{j+1/2}^{n} \right) - F^{-} \left( U_{j}^{n} \right) \right\}$$

(23)

and remains clearly stable under the same type of CFL assumption than (17).

Figure 1. Modified Riemann problems in the flux-vector splitting framework.

It is always possible to consider several higher-order extensions of this kind of numerical schemes using for instance the MUSCL [32] or PPM [33] reconstruction approaches as it is of use for classical homogeneous flux-vector splitting schemes.

3.2. Properties of the Proposed One-Dimensional Numerical Scheme

The main core of the scheme (23) is, therefore, the handling of the source term by means of the generalized nonconservative jump relations written in (22) which give rise to the states $U_{j \pm 1/2}^{\pm}$. Considering the definition of $\Phi$ (15), we can state the following result.

**Lemma 2.** Assume that the function $U \mapsto dF(U)^{-1}G(U)$ is locally Lipschitz, then the two Riemann problems (22) are equivalent to the following ones:

$$U_{j}^{n+1} + F^{+}(U_{j}^{n}) = 0,$$

$$(x, t) \in \left[ \left( j - \frac{1}{2} \right) \Delta x, \left( j + \frac{1}{2} \right) \Delta x \right] \times [n\Delta t, (n + 1)\Delta t]$$

$$U^{+}(x, n\Delta t) = \begin{cases} U_{j-1/2}^{+}, & \text{for } x \leq \left( j - \frac{1}{2} \right) \Delta x, \\ U_{j}^{n}, & \text{for } x > \left( j - \frac{1}{2} \right) \Delta x, \end{cases}$$

$$U_{j-1/2}^{+} = \tilde{U}^{+}(\Delta x), \quad \text{with } \left\{ F \left( \tilde{U}^{+} \right)_{x} = G \left( \tilde{U}^{+} \right) \tilde{a}_{x}, \right\}$$

and

$$U_{j}^{n+1} + F^{-}(U_{j}^{n}) = 0,$$

$$(x, t) \in \left[ \left( j - \frac{1}{2} \right) \Delta x, \left( j + \frac{1}{2} \right) \Delta x \right] \times [n\Delta t, (n + 1)\Delta t]$$

$$U^{-}(x, n\Delta t) = \begin{cases} U_{j}^{n}, & \text{for } x < \left( j + \frac{1}{2} \right) \Delta x, \\ U_{j+1/2}^{-}, & \text{for } x \geq \left( j + \frac{1}{2} \right) \Delta x, \end{cases}$$

$$U_{j+1/2}^{-} = \tilde{U}^{-}(-\Delta x), \quad \text{with } \left\{ F \left( \tilde{U}^{-} \right)_{x} = G \left( \tilde{U}^{-} \right) \tilde{a}_{x}, \right\}$$

(24)
PROOF. We just have to check that (24) and (22) share the same stationary contact discontinuities. Since $H(U) = G(U)$ is locally Lipschitz, the profiles $\tilde{U}$ in (24) are smooth and we can write for instance: $F^+(\tilde{U}) = G^+(\tilde{U})\tilde{a}_x$. It remains to notice that $\tilde{a}_x \equiv 1$ and to integrate between $x = 0$ and $x = \Delta x > 0$ to derive the stationary generalized jump relation:

$$0 = [\tilde{F}^+(U)] - \int_0^{\Delta x} \tilde{G}^+(\tilde{U}(x)) \, dx = \int_0^1 \left( \frac{dF^+}{0} - \frac{G^+}{0} \right) (\Phi(s; V_L, V_R)) \cdot \frac{\partial \Phi}{\partial s} (s; V_L, V_R) \, ds$$

Consequently, the scheme (23) reduces to the following one:

$$U^{n+1}_j = U^n_j - \frac{\Delta x}{\Delta t} \left\{ F^+ (U^n_j) - F^+ (U^{n+1/2}_j) + F^- (U^{n+1/2}_j) - F^- (U^n_j) \right\},$$

where $U^{n+1/2}_j$ and $U^{n-1/2}_j$ are given by

$$U^{n+1/2}_j = \bar{U}^+(\Delta x), \quad \text{with} \quad \begin{cases} F(\bar{U}^+) = G(\bar{U}^+) \bar{a}_x, \\ \bar{U}^+(0) = U^n_{j-1} \end{cases}$$

$$U^{n-1/2}_j = \bar{U}^-(\Delta x), \quad \text{with} \quad \begin{cases} F(\bar{U}^-) = G(\bar{U}^-) \bar{a}_x, \\ \bar{U}^-(0) = U^n_{j+1} \end{cases}$$

for which the following property holds.

**Theorem 3.** Under the hypotheses of Lemma 1 and the homogeneous CFL condition (17), the scheme (25) is well balanced in the sense of Definition 1: if the initial states $(U^0_j)_{j \in \mathbb{Z}}$ are such that $F(U^0_{j+1}) - F(U^0_j) = [G(U)\Delta x]^+_s$ for all $j \in \mathbb{Z}$ and $\Delta x > 0$, then we have $U^{n+1}_j = U^n_j$, for all $(j,n) \in \mathbb{Z} \times \mathbb{N}$.

Written like (25), this scheme appears clearly as a very natural extension of the scalar scheme (3),(4). Moreover, one easily sees that it remains stable under the same CFL condition (17) since the action of the sources is handled as generalized jump relations without any consequence upon the choice of the time step $\Delta t$.

As an illustrative example, we consider the shallow water equations with topography (see, e.g., [34]) modeling the flow of an incompressible fluid in a channel of rectangular cross-section

$$h_t + (hu)_x = 0,$$

$$(hu)_t + \left( hu^2 + \frac{h^2}{2} \right)_x = -h_q_x, \quad x \in \mathbb{R}, \quad t > 0.$$  \hspace{2cm} (26)

For this system, $U = (h, hu) \in \mathbb{R}_+^* \times \mathbb{R}$ is the unknowns vector, $F(U) = (hu, hu^2 + h^2/2)$ is the fluxes vector and $G(U) = (0, -h_q_x)$ is the source term. The notations are classical: $h$ denotes the sum of the free surface elevation and the undisturbed depth of fluid, $u$ is the velocity of the fluid, and the given function $q(x)$ describes the topography of the bottom. In this particular case, it is even not necessary to introduce any new function in order to derive a WB scheme since we already have an $x$-derivative in the right-hand side of (26). Consequently, it is straightforward to derive a nonconservative reformulation associated to the family of paths $\Phi$

$$h_t + (hu)_x = 0,$$

$$(hu)_t + \left( hu^2 + \frac{h^2}{2} \right)_x + [h_q_x]_x = 0, \quad \text{with} \quad h_q_x = 0.$$  \hspace{2cm} (27)

The steady jump relations are given by integration along the microscopic profile which satisfies (see (15))

$$(hu)_x = 0,$$

$$\left( hu^2 + \frac{h^2}{2} \right)_x + h_q_x = 0.$$
This leads to a particular case of the Theorem of Bernoulli:

\[
|hu| = 0,
\]

\[
\frac{u^2}{2} + h + q = 0.
\]  

(28)

Then, scheme (25) reduces to

\[
U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x} \left\{ F^+(U^n_j) - F^+(U^n_{j-1/2}) + F^-(U^n_{j+1/2}) - F^-(U^n_j) \right\}.
\]  

(29)

In this expression, \(F^\pm\) are some appropriate splitted fluxes for system (26) [20,30,35] and relations (28) hold between the states \(U^n_{j-1}, U^n_{j-1/2}\) on the left side and \(U^n_{j+1/2}, U^n_{j+1}\) on the right side (see Figure 1).

We will compare this approach with the very classical one given by the following scheme:

\[
U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x} \left\{ F^+(U^n_j) - F^+(U^n_{j-1}) + F^-(U^n_{j+1}) - F^-(U^n_j) \right\} - \Delta t \left( \frac{0}{h^a_j q_c(j \Delta x)} \right).
\]  

(30)

We performed some numerical runs on a standard test case where \(q(x) = Q_0(1 - \tanh(R(x - 1/2)))\) and \(R = 10\) (cf., e.g., [36]):

\[
Q_0 = 2.5,
\]

\[
u_0 = 0.6,
\]

\[
h + q = 10.
\]

The numerical results are shown in Figures 2 and 3 displaying, respectively, the free surface elevation over the topography and the mass flow rate at time \(T = 0.12\) for both schemes (29),(30). We ran the WB scheme with \(\Delta x = 0.02\) (50 grid points) and the classical one with \(\Delta x = 0.02, \Delta x = 0.005\). We kept constant the fraction \(\Delta t / \Delta x = 0.04\) for both of them. Therefore, it is possible to compare the numerical results, especially the consistency with theoretical relations (28). It is noticeable on Figure 2 that the classical approach converges to the WB one as the grid is refined. Looking at the mass flow rates on Figure 3, one sees clearly that the classical scheme exhibits a spurious bump at the location of the topography variation. This feature disappears as \(\Delta x\) is decreased. On the other hand, the WB scheme is far more accurate, even on a grid four times coarser as it is shown on Figure 3.
We close this section with a remark on the consistency of this approximating process. Considering linearizations of the jump relations, we get

\[
U_{j-1/2}^- = U_{j-1} + \Delta x dF(U_{j-1})^{-1} G(U_{j-1}) + o(\Delta x),
\]
\[
U_{j+1/2}^+ = U_{j+1} - \Delta x dF(U_{j+1})^{-1} G(U_{j+1}) + o(\Delta x).
\]

Inserting these values in the numerical fluxes, one gets

\[
U_{j+1}^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} \left( F^+ (U_j^n) - F^+ (U_{j-1}^n) + F^- (U_{j+1}^n) - F^- (U_j^n) \right)
\]
\[
+ \Delta t \left( \frac{dF^+ (U_j^n)}{G^+ (U_{j-1}^n)} dF (U_j^n)^{-1} G (U_{j-1}^n) + \frac{dF^- (U_{j+1}^n)}{G^- (U_{j+1}^n)} dF (U_{j+1}^n)^{-1} G (U_{j+1}^n) \right) + o(\Delta x).
\]

Some various one-dimensional numerical tests have been carried out to evaluate the performances of such an algorithm. Since the stiffness of the sources has no influence on the mesh size because of the nonconservative formulation, one can consider a very wide range of problems. One restriction is of course the transonic regimes in which the Jacobian of the fluxes \( dF(U) \) becomes singular. We refer for example to \([15,24]\) for some other computational results.

### 3.3. How to Extend to Two-Dimensional Problems?

At this stage, it is convenient to move back to the very simple case of the scalar 2D advection equation

\[
\begin{align*}
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} &= \alpha u, \quad \text{with } (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\
u(x,y,0) &= u_0 \in C^1(\mathbb{R}^2).
\end{align*}
\]

Its exact solution is obvious: \( u(x,y,t) = u_0(x-at,y-b) e^{\alpha t} \). In order to extend the preceding ideas, the first step is to introduce a smooth function \( K \in C^1(\mathbb{R}^2) \) whose divergence is \( \alpha \) to
rewrite (31) as
\[(\ln |u|)_t + a(\ln |u| - K)_x + b(\ln |u| - K)_y = 0, \quad K_t = 0.\] (32)

The simplest choice is given by \(K(x, y) = (\alpha/2)(x + y)\), but considering the case where \(b\) is very close to zero, one notices that \(K\) should depend on the velocity field \(\vec{V} = (a, b)\). A better choice is, therefore,

\[K(x, y) = \frac{\alpha}{\sigma^2 + b^2} (a^2 x + b^2 y).\] (33)

Assuming we work on a Cartesian grid, the 1D elementary problems which are about to be solved in each direction \(Ox, Oy\) for (31) are of the following type:

\[(\ln |u|)_t + a(\ln |u| - K)_x = 0, \quad K_t = 0\]

and

\[(\ln |u|)_t + b(\ln |u| - K)_y = 0, \quad K_t = 0.\] (34)

Now, considering the nonlinear scalar conservation law [12]

\[u_t + f(u)_x + g(u)_y = h(u), \quad u(x, y, 0) = u_0(x, y) \in (L^\infty \cap BV) (\mathbb{R}^2),\] (35)

we propose to split the source term the following way:

\[h(u) = \frac{h(u)}{f'(u)^2 + g'(u)^2} (f'(u)^2.K_x + g'(u)^2.K_y), \quad \text{with } K(x, y) = x + y.\] (36)

Consequently, on a similar Cartesian grid, the 1D elementary problems rewrite as

\[u_t + f(u)_x = h(u).\frac{f'(u)^2}{f'(u)^2 + g'(u)^2}.K_x, \quad K_t = 0\] (37)

and

\[u_t + g(u)_y = h(u).\frac{g'(u)^2}{f'(u)^2 + g'(u)^2}.K_y, \quad K_t = 0.\]

And we are back in the preceding framework designed for scalar one-dimensional problems (3) and (4). Of course, the jump relations are a bit more intricate in this case because of the new terms one has to take into account in order to follow the propagation directions. The extension to systems is carried out with the same ideas using the nonconservative formulation. The next section is devoted to the study of an example for which these computations can be achieved.

4. EXAMPLE: A 2D SIMPLIFIED TWO-PHASE FLOW MODEL

4.1. The Physical Model

Throughout this section, we will be interested in the numerical approximation of the following two-dimensional system:

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho c)_t + (\rho uc)_x + (\rho uc)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p(\rho, c))_x + (\rho uv)_y &= -k(x, y)\rho u|u|, \\
(\rho v)_t + (\rho vu)_x + (\rho v^2 + p(\rho, c))_y &= 0.
\end{align*}
\] (38)
This kind of problem belongs to the class studied in, e.g., [35,37–39]. It models the flow of two species sharing the same volumetric velocity. In this work, we will mainly consider the case of a very simplified system for a vapor/water mixture as it may be encountered in nuclear reactor cores. Therefore, the unknowns \((\rho, c, u, v) \in \mathbb{R}_+^4 \times [0,1] \times \mathbb{R}^2\) denote, respectively, the density of the mixture, the mass fraction of the vapor, and the two components of the common velocity vector. The global pressure law is usually given by a perfect gas law depending on the vapor fraction only and vanishing when \(c = 0\):

\[
p(\rho, c) = (\rho_{\text{vapor}})^\gamma, \quad \text{with} \quad \rho_{\text{vapor}} = \frac{1.6 \rho c}{1.0 - \rho(1 - c)} \quad \text{and} \quad \gamma = 2.
\]

For technical reasons, one may choose to work with the following one whose expression is easier to handle in a flux-splitting approach and whose graph is quite similar to the original as soon as the vapor fraction isn't too close to zero:

\[
p(\rho, c) = \rho^2 \sqrt{c}.
\]  

(39)

On the right-hand side, the source term models a damping effect due to microscopic obstacles whose characteristic scale is much smaller than the one of the macroscopic flow. The smooth function \(k(x, y) \geq 0\) is used to localize its effects in a particular region of the considered domain. A concrete example is given by very thin grids placed in a square sectioned duct. In this case, their presence will be reflected by a nonzero value of the coefficient \(k\). We will consequently use the following notations:

\[
U = (\rho, \rho c, \rho u, \rho v),
\]

\[
F(U) = (\rho u, \rho v c, \rho u^2 + \rho, \rho u v),
\]

\[
G(U) = (\rho v, \rho u c, \rho u v, \rho v^2 + \rho),
\]

\[
H(U) = (0, 0, -\rho u |u|, 0).
\]

And system (38) rewrites in a condensed form as

\[
U_t + F(U)_x + G(U)_y = k(x, y).H(U).
\]

It is proved in, e.g., [35] that this system is strictly hyperbolic. In order to use the results of the former section, the first step is to introduce an appropriate decomposition of the flux functions \(F\) and \(G\). The forthcoming result is an immediate consequence of [35].

**Theorem 4.** There exists splitted fluxes \(F^\pm, G^\pm\) for system (38),(39) which are differentiable and satisfy the following properties:

\[
\forall U \in (\mathbb{R}_+^4 \times [0,1] \times \mathbb{R}^2), \begin{cases}
\Lambda(\text{d}F^\pm(U)) \subset \mathbb{R}^\pm \text{ and } \Lambda(\text{d}G^\pm(U)) \subset \mathbb{R}^\pm,
F^+(U) + F^-(U) = F(U) \text{ and } G^+(U) + G^-(U) = G(U).
\end{cases}
\]

They are defined by

\[
F^+(U) = \begin{cases}
F(U), & \text{if } u \geq a, \\
\begin{pmatrix}
M^+ \\
M^+ c \\
M^+ (\frac{u}{2} + a) \\
M^+ v
\end{pmatrix}, & \text{if } |u| < a, \text{ and } F^-(U) = \begin{pmatrix}
\frac{M^-}{u} \\
\frac{M^- c}{u} \\
\frac{M^- (\frac{u}{2} - a)}{u} \\
M^- v
\end{pmatrix}, & \text{if } |u| < a,
\end{cases}
\]

\[
G^+(U) = \begin{cases}
G(U), & \text{if } u \geq a, \\
\begin{pmatrix}
N^+ \\
N^+ c \\
N^+ u \\
N^+ (\frac{u}{2} + a)
\end{pmatrix}, & \text{if } |u| < a, \text{ and } G^-(U) = \begin{pmatrix}
\frac{N^-}{u} \\
\frac{N^- c}{u} \\
\frac{N^- u}{u} \\
N^- (\frac{u}{2} - a)
\end{pmatrix}, & \text{if } |u| < a,
\end{cases}
\]

(41)

\[
\text{if } u \leq -a,
\]

\[
\text{if } u \leq -a.
\]
The quantity \( a = \sqrt{\frac{2p(\rho,c)}{\rho}} \) denotes the sound speed in the mixture and \( M^\pm, N^\pm \) are some convenient splittings of the momentum:

\[
M^+ = \frac{\rho}{4a}(u + a)^2, \quad N^+ = \frac{\rho}{4a}(v + a)^2, \\
M^- = \frac{\rho}{4a}(u - a)^2, \quad N^- = \frac{\rho}{4a}(v - a)^2.
\]

### 4.2. Nonconservative Reformulation and Jump Relations

At this level, we introduce a Cartesian discretization which is determined by the space and time steps, respectively, denoted \( \Delta x, \Delta y, \Delta t \). It will be also convenient to introduce the stiffness functions

\[
k^{\Delta x}(x, y) = \int^x k(s, y) \, ds \quad \text{and} \quad k^{\Delta y}(x, y) = \int^y k(x, s) \, ds. \tag{42}
\]

Following the ideas proposed in the preceding section, we split the source term in the following way:

\[
H(U) = \left( \frac{M^2}{M^2 + N^2} + \frac{N^2}{M^2 + N^2} \right) H(U),
\]

where \( M = \rho u \) and \( N = \rho v \). In order to construct a flux splitting scheme, we are about to solve in each cell one-dimensional Riemann problems for the following two systems:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho c)_t + (\rho uc)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x + \left[ \rho u^2 \left( \frac{M^2}{M^2 + N^2} + \frac{N^2}{M^2 + N^2} \right) \right]_{\Phi} &= 0, \\
(\rho v)_t + (\rho vu)_x &= 0 \quad \text{and}
\end{align*}
\]

and

\[
\begin{align*}
\rho_t + (\rho v)_y &= 0, \\
(\rho c)_t + (\rho uc)_y &= 0, \\
(\rho u)_t + (\rho vu)_y + \left[ \rho u^2 \left( \frac{N^2}{M^2 + N^2} \right) \right]_{\Phi} &= 0, \\
(\rho v)_t + (\rho v^2 + p)_y &= 0. \tag{44}
\end{align*}
\]

![Figure 4. Duct geometry for the first numerical test.](image)
Of course, we keep on using the same regularizing path (15) to handle the nonconservative terms. As any two-dimensional Godunov-type scheme does, this splitting is likely to upset the balance between the fluxes in each $x, y$ direction. On the other hand, it is commonly used and the numerical results (see, e.g., Figures 5 and 9) suggest that this error remains low compared with the one coming from the centered discretization of the right-hand side.

To derive the states on each side of the cell interfaces which are to be used in the numerical fluxes, we need the steady-state jump relations.

**Lemma 3.** For piecewise constant steady-states, the following jump relations hold.
For system (43): $[\rho u] = [c] = [v] = 0$ and
\[
-M^2 \ln(\rho) + [p_1] - \frac{v^2}{2} [\rho^2] + \frac{v^2}{M^2} [p_2] = -M |M| [k^{\Delta y}] .
\] (45)

For system (44): $[\rho v] = [c] = [v] = 0$ and
\[
v \left[ \frac{1}{u} \right] - \frac{1}{v} [u] = \text{sgn}(u) [k^{\Delta y}],
\] (46)

where $p_1, p_2$ denote, respectively, antiderivatives of $\rho \to \rho \frac{\partial}{\partial \rho}$ and $\rho \to \rho^3 \frac{\partial}{\partial \rho}$ with $c$ kept constant.
PROOF.

(i) System (43):

\[
\left( \frac{M^2}{\rho} + p(\rho, c_0) \right)_x = -\frac{M^2}{\rho^2} \rho_x + p \rho_x = -M|\lambda| \frac{M^2}{\rho (M^2 + \rho^2 v_0^2)} (k^{\Delta x})_x .
\]

In the case where \( M \neq 0 \),

\[
\left( -\frac{M^2}{\rho^2} \rho_x + p_0 \rho_x \right) \left( \rho + \rho^3 \frac{v_0^2}{M^2} \right) = -M|\lambda| (k^{\Delta x})_x .
\]

And

\[
-M^2 \rho_x + (\rho p) \rho_x - v_0^2 \left( \frac{P^2}{2} \right)_x + \frac{v_0^2}{M^2} \rho_3 p \rho_x = -M|\lambda| (k^{\Delta x})_x .
\]

We introduce: \( p_1(\rho, c_0) \) such that \( (p_1)_\rho = \rho p_\rho \) and \( p_2(\rho, c_0) \) such that \( (p_2)_\rho = \rho^3 p_\rho \).

Integrating with respect to \( x \), one gets (45).

(ii) System (44): we have \( [\rho v] = [c] = 0 \). The conservation of \( Q = \rho v^2 + p \) implies \( [v] = [\rho] = 0 \) and

\[
v_y u_y = -u|\rho| \frac{v_0^2}{u^2 + v^2} (k^{\Delta y})_y .
\]

Multiplying by \( (1 + u^2/v^2) \) and \( 1/|u| \), one gets:

\[
v_y \frac{u_y}{|u|} \left( 1 + \frac{u^2}{v^2} \right) = \text{sgn}(u) \left( v \left( \frac{1}{u} \right)_y - \left( \frac{1}{v} \right)_y \right) = (k^{\Delta y})_y .
\]

Since \( v = v_0 \), integrating with respect to \( y \) yields relation (46).

If we assume that the initial data is discretized in the following classical way:

\[
U^0_{i,j} = \frac{1}{\Delta x \Delta y} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} U^0(x, y) \, dx \, dy,
\]

we are ready to write down a WB numerical scheme for (38) and (39):

\[
\begin{align*}
U_{i,j}^{n+1} &= U_{i,j}^n - \frac{\Delta t}{\Delta x} \left\{ F^+ \left( U_{i,j}^n \right) - F^+ \left( U_{i-1/2,j}^n \right) + F^- \left( U_{i+1/2,j}^n \right) - F^- \left( U_{i,j}^n \right) \right\} \\
&\quad - \frac{\Delta t}{\Delta y} \left\{ G^+ \left( U_{i,j}^n \right) - G^+ \left( U_{i,j-1/2}^n \right) + G^- \left( U_{i,j+1/2}^n \right) - G^- \left( U_{i,j}^n \right) \right\} .
\end{align*}
\]
As in the one-dimensional case (29), the modified states $U_{i\pm1/2,j}^\pm$ and $U_{i,j\pm1/2}^\pm$ are prescribed by the jump relations (45) and (46). These nonlinear equations may be solved for example by means of a Newton iterative algorithm.

4.3. Numerical Results

We are going to display numerical computations achieved for problem (38) in some industrially relevant situations. As a first example, we consider a duct as in Figure 4. We simulate its walls
We present the steady values obtained around time $t \approx 10$ for the axial velocity and the transverse velocity on Figures 6 and 7. The parameters used for this run are $\Delta x = 0.02$, $\Delta y = 0.02$, $\Delta t = 0.003$. We define $U^n = (U^n_{i,j})_{i,j}$ with $U^n_{i,j} = (\rho^n_{i,j}, (\rho e)^n_{i,j}, (pu)^n_{i,j}, (pv)^n_{i,j})$ the vector of the conservative variables at time $n \Delta t$. We compared the decay of the residues for the WB scheme (47) with the one coming out of a classical flux-splitting approach with a centered discretization of the source terms (compare with (30)): 

$$
U^{n+1}_{i,j} = U^n_{i,j} - \frac{\Delta t}{\Delta x} \left\{ F^+ (U^n_{i,j}) - F^+ (U^n_{i-1,j}) + F^- (U^n_{i+1,j}) - F^- (U^n_{i,j}) \right\}
$$

$$
- \frac{\Delta t}{\Delta y} \left\{ G^+ (U^n_{i,j}) - G^+ (U^n_{i,j-1}) + G^- (U^n_{i,j+1}) - G^- (U^n_{i,j}) \right\}
$$

$$
\Delta t \cdot k(i \Delta x, j \Delta y) \cdot \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
$$

In Figure 5, we display also the history of the residues in the infinity norm $\|U^{n+1} - U^n\|_\infty$ for this test case on two different grids: a coarse one ($\Delta x = \Delta y = 0.02$) and a finer one ($\Delta x = \Delta y = 0.01$). This comparison shows clearly the advantages of this nonconservative algorithm.

The second numerical run consists in solving system (38) in a situation where the source terms are confined in some small regions of the computational domain. The physical motivation may be the study of the effects of very thin and localized grids onto the whole flow (see Figure 8). We choose $k(x, y) = 20 \cdot \mathbb{1}_{x-0.6y<0.15} + 50 \cdot \mathbb{1}_{x-1.2y>0.15}$.

The initial and left boundary data are those corresponding to a subsonic uniform flow 

$$
\rho_0 = 2,
$$

$$
\rho_0 = 0.5,
$$

$$
u_0 = 0.5,
$$

$$
\nu_0 = 0
$$

and 

$$ c_{\text{left}} = 0.5, $$

$$ (pu)_{\text{left}} = 1, $$

$$ \nu_{\text{left}} = 0. $$

\[1\text{We do not claim anything concerning singular source terms. In the present case, this choice for } k \text{ corresponds to a possible discretization of a smooth but very localized sink term on a coarse computational grid. We refer to, e.g., [40] concerning an analysis of nonconservative numerical schemes.} \]
We display the values at numerical steady-state for the pressure and the axial velocity for both schemes on different grids on Figures 10-13. Looking at Figures 11 and 12 one notices that the classical approach does not capture accurately the pressure jumps, even with a fine computational grid. This brings spurious oscillations on the axial velocity in these regions of the flow. Conversely, the axial velocity computed by the WB scheme is free from any oscillations. As before, we also compare the residues decay in the infinity norm with the one obtained by means of a classical approach with a centered source terms discretization: see Figure 9. We used two sets of parameters with both numerical schemes: $\Delta x = 0.04$, $\Delta y = 0.02$, $\Delta t = 0.007$ (coarse grid) and $\Delta x = 0.02$, $\Delta y = 0.01$, $\Delta t = 0.003$ (fine grid). Once again, it is clear that the proposed nonconservative treatment of the sources is better suited for this kind of delicate computations.
5. CONCLUSION

We proposed in this paper a new way to process source terms for hyperbolic systems of balance laws in one or two space dimensions. It mainly relies on a reformulation of these zero-order terms as a vector of nonconservative products which are regularized by integral curves of the steady state equations. This provides a way to solve homogeneous Riemann problems and a Godunov type scheme can be deduced. In order to avoid intricate computations of elementary solutions in each computational cell, a simpler flux-splitting technique has been developed. Numerical results reveal practical evidence of the nice behaviour of this kind of approach.
APPENDIX

NONCONSERVATIVE PRODUCTS AND LOCALLY LIPSCHITZIAN PATHS

The aim of the theory recalled here is to give a precise mathematical sense to distributions products $A(W)W$ where $W \in [BV(\mathbb{R})]^N$ and $W \mapsto A(W)$ is a smooth locally bounded map. After the work performed by Colombeau and LeRoux [21], DalMaso, LeFloch and Murat [22] proposed an interpretation of such ambiguous terms using a family of paths drawn in the phases space $\Omega \subset \mathbb{R}^N$. The equivalence between these two concepts has been shown in [41]. See also the recent work of LeFloch and Tzavaras [23].

**DEFINITION 2.** A family of paths $\Phi$ in $\Omega \subset \mathbb{R}^N$ is a smooth map $[0,1] \times \Omega \rightarrow \Omega$ satisfying

- $\Phi(0,W_L,W_R) = W_L$ and $\Phi(1,W_L,W_R) = W_R$,
- $\forall \mathcal{V}$ bounded in $\Omega$, $\exists k$ such that $\forall s \in [0,1], \forall (V_L, V_R) \in \mathcal{V}$
  \[
  \left| \frac{\partial \Phi}{\partial s} (s; V_L, V_R) \right| \leq k |V_L - V_R|,
  \]
- $\forall \mathcal{V}$ bounded in $\Omega$, $\exists K$ such that $\forall s \in [0,1], \forall (V_L', V_R') \in \mathcal{V}$
  \[
  \left| \frac{\partial \Phi}{\partial s} (s; V_L', V_R') - \frac{\partial \Phi}{\partial s} (s; V_L^2, V_R^2) \right| \leq K \left( |V_L^1 - V_L^2| + |V_R^1 - V_R^2| \right).
  \]

We recall now a fundamental result from [22].

**THEOREM 5.** (See [22].) Let $W \in BV([a,b], \mathbb{R}^N)$ and $A : \mathbb{R}^N \times [a,b] \rightarrow \mathbb{R}$ a locally bounded integrable function, i.e.,

- $\forall X \in \mathbb{R}^N$ bounded, $\exists C > 0$ such that $\forall W \in X$, $\forall x \in [a,b]$, $|A(W,x)| \leq C$.

There exists a unique Borel measure $\mu$ on $[a,b]$ characterized by the following properties.
- If $x \mapsto W(x)$ is continuous on an open set $B \subset [a,b]$
  \[
  \mu(B) = \int_B A(W,x) \frac{\partial W}{\partial x} = \int_B A(W(x),x) \frac{\partial W(x)}{\partial x} \, dx.
  \]
- If $x_0 \in [a,b]$ is a discontinuity point of $x \mapsto W(x)$, then
  \[
  \mu(x_0) = \left\{ \int_0^1 A\left( \Phi \left(s; W(x_0^-), W(x_0^+)\right), x_0 \right) \frac{\partial \Phi}{\partial s} \left(s; W(x_0^-), W(x_0^+)\right) \, ds \right\} \delta(x_0),
  \]
  where $\delta(x_0)$ denotes the Dirac mass at the point $x_0$.

The Borel measure $\mu$ is called nonconservative (NC) product and is usually written $[A(W)W]_{\Phi}$. The authors of [22] found again the classical results of the usual theory for conservative strictly hyperbolic systems, especially the structure of the Riemann problem for systems written in nonconservative form which is still composed of $(N+1)$ constant states separated by $N$ simple waves. These are on one hand shocks or rarefaction waves if the field is genuinely nonlinear (GNL), on the other one, contact discontinuities if the field is linearly degenerate (LD). The main difference comes from the fact that everything which deals with discontinuities ($W^-, W^+$) depends explicitly on the path $\Phi$ through the generalized Rankine-Hugoniot relations

\[
\int_0^1 (\sigma. Id - A\left( \Phi \left(s; W^-, W^+\right)\right)) \frac{\partial \Phi}{\partial s} \left(s; W^-, W^+\right) \, ds = 0,
\]

where $\sigma$ denotes the speed of the singularity in the $(x,t)$ plane.
REFERENCES


